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Two methods for optimal investment with trading strategies of finite variation*

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Two methods for designing optimal portfolios are proposed. In order to reduce the variation in the asset holdings and hence the eventual proportional transaction costs, the trading strategies of these portfolios are constrained to be of a finite variation. The first method minimizes an upper bound on the discrete-time logarithmic error between a reference portfolio and the one with a constrained trading strategy, and thus penalizes the shortfall only. A quadratic penalty on the logarithmic variation of the trading strategy is also included in the objective functional. The second method minimizes a sum of the discrete-time log-quadratic errors between the asset holding values of the constrained portfolio and a certain reference portfolio, which results in tracking the reference portfolio. The optimal trading strategy is obtained in an explicit closed form for both methods. Simulation examples with the log-optimal and the Black-Scholes replicating portfolios as references, show smoother trading strategies for the new portfolios and a significant reduction in the eventual proportional transaction cost. The performance of the new portfolios are very close to their references in both cases.

Keywords:
Single-period dynamic optimization, differentiable trading strategies, eventual proportional transaction cost.

1. Introduction

One of the main problems of mathematical finance is achieving some pre-specified objective via the portfolio selection, i. e. the trading of the assets. An example of such an objective is the optimal wealth growth for the investor. The best criterion to maximize in this case is the logarithm of the terminal wealth, and the obtained portfolio is termed *log-optimal*; see Merton (1969), Korn (1997), Luenberger (1998). Another example of the objective is the replication of a contingent claim; see Black & Scholes (1973), Merton (1973), Wilmott (1998), Bingham & Kiesel (1999). The minimal initial wealth required to achieve such a replication represents the *price* of the claim and the replicating portfolio is said to *hedge* the claim.
Many of the known portfolio selection methods, including the above examples, assume an idealized market where there is no transaction cost. This simplifies greatly the analysis and in special cases gives explicit solutions. But when applied in practice, transaction costs are always incurred and such trading strategies may lead to a very large transaction cost.
There are three main existing approaches to dealing with the problem of transaction cost. The first approach includes the transaction cost explicitly in the model; see Davis & Norman (1990), Davis *et al.* (1993), Shreve & Soner (1994), Wilmott (1998), Korn (1997), and the references therein. Replication

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portfolio strategies for hedging under proportional transaction costs have been discussed in Lai & Lim (2009). When the costs are small, perturbation theory has been employed in Mokkhavesa & Atkinson (2002) to derive a solution to the portfolio selection problem for a broad class of utility functions for a single risky asset case. In all these cases, the trading strategy allows allows infinite variation. The second approach also includes the transaction cost explicitly in the model and further uses a trading strategy of a finite variation; see Kabanov (1999), Kabanov & Last (2002). The third approach is developed in the context of option pricing, and is the most recent one; see Brodie *et al.* (1998), Soner & Touzi (2000), Soner (2007), Cheridito *et al.* (2005). It assumes that there is no transaction cost in the market, and introduces a constraint on the diffusion coefficient of the trading strategy. If the number of shares held for the asset i is denoted by $v_i(t)$ and its corresponding equation is given as

$$dv_i(t) = v_i^{(1)}(\cdot)dt + v_i^{(2)}(\cdot)dW_i, \tag{1.1}$$

where $W_i(t)$ is a standard brownian motion, then this approach imposes the constraint $\Gamma_*(\cdot) \leqslant v_i^{(2)} \leqslant \Gamma^*(\cdot)$. Here the bounds $\Gamma_*(\cdot)$ and $\Gamma^*(\cdot)$ can depend on the stock price and are user-specified. We interpret the introduction of this constraint as an attempt to reduce the variability of the trading strategy, and thus the eventual transaction cost. A similar approach with purely deterministic volatility function is followed in Company *et al.* (2010), where the authors solve a non-linear Black-Scholes equation numerically.

In this paper we propose an alternative approach to dealing with the problem of proportional transaction cost. We also assume, as in the third approach above, that there is no transaction cost in the market, but constrain the trading strategy to be of a *finite variation*. This means that we constrain the diffusion term $v_i^{(2)}$ to be identically zero almost surely. Such a trading strategy also has a finite first variation. Thus, this class of a trading strategy can be seen as a subset of the trading strategies of both the second and the third approach above. We further constrain the trading strategy to be positive in this paper.

An additional element of our approach is to use criteria that penalize the logarithmic variation of the trading strategy. The proportional transaction cost is proportional to the variation of the asset holdings, and thus penalizing a "proxy" of such a variation gives the investor the means for trade off between a higher profit and a lower eventual proportional transaction cost. It is the combination of a differentiable trading strategy and the penalization of the variation that leads to a significant reduction in the eventual proportional transaction cost. This can be seen as an *implicit* approach to dealing with the problem of proportional transaction cost. In this respect, it is similar to the method of Gamma constraints as mentioned above.

Various different objectives to be achieved can be imposed on the constrained portfolio. In this paper, we propose to use an already designed portfolio as a reference to our constrained portfolio. This is done with an expectation that the constrained portfolio will behave very closely or perhaps outperform such a reference portfolio, and will have a lower eventual proportional transaction cost. In order to obtain explicit and closed-form results and to permit a broad class of market and reference models, the proposed optimization is in discrete time and over a single period.

There are several distinct advantages to this approach in comparison with the existing approaches. For example, the method of Davis & Norman (1990), Akian *et al.* (1995), can only be applied to 2 or 3 risky assets since the computational effort for a large number of assets is prohibitively high. In our methods, the solutions are in an explicit closed form and are easily implemented. Another prominent class of methods of using Gamma constraints as advocated in Brodie *et al.* (1998), Soner & Touzi (2000), Soner (2007), Cheridito *et al.* (2005), often leads to a higher initial option price than the Black Scholes price, whereas our method is demonstrated to give a lower eventual proportional transaction costs for the exact initial (replication) price.

We propose two new criteria for the optimal investment, which result in two different methods. In the first method, an upper bound on the discrete-time logarithmic error is minimized, which penalizes only the shortfall with respect to the reference portfolio. This is achieved by first making its variance zero, and then minimizing its mean. The optimal trading strategy is derived in an explicit closed form. A simulation example for the log-optimal portfolio as a reference shows a significant reduction in the eventual proportional transaction cost. In the second method, a sum of discrete-time log-quadratic errors between asset holding values of the constrained portfolio and the reference portfolio is minimized. This means that we view each asset holding value as a reference and try to track it, rather than tracking the portfolio value. Thus, this kind of a portfolio can be seen as a *replicating* portfolio. The optimal trading strategy is derived in an explicit closed form for this case as well. A modified version of this method is used to track the Black-Scholes replicating portfolio. A simulation example shows that the performance of the new portfolio is very close to that of the Black-Scholes replicating portfolio, while having an identical initial value and a significantly lower eventual proportional transaction cost.

The rest of the paper is organized as follows. In Sec. 2 a general model of asset prices is used to derive the dynamics of a self-financing portfolio with a positive differentiable trading strategy. Such a model is linear in control variables, which in this case are the logarithmic variations of the trading strategies. There are also no explicit constraints on either the state or the control variables. The first method for designing optimal portfolios is presented in Sec. 3. Here an upper bound of the discrete-time logarithmic error between the reference and the constrained portfolios is derived, and used as a criterion for optimal investment. The optimization task is formulated as a control problem with an additional quadratic penalty on the controls, and solved in an explicit closed form. A simulation example illustrates a significant reduction in the eventual proportional transaction cost as compared to the log-optimal portfolio. In Sec. 4 the second method for optimal investment is proposed. The optimality criterion is a sum of the discrete-time log-quadratic errors of the asset holding values. The dynamics of such errors are derived for the general references. The control problem also includes a quadratic control penalty and is solved in an explicit closed form. The optimal trading strategy can then be obtained from such controls. A modified version of this approach that allows borrowing is used to track the Black-Scholes replicating portfolio for a European Call option. Simulation results show that for almost the same performance of wealth and an identical initial value, the new portfolio has a lower eventual proportional transaction cost.

2. Market model and the portfolio with a trading strategy of a finite variation.

We study a market consisting of a single risk-free asset $S_0(t)$, and n risky assets $S_i(t)$, $i = 1,2,...,n$, the prices of which are given in the following form; see, e. g. Björk (2004):

$$
\begin{aligned}
dS_0(t) &= r(t)S_0(t)dt, && (2.1)\\
dS_i(t) &= S_i(t)\left[\mu_i(t,S(t))dt + \sum_{j=1}^{m} \sigma_{ij}(t,S(t))dW_j(t)\right] &&\\
&= S_i(t)[\mu_i(t,S(t))dt + \sigma_i(t,S(t))dW] && (2.2)
\end{aligned}
$$

where $S(t) = [S_1(t),...,S_n(t)]'$, $S_i(0) > 0$, $i = 0,1,...,n$. The risk-free interest rate $r(t)$ is a continuous and deterministic function of time[1], the drift $\mu_i(t,S)$ and the volatility $\sigma_{ij}(t,S)$ are assumed to satisfy the conditions that ensure the strict positivity of asset prices; see, e. g. Cvitanić & Ma (1996). The

[1]The results of this paper will not change even if $r(t) = r(t,S_0,S)$.

volatility matrix $\sigma(t,S)$ is of order $(n\times m)$ and has vectors $\sigma_i(t,S)$ as rows. We do *not* assume that the matrix $\sigma(t,S)\sigma'(t,S)$ is positive definite, an assumption encountered in all of previous work on transaction cost; see Korn (1997) and the references therein. The uncertainty is due to a m-dimensional standard Brownian motion $W(t)$. For simplicity of notation, we shall frequently write μ_i and σ_{ij} rather that indicating explicitly their dependencies on the time and asset prices. Equations for asset price logarithms $\ln[S_i(t)]$ are found by applying Ito's lemma to (2.1) and (2.2) to obtain

$$\begin{aligned} d\ln[S_0(t)] &= r(t)dt, && (2.3)\\ d\ln[S_i(t)] &= [\mu_i(t,S) - (1/2)\sigma_i(t,S)\sigma_i'(t,S)]dt + \sigma_i(t,S)dW(t). && (2.4) \end{aligned}$$

The *trading strategy* is defined as an adapted real-valued process $[v_0(t),\dots,v_n(t)]'$, that satisfies the standard integrability conditions; see e. g.(Bingham & Kiesel (1999), Sec. 6.1). Here $v_i(t)$ denotes the number of shares of asset i held by the investor. The portfolio value (investors total wealth) $y(t)$ is given by

$$y(t) = \sum_{i=0}^{n} v_i(t)S_i(t) = \sum_{i=0}^{n} y_i(t). \tag{2.5}$$

Here $y_i(t)$, $i = 0,1,...,n$, denotes the value of the holdings per asset. A portfolio is self-financing if the change in its value occurs only due to price changes, and is described by

$$dy(t) = \sum_{i=0}^{n} v_i(t)dS_i(t) \tag{2.6}$$

We shall constrain the trading strategy to be positive and thus make the following assumption.
Assumption A1. The borrowing and the short-selling is not permitted, i. e. $v_i(t) > 0$, a.s, $i = 0,1,2,...,n$.

Most of the known portfolio selection methods give a trading strategy of infinite first variation due to the dW term in their equation; see, e. g. (Bingham & Kiesel (1999), Sec. 5. 3. 2). When applied to a real-world situation, where there is always some transaction cost, a discrete-time approximations of such a strategy may lead to a very large eventual transaction cost. Hence we constrain the trading strategy further to be differentiable and thus of finite variation as follows.
Assumption A2. The elements of the trading strategy $v(t) = [v_0(t),\dots,v_n(t)]'$ are differentiable and defined as

$$d\ln[v_i(t)] = u_i(\cdot)dt, \tag{2.7}$$

where $i = 0,1,2,...,n$, and the scalars $u_i(\cdot)$ are adapted and continuous functions.

We next develop the continuous-time and discrete-time models of the self-financing portfolio with a trading strategy that satisfies the above assumptions.

LEMMA 2.1 A portfolio is self-financing under assumption (**A2**) if

$$\sum_{i=0}^{n} y_i(t)u_i(t)dt = 0. \tag{2.8}$$

Proof. Applying Ito's lemma to (2.5) under assumption (**A2**) one obtains

$$dy(t) = \sum_{i=0}^{n} v_i(t)dS_i(t) + \sum_{i=0}^{n} S_i(t)dv_i(t). \tag{2.9}$$

The equations (2.7) give $dv_i(t) = v_i(t)u_i(t)dt$, and by comparing (2.9) to the self-financing equation (2.6), we obtain

$$\sum_{i=0}^{n} S_i dv_i = \sum_{i=0}^{n} S_i(t)v_i(t)u_i(t)dt = \sum_{i=0}^{n} y_i(t)u_i(t)dt = 0 \tag{2.10}$$

□

LEMMA 2.2 Let $x_i(t) = \ln[y_i(t)]$, $i = 0,1,2...,n$. For a self-financing portfolio, under the assumptions (**A1**) and (**A2**), the following holds

$$dx_0(t) = -\sum_{i=1}^{n} e^{x_i(t)-x_0(t)} u_i(t)dt + rdt \tag{2.11}$$

$$dx_i(t) = [u_i(t) + \mu_i(t) - (1/2)\sigma_i(t)\sigma_i'(t)]dt + \sigma_i(t)dW(t) \tag{2.12}$$

Proof. First consider the case when $i = 1,2,...,n$. Taking the logarithm of $y_i(t) = v_i(t)S_i(t)$, which is allowed due to the assumption (**A1**), we obtain $\ln[y_i(t)] = \ln[v_i(t)] + \ln[S_i(t)]$. Its differential is $d\ln[y_i(t)] = d\ln[v_i(t)] + d\ln[S_i(t)]$, which after substituting (2.7) and (2.4) gives equations (2.12). Similarly we obtain the dynamics of $\ln[y_0(t)]$ as

$$d\ln[y_0(t)] = d\ln[v_0(t)] + d\ln[S_0(t)] = -\sum_{i=1}^{n} e^{\ln[y_i(t)]-\ln[y_0(t)]} u_i(t)dt + rdt,$$

where we have used the self-financing constraint (2.8) in the form

$$d\ln[v_0(t)] = -\sum_{i=1}^{n} e^{\ln[y_i(t)]-\ln[y_0(t)]} u_i(t)dt. \tag{2.13}$$

□

REMARK 2.1 Equations (2.11) and (2.12) represent the continuous-time state-space model of a self-financing portfolio with a positive differentiable trading strategy[2]. Note that there are no explicit constraints on the state variables $x_i(t)$, $i = 0,1,...,n$, or on the control variables $u_i(t)$, $i = 1,2,...,n$.

Using the Euler approximation[3] with a sufficiently small sampling time T, we obtain the discrete-time form of (2.11) and (2.12) as

$$x_0(k+1) = x_0(k) - \sum_{i=1}^{n} e^{x_i(k)-x_0(k)} u_i(k)T + rT, \tag{2.14}$$

$$x_i(k+1) = x_i(k) + [u_i(k) + \mu_i(k) - (1/2)\sigma_i(k)\sigma_i'(k)]T + \sigma_i(k)e(k)\sqrt{T}, \tag{2.15}$$

where $e(k) = [e_1(k),...,e_m(k)]'$ is a vector of zero mean, unit variance, i. i. d. Gaussian random variables for each k. The model (2.14) and (2.15) can be written in the following more convenient matrix form

$$x(k+1) = x(k) + A(k,x(k))u(k)T + D(k)T + \Sigma e(k)\sqrt{T} \tag{2.16}$$

[2]In general, part of this model are also the risky asset price dynamics (2.2).

[3]Similarly one can use other forward approximation schemes; see, e. g. Kloden & Platen (1992).

where $x(k) = [x_0(k), ..., x_n(k)]'$, $u(k) = [u_1(k), ..., u_n(k)]'$. Denoting by $\beta(x,k) = [-e^{x_1(k)-x_0(k)}, ..., -e^{x_n(k)-x_0(k)}]$, $\underline{0}_m$ an $1 \times m$-vector of zeros, I_n the n-th order identity matrix, we can express the matrices in (2.16) as follows

$$\begin{aligned} A(k,x(k)) &= \begin{bmatrix} \beta(x,k) \\ I_n \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \underline{0}_m \\ \sigma \end{bmatrix}, \\ D(k) &= [\ r(k), \quad \mu_1 - (1/2)\sigma_1(k)\sigma_1'(k), \quad \cdots \quad, \mu_n - (1/2)\sigma_n(k)\sigma_n'(k)\ ]'. \end{aligned} \tag{2.17}$$

Equation (2.16) will be used in the following sections as a model of a self-financing portfolio with positive trading strategies of finite variation.

REMARK 2.2 Note that the matrix $A(k,x(k))$ is of a full rank, and thus its columns are linearly independent. This means that for a positive definite matrix R, the matrix quadratic form $A'RA$ will always be positive definite. This fact will be used later in the proof of Theorem 4.1.

The total wealth of the constrained portfolio $y(k+1)$ can be easily obtained from (2.14) and (2.15) as

$$y(k+1) = \sum_{i=0}^{n} e^{x_i(k+1)} \tag{2.18}$$

3. The first method: an upper bound on the log-error.

The aim to be achieved with the constrained portfolio of the previous section is to either track closely or outperform some already designed reference portfolio that has a positive trading strategy; see, e. g. Cvitanić & Karatzas (1992), Cvitanić & Karatzas (1993), Korn (1997), Karatzas & Shreve (1991), and the references therein, for possible examples of such reference portfolios. This is done with the aim of obtaining a lower eventual proportional transaction cost due to the differentiable trading strategy of the constrained portfolio. In this section the criterion of optimality will be an upper bound on the *discrete-time logarithmic error* $e_l(k+1)$ between the two portfolios

$$e_l(k+1) = \ln[y^r(k+1)] - \ln[y(k+1)]. \tag{3.1}$$

Here $y^r(k+1)$ is the value of the self-financing reference portfolio at time $T(k+1)$ with its logarithm defined below, and $y(k+1)$ is the value of the constrained tracking portfolio (2.18). Minimization of this criterion penalizes only the shortfall of the constrained portfolio with respect to the reference portfolio. As we see later, by working with an upper bound to $e_l(k+1)$ rather than $e_l(k+1)$ itself, simple explicit closed-form solutions are obtained.

Denoting for the reference portfolio $v_i^r(t)$-the number of shares held for the asset i, $\alpha_i^r(t)$-the fraction of the wealth allocated to asset i, and for which it holds $\alpha_0^r + \alpha_1^r + ... + \alpha_n^r = 1$, we can derive the dynamics of its value $y^r(t)$ as

$$\begin{aligned}
dy^r(t) &= \sum_{i=0}^{n} v_i^r(t) dS_i(t) = v_0^r r S_0 dt + \sum_{i=1}^{n} v_i^r S_i(\mu_i dt + \sigma_i dW) \\
&= y^r \left[\alpha_0^r r dt + \sum_{i=1}^{n} \alpha_i^r (\mu_i dt + \sigma_i dW) \right] \\
&= y^r \left[\left(1 - \sum_{i=1}^{n} \alpha_i^r \right) r dt + \sum_{i=1}^{n} \alpha_i^r (\mu_i dt + \sigma_i dW) \right] \\
&= y^r \left[r dt + \sum_{i=1}^{n} \alpha_i^r (\mu_i - r) dt + \sum_{i=1}^{n} \alpha_i^r \sigma_i dW \right]. \qquad (3.2)
\end{aligned}$$

Denoting by $\alpha = [\alpha_1^r, ..., \alpha_n^r]'$, $M = [\mu_1 - r, ..., \mu_n - r]'$, and applying Ito's lemma to (3.2), we obtain the equations for $\ln[y^r(t)]$ and $\ln[y^r(k+1)]$ as

$$\begin{aligned}
d\ln[y^r(t)] &= [r + \alpha' M - 0.5\alpha'\sigma\sigma'\alpha]dt + \alpha'\sigma dW, \\
\ln[y^r(k+1)] &= \ln[y_r(k)] + [r + \alpha' M - 0.5\alpha'\sigma\sigma'\alpha]T + \alpha'\sigma e(k)\sqrt{T} \qquad (3.3)
\end{aligned}$$

An upper bound on $e_l(k+1)$ can be found using Jensen's inequality (see e. g. Roberts & Varberg (1973)), as follows. Let $\gamma_i(k+1)$, $i = 0, 1, ..., n$, be variables such that $0 \leqslant \gamma_i(k+1) \leqslant 1$, and $\gamma_0(k+1) + ... + \gamma_n(k+1) = 1$. Then the Jensen's inequality gives the following for each k

$$\ln[y(k+1)] \geqslant \sum_{i=0}^{n} \gamma_i(k+1) \ln[y_i(k+1)]. \qquad (3.4)$$

An upper bound on the logarithmic error $e_u(k+1) \geqslant e_l(k+1)$ can thus be expressed as

$$e_u(k+1) = \ln y^r(k+1) - \sum_{i=0}^{n} \gamma_i(k+1) \ln[y_i(k+1)] = \ln y^r(k+1) - \sum_{i=0}^{n} \gamma_i(k+1) x_i(k+1) \qquad (3.5)$$

The aim now is to minimize this upper error bound. One possibility is to minimize its mean and variance. We do so by first selecting the variables $\gamma_i(k+1)$, $i = 0, 1, ..., n$, such that the variance of $e_u(k+1)$ is zero, and then minimize its mean. We stress that this is only one of many possibilities and it is also supported by simulation results.

THEOREM 3.1 For $k = 0, 1, ...$, the variance of $Var[e_u(k+1)|x(k)]$ is equal to zero if

$$\gamma_i(k+1) = \alpha_i^r(k) \qquad (3.6)$$

for each $i = 0, 1, ..., n$.

Proof. Substituting $\ln[y^r(k+1)]$ from (3.3) in (3.5), together with $x_0(k+1)$ and $x_i(k+1)$, $i = 1, 2, ..., n$, from (2.14) and (2.15), and taking the variance, we obtain

$$Var[e_u(k+1)|x(k)] = \sum_{j=1}^{m} \left\{ \sum_{i=1}^{n} [\alpha_i^r(k) - \gamma_i(k+1)] \sigma_{ij} \right\}^2 T. \qquad (3.7)$$

It is clear that a sufficient condition for (3.7) to be equal to zero is for (3.6) to hold. □

LEMMA 3.1 If the volatility matrix $\sigma(t)$ is square and non-singular, then the condition (3.6) is also necessary for $Var[e_u(k+1)|x(k)] = 0$.

Proof. The necessary conditions for (3.7) to be zero are

$$\sum_{i=1}^{n} [\alpha_i^r(k) - \gamma_i(k+1)]\,\sigma_{ij} = 0 \tag{3.8}$$

for every $j = 1,2,...,n$. This system of equations can also be written as

$$\sigma' N = 0, \tag{3.9}$$

where $N = [\alpha_1^r(k) - \gamma_1(k+1), \ldots, \alpha_n^r(k) - \gamma_n(k+1)]'$. Since the volatility matrix σ is assumed square and nonsingular, then the system of equations (3.9) has a unique solution given by $N = 0$, which gives (3.6) for each k. $\square$

The expected value of $e_u(k+1)$ will be minimized if we maximize the following

$$\mathbb{E}\{\sum_{i=0}^{n} \gamma_i(k+1)\ln[y_i(k+1)]|x(k)\} = \mathbb{E}\{\sum_{i=0}^{n} \alpha_i^r(k) x_i(k+1)|x(k)\} = \mathbb{E}[\alpha'(k)x(k+1)|x(k)], \tag{3.10}$$

where we have substituted the relations (3.6).

3.1 *Optimal trading strategies.*

In order to give the investor the means for trade off between a lower eventual transaction cost and a higher profit, and have a well defined optimization problem, we extend the criterion (3.10) to include a quadratic penalty on the logarithmic rates of change of trading strategies $u_i(k)$. The penalty matrix for controls in general can be a function of the state $x(k)$ to reflect a wealth dependent penalization. The resulting optimization problem can be stated as follows.

Portfolio control problem I (PCP-I). *Let $B(k,x(k)) \in \mathbb{R}^{n\times n}$ be a given positive definite matrix, possibly state dependent. Find the control $u(k)$, $k = 0,1,...$, that minimizes the following objective*

$$V^{(1)}(k,x(k),u(k)) = \mathbb{E}\left[\frac{1}{2}u'(k)B(k,x(k))u(k)T - \alpha'(k)x(k+1)|x(k)\right], \tag{3.11}$$

where $x(k+1)$ is given in (2.16) and $\alpha(k) = [\alpha_1^r(k),...,\alpha_n^r(k)]'$.

THEOREM 3.2 The solution to the PCP-I always exists, is unique and for every $k = 0,1,2,...$, is given by

$$u^*(k) = B^{-1}(k,x(k))A'(k,x(k))\alpha(k), \tag{3.12}$$

where $A(k,x(k))$ is defined in (2.17).

Proof. Substituting $x(k+1)$ from (2.16) into (3.11) and taking the expectation conditional on $x(k)$, we obtain

$$V^{(1)} = \frac{1}{2}u'(k)B(k,x(k))u(k)T - \alpha'(k)[x(k) + A(k,x(k))u(k)T + D(k)T]. \tag{3.13}$$

Differentiating with respect to $u(k)$ and equating to zero gives

$$\begin{aligned} \frac{\partial V^{(1)}}{\partial u(k)} &= B(k,x(k))u(k)T - A'(k,x(k))\alpha(k)T = 0 \\ \Rightarrow\ u^*(k) &= B^{-1}(k,x(k))A'(k,x(k))\alpha(k). \end{aligned} \tag{3.14}$$

Due to the assumed positive definite property of $B(k,x(k))$, the unique control law (3.14) always exists and represents the required optimum since the Hessian of $V^{(1)}$ is positive definite. □

REMARK 3.1 Optimal controls (3.12) contain the fractions of wealth $\alpha(k) = [\alpha_1^r(k),...,\alpha_n^r(k)]'$ of the reference portfolio. This means that one needs to first design the reference portfolio under the no short-selling constraints before implementing (3.12).

In the special case of the control penalization matrix $B(k,x(k)$ being of diagonal form, the optimal controls (3.12) reduce to

$$u_i^*(k) = \frac{1}{b_i}\left[\alpha_i^r(k) - \alpha_0^r(k)e^{x_i(k)-x_0(k)}\right], i = 1,2,...,n. \tag{3.15}$$

where b_i, $i = 1,2,...,n$, are the diagonal elements of B. The optimal trading strategies $v_i^*(k+1)$, $i = 0,1,...,n$, for $k = 0,1,...,n$, are found by applying the Euler's approximation to (2.7) and (2.13), and using the optimal controls $u_i^*(k)$ from (3.12) to obtain

$$v_i^*(k+1) \quad = \quad v_i^*(k)e^{u_i^*(k)T}, \quad i = 1,2,...,n. \tag{3.16}$$

$$v_0^*(k+1) \quad = \quad v_0^*(k)e^{u_0^*(k)T}, \quad u_0^*(k) = -\sum_{i=1}^{n} e^{x_i(k)-x_0(k)}u_i^*(k). \tag{3.17}$$

Equations (3.16) and (3.17), do not give an answer on how to make the initial optimal selection $v_i^*(0)$, $i = 0,1,...,n$, which is understandable since the optimization has been carried out with respect to the logarithmic rates of change rather than the quantities themselves. Thus, we make the initial selection identical to the reference portfolio, which gives $e_l(0) = \ln[y^r(0)] - \ln[y(0)] = 0$. In this case we have

$$v_i^*(0) = \frac{\alpha_i^r y(0)}{S_i(0)}, i = 0,1,...,n. \tag{3.18}$$

REMARK 3.2 Note that the controls in (3.12) will have the same form for any value of T. In particular, as $T \to 0$, they will represent a continuous control with k replaced by t. The continuous-time optimal trading strategies will have a finite variation and are derived by solving equations in (2.7) and (2.13) with the initial conditions given by (3.18).

The optimal variation (3.12) does not depend explicitly on the market parameters. Such an information is contained in the fractions of wealth $\alpha(k)$ of the reference portfolio. This means that this approach is also applicable to the market with uncertain parameters, i. e. the parameters are interval numbers of the type $r(t) \in [r^-(t), r^+(t)]$, $\mu_i(t,S) \in [\mu_i^-(t,S), \mu_i^+(t,S)]$, and $\sigma_{ij}(t,S) \in [\sigma_{ij}^-(t,S), \sigma_{ij}^+(t,S)]$; see, e. g. Ahn *et al.* (1997), Ahn *et al.* (1999), Wilmott (1998), Wilmott & Oztukel (1998), for the time-varying uncertain parameters. In this case, the design of the reference portfolio deals with the parameter uncertainty.

3.1.1 *Constrained portfolio control.* In some applications it could be required to have a certain guarantee on the quality of performance. One approach to achieving this is to place a constraint on the upper bound of the logarithmic error, i. e. to require that $e_u(k+1) \leqslant \varepsilon(k)$, $k = 0,1,2,...$, a. s., where $\varepsilon(k)$ is some pre-specified positive variable. If we select the coefficients $\gamma(k+1)$ as in (3.6), than the upper bound becomes

$$\begin{aligned} e_u(k+1) \quad &= \quad \ln[y^r(k)] + [r + \alpha'(k)M(k) - 0.5\alpha'(k)\sigma(k)\sigma'(k)\alpha(k)]T \\ &- \quad \alpha'(k)[x(k) + A(k,x(k))u(k)T + D(k)T]. \end{aligned} \tag{3.19}$$

The constrained optimization is stated as follows:

$$\min_{u(k)} \left\{ \frac{1}{2} u'(k)B(k,x(k))u(k)T - \alpha'(k)[x(k)+A(k,x(k))u(k)T+D(k)T] \right\} \tag{3.20}$$

$$\begin{aligned} & s.t.: \\ & -\alpha'(k)A(k,x(k))u(k)T \leqslant \varepsilon(k) - [r+\alpha'(k)M(k) - 0.5\alpha'(k)\sigma(k)\sigma'(k)\alpha(k)]T \\ & -\ln[y^r(k)] + \alpha'(k)[x(k)+D(k)T] \end{aligned} \tag{3.21}$$

This is a quadratic programming problem that can be solved numerically for each k.

3.2 *Bounds on the trading strategies and the problem of restricting the number of shares.*

Using the explicit optimal controls (3.15) for the case of a diagonal control penalization, we can derive bounds on the logarithmic variations of the optimal trading strategies $\Delta \ln v_i^*(k) = u_i^*(k)$, $i = 0,1,2,...,n,$, as follows.

LEMMA 3.2 Lower and upper bounds on the optimal logarithmic changes $\Delta \ln v_i^*(k)$, $i = 0,1,2,...,n$, are

$$\frac{-\max[\alpha_i^r(k)]y(k)}{\min(b_i)y_0(k)} \leqslant \Delta \ln v_0^*(k) \leqslant \frac{\alpha_0^r(k)}{\min(b_i)} \frac{y^2(k)}{y_0^2(k)} \tag{3.22}$$

$$\frac{-\alpha_0^r(k)}{b_i} \frac{y_i(k)}{y_0(k)} \leqslant \Delta \ln v_i^*(k) \leqslant \frac{\alpha_i^r(k)}{b_i} \tag{3.23}$$

where $\max[\alpha_i^r(k)]$ and $\min(b_i)$ represent the maximum $\alpha_i^r(k)$ and minimum b_i for $i = 1,2,...,n$, respectively.

Proof. We first prove (3.22). The lower bound is found by starting from the discrete form of (2.10) and making use of (3.15) as

$$\begin{aligned} 0 &= \sum_{i=0}^{n} y_i(k)\Delta \ln v_i^*(k) \leqslant \sum_{i=1}^{n} y_i(k)\frac{\alpha_i^r}{b_i} + y_0(k)\Delta \ln v_0^*(k) \\ &\leqslant \frac{\max(\alpha_i^r)}{\min(b_i)} \sum_{i=1}^{n} y_i(k) + y_0(k)\Delta \ln v_0^*(k) = \frac{\max(\alpha_i^r)}{\min(b_i)}[y(k)-y_0(k)] + y_0(k)\Delta \ln v_0^*(k). \end{aligned}$$

$$\Delta \ln v_0^*(k) \geqslant \frac{\max(\alpha_i^r)}{\min(b_i)}\left[1 - \frac{y(k)}{y_0(k)}\right] \geqslant \frac{-\max(\alpha_i^r)y(k)}{\min(b_i)y_0(k)}.$$

Similarly, we find the upper bound as

$$\begin{aligned} 0 &= \sum_{i=0}^{n} y_i(k)\Delta \ln v_i^*(k) \geqslant \sum_{i=1}^{n} \frac{-\alpha_0^r y_i^2(k)}{b_i y_0(k)} + y_0(k)\Delta \ln v_0^*(k) \\ &\geqslant -\frac{\alpha_0^r}{\min(b_i)y_0(k)} \sum_{i=1}^{n} y_i^2(k) + y_0(k)\Delta \ln v_0^*(k) \end{aligned}$$

$$\Delta \ln v_0^*(k) \leqslant \frac{\alpha_0^r}{\min(b_i)} \frac{y^2(k)}{y_0^2(k)}$$

Bounds in (3.23) follow directly from (3.16) and (3.15). □

An important application of the upper bounds is when we restrict the number of shares per asset, where for some deterministic $M_i(k+1)$ it is required that $v_i^*(k+1) \leqslant M_i(k+1)$, $k = 0,1,2,...,$, and $i = 0,1,2,...,n$. The penalty coefficients $b_i(k)$, $i = 1,2,...,n$, can be selected as follows in order for such a constraint to hold. First note that the upper bounds in (3.22) and (3.23) can be expressed as

$$v_0^*(k+1) \quad \leqslant \quad v_0^*(k)\exp\left[\frac{\alpha_0^r(k)y^2(k)}{\min(b_i)y_0^2(k)}\right] \tag{3.24}$$

$$v_i^*(k+1) \quad \leqslant \quad v_i^*(k)\exp\left(\frac{\alpha_i^r(k)}{b_i}\right), i = 1,2,...,n. \tag{3.25}$$

By comparing these with $v_i^*(k+1) \leqslant M_i(k+1)$, it can be seen that sufficient conditions for $b_i(k)$, $i = 1,2,...,n$, to satisfy for every $k = 1,2,...,$ are

$$\frac{\alpha_0^r(k)y^2(k)}{\min(b_i)y_0^2(k)} \leqslant \ln\left[\frac{M_0(k+1)}{v_0^*(k)}\right] \tag{3.26}$$

$$\frac{\alpha_i^r(k)}{b_i} \leqslant \ln\left[\frac{M_i(k+1)}{v_i^*(k)}\right] \tag{3.27}$$

For the special case of constant fractions of wealth for the reference portfolio $\alpha_i^r(k) = \alpha_i^r$, constant penalty coefficients b_i, an unrestricted number of shares in the risk free asset (e. g. the bank account), and a constant restriction on the remaining assets $M_i(k+1) = M_i$, $i = 1,2,...,n$, we have the following

LEMMA 3.3 Let the initial selection be such that $v_i^*(0) < M_i$ for every $i = 1,2,...,n$. Then the upper constraints $v_i^*(k) \leqslant M_i$ are satisfied for every $k = 1,2,...,$ if

$$b_i \geqslant \frac{\alpha_i^r}{\ln\left[\frac{M_i}{v_i(0)}\right]} \tag{3.28}$$

Proof. Referring to (3.25), for $k = 1,2,...,$ we have

$$v_i^*(k) \leqslant v_i^*(0)\exp\left(\frac{k\alpha_i^r}{b_i}\right).$$

A sufficient condition for $v_i^*(k) < M_i$ is

$$v_i^*(0)\exp\left(\frac{k\alpha_i^r}{b_i}\right) \leqslant M_i$$

$$\exp\left(\frac{\alpha_i^r}{b_i}\right) \leqslant \left[\frac{M_i}{v_i^*(0)}\right]^{\frac{1}{k}}$$

Due to assumption $[M_i/v_i(0)] > 1$, the above inequality yields

$$\exp\left(\frac{\alpha_i^r}{b_i}\right) \leqslant \left[\frac{M_i}{v_i^*(0)}\right]$$

for every $k = 1,2,...,$ and hence the result in (3.28). □

One solution to the problem of having $v_i^*(0) < M_i$, $i = 1,2,...,n$, is to design a reference portfolio that satisfies the condition

$$\alpha_i^r < \frac{S_i(0)M_i}{y(0)}, \quad i = 1,2,...,n.$$

3.3 *Example: pseudo-log-optimal portfolio.*

The reference portfolio in this example is selected to be the log-optimal portfolio. Such portfolios are the best to use when the aim of investment is optimal wealth growth. These were introduced in Kelly (1956) and Latane (1959) for the case of discrete-time static portfolios and more fully developed in Breiman (1961). A similar optimization problem in a market with transaction cost is given in Iyengar & Cover (1998). The log-optimal portfolio in a continuous-time dynamic case was introduced in Merton (1969), and its versions with convex constraints and transaction cost can be found in textbooks such as Karatzas & Shreve (1991), Korn (1997), and the references therein.

Let us consider a market having a bank account $S_0(t)$ and a single stock $S_1(t)$ with the following dynamics

$$dS_0(t) = rS_0(t)dt, \tag{3.29}$$

$$dS_1(t) = S_1(t)(\mu dt + \sigma dW_1). \tag{3.30}$$

We assume that the parameters are constant and have these numerical values: $r = 0.04$, $\mu = 0.05$, and $\sigma = 0.25$. The initial investors wealth and the initial asset prices are assumed as $y(0) = S_0(0) = S_1(0) = 1$. The fraction of wealth invested in the stock for the log-optimal portfolio $\alpha_1^r(k)$ is given as Merton (1969):

$$\alpha_1^r(k) = \alpha_1^r = \frac{\mu - r}{\sigma^2} = 0.16,$$

which clearly satisfies the no short-selling constraint. The initial selection for both the portfolios (the log-optimal and the constrained one) will thus be $v_0^*(0) = 0.84$, $v_1^*(0) = 0.16$. The control law (3.15) with a sampling time of $T = 0.004$, becomes

$$u_1^*(k) = \frac{1}{b_1}\left[0.16 - 0.84\frac{v_1(k)S_1(k)}{v_0(k)S_0(k)}\right]. \tag{3.31}$$

Let us also have two different values for the penalty coefficient, $b_1^{(1)} = 0.05$, $b_1^{(2)} = 0.5$, and denote the corresponding trading strategies for the stock of the constrained portfolio as $v_1^{(1)}(k)$ and $v_1^{(2)}(k)$. In a market with no transaction cost, for one realization of the stock price, the trading strategies for the stock of the log-optimal $v_1^r(k) = [\alpha_1^r(k)y^r(k)]/S_1(k)]$, and the constrained portfolios $v_1^{(1)}(k)$, $v_1^{(2)}(k)$, are shown in Fig. 1. The trading takes place during the interval of time [0, 10]. The total portfolio wealth is shown in Fig. 2, where one can notice an almost undistinguishable behavior of the portfolios. This is the reason why we propose to call this constrained portfolio *pseudo-log-optimal*. In Fig. 3, the end period portfolio wealth is enlarged. The eventual transaction costs that would have accumulated at time $(k+1)T$ for the log-optimal $C_l(k+1)$ and pseudo-log-optimal $C_p(k+1)$ portfolios, are assumed to be:

$$C_l(k+1) = C_l(k) + 0.01\alpha_1^r \mid y^r(k+1) - y^r(k)S_1(k+1)/S_1(k) \mid$$

$$C_p(k+1) = C_p(k) + 0.01 \mid v_1^*(k+1) - v_1^*(k) \mid S_1(k)$$

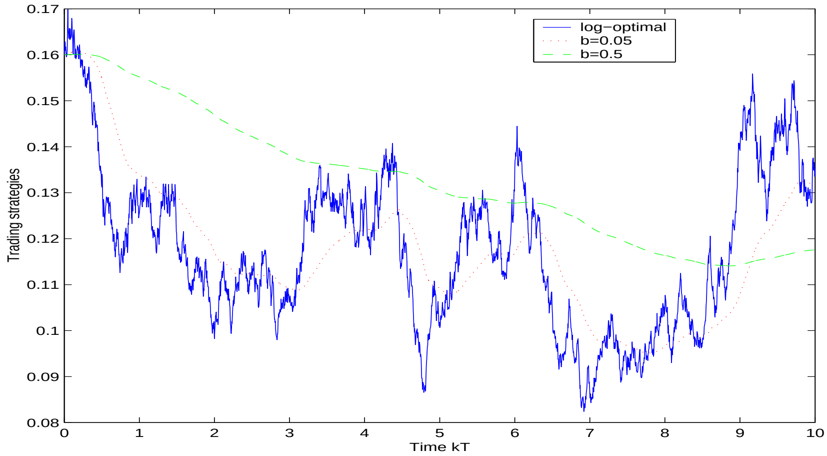

FIG. 1. Trading strategies for the stock.

with $C_l(0)$=$C_p(0)$=$0.01v_1^*(0)S_1(0)$. This corresponds to a charge of 1% of the total transaction value of buying or selling the stock, and no transaction cost for the bank account. At the end of the trading period, the wealth y_f^l, $y_f^{(1)}$, $y_f^{(2)}$, and the eventual proportional transaction cost C_f^l, $C_f^{(1)}$, $C_f^{(2)}$, of the log-optimal, pseudo-log-optimal with $b_1^{(1)}$, and pseudo-log-optimal with $b_1^{(2)}$, respectively, are:

$$\begin{aligned}
\mathbf{Log-optimal} \quad &: \quad y_f^l = 1.61983, \quad C_f^l = 0.05743 \\
\mathbf{b_1^{(1)}} \quad &: \quad y_f^{(1)} = 1.61128, \quad C_f^{(1)} = 0.00488 \\
\mathbf{b_1^{(2)}} \quad &: \quad y_f^{(2)} = 1.58572, \quad C_f^{(2)} = 0.00258
\end{aligned}$$

This shows that for almost the same final wealth, the eventual transaction cost is more than 11 and 22 times smaller for the pseudo-log-optimal portfolios in comparison with the log-optimal one. Further the differences between the final wealth and the total eventual transaction cost is higher for the pseudo-log-optimal portfolios.

The average results of several realizations are similar to the above single realization. The average of differences $(y_f^l - y_f^{(1)})$, $(y_f^l - y_f^{(2)})$, and the average of ratios $C_f^l/C_f^{(1)}$, $C_f^l/C_f^{(2)}$, for 100 realizations of the stock price, are given below

$$\begin{aligned}
&average \quad (y_f^l - y_f^{(1)}) = 0.000537, \quad average \quad C_f^l/C_f^{(1)} = 11.51515, \\
&average \quad (y_f^l - y_f^{(2)}) = 0.001655, \quad average \quad C_f^l/C_f^{(2)} = 22.47635.
\end{aligned}$$

4. The second method: tracking portfolios.

In this section the aim is to track as closely as possible an already designed reference portfolio with the portfolio introduced in Sec. 2. The criterion for the quality of the tracking is a quadratic form in the discrete-time *log-square errors* of individual asset holding values, i. e. the square of the logarithmic difference $\lambda_i(k+1) = \ln[y_i(k+1)] - \ln[y_i^r(k+1)]$, $i = 0,1,...,n$, where $\ln[y_i^r(k+1)] = a_i(k+1)$ is the logarithm of the value of the holdings for asset i of the reference portfolio. Thus we view each asset

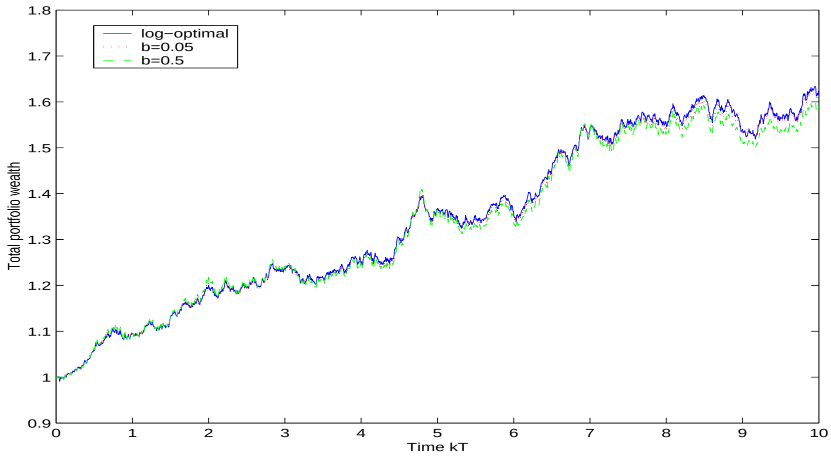

FIG. 2. Total portfolio wealth during the trading period.

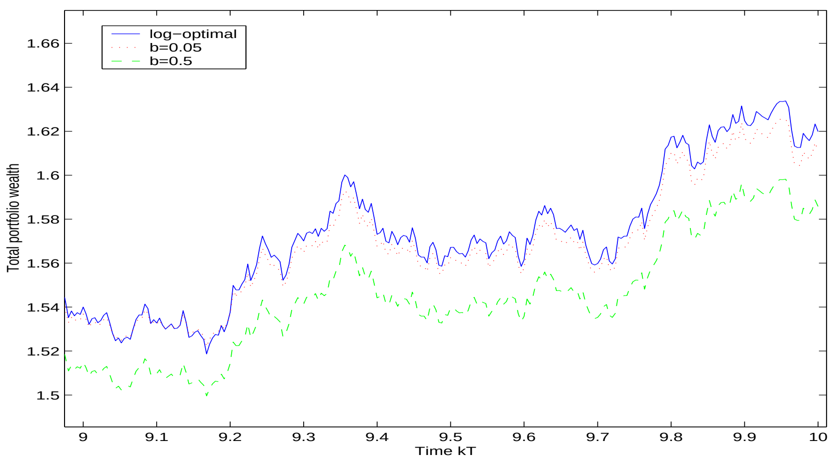

FIG. 3. Total portfolio wealth at the end of the trading period.

holding values as references and try to track them rather than track the total value of the reference portfolio. The resulting portfolios can thus be seen as *replicating* portfolios, since their trading strategies are trying to replicate those of the reference portfolio.
The general form of the dynamics of $\ln[y_i^r(t)] = a_i(t)$ is given as

$$da_i(t) = g_i(t,S)dt + h_i(t,S)dW, \quad i = 0,1,\ldots,n,$$

where the scalars $g_i(t,S)$ and the row vectors $h_i(t,S)$ are known functions of time and asset prices. The Euler approximation gives the following discrete form

$$a_i(k+1) = a_i(k) + g_i(k)T + h_i(k)e(k)\sqrt{T}, \quad i = 0,1,\ldots,n.$$

For simplicity, we have used the notations $g_i(k)$ and $h_i(k)$ rather than $g_i(kT,S(kT))$ and $h_i(kT,S(kT))$. The dynamics of the vector of the logarithmic errors $\lambda(k) = [\lambda_0(k), \ldots,\lambda_n(k)]'$ is obtained by taking the difference between the constrained portfolio model (2.16) and the above references, which gives

$$\lambda(k+1) = \lambda(k) + A(x,k)u(k)T + [D(k) - G(k)]T + [\Sigma(k) - H(k)]e(k)\sqrt{T}, \tag{4.1}$$

where $G(k)$ is a column vector with $g_i(k)$ as elements, and $H(k)$ is a $(n+1)\times m$ matrix with vectors $h_i(k)$ as rows. We can now give the formulation of the optimal investment problem.

Portfolio control problem II (PCP-II). *Let $B(k,x(k)) \in \mathbb{R}^{n\times n}$ and $Q(k,x(k)) \in \mathbb{R}^{(n+1)\times(n+1)}$ be two given symmetric positive semi-definite matrices, possibly state dependent and at least one being positive definite. Find the controls $u(k)$, $k = 0,1,\ldots$, that minimize the following objective*

$$V^{(2)}(k,\lambda(k),u(k)) = \frac{1}{2}\mathbb{E}\left[u'(k)B(k,x(k))u(k)T + \lambda'(k+1)Q(k,x(k))\lambda(k+1)|\lambda(k)\right], \tag{4.2}$$

where $\lambda(k+1)$ is given by (4.1).

THEOREM 4.1 The solution to the PCP-II always exists, is unique and for every $k = 0,1,\ldots$, is given by

$$u^*(k) = -(B + A'QAT)^{-1}A'Q[\lambda(k) + (D-G)T] \tag{4.3}$$

Proof. Substituting the expression for error dynamics (4.1) in (4.2) and taking the expectation, we obtain

$$\begin{aligned} V^{(2)} &= 0.5\{u'BuT + \lambda'Q\lambda + \lambda'QAuT + \lambda'Q(D-G)T + u'A'Q\lambda T + u'A'QAuT^2 \\ &+ u'A'Q(D-G)T^2 + (D'-G')Q\lambda T + (D'-G')QAuT^2 + (D'-G')Q(D-G)T^2 \\ &+ tr[(\Sigma'-H')Q(\Sigma-H)]T\}, \end{aligned} \tag{4.4}$$

where $tr(\cdot)$ denotes the trace of a matrix. Differentiating $V^{(2)}$ with respect to $u(k)$ and equating it to zero gives

$$\begin{aligned} \frac{\partial V^{(2)}}{\partial u(k)} &= Bu(k)T + 0.5A'Q\lambda T + 0.5A'Q\lambda T + A'QAT^2u(k) \\ &+ 0.5A'Q(D-G)T^2 + 0.5A'Q(D-G)T^2 = 0 \\ \Rightarrow u^*(k) &= -(B + A'QAT)^{-1}A'Q[\lambda(k) + (D-G)T]. \end{aligned}$$

Due to our assumption on matrices B and Q, the matrix $B+A'QAT$ is always positive definite and so is the Hessian of $V^{(2)}$. Thus the obtained control law always exists, is unique, and it represents the required optimum. $\square$

For a positive definite matrix B and considering the limiting case of $T \rightarrow 0$, the control law (4.3) becomes $u^*(t) = -B^{-1}A'Q\lambda(t)$ and the corresponding continuous-time trading strategy has a finite first variation. This control law does not depend explicitly on the market parameters.

In the special case of a market with a bank account $S_0(t)$ and a single stock $S_1(t)$ given by (3.29) and (3.30), and penalty matrices selected as $B = b_1$, $Q = diag(q_0, q_1)$, the control law (4.3) reduces to the following form that will be useful later:

$$u_1^*(k) = \frac{q_0 e^{x_1 - x_0}[x_0 - a_0 + rT - g_0 T] - q_1[x_1 - a_1 + (\mu - 0.5\sigma^2 - g_1)T]}{q_0 T e^{x_1 - x_0(k)} + q_1 T + b_1}. \tag{4.5}$$

The optimal trading strategy is found by substituting the optimal controls $u_i^*(k)$, $k = 0, 1, 2, \ldots$, from (4.3) in (3.16) and (3.17). The initial selection is made identical to the reference portfolio. Similarly to the Sec. 3.1.1, in order to have a certain guarantee on the quality of tracking, one can solve a constrained portfolio control problem of a quadratic programming type with (3.20) replaced by (4.4), and the elements of the vector $\alpha(k) = [\alpha_1^r(k), \ldots, \alpha_n^r(k)]'$ in (3.21) be given as

$$\alpha_i^r(k) = \frac{y_i^r(k)}{\sum_{j=0}^n y_j^r(k)} = \frac{e^{a_i(k)}}{\sum_{j=0}^n e^{a_j(k)}}$$

4.1 *Example: Black-Scholes replicating portfolio as a reference.*

As a reference we will use the well-known Black-Scholes replicating portfolio for a European Call option on a single stock as described in Black & Scholes (1973), Merton (1973). This is an interesting example since it shows how the original formulation of the optimal portfolio can be modified to deal with the case when *borrowing*[4] is allowed, and also the resulting optimal portfolio is of a practical importance. In a market given by (3.29) and (3.30), the price of a European call option $C(S_1, t)$, i. e. the value of the Black-Scholes replicating portfolio, is given as Wilmott (1998)

$$C(S_1, t) = S_1 N(d_1) - E e^{-r(T_e - t)} N(d_2), \tag{4.6}$$

where E is the exercise price, T_e the expiry time, and

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-0.5y^2} dy \tag{4.7}$$

$$d_2(S_1, t) = \frac{\ln \frac{S_1}{E} + (r - 0.5\sigma^2)(T_e - t)}{\sigma\sqrt{T_e - t}}, \quad d_1(S_1, t) = d_2(S_1, t) + \sigma\sqrt{T_e - t} \tag{4.8}$$

The number of shares in the cash v_0^{BS} and in the stock v_1^{BS} (the trading strategy) for this reference portfolio are obtained from (4.6) as

$$v_0^{BS} = \frac{-E e^{-r(T_e - t)} N(d_2)}{S_0(t)}, \tag{4.9}$$

$$v_1^{BS} = N(d_1). \tag{4.10}$$

[4]Similarly we can deal with the European Put option, in which case the short selling of the stock is allowed.

4.1.1 *Model of the constrained portfolio with* $\mathbf{v_0(t)} < \mathbf{0}$. It can be seen from (4.9) that the number of shares in the cash is always negative. Thus we cannot apply the method of this section directly. One approach to achieving a log-quadratic tracking is to first constrain the number of shares in the cash of the tracking portfolio $v_0(t)$ to also be negative; $v_0(t) < 0$. This means that both the reference portfolio and the replicating portfolio will have negative values of the holdings in the cash. Thus we propose to try and match the logarithms of the *negative* of the values of the holdings in the cash, which are well defined in this case.

Under the assumption of $v_0(t) < 0$, and following the basic steps of Sec. 2, we derive the dynamics of the portfolio with a differentiable trading strategy, which will be only slightly different from the one of Sec. 2. The value of the holding in the cash is $y_0(t) = v_0(t)S_0(t)$, and the logarithm of its negative value is well defined and given as

$$\begin{aligned} \ln[-y_0(t)] &= \ln[-v_0(t)]+\ln[S_0(t)] \Rightarrow d\ln[-y_0(t)] = d\ln[-v_0(t)]+d\ln[S_0(t)], && (4.11)\\ \ln[y_1(t)] &= \ln[v_1(t)]+\ln[S_1(t)] \Rightarrow d\ln[y_1(t)] = d\ln[v_1(t)]+d\ln[S_1(t)]. && (4.12) \end{aligned}$$

The differentiability constraint on the trading strategy is

$$d\ln[-v_0(t)] = u_0(\cdot)dt, \quad d\ln[v_1(t)] = u_1(\cdot)dt.$$

The self-financing constraint (2.10) now gives

$$S_0dv_0 + S_1dv_1 = 0 \Rightarrow S_1dv_1 = S_0d(-v_0) \Rightarrow S_1v_1u_1dt = S_0(-v_0)u_0dt \Rightarrow u_0 = e^{x_1-x_0}u_1,$$

where $x_0 = \ln(-y_0)$ and $x_1 = \ln(y_1)$. The dynamics of a self-financing portfolio is now obtained from (4.11) and (4.12) as

$$\begin{aligned} dx_0(t) &= u_0dt + rdt = e^{x_1-x_0}u_1dt + rdt, && (4.13)\\ dx_1(t) &= u_1dt + (\mu - 0.5\sigma^2)dt + \sigma dW_1. \end{aligned}$$

The references in this case are the logarithm of the negative value of the holdings in the cash $a_0 = \ln(-v_0^{BS}S_0)$ and the logarithm of the value of the holdings in the stock $a_1 = \ln(v_1^{BS}S_1)$. We select the objective to be minimized as

$$V^{(3)} = \frac{1}{2}\mathbb{E}\{b_1u_1^2(k) + q_0[x_0(k+1) - a_0(k+1)]^2 + q_1[x_1(k+1) - a_1(k+1)]^2\}.$$

The only difference between this optimization problem and PCP-II is in the dynamics for $x_0(t)$. If (4.13) is compared to (2.11), the only difference is that (4.13) does not have a minus sign in front of the exponential term. This means we can write the optimal control law for this problem $u_1^*(k)$ directly from (4.5) by only placing a minus sign in front of the exponential in the numerator (since the exponential in denominator is squared) to obtain

$$u_1^*(k) = \frac{-q_0e^{x_1-x_0}[x_0-a_0+rT-g_0T]-q_1[x_1-a_1+(\mu-0.5\sigma^2-g_1)T]}{q_0Te^{x_1-x_0(k)}+q_1T+b_1}. \qquad (4.14)$$

4.1.2 *Deriving the dynamics of the references.* Here we derive the discrete-time dynamics of the references a_0 and a_1, which are defined as

$$\begin{aligned} a_0 &= \ln(-v_0^{BS}S_0) = \ln[Ee^{-r(T_e-t)}N(d_2)] = \ln(E)-r(T_e-t)+\ln[N(d_2)], && (4.15)\\ a_1 &= \ln(v_1^{BS}S_1) = \ln[S_1N(d_1)] = \ln(S_1)+\ln[N(d_1)]. && (4.16) \end{aligned}$$

In order to find the differentials of these references we shall make frequent use of the Ito's lemma and make many elementary calculations. To derive the dynamics of $d_2 = d_2(t, \ln[S])$ we need the following partial derivatives:

$$\frac{\partial d_2}{\partial t} = \frac{\ln(\frac{S}{E}) - (r - 0.5\sigma^2)(T_e - t)}{2\sigma(T_e - t)^{1.5}}, \quad \frac{\partial d_2}{\partial \ln(S)} = \frac{1}{\sigma\sqrt{T_e - t}}, \quad \frac{\partial^2 d_2}{\partial \ln(S)^2} = 0.$$

Applying Ito's lemma to (4.8) and substituting the above partial derivatives, we obtain

$$\begin{aligned} d(d_2) &= \left[\frac{\partial d_2}{\partial t} + \frac{\partial d_2}{\partial \ln(S)}(\mu - 0.5\sigma^2)\right] dt + \frac{\partial d_2}{\partial \ln(S)} \sigma dW_1 \\ &= \left[\frac{\ln[\frac{S}{E}] - (r - 0.5\sigma^2)(T_e - t)}{2\sigma(T_e - t)^{1.5}} + \frac{(\mu - 0.5\sigma^2)}{\sigma\sqrt{T_e - t}}\right] dt + \frac{dW_1}{\sigma\sqrt{T_e - t}} \\ &= m_2 dt + s_2 dW_1. \end{aligned}$$

Next we need the differential of $\ln[N(d_2)]$ which, together with its partial derivatives, is given as

$$\begin{aligned} \ln[N(d_2)] &= \ln\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2} e^{-0.5y^2} dy\right], \quad \frac{\partial \ln[N(d_2)]}{\partial t} = 0, \\ \frac{\partial \ln[N(d_2)]}{\partial d_2} &= \frac{e^{-0.5d_2^2}}{N(d_2)\sqrt{2\pi}}, \quad \frac{\partial^2 \ln[N(d_2)]}{\partial d_2^2} = \frac{-\left[e^{-0.5d_2^2} d_2 N(d_2)\sqrt{2\pi} + e^{-d_2^2}\right]}{2\pi[N(d_2)]^2} \end{aligned}$$

Applying Ito's lemma for this case as well, we obtain

$$d\ln[N(d_2)] = \left[\frac{\partial \ln[N(d_2)]}{\partial d_2} m_2 + 0.5 \frac{\partial^2 \ln[N(d_2)]}{\partial d_2^2} s_2^2\right] dt + \frac{\partial \ln[N(d_2)]}{\partial d_2} s_2 dW_1 \tag{4.17}$$

The differential and difference equations of the first reference (4.15) are

$$\begin{aligned} da_0(t) &= r dt + d\ln[N(d_2)], \\ a_0(k+1) &= a_0(k) + rT + \Delta \ln[N(d_2)], \end{aligned} \tag{4.18}$$

where $\Delta \ln[N(d_2)]$ is obtained by applying Euler's method to (4.17), and in particular note that the term $(T_e - t)$ in m_2 and s_2 becomes $(T_e - kT)$ in this case. To calculate the optimal control law u_1^* in (4.14), we need the drift term $g_0(k)$ of (4.18), which is

$$g_0(k) = r + \frac{\partial \ln[N(d_2)]}{\partial d_2} m_2 + 0.5 \frac{\partial^2 \ln[N(d_2)]}{\partial d_2^2} \frac{1}{\sigma^2(T_e - kT)}$$

The dynamics of the second reference (4.16) requires the differential of $\ln[N(d_1)]$, which can be obtained from that of $\ln[N(d_2)]$ by substituting d_2 with d_1, and $(r - 0.5\sigma^2)$ with $(r + 0.5\sigma^2)$. The differential and difference equations of the second reference are

$$\begin{aligned} da_1(t) &= d\ln[S_1(t)] + d\ln[N(d_1)] = (\mu - 0.5\sigma^2)dt + \sigma dW_1 + d\ln[N(d_1)], \\ a_1(k+1) &= a_1(k) + (\mu - 0.5\sigma^2)T + \sigma e_1(k)\sqrt{T} + \Delta \ln[N(d_1)], \end{aligned} \tag{4.19}$$

and the drift term of (4.19) can be obtained as

$$g_1(k) = (\mu - 0.5\sigma^2) + \frac{\partial \ln[N(d_1)]}{\partial d_1} m_1 + 0.5 \frac{\partial^2 \ln[N(d_1)]}{\partial d_1^2} \frac{1}{\sigma^2 (T_e - kT)},$$

where m_1 is the same as m_2 with $(r - 0.5\sigma^2)$ substituted with $(r + 0.5\sigma^2)$.
Note that $u_1^*(k)$ in (4.14) depends explicitly on the drift μ through the parameters m_1 and m_2, which makes the tracking portfolio *not* indifferent to it as is the case with the replicating portfolio.

4.1.3 *Simulation results.* Let the market parameters, the sampling time, and the transaction cost structure be selected as in the previous section. Also let the option parameters be $T_e = 10$, $E = 2$, and the tracking portfolio parameters be $q_0 = q_1 = 100$, $b_1 = 50$. For one realization of the stock price, the value processes of the two portfolios are shown in Fig. 4, and it can

be noticed that these are almost identical. The corresponding trading strategies for the cash and the stock are shown in Fig. 5 and 6, respectively. Similarly to the previous example, the trading strategy of the tracking portfolio is smooth and it is intuitively clear that this will result in a lower eventual transaction cost. At the end of the trading period, the wealth y_f^{BS}, y_f^{Tr}, and the eventual proportional transaction cost C_f^{BS}, C_f^{Tr}, for the Black-Scholes and the tracking portfolios, respectively, are

$$\begin{aligned} \textbf{Black} - \textbf{Scholes} \quad &: \quad y_f^{BS} = 0.77251, \quad C_f^{BS} = 0.29004, \\ \textbf{Tracking} \quad &: \quad y_f^{Tr} = 0.71526, \quad C_f^{Tr} = 0.03828. \end{aligned}$$

Thus this example illustrates that when the tracking portfolio is used to hedge an option, it will have an *identical* initial value to the Black-Scholes replicating portfolio, almost the same total value throughout the trading period, and a lower eventual transaction cost (more than 7 times lower in this example).
The average difference $[y_f^{BS} - y_f^{Tr}]$, and the average ratio C_f^{BS}/C_f^{Tr}, for 100 realization of the stock price, are obtained as

$$average \quad [y_f^{BS} - y_f^{Tr}] = 0.0118, \quad average \quad C_f^{BS}/C_f^{Tr} = 7.2645.$$

4.1.4 *Open problems* There are two important open questions regarding this approach to option hedging. Due to borrowing, the value of the tracking portfolio can become negative and thus one should use this approach with care. This problem does not occur if the reference option satisfies the positivity constraint on the trading strategy (see, e. g. Cvitanić & Karatzas (1993)), in which case the tracking portfolio that also satisfies such a constraint is used and its value is positive. The second problem is that of the terminal wealth, since there is no guarantee that the tracking portfolio will be identical to the replicating portfolio at the terminal time, as the above example illustrates. When the value of the tracking portfolio is *higher* than that of the reference portfolio just before the last trading step, then a simple ad-hoc solution is as follows: first equalize the portfolio values by consuming the excess wealth of the tracking portfolio, and then employ an identical trading strategy to that of the replicating portfolio at the very last trading step. A better solution to this problem is required, and also a solution for the case of the tracking portfolio having a *lower* value to that of the reference portfolio.

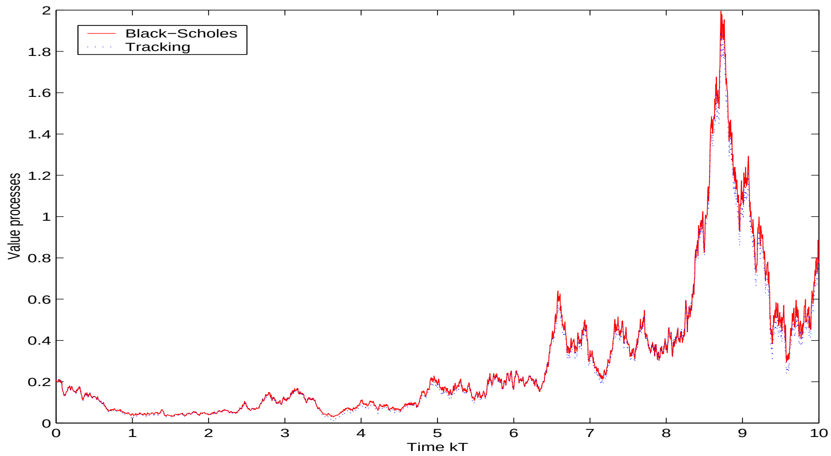

FIG. 4. Value processes for the Black-Scholes and the tracking portfolios.

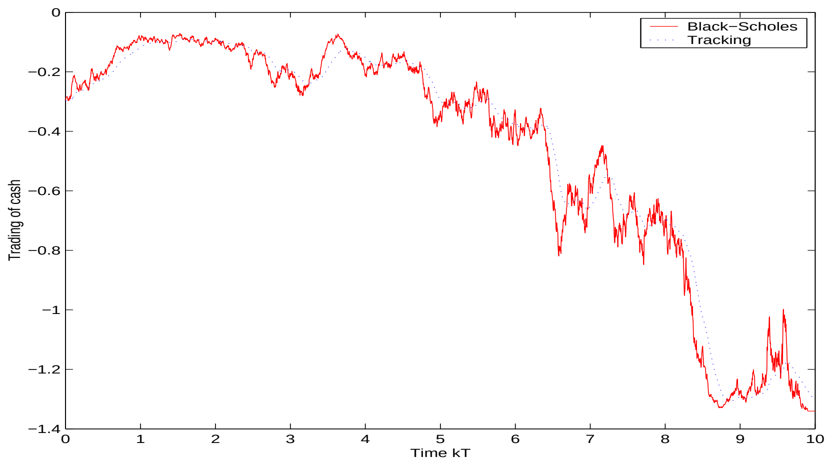

FIG. 5. The trading strategies for the cash.

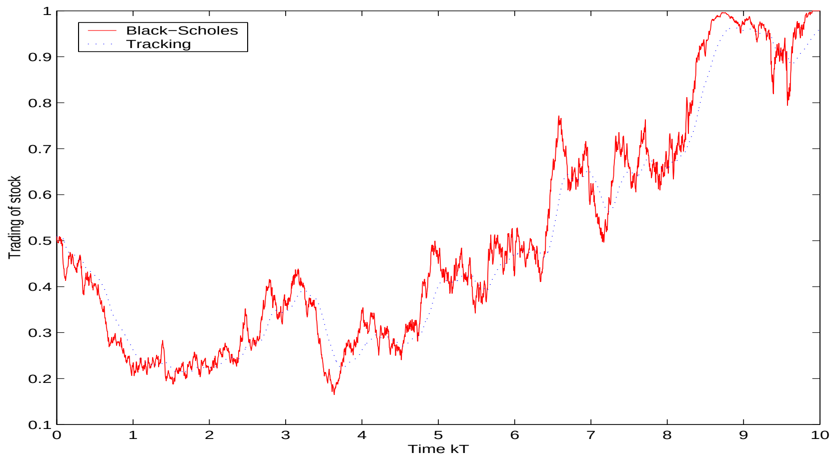

FIG. 6. The trading strategies for the stock.

5. Conclusions

Two simple and very general methods for designing optimal portfolios are proposed. In order to reduce the eventual proportional transaction cost the trading strategy is constrained to be differentiable and thus of finite variation. In the first method, an upper bound on the discrete-time log-error between the reference and the constrained portfolios is minimized, by which the shortfall with respect to such a reference portfolio is penalized. The criterion also has a quadratic penalty of the logarithmic rates of change of the trading strategy. This gives the investor the means for trade off between a lower eventual transaction cost and a higher profit, and also ensures the existence and uniqueness of the optimal solution. The trading strategy is obtained in an explicit closed-form, and a simulation example illustrates a significant reduction in the eventual proportional transaction cost as compared to the log-optimal portfolio. In the second method, optimal tracking is achieved by using a sum of discrete-time log-quadratic errors of the asset holding values. The optimal trading strategy is obtained in an explicit closed-form for this approach as well, and a simulation example shows the use of a tracking portfolio as a hedging strategy for options.

In terms of the performance, simulation results show no significant difference between these two methods. On the other hand, the first method can also be applied to the markets with interval parameters, whereas the present form of the discrete-time version of the second method can not, since the trading strategy depends explicitly on the market parameters. A modified version of the second method can also be applied when borrowing is allowed, in which case the value of the tracking portfolio can be negative. This means the first method can not be applied in this case since the log-error can not be defined.

In both of these methods the reference portfolio needs to be designed first, and it is not necessary for the design method being used to include the transaction cost explicitly in the model. This means that one should be encouraged to develop such methods, since our approach increases their practical relevance.

Comparison of the second method as applied to option replication, with results of Korn (1998), Martellini & Priaulet (2002), and Whalley & Wilmott (1999), is an important future work. Furthermore, the model of a self-financing portfolio with a positive differentiable trading strategy proposed in this paper is of a general use, e. g. a different objective from the ones used in this paper can be employed with it. In particular, formulation of objective functions for various portfolio selection problems with constraints (such as the constraint on Capital-at-Risk discussed in Atkinson & Papakokkinou (2005)), and inclusion of more general asset price models in our framework are important topics for further research.

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