High-order in time Discontinuous Galerkin Finite Element methods for linear wave equations

A Thesis Submitted for the Degree of Doctor of Philosophy

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Abstract

In this thesis we analyse the high-order in time discontinuous Galerkin finite element method (DGFEM) for second-order in time linear abstract wave equations.

Our abstract approximation analysis is a generalisation of the approach introduced by Claes Johnson (in Comput. Methods Appl. Mech. Engrg., 107:117-129, 1993), writing the second order problem as a system of first order problems. We consider abstract spatial (time independent) operators, high-order in time basis functions when discretising in time; we also prove approximation results in case of linear constraints, e.g. non-homogeneous boundary data. We take the two steps approximation approach i.e. using high-order in time DGFEM; the discretisation approach in time introduced by D Schötzau (PhD thesis, Swiss Federal institute of technology, Zürich, 1999) to first obtain the semi-discrete scheme and then conformal spatial discretisation to obtain the fully-discrete formulation. We have shown solvability, unconditional stability and conditional a priori error estimates within our abstract framework for the fully discretized problem.

The skew-symmetric spatial forms arising in our abstract framework for the semi- and fully-discrete schemes do not fullfill the underlying assumptions in D. Schötzau's work. But the semi-discrete and fully discrete forms satisfy an Inf-sup condition, essential for our proofs; in this sense our approach is also a generalisation of D. Schötzau's work. All estimates are given in a norm in space and time which is weaker than the Hilbert norm belonging to our abstract function spaces, a typical complication in evolution problems.

To the best of the author's knowledge, with the approximation approach we used, these stability and a priori error estimates with their abstract structure have not been shown before for the abstract variational formulation used in this thesis.

Finally we apply our abstract framework to the acoustic and an elasto-dynamic linear equations with non-homogeneous Dirichlet boundary data.

Operators used in Part I	A	:	see Definition 2.2 on p. 6.
	$\hat{\Pi}$	:	see Definition 3.11 on p. 58.
	$\hat{A}$	:	see Definition 2.6 on p. 12.
	$\tilde{\Pi}^r$	:	see Definition 3.12 on p. 59.
Operators used in Part II	A	:	see Definition 5.1 on p. 101.
	$\hat{\Pi}$	:	see Definition 6.6 on p. 146.
Operators used in both parts	Π	:	see Definition 3.12 on p. 59.
	$\mathcal{A}$ and $\mathcal{B} \in \mathbb{C}^{(r+1)\times(r+1)}$	:	see Definition 3.7 on p. 29.

Table 1: Summary of all abstract operators

Abstract spatial spaces used in Part I	Z	:	see Definition 2.1 on p. 6.
	D_A	:	see Definition 3.13 on p. 60.
Abstract spatial spaces in Part II	Z and D^A	:	see Definition 5.1 on p. 101.
Abstract spatial space used in both parts	H	:	see Definition 2.1 on p. 6.

Table 2: Summary of all abstract spatial spaces

Common Notations	$\mathbb{C}$	:	The field of complex numbers.
	$\mathbb{R}$	:	The set of Real numbers.
	$\mathbb{N}_0$	:	The set of Natural numbers.
	$J = (0, T)$	:	The time interval for $T < \infty$.
	$\bar{J} = [0, T]$	:	The closed time interval.
	$\mathbf{0}$	:	The vector-valued zero function i.e. $\mathbf{0} = \binom{0}{0}$.
	N	:	The number of time steps.
	$k = \frac{T}{N}$	:	The time step size.
	$\underline{r} = \{r_n\}_{n=0}^{N-1}$	:	Vector of the temporal orders over all time steps.
	$\mathcal{N}$	:	Partition of the time interval J into N time steps.
	U_-^n	:	The left-sided limits for $n = 0, \cdots, N-1$.
	U_+^n	:	The right-sided limits for $n = 0, \cdots, N-1$.
	$[U] = U_+^n - U_-^n$	:	Jump of vector-valued function U across time node t_n.

Table 3: Summary of all common notations and math symbols

Chapter 1

Introduction

In [30], Claes Johnson proved a priori and a posteriori error estimates for linear wave equation (with homogeneous boundary data) based on using space-time finite element discretisations with basis functions which are continuous in space and discontinuous in time.

In this thesis, the approach introduced by Claes Johnson [30] will be generalised by:

- Analysing abstract linear second-order in time evolution problem with abstract linear (time independent) spatial operator. This covers e.g. scalar acoustic wave equation and elastic wave equation governed by Lamé-Navier equations of linear elasticity theory.

- Rewriting the abstract problem as a first-order system in time and analysing theoretically its equivalent variational formulation in general abstract Hilbert spaces and showing its solvability, stability and a priori error estimates.

- Introducing a version of a variational formulation of the first-order system in time with linear constraints and showing its solvability, stability, and a priori error estimates. The linear constraints model e.g. in-homogeneous boundary data.

- Using the two step discretisation approach in the analysis of both versions, with firstly, discretising in time by applying the high-order in time DGFEM approach introduced by D Schötzau [44] to get the semi-discrete scheme. Secondly, using a conformal spatial discretisation to get the fully-discrete scheme.

In the area of approximating the wave equation, there are two basic solution concepts. One, is to deal directly with second-order in time problem in the sense of a single field formulation. Two, is rewriting the problem as a first-order system in time, rendering a two field formulation. There are two approaches to approximate wave equation (as single or two field formulation) in space and time. The first one is the semi-discrete approach, by first discretising in space and then use any time stepping scheme to get the fully discretised one, see e.g. [5], [14], [47].
The second one is to discretise in space and time simultaneously to get the fully-discrete scheme, see e.g. [40], [25], [30], [37]. This approach is usually more complex and complicated specially if the non-conformal approximation methods are used in both space and time, see [25].

Many approaches of the Discontinuous Galerkin method for the second-order in time evolution problem have been emerged, where discontinuities may be considered in the spatial discretisation, in the temporal discretisation, or even both, see [40, 1. Introduction] and the references therein.

Talking about high-order in time DGFEM method as well as the two field formulation since they are considered in this text. It may come to the reader that this approach may not be the best since:

- It is dissipative [30] via the discontinuity at the time nodes; jumps, in comparison to CG method in time.

But:

- Still the total energy is bounded by the given data.

- Moreover, the resulting systems in this approach after discretising in time are diagonalisable and in practice can be solved in parallel and this make it more favourable.

Recently, *Banks et al.* 2014, [6], demonstrated the practicality of the introduced approach by D Schötzau [44] for a collection of common and important linear wave equation problems without any stability or error analysis.
This thesis provides general abstract theorems on stability and a priori error estimates which come to explain and illustrate the encountered observation in the numerical experiments done in [6], e.g. the reasons why the computed convergence rates change whenever different interpolants are used to approximate the continuous solutions of a formulation which have linear constraints (e.g. inhomogeneous boundary data) and the choice of the conformal finite dimensional spaces with the order of the bases function in space.

There are mainly two obstacles faced while carrying out this research:

1. The resulting forms of the semi and fully discrete systems are skew symmetric and not as the case in D Schötzau [44].

2. The stability and a priori error estimates are given in a non matching norm of the abstract product space i.e. stability and a priori error estimates are given in a normed-product abstract space.

For the first obstacle, it is resolved. Since the resulting forms in the local time-stepping schemes have a structure which allow them to satisfy the Inf-sup condition to be able to get the desired complete proofs.
For the second obstacle, that is the best which can be achieved. Since the resulting inner products of the coefficient of our solution after discretising in time and in space are defined in the $E-$energy norms of the product spaces.

In Literature, Runge Kutta methods are used to discretize time integral problems, where only implicit Runge Kutta methods can be A-stable. This method is applied to stiff systems which are already been discretised in space. Usually, the resulting matrices when using A-stable Runge Kutta are either lower triangle matrices or full, see [20].
whereas in our approach, after we discretize in time using DGFE method, the resulting semi-discrete schemes are an implicit time stepping schemes and the matrices after time basis transformation are diagonalizable and also can be solved in parallel which is more favourable.

The outline of the thesis is as follows:

1. Part I is about general abstract theory analysis of High-order in time DGFEM for abstract linear wave equation. It includes three chapters in addition to the introduction. The second chapter discusses the existence, uniqueness, and stability estimates of the Abstract Linear second-order

in time evolution equation. The third chapter discusses the discretisation of the variational formulation of the first-order system in time with using high-order in time GDFEM with conformal spatial discretisation. It includes existence, uniqueness, stability estimates for the semi and fully discrete formulations and ends with a priori error estimates given in Theorem 3.6 on p. 61. In Corollary 3.5 on p. 65 with assuming higher regularity such that $U \in H^2(J; Z\times Z)$, $Au_1 \in H^2(J;H)$, the a priori error estimate in the last left sided limit reads

$$\begin{aligned}\|(U - U_{DG}^h)_-^N\|_E \leq \|(U - \hat{\Pi}U)_-^N\|_E + \sqrt{2}\sum_{n=1}^{N}\|(u_2 - \hat{\Pi}u_2)_-^n\|_H \\ + \left(\frac{2\sqrt{\mathcal{P}_{r_n}}\mathcal{C}}{k} + \frac{\max(1,M)\mathcal{C}}{\sqrt{\min(\alpha,1)}}\right)\Big(C(\frac{k}{2})^2\frac{1}{\max\{1,r\}^2}\|u_2\|_{H^2(J;Z)} \\ + C\max\{1,r\}^{1/2}\|u_2 - \hat{\Pi}u_2\|_{L^2(J;Z)} \\ + C\max\{1,r\}^{-1/2}(\frac{k}{2})^{1/2}\sum_{n=1}^{N}\|(u_2 - \hat{\Pi}u_2)_-^n\|_Z\Big) \\ + \frac{\max(1,M)\mathcal{C}}{\sqrt{\min(\alpha,1)}}\Big(C(\frac{k}{2})^2\frac{1}{\max\{1,r\}^2}\|Au_1\|_{H^2(J;H)} \\ + C\max\{1,r\}^{1/2}\|A(u_1 - \hat{\Pi}u_1)\|_{L^2(J;H)} \\ + C\max\{1,r\}^{-1/2}(\frac{k}{2})^{1/2}\sum_{n=1}^{N}\|A(u_1 - \hat{\Pi}u_1)_-^n\|_H\Big),\end{aligned} \tag{1.1}$$

and with such assumptions we obtain the convergence result in h and k :

$$\|(U - U_{DG}^h)_-^N\|_E = \mathcal{O}(h^2 + k^{-1}h^2 + k + k^2 + k^{-1/2}h^2), \tag{1.2}$$

shown in Corollary 3.6 on p. 66. Also, a local a priori error estimate in the $L^2(I;E)$-norm shown in Theorem 3.7 on p. 68 and a global a priori error estimate in the $L^2(J;E)$-norm in Lemma 3.24 on p. 74 with taking the maximum over time steps. The fourth chapter discusses existence, uniqueness, and stability estimates of a first-order in time variational formulation but this time with a general vector valued functional.

2. Part II is about Abstract theory analysis of High-order in time DGFEM for abstract linear wave equation with linear constrains. It includes two chapters. The first chapter(chapter 5) discusses the existence, uniqueness, and stability estimates of linear variational formulation of second order in time with linear constraints, where chapter three from Part I is used. The second chapter (chapter 6) discusses the discretisation of the variational formulation of the first-order system in time with using high-order in time GDFEM with conformal spatial discretisation. It includes existence, uniqueness, stability estimates for the semi and fully discrete formulations with linear constraints and ends with a last left sided limit a priori error estimate given in Theorem 6.8 on p. 149, a local a priori error estimate in the $L^2(I;X)$ norm on Theorem 6.9 on p. 152.

3. Part III includes one chapter which is about application and conclusion. It shows that the abstract frame work of the shown theory in Part I and II does fit the linear Acoustic wave equation and the elastic wave equation.

4. The appendix includes all the fundamental definitions and theorems used throughout the thesis.

Part I: General abstract theory analysis of High-order in time DGFEM for abstract linear wave equation

Chapter 2

Linear second-order in time evolution equation

This chapter discusses the linear second-order in time evolution equation and the main theorem of its solvability. With using the high-regularity of the problem, it will be rewritten as a first-order system in time. The variational formulation of the latter is also formulated and its stability, existence, and uniqueness are shown.

2.1 Abstract Linear Second-order in time evolution equation

Definition 2.1 (The abstract spaces) *Let Z and H be Hilbert spaces (see Definition A.16 on p. 176), over the field $\mathbb{C}$ and have the following properties:*

$$Z \subset H, (Z \text{ is a subspace of } H), \; Z \text{ dense in } H, \; Z \text{ is separable (see Definition A.5 on p. 172)}, \tag{2.1}$$

and they satisfy the Gelfand triple properties i.e. $(Z \subset H \subset Z')$, where Z' is the dual space of Z (see Definition A.17 on p. 176).

Definition 2.2 (The abstract linear spatial operator) *Let Z be the Hilbert space given in Definition 2.1 with its dual space Z'. Let $a : Z \times Z \to \mathbb{C}$ be a Hermitian form which satisfies the properties given in Theorem A.3 on p. 178:*

$$\begin{aligned} |a(u,v)| &\le M \parallel u \parallel_Z \parallel v \parallel_Z, \qquad && \forall \quad u, v \in Z, \\ a(u,u) &\ge \alpha \parallel u \parallel_Z^2, && \forall \quad u \in Z, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} a(u,v) &= a(v,u), \qquad && \textit{for real functions } u, v \in Z, \; \textit{(Symmetric)}, \\ a(u,v) &= \overline{a(v,u)}, && \textit{for complex functions } u, v \in Z, \; \textit{(Conjugate symmetric)}, \end{aligned}$$

then an elliptic spatial operator (see Corollary A.9 on p. 178) $A \in L(Z, Z')$ is defined:

$$\langle Au, v \rangle_{Z' \times Z} := a(u,v), \forall \, u, v \in Z,$$

Remark 2.1 *The operator A given in Definition 2.2 is denoted as $A(t)$ in [34] and $L(t)$ in [50]. In these references it is considered to be time dependent whereas in this text it is time independent.*

Problem 2.1.1 (Abstract linear second-order in time evolution problem) *Given the spatial abstract operator in Definition 2.2 the following second-order in time evolution equation (abstract hyperbolic differential equation) is considered:*

$$\begin{aligned} &\partial_t^2 u + Au = f, \\ &u|_{t=0} = u_0, \ \partial_t u|_{t=0} = \bar{u}_0, \end{aligned} \tag{2.3}$$

see [34, 1.2 Problem Statement. An existence and uniqueness result] and [50, 17 The existence and uniqueness of the solution].

Remark 2.2

In this thesis, the considered function u, initial, and forcing data of Problem 2.1.1 are real, but due to technical reasons which appear in the time discretisation steps where the diagonalisation process is only done in complex arithmetic, the abstract Hilbert spaces are chosen to be over the field $\mathbb{C}$.

2.1.1 Main theorem on linear second-order in time evolution equation

Theorem 2.1 (Existence and uniqueness of second-order in time evolution problem) *Let Z and H be the Hilbert spaces given in Definition 2.1. With a form $a(\,,)$ which satisfies the properties in Theorem A.3 on p. 178:*

$$\begin{aligned} |a(u,v)| &\leq M \parallel u \parallel_Z \parallel v \parallel_Z, \qquad \forall \quad u,v \in Z, \\ a(u,u) &\geq \alpha \parallel u \parallel_Z^2, \qquad \forall \quad u \in Z, \end{aligned} \tag{2.4}$$

and let f, u_0, and $\bar{u}_0$ in (2.3) be given with

$$f \in L^2(J;H),\, u_0 \in Z,\, \bar{u}_0 \in H. \tag{2.5}$$

Then there exists a unique function u satisfying (2.3) and

$$u \in L^2(J;Z),\, \partial_t u \in L^2(J;H). \tag{2.6}$$

Detailed proofs of uniqueness and existence of the solution of Problem 2.1.1 are given in [52, Section 24.1]. Proofs of uniqueness and existence of Problem 2.1.1, with operator $A(t)$, dependent in time, can be found in [34, 1.2. Problem Statement. An Existence and Uniqueness], [33, 8.2 Existence and Uniqueness Theorem], and in [50, 17 The existence and uniqueness of the solution].

Remark 2.3 *[33, Remark 8.2] If (2.6) holds then $Au \in L^2(J;Z')$ so that (2.3) implies*

$$\partial_t^2 u \in L^2(J;Z'). \tag{2.7}$$

Theorem 2.2 (Existence and uniqueness with higher regularity) *[50, c.f. Theorem 30.1]*
For Problem 2.1.1 on p. 7 with the given initial data u_0 and $\bar{u}_0$, the properties of the abstract Hilbert spaces Z and H given in Definition 2.1 on p. 6, and supposing that

$$f \in W^{k-1,2}(J;H),\ k \geq 1,\ \textit{(see Definition A.23 on p. 181)},\ \textit{and } u_0 \in Z,\ \bar{u}_0 \in Z. \tag{2.8}$$

Then the solution u of Problem 2.1.1 satisfies

$$u \in W^{k-1,2}(J;Z),\, \partial_t^k u \in L^2(J;H). \tag{2.9}$$

Detailed proof of Theorem 2.2 is in [50, c.f. Theorem 30.1].

Corollary 2.1 (Important higher regularity results)
For Problem 2.1.1 on p. 7 and based on Theorem 2.2 the following hold:

1) *For $k = 1$, then*

$$f \in W^{0,2}(J;H) := L^2(J;H), \text{ and } u_0 \in Z, \bar{u}_0 \in Z \subset H. \tag{2.10}$$

Thus the solution u of Problem 2.1.1 satisfies

$$u \in L^2(J;Z),\, \partial_t u \in L^2(J;H), \text{ and } \partial_t^2 u \in L^2(J;Z'). \tag{2.11}$$

2) *For $k = 2$, then*

$$f \in W^{1,2}(J;H) = H^1(J;H) := \{v | v \in L^2(J;H), \partial_t v \in L^2(J;H)\}, \text{ and } u_0 \in Z, \bar{u}_0 \in Z. \tag{2.12}$$

Thus the solution u of Problem 2.1.1 satisfies

$$u \in H^1(J;Z) := \{v | v \in L^2(J;Z),\, \partial_t v \in L^2(J;Z)\},\, \partial_t^2 u \in L^2(J;H). \tag{2.13}$$

Detailed proof of Corollary 2.1 also is in [50, c.f. Theorem 30.1].

2.2 The first-order system in time

In this section, Problem 2.1.1 on p. 7; the second-order on time evolution problem, with the higher regularity results in (2.13) in Corollary 2.1 on p. 8 will be rewritten as a first-order system in time. The latter with the use of Definition 2.6 on p. 12 is given in Problem 2.2.1 on p. 12 in terms of two unknowns introduced in Definition 2.5 on p. 10.
The equivalence proof between Problems 2.1.1 and 2.2.1 is given in Lemma 2.3 on p. 12. The normed product space $(Z \times Z, \|\cdot\|_X)$ is given in Definition 2.3 and an equivalent norm to $\|\cdot\|_X$ on $Z \times Z$ in Definition 2.4 on p. 8. Also an inner product in the normed product space $(Z \times Z, \|\cdot\|_E)$ is introduced in Corollary 2.2 on p. 10. These Definitions and Corollary will be used in both parts of the thesis.

Definition 2.3 (New normed product space) *Let Z and H be the Hilbert spaces over the field $\mathbb{C}$ given in Definition 2.1 on p. 6.*
$Z \times Z$ is a normed product (see Definition A.6 on p. 172) space and the norm in this space is defined as

$$\| U \|_X := \sqrt{\| u_1 \|_Z^2 + \| u_2 \|_H^2}, \text{ for } U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in Z \times Z. \tag{2.14}$$

Remark 2.4 *In Theorem A.3 on p. 178, the Hermitian form $a : Z \times Z \to \mathbb{C}$ with real functions is symmetric, bilinear, and positive definite:*

$$\begin{aligned} a(u,v) &= a(v,u), && \forall \quad u,v \in Z, \\ |a(u,v)| &\leq M \| u \|_Z \| v \|_Z, && \forall \quad u,v \in Z, \\ a(u,u) &\geq \alpha \| u \|_Z^2, && \forall \quad u \in Z. \end{aligned}$$

With defining

$$\| u \|_a := a(u,u), \text{ for } u \in Z,$$

then $(Z, a(\,,\,))$ is a normed space.

Definition 2.4 *Let $Z \times Z$ be the normed product space given in Definition 2.3. With using Remark 2.4 the following norm is defined in the $Z \times Z$ normed product space*

$$\| U \|_E := \sqrt{a(u_1, u_1) + \| u_2 \|_H^2}, \ \textit{for } U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in Z \times Z. \tag{2.15}$$

Lemma 2.1 (Equivalent norms in the $Z \times Z$ normed product space) *Let Z and H be the Hilbert spaces over the field $\mathbb{C}$ given in Definition 2.1 on p. 6. The normed product space $Z \times Z$ given in Definition 2.3 have two equivalent norms given in Definitions 2.3 and 2.4 in (2.14) and (2.15), respectively such that for $U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in Z \times Z$*

$$\| U \|_X \leq \frac{1}{\sqrt{\min(\alpha, 1)}} \| U \|_E, \ \textit{and} \ \| U \|_E \leq \sqrt{\max(M, 1)} \| U \|_X .$$

Moreover

$$\| U \|_E \leq \sqrt{\max(M, c^2)} \| U \|_{Z \times Z}, \tag{2.16}$$

and

$$\| U \|_X \leq \sqrt{\max(1, c^2)} \| U \|_{Z \times Z} . \tag{2.17}$$

For real constants $\alpha \in \mathbb{R}$ and $M < \infty$ correspond to the lower and upper bounds of the form $a(,)$ given in Theorem A.3 on p. 178, respectively, and c is the inclusion constant between the H and Z norms inequality i.e. $\| u \|_H \leq c \| u \|_Z \ \forall u \in Z$ (see [22, (6.)]) and for $U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in Z \times Z$:

$$\| U \|_E = \sqrt{a(u_1, u_1) + \| u_2 \|_H^2}, \ \| U \|_X = \sqrt{\| u_1 \|_Z^2 + \| u_2 \|_H^2}, \ \textit{and} \ \| U \|_{Z \times Z} = \sqrt{\| u_1 \|_Z^2 + \| u_2 \|_Z^2},$$

Proof:
Starting with the definition of $\| \cdot \|_E$ in (2.15), for any $U = \binom{u_1}{u_2} \in Z \times Z$, and using the *ellipticity* of the form $a(,)$ given in Theorem A.3 on p. 178 with $\alpha \neq 0$, then

$$\| U \|_E^2 = a(u_1, u_1) + \| u_2 \|_H^2 \geq \alpha \| u_1 \|_Z^2 + \| u_2 \|_H^2 \geq \min(\alpha, 1) \overbrace{\Big(\| u_1 \|_Z^2 + \| u_2 \|_H^2 \Big)}^{= \| U \|_X^2}$$
$$\Longrightarrow \| U \|_X \leq \frac{1}{\sqrt{\min(\alpha, 1)}} \| U \|_E,$$

and when using the *continuity* of the form $a(,)$, then

$$\| U \|_E^2 = a(u_1, u_1) + \| u_2 \|_H^2 \leq M \| u_1 \|_Z^2 + \| u_2 \|_H^2 \leq \max(M, 1) \overbrace{\Big(\| u_1 \|_Z^2 + \| u_2 \|_H^2 \Big)}^{=: \| U \|_X^2}$$
$$\Longrightarrow \| U \|_E \leq \sqrt{\max(M, 1)} \| U \|_X .$$

Moreover, with using again the *continuity* of the form $a(,)$ and finally the inequality

$$\| u \|_H \leq c \| u \|_Z, \ \ \forall u \in Z,$$

imply

$$\| U \|_E^2 = a(u_1, u_1) + \| u_2 \|_H^2 \leq M \| u_1 \|_Z^2 + c^2 \| u_2 \|_Z^2 \leq \max(M, c^2) \Big(\| u_1 \|_Z^2 + \| u_2 \|_Z^2 \Big)$$
$$\Longrightarrow \| U \|_E \leq \sqrt{\max(M, c^2)} \| U \|_{Z \times Z},$$

and

$$\| U \|_X^2 = \| u_1 \|_Z^2 + \| u_2 \|_H^2 \leq \| u_1 \|_Z^2 + c^2 \| u_2 \|_Z^2 \leq \max(1, c^2)\Big(\| u_1 \|_Z^2 + \| u_2 \|_Z^2 \Big)$$

$$\Longrightarrow \| U \|_X \leq \sqrt{\max(1,c^2)} \, \| U \|_{Z\times Z} \, . \qquad \square$$

Corollary 2.2 (The $E-$inner product in the $Z \times Z$ normed-product space)
Let Z and H be Hilbert spaces over the field $\mathbb{C}$ given in Definition 2.1 on p. 6. For the normed product space $(Z \times Z, \| \cdot \|_E)$ given in Definition 2.4 on p, 9, then the following inner product in $(Z \times Z, \| \cdot \|_E)$ is defined

$$(U, V)_E := a(u_1, v_1) + (u_2, v_2)_H, \qquad \textit{for all } \; U, V \in Z \times Z, \tag{2.18}$$

such that

(1) $(\,,)_E$ in (2.18) is linear, symmetric, continuous, and positive definite for all real vector-valued functions $U, V \in Z \times Z$.

(2) From (1), if the vector-valued functions $U, V \in Z \times Z$ are complex, then all properties still holds. Only the symmetry is replaced by the conjugate-symmetry:

$$(U, V)_E = \overline{(V, U)_E}.$$

where the Hermitian form $a(\,,)$ satisfies the properties given in Theorem A.3 on p. 178:

$$\begin{aligned} a(u,v) &= a(v,u), && \textit{for real } u, v \in Z \textit{ or} \\ a(u,v) &= \overline{a(v,u)}, && \textit{for complex } u, v \in Z \\ |a(u,v)| &\leq M \| u \|_Z \| v \|_Z, && \forall \quad u, v \in Z, \\ a(u,u) &\geq \alpha \| u \|_Z^2, && \forall \quad u \in Z. \end{aligned}$$

Proof:
Firstly, for real vector-valued functions; $U, V \in Z \times Z$, and based on the definition of the Hermitian form $a(\,,)$ which is bilinear, symmetric, and positive definite as given in Theorem A.3 on p. 178 and the Hilbert space $(H, \| \cdot \|_H)$, then

$$(U, V)_E = a(u_1, v_1) + (u_2, v_2)_H = a(v_1, u_1) + (v_2, u_2)_H = (V, U)_E.$$

Secondly, if the vector-valued functions $U, V \in Z \times Z$ are complex, and ones again from the properties of $a(\,,)$ and $(H, \| \cdot \|_H)$ only the symmetry will be replaced by the conjugate-symmetry:

$$(U, V)_E = a(u_1, v_1) + (u_2, v_2)_H = \overline{a(v_1, u_1)} + \overline{(v_2, u_2)_H} = \overline{(V, U)_E},$$

and that completes the proof. $\square$

Definition 2.5 (The two new unknown functions)
The following functions are introduced

$$\begin{aligned} u_1 &:= u, \; \textit{the solution of } (2.3), \; \textit{and} \\ u_2 &:= \partial_t u. \end{aligned} \tag{2.19}$$

Now, to rewrite Problem 2.1.1 on p. 7 as a first-order system in time with using the higher regularity results for the given forcing and initial data in (2.13) given in Corollary 2.1 on p. 8:

$$f \in H^1(J;H)\, i.e.\, f \in L^2(J;H) \subset L^2(J;Z') \text{ and } \partial_t^2 u \in L^2(J;H) \subset L^2(J;Z'),\ (\textit{Gelfand triple}),$$

and using the two functions introduced in Definition 2.5 with the knowledge that:

- The operator $A \in L(Z, Z')$ given in Definition 2.2 on p. 6, that is time independent and invertible.
- $u_2 = \partial_t u_1 \in L^2(J;Z)$ from (2.19) and $\partial_t u_2 \in L^2(J;H) \subset L^2(J;Z')$.

then rewriting Problem 2.1.1 in terms of the two functions u_1 and u_2 introduced in Definition 2.5 will be:

$$\begin{aligned} \partial_t A u_1 - A u_2 &= 0, \\ \partial_t u_2 + A\, u_1 &= f, \end{aligned} \tag{2.20}$$

with given initial data $u_1|_{t=0} = u_{1,0} := u_0 \in Z,\ \ u_2|_{t=0} = u_{2,0} := \bar{u}_0 \in Z$.

Remark 2.5 *The system in (2.20) is a generalisation of the one in [30], since the abstract spatial operator A with properties given in Definition 2.2 on p. 6 covers not just the Laplace operator as in [30], but all linear time-independent elliptic spatial operators with sufficiently regular coefficients. For example:*

- *The (scalar) diffusion operator*
$$Au := -\mathrm{div}(a\nabla u),$$
occurring in the acoustic wave equation.
- *The (vectorial) Lamé-Navier operator*
$$A\mathbf{u} := -\mu\Delta\mathbf{u} + (\lambda+\mu)\nabla \mathrm{div}\, \mathbf{u}.$$

Lemma 2.2 *Let Z be the separable Hilbert space over the field $\mathbb{C}$ given in Definition 2.1 on p. 6. Then the product $Z \times Z$ with a norm*

$$\| U \|_{Z\times Z} := \sqrt{(U,U)_{Z\times Z}}, \tag{2.21}$$

is a separable Hilbert space. The Bochner space $L^2(J; Z\times Z)$ with the norm

$$\| U \|_{L^2(J;Z\times Z)} := \sqrt{(U,U)_{L^2(J;Z\times Z)}}, \tag{2.22}$$

is also a Hilbert space.

Proof:
Firstly, from Definition 2.1 Z with the norm $\| u \|_Z := \sqrt{(u,u)_Z}$ for any $u \in Z$ is a separable Hilbert space, then also holds that the product $Z \times Z$ with the following norm

$$\| U \|_{Z\times Z} = \sqrt{\| u_1 \|_Z^2 + \| u_2 \|_Z^2} = \sqrt{(u_1,u_1)_Z + (u_2,u_2)_Z} = \sqrt{(U,U)_{Z\times Z}},$$

is also a Hilbert space and separable, see [7, 1.3 Hilbert Spaces from Hilbert spaces].
Secondly, since the product $Z \times Z$ is a separable Hilbert space as shown earlier in this proof then the Bochner space $L^2(J; Z \times Z)$ induced with the following

$$\| U \|_{L^2(J;Z\times Z)} := \sqrt{(U,U)_{L^2(J;Z\times Z)}} = \sqrt{\int_J (U,U)_{Z\times Z} dt}, \tag{2.23}$$

is also a Hilbert space and also separable, (see [11, 1.7 Sobolev Spaces Involving Time] and [39, Remark 10.1.10]), and that completes the proof. □

Now, it comes to couple the latter system (2.20) in vector-valued form after using the following definition

Definition 2.6 (The matrix spatial operators)
Let Z be the Hilbert space given in Definition 2.1 on p. 6. Let $A \in L(Z \times Z')$ be the spatial (time independent) operator given in Definition 2.2 on p. 6. The following matrix operators are defined for $U := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in L^2(J; Z \times Z)$:

$$\bar{A} := \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}, \bar{A} : L^2(J; Z \times Z) \to L^2(J; Z' \times Z), \text{ where } \bar{A}U := \begin{pmatrix} Au_1 \\ u_2 \end{pmatrix} \in L^2(J; Z' \times Z),$$

and

$$\hat{A} := \begin{pmatrix} 0 & -A \\ A & 0 \end{pmatrix}, \hat{A} : L^2(J; Z \times Z) \to L^2(J; Z' \times Z') \text{ where } \hat{A}U := \begin{pmatrix} -Au_2 \\ Au_1 \end{pmatrix} \in L^2(J; Z' \times Z'). \tag{2.24}$$

Remark 2.6 *The operator $\hat{A}$ given in Definition 2.6 is skew-symmetric since*

$$-\hat{A} = \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix} = \hat{A}^T, \text{ where } \hat{A}^T \text{ means the transpose of } \hat{A}.$$

Then the system (2.20) reads:

Problem 2.2.1 (The first-order system in time)
Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let $\bar{A}$, and $\hat{A}$ be the matrix operators given in Definition 2.6.
Find $U \in L^2(J; Z \times Z)$, with $\partial_t U \in L^2(Z \times H) \subset L^2(J; Z \times Z')$:

$$\partial_t \overbrace{\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}}^{\bar{A}=} U + \overbrace{\begin{pmatrix} 0 & -A \\ A & 0 \end{pmatrix}}^{\hat{A}=} U = \bar{A} \begin{pmatrix} 0 \\ f \end{pmatrix}, \text{ a.e.}, \tag{2.25}$$

with initial data $U_0 = \begin{pmatrix} u_{1,0} \\ u_{2,0} \end{pmatrix} := \begin{pmatrix} u_0 \\ \bar{u}_0 \end{pmatrix} \in Z \times Z$, and $f \in L^2(J; H) \subset L^2(J; Z')$ which are the same initial and forcing data given in (2.3) on p. 7 which satisfy the higher regularity results in (2.13) in Corollary 2.1 on p. 8.

Lemma 2.3 (Equivalence of the evolution problem with the first-order system in time)
Problem 2.1.1 on p. 7 given that its solution satisfies the higher regularity in (2.13) in Corollary 2.1 on p. 8 is equivalent to Problem 2.2.1.

Proof:
Firstly, starting from Problem 2.1.1 on p. 7, and setting

$$\begin{aligned} &u_1 := u, \text{ the solution of Problem 2.1.1, and} \\ &u_2 := \partial_t u, \text{ which satisfy the regularity results in (2.13) in Corollary 2.1 on p. 8,} \end{aligned} \tag{2.26}$$

then

$$\partial_t \overbrace{(\partial_t u_1)}^{=:u_2} + A\, u_1 = f,$$

where the initial data

$$\overbrace{u}^{u_1:=}|_{t=0} = \overbrace{u_0}^{=:u_{1,0}}, \text{ and } \overbrace{\partial_t u_1}^{u_2=}|_{t=0} = \overbrace{\bar{u}_0}^{=:u_{2,0}},$$

and the *Bochner* spaces (see Definition A.22 on p. 181) of the latter functions in (2.26) are:

$$\begin{aligned} u_1 &\in L^2(J;Z); \text{ since } u_1 := u, \text{ and that } u \in L^2(J;Z), \\ \partial_t u_1 &\in L^2(J;Z); \text{ since } u_1 := u, \text{ and that } \partial_t u \in L^2(J;Z), \\ u_2 &\in L^2(J;Z); \text{ since } u_2 = \partial_t u_1, \text{ and that } \partial_t u_1 \in L^2(J;Z), \text{ and} \\ \partial_t u_2 &\in L^2(J;H); \text{ since } u_2 := \partial_t u, \text{ and that } \partial_t \overbrace{(\partial_t u)}^{u2:=\partial_t u} = \partial_t^2 u \in L^2(J;H), \end{aligned} \tag{2.27}$$

with $Au_2 = \partial_t A u_1 \implies (\partial_t A u_1 - Au_2 = 0)$, thus using the matrix operators in Definition 2.6 on p. 12 i.e. for $U = \binom{u_1}{u_2}$

$$\partial_t \overbrace{\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}}^{\bar{A}=} U + \overbrace{\begin{pmatrix} 0 & -A \\ A & 0 \end{pmatrix}}^{\hat{A}=} U = \begin{pmatrix} 0 \\ f \end{pmatrix},$$

then

$$\text{Problem 2.1.1} \implies \text{Problem 2.2.1}, \text{ with } u = u_1 \text{ and } \partial_t A u_1 = A u_2. \tag{2.28}$$

Secondly, starting from Problem 2.2.1, and

$$\partial_t A u_1 = A u_2 \iff \partial_t u_1 = u_2, \text{ since } A \text{ is invertible, then}$$

$$\partial_t \overbrace{u_2}^{\partial_t u_1} + A\, u_1 = f,$$

with initial data

$$\overbrace{u_1}^{u=}|_{t=0} = \overbrace{u_{1,0}}^{=u_0}, \text{ and } \overbrace{u_2}^{\partial_t u_1=}|_{t=0} = \overbrace{u_{2,0}}^{=\bar{u}_0},$$

such that $u_1 = u$ and that completes the proof. □

2.2.1 The variational formulation of the first-order system in time

Definition 2.7 (The two new dual pairings) *Let Z be the Hilbert space given in Definition 2.1 on p. 6. Let $A \in L(Z \times Z')$ be the spatial time independent operator given in Definition 2.2 on p. 6:*

$$\langle Au, v\rangle_{Z'\times Z} = a(u,v), \forall\, u, v \in Z.$$

With the matrix spatial operators given in Definition 2.6 on p. 12 we define

$$\begin{aligned}
\int_J \langle \overbrace{\begin{pmatrix} 0 & -A \\ A & 0 \end{pmatrix}}^{\hat{A}} U, V\rangle_{Z'\times Z'\times Z\times Z}\, dt &= \int_J \Big\{ -\langle Au_2, v_1\rangle_{Z'\times Z} + \langle A\, u_1, v_2\rangle_{Z'\times Z}\Big\} dt \\
&= \int_J \{ -a(u_2, v_1) + a(u_1, v_2)\} dt =: \int_J \hat{a}(U,V) dt, \\
\int_J \langle \partial_t \overbrace{\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}}^{\bar{A}=} U, V\rangle_{Z'\times Z'\times Z\times Z}\, dt &= \int_J \Big\{ -\langle \partial_t A u_1, v_1\rangle_{Z'\times Z} + \langle \partial_t\, u_2, v_2\rangle_{Z'\times Z}\Big\} dt \\
&= \int_J \{a(\partial_t u_1, v_1) + \langle \partial_t\, u_2, v_2\rangle_{Z'\times Z}\} dt,
\end{aligned} \tag{2.29}$$

for all $U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in L^2(J; Z\times Z)$.

Remark 2.7 *From Remark 2.6 on p. 12 and the symmetry property of the form $a(\,,)$, also the form $\hat{a}(U,V)$ is skew-symmetric i.e.*

$$-\hat{a}(U,V) = \hat{a}(V,U).$$

Deriving the variational formulation for Problem 2.2.1 on p. 12, starts by choosing a test vector-valued function $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in L^2(J; Z\times Z)$, testing Problem 2.2.1 with it. With using the dual pairings given in (2.29) in Definition 2.7, then

$$\int_J \Big\{ \langle \partial_t \bar{A} U, V\rangle_{Z'\times Z'\times Z\times Z} + \langle \hat{A} U, V\rangle_{Z'\times Z'\times Z\times Z}\Big\} dt = \int_J \langle \begin{pmatrix} 0 \\ f \end{pmatrix}, V\rangle_{Z'\times Z'\times Z\times Z} dt.$$

In component-wise it would be

$$\int_J \{a(\partial_t u_1, v_1) + \langle \partial_t u_2, v_2\rangle_{Z'\times Z} - a(u_2, v_1) + a(u_1, v_2)\} dt = \int_J \langle f, v_2\rangle_{Z'\times Z} dt. \tag{2.30}$$

Then with defining the following form with the use of $\hat{a}(\,,)$ given in Definition 2.7:

$$B(U,V) := \int_J \quad \{a(\partial_t u_1, v_1) + \langle \partial_t u_2, v_2\rangle_{Z'\times Z} + \hat{a}(U,V)\} dt, \tag{2.31}$$

and with given $\binom{0}{f} \in L^2(J; Z\times H)$ the functional (see Definition A.18 on p. 176) $F \in L^2(J; Z\times H)'$:

$$F(V) := \int_J (f, v_2)_H dt, \forall v_2 \in L^2(J, H), \tag{2.32}$$

where here all vector-valued functions in $L^2(J; Z\times Z)$ are also involved in the this linear map since $L^2(J; Z\times Z) \subset L^2(J; Z\times H)$.
The variational formulation of Problem 2.2.1 on p. 12 reads:

Problem 2.2.2 (The variational formulation of the first-order system in time)
Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let $B(,)$ be the form given in (2.31) and $F(\cdot)$ the functional in (2.32).
Find $U \in L^2(J; Z \times Z)$ with $\partial_t U \in L^2(J; Z \times H) \subset L^2(J; Z \times Z')$:

$$B(U, V) = F(V), \ \forall V \in L^2(J; Z \times Z), \tag{2.33}$$

for given initial data $U_0 = \begin{pmatrix} u_{1,0} \\ u_{2,0} \end{pmatrix} \in Z \times Z$ and $f \in H^1(J; H)$.

Lemma 2.4 (Equivalence of the first-order system in time with its formulation)
Problem 2.2.2 is equivalent to Problem 2.2.1 on p. 12.

Proof:
Firstly, starting with the choice of an arbitrary real vector-valued test function $V \in L^2(J; Z \times Z)$, and testing Problem 2.2.1 with it. After using the forms in Definition 2.7 on p. 14, and the given spaces of u_1, u_2 and their time partial derivatives in (2.27) on p. 13, then

$$\int_J \Big\{ \Big\langle \partial_t \overbrace{\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}}^{\bar{A}=} U, V \Big\rangle_{Z' \times Z' \times Z \times Z} + \Big\langle \overbrace{\begin{pmatrix} 0 & -A \\ A & 0 \end{pmatrix}}^{\hat{A}=} U, V \Big\rangle_{Z' \times Z' \times Z \times Z} \Big\} dt$$

$$= \int_J \Big\langle \overbrace{\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}}^{\bar{A}=} \begin{pmatrix} 0 \\ f \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Big\rangle_{Z' \times Z' \times Z \times Z} dt,$$

$$\Longrightarrow \int_J \Big\{ a(\partial_t u_1, v_1) + \langle \partial_t u_2, v_2 \rangle_{Z' \times Z} + \hat{a}(U, V) \Big\} dt = \int_J \langle f, v_2 \rangle_{Z' \times Z} dt, \qquad \forall V \in L^2(J; Z \times Z),$$

with given data $f \in H^1(J; H)$ i.e. $f \in L^2(J; H) \subset L^2(J; Z')$ and $U_0 \in Z \times Z$.
Secondly, starting with Problem 2.2.2 that is

$$\int_J \Big\{ a(\partial_t u_1, v_1) + \langle \partial_t u_2, v_2 \rangle_{Z' \times Z} + \hat{a}(U, V) \Big\} dt = \int_J \langle f, v_2 \rangle_{Z' \times Z} dt, \text{ for all } V \in L^2(J; Z \times Z).$$

After using the given forms in (2.29) in Definition 2.7 in the previous page such that

$$\int_J a(\partial_t u_1, v_1) dt = \int_J \langle \partial_t A u_1, v_1 \rangle_{Z' \times Z} dt,$$

then

$$\int_J \Big\{ \langle \partial_t \bar{A} U, V \rangle_{Z' \times Z' \times Z \times Z} + \langle \hat{A} U, V \rangle_{Z' \times Z' \times Z \times Z} \Big\} dt = \int_J \langle f, v_2 \rangle_{Z' \times Z} dt,$$

$$= \int_J \Big\langle \overbrace{\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}}^{\bar{A}=} \begin{pmatrix} 0 \\ f \end{pmatrix}, V \Big\rangle_{Z' \times Z' \times Z \times Z} dt,$$

$$\Longrightarrow \int_J \langle \partial_t \bar{A} U + \hat{A} U - \overbrace{\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}}^{\bar{A}=} \begin{pmatrix} 0 \\ f \end{pmatrix}, V \rangle_{Z' \times Z' \times Z \times Z} dt = 0.$$

Referring to Corollary A.10 on p. 179 i.e. for all $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in L^2(J; Z \times Z)$, then there exists

$$\partial_t \bar{A} U + \hat{A} U = \bar{A} \begin{pmatrix} 0 \\ f \end{pmatrix},$$

with having regular data; $f \in H^1(J; H)$, $U_0 := \binom{u_0}{\bar{u}_0} \in Z \times Z$ given in (2.13) in Corollary 2.1 on p. 8 and that completes the proof. □

2.2.2 Stability estimate, existence and uniqueness

This section starts with showing stability estimate of the solution of Problem 2.2.2 on p. 15 in Lemma 2.5 on p. 16 in the $(L^2(J; Z \times Z), \| \cdot \|_{L^2(J;E)})$-normed product space. This section ends with the existence and uniqueness proof for the solution of Problem 2.2.2 on p. 15.

Lemma 2.5 (The stability of the variational formulation of the first-order system in time) *Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let $A \in L(Z \times Z')$ be the spatial operator given in Definition 2.2 on p. 6. In the normed product space $(L^2(J; Z \times Z), \| \cdot \|_{L^2(J;E)})$, the solution of the variational formulation in (4.4) with a functional $F \in L^2(J; Z' \times Z')$ with forcing data $\binom{0}{f} \in L^2(J; Z \times H) \subset L^2(J; Z \times Z')$ and initial data $U_0 \in Z \times Z$, satisfies*

$$\| U \|_{L^2(J;E)} \leq T\sqrt{C} \big(\| f \|_{L^2(J;H)} + \| U_0 \|_E \big), \tag{2.34}$$

for

$$\| U \|^2_{L^2(J;E)} := \int_J \Big\{ \overbrace{a(u_1, u_1) + \| u_2 \|^2_H}^{\|U\|^2_E =} \Big\} dt, \ \forall U \in L^2(J; Z \times Z), \tag{2.35}$$

where C is a generic constant which does not depend on $T < \infty$.

Proof:
Starting with the variational formulation in (2.33) and considering the vector-valued test function to be

$$V = \begin{cases} \bar{V}(\tau), & \bar{V}(\tau) \in L^2((0,t); Z \times Z), \text{ for } t < T < \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

Then it becomes

$$\int_0^t \quad \{ a(\partial_\tau u_1, \bar{v}_1) + \langle \partial_\tau u_2, \bar{v}_2 \rangle_{Z' \times Z} + \hat{a}(U, \bar{V}) \} d\tau = F(\bar{V}) = \int_0^t (f, \bar{v}_2)_H d\tau. \tag{2.36}$$

Then with choosing $\bar{V} = U \in L^2((0,t); Z \times Z)$, which are all real and thus the form $a(\,,)$ is real and symmetry holds, applying the chain rule in $a(\partial_\tau u_1, u_1)$ and Theorem A.7 on p. 181 in $\langle \partial_\tau u_2, u_2 \rangle_{Z' \times Z}$:

$$a(\partial_\tau u_1, u_1) = \frac{1}{2} \frac{d}{d\tau} a(u_1, u_1), \text{ by the symmetry of } a(\,,), \text{ and}$$
$$\langle \partial_\tau u_2, u_2 \rangle_{Z' \times Z} = \frac{1}{2} \frac{d}{d\tau} \| u_2 \|^2_H, \text{ respectively,}$$

then using the *Cauchy inequality* with the *Bochner integral* norm in Definition A.22 on p. 181 with $p = 2$ with the Hilbert space $(L^2((0,t); Z \times H), \| \cdot \|_E)$, and *Young's inequality* (see Lemma A.2 on p. 181) in the r.h.s of (2.36), imply

$$\int_0^t \{ a(\partial_\tau u_1, u_1) + \langle \partial_\tau u_2, u_2 \rangle_{Z' \times Z} \overbrace{- a(u_2, u_1) + a(u_1, u_2)}^{=0} \} d\tau = \int_0^t (f, u_2)_H d\tau$$

$$\Longrightarrow \frac{1}{2}\int_0^t \frac{d}{d\tau}\{\overbrace{a(u_1,u_1)+ \parallel u_2 \parallel_H^2}^{=\|U\|_E^2}\}d\tau = \int_0^t (f,u_2)_H d\tau$$

$$\leq \sqrt{T} \parallel f \parallel_{L^2((0,t);H)} \frac{1}{\sqrt{T}} \parallel u_2 \parallel_{L^2((0,t);H)},$$

$$\leq \frac{1}{2}(T \parallel f \parallel^2_{L^2((0,t);H)}$$

$$+ \frac{1}{T} \underbrace{\parallel u_2 \parallel^2_{L^2((0,t);H)}}_{\leq \int_0^t a(u_1,u_1)d\tau + \parallel u_2 \parallel^2_{L^2((0,t);H)}})$$

$$= \frac{1}{2}(T \parallel f \parallel^2_{L^2((0,t);H)} + \frac{1}{T} \parallel U \parallel^2_{L^2((0,t);E)}),$$

with multiplying both sides with 2 and then using the *Gronwall's inequality* given in Theorem A.8 on p. 182, where

$$e^{\frac{t}{T}} \leq e^1 = C, \text{ for } t \leq T, \tag{2.37}$$

imply

$$\parallel U(t) \parallel_E^2 \leq C\Big(T \parallel f \parallel^2_{L^2((0,t);H)} + \parallel U_0 \parallel_E^2 \Big), \tag{2.38}$$

where C is a constant which does not depend on T. Now, taking the integral over $J = (0,T)$ of both sides in (2.38) and using the inequality

$$\int_0^t u^2 d\tau \leq \int_0^t u^2 d\tau + \int_t^T u^2 d\tau = \int_J u^2 dt, \ \forall t \in J = (0,T),\ 0 < t < T < \infty.$$

Knowing that $\parallel U_0 \parallel_E^2$ and $\int_J \parallel f \parallel_H^2 dt$ are constants in time, then

$$\parallel U \parallel^2_{L^2(J;E)} \leq C\big(T \underbrace{\int_J \int_0^t \parallel f \parallel_H^2 d\tau dt}_{\leq \int_J \int_J \parallel f \parallel_H^2 dt dt} + T \parallel U_0 \parallel_E^2 \big)$$

$$\leq T^2 C\big(\parallel f \parallel^2_{L^2(J;H)} + \parallel U_0 \parallel_E^2 \big)$$

$\Longrightarrow \parallel U \parallel_{L^2(J;E)} \leq T\sqrt{C}\big(\parallel f \parallel_{L^2(J;H)} + \parallel U_0 \parallel_E \big)$, after using Lemma A.3 on p. 181. □

Theorem 2.3 (Existence, uniqueness of the formulation of first-order system in time) *Let Z be Hilbert space given in Definition 2.1 on p. 6. There exists a unique real solution $U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in L^2(J; Z \times Z)$ which solves Problem 2.2.2 on p. 15.*

Proof:
Firstly, to show uniqueness, Let $U^1 = \binom{u_1^1}{u_2^1}$ and $U^2 = \binom{u_1^2}{u_2^2}$ be two solutions which solve Problem 2.2.2. Let $\bar{W} := U^1 - U^2$ which solves

$$\int_J \Big\{a(\partial_t \bar{w}_1, v_1) + \langle \partial_t \bar{w}_2, v_2 \rangle_{Z' \times Z} + \hat{a}(\bar{W}, V)\Big\}dt = 0, \forall V \in L^2(J; Z \times Z), \text{ with initial data } \bar{W}_0 = \mathbf{0}.$$

Then with choosing $V = \bar{W} \in L^2(J; Z \times Z)$, and repeating the same steps in the proof of Lemma 2.5 on p. 16 but this time with zero forcing and initial data i.e. $F = \mathbf{0}$ and $\bar{W}_0 = \mathbf{0}$:

$$\| \bar{W} \|^2_{L^2(J;E)} = 0 \Longrightarrow 0 = \bar{W} = U^1 - U^2, \text{ i.e. } U^1 = U^2. \tag{2.39}$$

Secondly, to show existence and from the given equivalence proofs in Lemmas 2.4 on p. 15 and 2.3 on p. 12, respectively i.e.

$$\text{Problem 2.2.2 on p. 15} \Longleftrightarrow \text{ Problem 2.1.1 on p. 7},$$

and from the existence of the solution of the latter as give in Theorem 2.2 with Corollary 2.1 on p. 8 and since $U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in L^2(J; Z \times Z)$ for

$$u_1 = u, \text{ the solution of Problem 2.1.1 on p. 7 and}$$
$$u_2 = \partial_t u,$$

the real functions given in Definition 2.5 on p. 10 which solve Problem 2.2.2 and satisfy (2.13) in Corollary 2.1 thus that imply the existence and that completes the proof. □

Chapter 3

High-order in time DGFEM with conformal spatial discretisation for the variational formulation of the first-order system in time

The main techniques related to the discretisation in time introduced by Dominik Schötzau (see [44]) are used in the coming sections. In addition to some techniques related to the proof of the a priori error estimate introduced by Claes Johnson approach [30]. These techniques would now be applied to approximate our continuous variational formulation of the first-order system in time by first discretising in time with using high-order in time DGFEM approach introduced by Dominik Schötzau [44] and then apply conformal spatial discretisation to the semi-discrete formulation to get the fully-discretised formulation.

3.1 The Generalisation of Schötzau theorem

In this section, the introduced high-order in time Discontinuous Galerkin Finite Element method, DGFEM for (scalar) abstract parabolic problem in [44] will now be extended to the (vector-valued) first-order system in time variational formulation of the abstract linear wave equation given in Problem 2.2.2 on p. 15.

Briefly, the idea of the extended approach is to approximate in time the continuous vector-valued solution $U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in L^2(J; Z \times Z)$ of Problem 2.2.2 on p. 15 by partitioning the time interval $J = (0, T)$ into N time steps and on each time step I_n for $n = 0, \cdots, N-1$, approximate the time bases by a polynomial in time of degree $r_n > 1$ whose coefficients are in the Hilbert-product space $(Z \times Z, \|\cdot\|_{Z\times Z})$. At any time node, the solution is allowed to be discontinuous, resulting the semi-discrete formulation. Then using a conformal spatial discretisation to get the fully-discretised formulation.

In this section, the semi-discrete spaces will be defined. Essential identity and lemmas needed to generalise the theorem in [44] will be discussed here with pointing out the differences between it and our generalised theorem in this thesis.

Definition 3.1 (Semi-discrete spaces in time)
Let $\mathcal{N}$ be a partition of the time interval $J = (0, T)$ such that the time interval J is partitioned into a

time mesh of N time steps so that $J = \cup_{n=0}^{N-1} I_n$, where

$$I_n = (t_n, t_{n+1}), \quad 0 \leq n \leq N-1.$$

The time step size $k_n := t_{n+1} - t_n$, with nodes $0 =: t_0 < t_1 \ldots < t_N := T$, and associating with each time interval I_n an approximation order $r_n \geq 0$, with storing these temporal orders in the vector $\underline{r} := \{r_n\}_{n=0}^{N-1}$. On the mesh $\mathcal{N} = \{I_n\}_{n=0}^{N-1}$, the following semi-discrete spaces are:

$$\mathcal{V}^{\underline{r}}(\mathcal{N}; Z \times Z) := \{U : J \to Z \times Z \,:\, U|_{I_n} \in P^{r_n}(I_n; Z \times Z), 0 \leq n \leq N-1\}. \tag{3.1}$$

$\mathcal{V}^{\underline{r}}(\mathcal{N}; Z \times Z)$ is a linear space consisting of piecewise polynomials in time with coefficients in $Z \times Z$. It is a subspace of $L^2(J; Z \times Z)$. $P^{r_n}(I_n; Z \times Z)$ denotes the space of r_nth order polynomials in time at I_n with coefficients in $Z \times Z$.

$$\mathcal{C}_b(\mathcal{N}; Z \times Z) := \{U : J \to Z \times Z \,:\, U|_{I_n} \in \mathcal{C}_b(I_n; Z \times Z), 0 \leq n \leq N-1\}, \tag{3.2}$$

is the space of $Z \times Z$-vector-valued functions which are bounded and piecewise continuous in time. $\mathcal{C}_b(I_n; Z \times Z)$ denotes the bounded continuous function on I_n.
If $\underline{r}$ is constant on each time interval, i.e. $r_n = r$, for all $0 \leq n \leq N-1$, the space in (3.1) will be simplified as $\mathcal{V}^r(\mathcal{N}; Z \times Z)$.

Remark 3.1
The definitions used here are the same as the ones in [44], with now considering $I_n = (t_n, t_{n+1})$, for $0 \leq n \leq N-1$ and with Hilbert-product spatial abstract spaces instead of a scalar spatial abstract Hilbert space. The time partitioning notations used here are the same as the ones used by C Johnson, [30].

Definition 3.2 *Let $\varphi_\epsilon^n, \epsilon > 0$, be the real-valued and continuous function that interpolates piecewise linearly the following values at the nodes $\{t_n, t_n + \epsilon, t_{n+1} - \epsilon, t_{n+1}\}$, for $0 \leq n \leq N-1$:*

$$\varphi_\epsilon^n(t) = \begin{cases} 0, & t \geq t_{n+1}, \\ 0, & t \leq t_n, \\ 1, & t \in [t_n + \epsilon, t_{n+1} - \epsilon], \end{cases} \tag{3.3}$$

which gives

$$\frac{d}{dt}\varphi_\epsilon^n(t) = \begin{cases} \frac{1}{\epsilon}, & \text{in } (t_n, t_n + \epsilon), \\ -\frac{1}{\epsilon}, & \text{in } (t_{n+1} - \epsilon, t_{n+1}), \\ 0, & \text{elsewhere.} \end{cases} \tag{3.4}$$

Definition 3.3 (The left/right-sided limits and jumps) *In the high-order in time DGFEM, the time polynomials on different time steps are not required to be continuous across the time nodes, thus the left/right-sided limits in $(Z \times Z, (\,,\,)_E)$ of a vector-valued function U at any given time node are defined as:*

$$U_-^n = \lim_{s \to 0, s > 0} U(t_n - s), \quad 0 < n \leq N-1,$$
$$U_+^n = \lim_{s \to 0, s > 0} U(t_n + s), 0 \leq n \leq N-1,$$

and the jump of U across t_n is

$$[U^n] := U_+^n - U_-^n, \quad 0 \leq n \leq N-1.$$

The restriction of U to I_n is denoted by U^n.

Definition 3.4 *Let Z be the Hilbert space given in Definition 2.1 on p. 6. For a time interval $J = (0,T)$, $T < \infty$, the following is defined*

$$\mathcal{Y}(J) := \left\{ U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \text{ such that } U \in L^2(J; Z \times Z), \partial_t U \in L^2(J; Z \times Z') \right\}. \tag{3.5}$$

Corollary 3.1 *Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let $U \in \mathcal{Y}(J)$, $J = (0,T)$ and $T < \infty$. For $U \in \mathcal{Y}(J)$ given in Definition 3.4, then*

$$U \in C([0,T]; Z \times H) := \text{ space of continuous functions from } [0,T] \to Z \times H.$$

Proof:
Starting from Definition 3.3 on p. 20, and for $U \in \mathcal{Y}(J)$ in component-wise would be

$$u_1 \in L^2(J; Z), \partial_t u_1 \in L^2(J; Z), \text{ and } u_2 \in L^2(J; Z), \partial_t u_2 \in L^2(J; Z'),$$

which imply that

$$\begin{aligned} &u_1 \in H^1(J; Z) := \{v | v \in L^2(J; Z), \partial_t v \in L^2(J; Z)\}, \\ &u_2 \in \mathcal{W}(J) := \left\{v | v \in L^2(J; Z), \partial_t v \in L^2(J; Z')\right\}, \text{ see [52, c.f. Proposition 23.23]}, \end{aligned}$$

due to that, then

$$u_1 \in C(\bar{J}; Z), \text{ and } u_2 \in C(\bar{J}; H).$$

Thus

$$U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in C(\bar{J}; Z \times H). \qquad \square$$

The following identity is going to be used in the proof of Theorem 3.1 on p. 23.

Lemma 3.1 (Essential identity)
Let Z be the Hilbert space given in Definition 2.1 on p. 6. Let

$$v(t,x) = \psi_\epsilon(t)\bar{v}, \text{ for } \bar{v} \in Z \text{ and } \psi_\epsilon(t) \in C_0^\infty(J), \tag{3.6}$$

where $C_0^\infty(J)$, denotes the C_0^∞-functions with compact support in J.
For $u \in \mathcal{W}(J) = \left\{v | v \in L^2(J; Z), \partial_t v \in L^2(J; Z')\right\}$ such that $\mathcal{W}(J) \subset C([0,T]; H)$, then

$$\int_J \langle \partial_t u, v \rangle_{Z' \times Z} dt = -\int_J (u, \bar{v})_H \frac{d}{dt}\psi_\epsilon(t) dt, \qquad \forall \bar{v} \in Z, \psi_\epsilon(t) \in C_0^\infty(J), \tag{3.7}$$

see Theorem A.7 on p. 181 and [52, c.f. Proposition 23.20].

Lemma 3.2 (Important Identity) *Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let $\mathcal{N} = \{I_n\}$ be the partition of $J = (0,T)$ for $n = 0, \cdots, N-1$. Let $\mathcal{Y}(J)$ be the new space given in Definition 3.3 on p. 20.*

For $U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathcal{Y}(J) = \left\{ U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \text{ such that } U \in L^2(J; Z \times Z), \partial_t U \in L^2(J; Z \times Z') \right\}$*, then at a time interval* $I_n = (t_n, t_{n+1})$

$$\int_{I_n} a(\partial_t u_1, (\varphi_\epsilon^n(t)\bar{v}_1))dt + \int_{I_n} \langle \partial_t u_2, (\varphi_\epsilon^n(t)\bar{v}_2)\rangle_{Z' \times Z} dt = (U_-^{n+1}, V_-^{n+1})_E - (U_+^n, V_+^n)_E, \tag{3.8}$$

$\forall V(t,x) = \varphi_\epsilon^n(t)\bar{V}$*, for* $\bar{V} = \binom{\bar{v}_1}{\bar{v}_2} \in Z \times Z, \varphi_\epsilon^n(t) \in C_0^\infty(I_n)$*, the time basis given in Definition 3.2 on p. 20 where* $C_0^\infty(I_n)$ *denotes the* C_0^∞*-functions with compact support in* I_n *and*

$$(U, V)_E = a(u_1, v_1) + (u_2, v_2)_H, \quad \forall U, V \in Z \times Z. \tag{3.9}$$

Remark 3.2 *In comparison to the arguments in the proof of [44, Lemma 1.3], the second term in (3.8) on p. 22 that is*

$$\int_{I_n} \langle \partial_t u_2, \varphi_\epsilon^n(t)\bar{v}_2)\rangle_{Z' \times Z} dt,$$

follows the same arguments in [44, Lemma 1.3]. Whereas, the first term in (3.8) is the additional term to what was discussed in [44]. The form a(,) is time-independent inner-product and the following lines in the proof of Lemma 3.2 on p. 21 will show how it can be treated with the second term rendering the E-inner product of left/right-sided limits of the trail and test vector-valued functions at a given time interval I_n *for* $n = 0, \cdots, N-1$.

Proof:
Given Corollary 3.1 on p. 21 which states that the vector-valued function $U \in \mathcal{Y} \subset C(\bar{J}; Z \times H)$ and for $J = \cup_{n=0}^{N-1} I_n$, then $U \in C(\bar{I}_n; Z \times H)$.
Firstly, the arguments in the proof of [44, Lemma 1.3] with the Essential identity 3.1 on p. 21 are recalled for the second term in (3.8) such that

$$\int_{I_n} \langle \partial_t u_2, v_2 \rangle_{Z' \times Z} dt = -\int_{I_n} (u_2, \bar{v}_2)_H \frac{d}{dt}\varphi_\epsilon^n(t)dt = (u_{2,-}^{n+1}, v_{2,-}^{n+1})_H - (u_{2,+}^n, v_{2,+}^n)_H. \tag{3.10}$$

Secondly, with knowing that the form $a(,)$ is a time independent inner product, then with integrating by parts then

$$\int_{I_n} a(\partial_t u_1, \bar{v}_1\varphi_\epsilon^n(t))dt = a(u_1, \bar{v}_1\varphi_\epsilon^n(t))|_{\partial I_n} - \int_{I_n} a(u_1, \partial_t(\bar{v}_1\varphi_\epsilon^n(t)))dt$$

$$\Longrightarrow \int_{I_n} a(\partial_t u_1, \bar{v}_1\varphi_\epsilon^n(t))dt = a(u_1(t_{n+1}), \bar{v}_1\varphi_\epsilon^n(t_{n+1})) - a(u_1(t_n), \bar{v}_1\varphi_\epsilon^n(t_n)) - \int_{I_n} a(u_1, \partial_t(\bar{v}_1\varphi_\epsilon^n(t)))dt.$$

Now, in (3.3) in Definition 3.2 on p. 20 the time basis function φ_ϵ^n has zero values at the nodes t_n and t_{n+1}, thus

$$\int_{I_n} a(\partial_t u_1, \bar{v}_1\varphi_\epsilon^n(t))dt = -\int_{I_n} a(u_1, \partial_t(\bar{v}_1\varphi_\epsilon^n(t)))dt,$$

with taking the limit at $\epsilon \to 0$, using Lebesgue differentiation theorem, see [46, Theorem 1.17], applying the product rule of differentiation in the second term that is

$$-\int_{I_n} a(u_1, \partial_t(\bar{v}_1\varphi_\epsilon^n(t)))dt,$$

and the left/right-sided limits given in Definition 3.3 in (3.3) on p. 20 in Hilbert space $(Z, a(\,,\,))$, and then for $I_n = (t_n, t_{n+1})$ and with using the results from (3.4) in Definition 3.2 on p. 20 and that $\partial_t \bar{v}_1 = 0$ since it is a coefficient in Z, yield to

$$
\begin{aligned}
\int_{I_n} a(\partial_t u_1, \bar{v}_1 \varphi_\epsilon^n(t))dt &= -\int_{I_n} \{a(u_1, \overbrace{\partial_t \bar{v}_1}^{=0})\varphi_\epsilon^n(t) + a(u_1, \bar{v}_1)\partial_t \varphi_\epsilon^n(t)\}dt \\
\Longrightarrow \int_{I_n} a(\partial_t u_1, (\bar{v}_1 \varphi_\epsilon^n(t)))dt &= -\int_{I_n} a(u_1 \bar{v}_1)\partial_t \varphi_\epsilon^n(t)dt \\
&= -\Big(\frac{1}{\epsilon}\int_{t_n}^{t_n+\epsilon} a(u_1, \bar{v}_1)dt + \int_{t_n+\epsilon}^{t_{n+1}-\epsilon} a(u_1, \bar{v}_1)\big(\partial_t \varphi_\epsilon(t) = 0\big)dt \\
&\qquad - \frac{1}{\epsilon}\int_{t_{n+1}-\epsilon}^{t_{n+1}} a(u_1, \bar{v}_1)dt\Big) \\
&= -\Big(\frac{1}{\epsilon}\int_{t_n}^{t_n+\epsilon} a(u_1, \bar{v}_1)dt - \frac{1}{\epsilon}\int_{t_{n+1}-\epsilon}^{t_{n+1}} a(u_1, \bar{v}_1)dt\Big)
\end{aligned}
$$

$$
\Longrightarrow \int_{I_n} a(\partial_t u_1, v_1)dt = a(u_{1,-}^{n+1}, v_{1,-}^{n+1}) - a(u_{1,+}^{n}, v_{1,+}^{n}). \tag{3.11}
$$

Finally, with adding (3.11) to (3.10), that completes the proof. □

The following problem is considered to derive the generalisation of [44, Lemma 1.3].

Problem 3.1.1
Let Z and H be the Hilbert spaces over the field $\mathbb{C}$, given in Definition 2.1 on p. 6. Let $\mathcal{Y}(J) = \{U | U \in L^2(J; Z \times Z), \partial_t U \in L^2(J; Z \times Z')\}$ be the Bochner space given in Definition 3.3 on p. 20. Let $\hat{a}(\,,\,)$ be the skew-symmetric form given in Definition 2.7 on p. 14:

$$
\hat{a}(U, V) = -a(u_2, v_1) + a(u_1, v_2), \forall U, V \in Z \times Z.
$$

Find real vector-valued function $U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathcal{Y}(J)$:

$$
\begin{aligned}
&\int_J \{a(\partial_t u_1, v_1) + \langle \partial_t u_2, v_2\rangle_{Z' \times Z} + \hat{a}(U, V)\}dt = \int_J \langle f, v_2\rangle_{Z' \times Z} dt \\
&\forall V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathcal{Y}_0(J) := \{V \in \mathcal{Y}(J) : V(0, \cdot) = 0, \text{ and } V(T, \cdot) = 0\},
\end{aligned} \tag{3.12}
$$

with given initial data $U_0 \in Z \times Z$ and forcing data $f \in H^1(J; H)$.

The formulation in (3.12) is equivalent to Problem 2.2.2 on p. 15 since U the vector-valued solution of (3.12) satisfies the higher regularity of the solution of Problem 2.2.2 and also the solution of the latter is in the space $\mathcal{Y}(J)$ for all smooth enough vector-valued test function.

Remark 3.3 *Only real trial and test vector-valued functions are considered in (3.12).*

Theorem 3.1 (Generalised Schötzau theorem) *Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let $(\,,\,)_E$ be the inner product given in Corollary 2.2 on p. 10:*

$$
(U, V)_E = a(u_1, v_1) + (u_2, v_2)_H, \quad \text{for all } U, V \in Z \times Z,
$$

The solution $U \in \mathcal{Y}(J)$ of (3.12) satisfies

$$\sum_{n=0}^{N-1} \int_{I_n} \{a(\partial_t u_1, v_1) + \langle \partial_t u_2, v_2 \rangle_{Z' \times Z} + \hat{a}(U,V)\} dt + \sum_{n=1}^{N-1} ([U]^n, V_+^n)_E + (U_+^0, V_+^0)_E$$
$$= \sum_{n=0}^{N-1} \int_{I_n} \langle f, v_2 \rangle_{Z' \times Z}\, dt + (U_0, V_+^0)_E,$$
$$\forall V \in \mathcal{C}_b(\mathcal{N}; Z \times Z), \tag{3.13}$$

where $\mathcal{C}_b(\mathcal{N}; Z \times Z)$ is the $Z \times Z$-valued functions which are bounded and piecewise continuous in time. $U_0 = \binom{u_{1,0}}{u_{2,0}}$ is the given initial data, $\mathcal{Y}(J) = \{U | U \in L^2(J; Z \times Z), \partial_t U \in L^2(J; Z \times Z')\}$ is the new space given in Definition 3.3 on p. 20, and $\hat{a}(U,V) = -a(u_2, v_1) + a(u_1, v_2)$ is the skew-symmetric form given in Definition 2.7 on p. 14.

Proof:
Firstly, integrating by parts the first two terms in (3.12) on p. 23, with $V \in \mathcal{Y}_0(J)$ for

$$\mathcal{Y}_0(J) = \{V \in \mathcal{Y}(J) : V(0,\cdot) = 0, \text{ and } V(T,\cdot) = 0\},$$

it yields

$$\overbrace{(a(u_1, v_1) + \langle u_2, v_2 \rangle_{Z' \times Z})|_{\partial J}}^{=0} - \int_J \{a(u_1, \partial_t v_1) + \langle u_2, \partial_t v_2 \rangle_{Z \times Z'}\} dt + \int_J \hat{a}(U,V) dt = \int_J \langle f, v_2 \rangle_{Z' \times Z}\, dt,$$
$$\Longrightarrow - \int_J \{a(u_1, \partial_t v_1) + \langle u_2, \partial_t v_2 \rangle_{Z \times Z'}\} dt + \int_J \hat{a}(U,V) dt = \int_J \langle f, v_2 \rangle_{Z' \times Z} dt, \quad \forall V \in \mathcal{Y}_0(J; Z \times Z).$$

Partitioning the time interval $J = (0,T)$ into subintervals $\{I_n\}_{n=0}^{N-1}$, and letting the test vector-valued function $V(t,x) = \varphi_\epsilon^n(t)\bar{V}$, for $\bar{V} \in Z \times Z$ and $\varphi_\epsilon^n(t) \in C_0^\infty(I_n)$, be the time basis given in Definition 3.2 on p. 20. By the density of $(C_0^\infty(J) \otimes Z \times Z)$ in $\mathcal{Y}_0(J; Z \times Z)$, then in every time interval I_n we have

$$- \int_{I_n} \{a(u_1, \partial_t(\bar{v}_1 \varphi_\epsilon^n(t))) + \langle u_2, \partial_t(\bar{v}_2 \varphi_\epsilon^n(t)) \rangle_{Z \times Z'}\} dt + \int_{I_n} \hat{a}(U, \bar{V}\varphi_\epsilon^n(t)) dt = \int_{I_n} \langle f, \bar{v}_2 \varphi_\epsilon^n(t) \rangle_{Z' \times Z} dt,$$

and after applying the product rule of differentiation and since v_1 and v_2 in

$$\partial_t(\bar{v}_1 \varphi_\epsilon^n(t)), \text{ and } \partial_t(\bar{v}_2 \varphi_\epsilon^n(t)), \text{ respectively are coefficient in } Z \text{ which are constants in time,}$$

then that yield

$$- \int_{I_n} a(u_1, \bar{v}_1) \frac{d}{dt}\varphi_\epsilon^n(t) dt - \int_{I_n} \langle u_2, \bar{v}_2 \rangle_{Z \times Z'} \frac{d}{dt}\varphi_\epsilon^n(t) dt + \int_{I_n} \hat{a}(U, \bar{V}\varphi_\epsilon^n(t)) dt = \int_{I_n} \langle f, \bar{v}_2 \varphi_\epsilon^n(t) \rangle_{Z' \times Z} dt, \tag{3.14}$$

with applying the result in the proof of Lemma 3.2 on p. 21 after the integration by parts and with left/right-sided limits which exist in $(Z \times Z, (,)_E)$, yields

$$(U_-^{n+1}, V_-^{n+1})_E - (U_+^n, V_+^n)_E + \int_{I_n} \hat{a}(U, \bar{V}\varphi_\epsilon^n(t)) dt = \int_{I_n} \langle f, \bar{v}_2 \varphi_\epsilon^n(t) \rangle_{Z' \times Z} dt. \tag{3.15}$$

But, since in Lemma 3.2 on p. 21

$$\int_{I_n} a(\partial_t u_1, v_1)dt + \int_{I_n} \langle \partial_t u_2, v_2\rangle_{Z'\times Z} dt = (U_-^{n+1}, V_-^{n+1})_E - (U_+^n, V_+^n)_E, \text{ for all } V(t,x) = (\varphi_\epsilon^n(t)\bar{V}),$$

for $\bar{V} \in Z \times Z$ and $\varphi_\epsilon^n(t) \in C_0^\infty(I_n)$,

then

$$\int_{I_n} \{a(\partial_t u_1, v_1) + \langle \partial_t u_2, v_2\rangle_{Z'\times Z}\}dt + \int_{I_n} \hat{a}(U,V)dt = \int_{I_n} \langle f, v_2\rangle_{Z'\times Z} dt, \ \forall V \in \mathcal{C}_b(\mathcal{V}; Z \times Z),$$

and with summing over n for $0 \le n \le N-1$, gives

$$\sum_{n=0}^{N-1} \int_{I_n} \{a(\partial_t u_1, v_1) + \langle \partial_t u_2, v_2\rangle_{Z'\times Z} + \hat{a}(U,V)\}dt = \sum_{n=0}^{N-1} \int_{I_n} \langle f, v_2\rangle_{Z'\times Z} dt. \tag{3.16}$$

Now, after summing over n, for $0 \le n \le N-1$, and using the result in Corollary 3.1 on p. 21 which states that $U \in C(\bar{J}; Z \times H)$ and similarly $U \in C(\bar{I}_n; Z \times H)$, then $[U]^n = 0$, and with setting $U_-^0 = U_0 = \begin{pmatrix} u_{1,0} \\ u_{2,0} \end{pmatrix}$, thus (3.16) can be written as

$$\sum_{n=0}^{N-1} \int_{I_n} \{a(\partial_t u_1, v_1) + \langle \partial_t u_2, v_2\rangle_{Z'\times Z} + \hat{a}(U,V)\}dt + \sum_{n=1}^{N-1} ([U]^n, V_+^n)_E + (U_+^0, V_+^0)$$
$$= \sum_{n=0}^{N-1} \int_{I_n} \langle f, v_2\rangle_{Z'\times Z}\, dt + (U_0, V_+^0)_E,$$

$\forall V \in \mathcal{C}_b(\mathcal{N}; Z \times Z)$,

and that completes the proof. □

3.2 High-order in time DGFEM, semi-discrete formulation

Let $\mathcal{N} = \{I_n\}_{n=0}^{N-1}$ be a partition of $J = (0,T)$ for $T < \infty$. Let $\underline{r}$ be the vector which stores the temporal orders in $\mathcal{N}$. Using the semi-discrete space given in Definition 3.1 on p. 19 which is

$$\mathcal{V}^{\underline{r}}(\mathcal{N}; Z \times Z) = \Big\{ U : J \to Z \times Z \, : \, U|_{I_n} \in P^{r_n}(I_n; Z \times Z), 0 \le n \le N-1 \Big\}.$$

With using the defined E-inner product in Corollary 2.2 on p. 10:

$$(U,V)_E = a(u_1, v_1) + (u_2, v_2)_H, \quad \forall U, V \in Z \times Z,$$

such that

$$\begin{aligned}\sum_{n=0}^{N-1} ([U_{DG}]^n, V_+^n)_E &= \sum_{n=1}^{N-1} ([U_{DG}]^n, V_+^n)_E + ([U_{DG}]^0, V_+^0)_E \\ &= \sum_{n=1}^{N-1} ([U_{DG}]^n, V_+^n)_H + (U_{DG,+}^0, V_+^0)_H - (U_{DG,-}^0, V_+^0)_H \\ &= \sum_{n=1}^{N-1} ([U_{DG}]^n, V_+^n)_H + (U_{DG,+}^0, V_+^0)_H - (U_0, V_+^0)_H.\end{aligned}$$

Let $\hat{a}(\,,\,)$ be the skew-symmetric form given in Definition 2.7 on p. 14:

$$\hat{a}(U_{DG}, V) = -a(u_{2,DG}, v_1) + a(u_{1,DG}, v_2),$$

then we define the semi-discrete form:

$$B_{DG}(U_{DG}, V) := \sum_{n=0}^{N-1} \int_{I_n} \Big\{(\partial_t U_{DG}, V)_E + \hat{a}(U_{DG}, V)\Big\} dt + \sum_{n=1}^{N-1} ([U_{DG}]^n, V_+^n)_E + (U_{DG,+}^0, V_+^0)_E, \tag{3.17}$$

and for $f \in L^2(J; H)$, the linear functional (see Definition A.18 on p. 176) $F_{DG} : L^2(J;H) \to \mathbb{C}$ such that $F_{DG} \in (L^2(J;H))' \equiv L^2(J;H)$ where L^2 is a pivot space i.e. $(L^2)' = L^2$ (see [3, 5.3 Normal Subspaces of a pivot space]) and $H \equiv H'$ from the Gelfand triple $(Z \subset H \subset Z')$:

$$F_{DG}(V) := \sum_{n=0}^{N-1} \int_{I_n} (f, v_2)_H dt. \tag{3.18}$$

The high-order in time DGFEM for Problem 2.2.2 on p. 15 reads:

Problem 3.2.1 (Semi-discrete formulation)
Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let $B_{DG}(\,,\,)$ be the form given in (3.17) and $F_{DG}(\cdot)$ the linear functional in (3.18). Let $(\,,\,)_E$ be the inner product given in Corollary 2.2 on p. 10:

$$(U, V)_E = a(u_1, v_1) + (u_2, v_2)_H, \quad \forall U, V \in Z \times Z.$$

Let $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z \times Z) = \Big\{U : J \to Z \times Z \,:\, U|_{I_n} \in P^{r_n}(I_n; Z \times Z), 0 \le n \le N-1\Big\}$ be the semi-discrete space given in Definition 3.1 on p. 19.
Find $U_{DG} \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z \times Z)$ such that

$$B_{DG}(U_{DG}, V) = F_{DG}(V) + (U_{DG,-}^0, V_+^0)_E, \quad \forall\; V \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z \times Z), \tag{3.19}$$

with given initial condition $U_{DG,-}^0 = U_0 = \begin{pmatrix} u_{1,0} \\ u_{2,0} \end{pmatrix} \in Z \times Z$ and forcing data $f \in H^1(J; H)$.

The coming lemma shows an important identity which will be used in the proof of a priori error estimate.

Lemma 3.3 (Important identity for the semi-discrete form $B_{DG}(\,,\,)$)
Let $\hat{a}(\,,\,)$ be the skew-symmetric form given in Definition 2.7 on p. 14:

$$\hat{a}(U_{DG}, V) = -a(u_{2,DG}, v_1) + a(u_{1,DG}, v_2),$$

and $(\,,\,)_E$ be the inner product given in Corollary 2.2 on p. 10:

$$(U, V)_E = a(u_1, v_1) + (u_2, v_2)_H, \quad \forall U, V \in Z \times Z.$$

For the form $B_{DG}(\,,\,)$ defined in (3.17) on p. 26, there holds

$$B_{DG}(U, V) = \sum_{n=0}^{N-1} \int_{I_n} \Big\{ -(\partial_t V, U)_E - \hat{a}(V, U)\Big\} dt - \sum_{n=1}^{N-1} ([V]^n, U_-^n)_E + (V_-^N, U_-^N)_E, \tag{3.20}$$

for real vector-valued $U, V \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z \times Z) = \Big\{U : J \to Z \times Z \,:\, U|_{I_n} \in P^{r_n}(I_n; Z \times Z), 0 \le n \le N-1\Big\}$. (Similar results can be found in [30, c.f. Lemma 3.2].

Proof:
Starting with integrating by parts in the first term of the form $B_{DG}(\,,\,)$ in (3.17):

$$B_{DG}(U,V)=\sum_{n=0}^{N-1}\int_{I_n}\Big\{(\partial_t U,V)_E+\hat{a}(U,V)\Big\}dt+\sum_{n=1}^{N-1}([U]^n,V_+^n)_E+(U_+^0,V_+^0)_E, \tag{3.21}$$

then

$$\sum_{n=0}^{N-1}\int_{I_n}(\partial_t U,V)_E dt=\sum_{n=0}^{N-1}(U,V)_E|_{\partial I_n}-\sum_{n=0}^{N-1}\int_{I_n}(U,\partial_t V)_E dt,$$

for $I_n=(t_n,t_{n+1})$ and taking the left/right-sided limits at the nodes $(t_{n+1}-s)$ and (t_n+s) as $s\to 0$ yield

$$\begin{aligned}\sum_{n=0}^{N-1}(U,V)_E|_{\partial I_n}&=\sum_{n=0}^{N-1}\lim_{s\to 0}\Big((U(t_{n+1}-s),V(t_{n+1}-s))_E-(U(t_n+s),V(t_n+s))_E\Big)\\&=\sum_{n=0}^{N-1}\Big((U_-^{n+1},V_-^{n+1})_E-(U_+^n,V_+^n)_E\Big),\end{aligned}$$

and with rearranging the sum terms yields

$$\sum_{n=0}^{N-1}\int_{I_n}(\partial_t U,V)_E dt=(U_-^N,V_-^N)_E+\sum_{n=1}^{N-1}(U_-^n,V_-^n)_E-\sum_{n=0}^{N-1}(U_+^n,V_+^n)_E-\sum_{n=0}^{N-1}\int_{I_n}(U,\partial_t V)_E dt. \tag{3.22}$$

Plugging (3.22) back into the form $B_{DG}(U,V)$ in (3.21) and with rearranging the sum terms where $[U]^n=U_+^n-U_-^n$, and the use of the symmetry of the E-inner product and the skew-symmetry of the form $\hat{a}(U,V)$ such that $\hat{a}(U,V)=-\hat{a}(V,U)$ given in Remark 2.7 on p. 14, then

$$\begin{aligned}B_{DG}(U,V)&=(U_-^N,V_-^N)_E+\sum_{n=1}^{N-1}(U_-^n,V_-^n)_E-(U_+^0,V_+^0)_E-\sum_{n=1}^{N-1}(U_+^n,V_+^n)_E-\sum_{n=0}^{N-1}\int_{I_n}(\partial_t V,U)_E dt\\&\qquad-\sum_{n=0}^{N-1}\int_{I_n}\hat{a}(V,U)dt+\sum_{n=1}^{N-1}(U_+^n,V_+^n)_E-\sum_{n=1}^{N-1}(U_-^n,V_+^n)_E+(U_+^0,V_+^0)_E\\&=\sum_{n=0}^{N-1}\int_{I_n}\Big\{-(\partial_t V,U)_E-\hat{a}(V,U)\Big\}dt-\sum_{n=1}^{N-1}([V]^n,U_-^n)_E+(V_-^N,U_-^N)_E,\end{aligned}$$

and that completes the proof. □

The high-order in time DGFEM formulation can be interpreted as an implicit time marching scheme.

Problem 3.2.2 (Semi-discrete time-stepping scheme)
Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let $\hat{a}(\,,\,)$ be the skew-symmetric form given in Definition 2.7 on p. 14. Let $(\,,\,)_E$ be the inner product given in Corollary 2.2 on p. 10:

$$\begin{aligned}&\hat{a}(U,V)=-a(u_2,v_1)+a(u_1,v_2),\\&(U,V)_E=a(u_1,v_1)+(u_2,v_2)_H,\quad \forall U,V\in Z\times Z.\end{aligned}$$

Let $P^{r_n}(I_n; Z \times Z)$ be the semi-discrete space of r_nth order polynomials in time at I_n with coefficients in $Z \times Z$ given in Definition 3.1 on p. 19.
Find $U_{DG} \in P^{r_n}(I_n; Z \times Z)$ such that

$$\overbrace{\int_{I_n} \Big\{(\partial_t U_{DG}, V)_E + \hat{a}(U_{DG}, V)\Big\} dt + (U^n_{DG,+}, V^n_+)_E}^{b^n_{DG}(U_{DG},V):=} = \overbrace{\int_{I_n} (f, v_2)_H dt}^{F^n_{DG}(V):=} + (U^n_{DG,-}, V^n_+)_E, \tag{3.23}$$

for all $V \in P^{r_n}(I_n; Z \times Z)$, with $U^n_{DG,-}$ to be the initial value at the time step I_n, for $0 \leq n \leq N-1$, and given forcing data $f \in H^1(J; H)$.

Corollary 3.2 *The form $b^n_{DG}(\,,\,)$ in (3.23) in Problem 3.2.2 satisfies*

$$b^n_{DG}(U, V) = (U^{n+1}_-, V^{n+1}_-)_E - \int_{I_n} (U, \partial_t V)_E dt + \int_{I_n} \hat{a}(U, V) dt, \forall U, V \in P^{r_n}(I_n; Z \times Z).$$

Proof:
Starting with integrating by parts the first term of the form $b^n_{DG}(\,,\,)$ in (3.23) in Problem 3.2.2 and repeating the same steps in the proof of Lemma 3.3 on p. 26 but this time without sums i.e.

$$\begin{aligned} b^n_{DG}(U, V) &= (U^{n+1}_-, V^{n+1}_-)_E - (U^n_+, V^n_+)_E - \int_{I_n} (U, \partial_t V)_E dt + \int_{I_n} \hat{a}(U, V) dt + (U^n_+, V^n_+)_E \\ &= (U^{n+1}_-, V^{n+1}_-)_E - \int_{I_n} (U, \partial_t V)_E dt + \int_{I_n} \hat{a}(U, V) dt, \forall U, V \in P^{r_n}(I_n; Z \times Z). \end{aligned}$$

and that completes the proof. □

3.2.1 Decoupling, existence and uniqueness

The form $B_{DG}(\,,\,)$ in the semi-discrete formulation (3.19) on p. 26 and more precisely as given in Remark 2.7 on p. 14 the form $\hat{a}(\,,\,)$ is skew-symmetric and not symmetric as the case in [44]. Because of that, time decoupling technique introduced in [44] is used in order to show at the end that our semi-discrete formulation is uniquely solvable.

3.2.1.1 Time decoupling

Let $r_n \to r$, and $I_n \to I = (t_0, t_1)$, then for a generic time step I with a step size $k = t_1 - t_0$ and an approximation order r, the implicit time marching scheme in (3.23) is now written as

$$\begin{aligned} &\text{Find } U_{DG} \in P^r(I; Z \times Z) \text{ such that} \\ &\int_I \Big\{(\partial_t U_{DG}, V)_E + \hat{a}(U_{DG}, V)\Big\} dt + (U_{DG,+}(t_0), V_+(t_0))_E = \int_I (f, v_2)_H dt + (U^0_{DG,-}, V_+(t_0))_E, \\ &\text{for all } V \in P^r(I; Z \times Z), \text{ with } U^0_{DG,-} =: U_{DG,-}(t_0) \text{ the initial value at a given time step } I. \end{aligned} \tag{3.24}$$

Definition 3.5 (Time bases polynomials)
Let $P^r(t_0, t_1)$ be the space of polynomials in time up to an order r. Let $\{\varphi_j\}^r_{j=0}$ and $\{\psi_i\}^r_{i=0}$ be two bases in $P^r(t_0, t_1)$ and their transported variants $\{\hat{\varphi}_j\}^r_{j=0}$, $\{\hat{\psi}_i\}^r_{i=0} \in P^r(-1, 1)$, respectively similar

to the way represented by Dominik Schötzau [44, (4.2) and (4.3)] i.e. for Q to be the push-forward transformation:

$$t = Q(\hat{t}) = \frac{1}{2}(t_0 + t_1 + k\hat{t}), \tag{3.25}$$

then

$$\varphi_j \circ Q(\hat{t}) = \hat{\varphi}_j(\hat{t}) \quad \text{and} \quad \psi_i \circ Q(\hat{t}) = \hat{\psi}_i(\hat{t}), \qquad \text{for } i = 0, \cdots, r,$$

and the derivatives of the latter bases are

$$\frac{d\varphi_j}{dt} = \frac{2}{k}\frac{d\hat{\varphi}_j}{d\hat{t}}, \quad \text{and} \quad \frac{d\psi_i}{dt} = \frac{2}{k}\frac{d\hat{\psi}_i}{d\hat{t}},$$

where k is the time step size.

Definition 3.6 (Semi-discrete trial and test vector-valued polynomials)
In (3.24) The semi-discrete trial and test vector-valued polynomials $U_{DG}, V \in P^r(I; Z \times Z)$ with using high-order in time DGFEM can be uniquely written as:

$$U_{DG} = \sum_{j=0}^{r} \varphi_j(t) U_j \text{ and } V = \sum_{i=0}^{r} \psi_i(t) V_i,$$

for $\{\varphi_j(t)\}_{j=0}^r$, $\{\psi_i(t)\}_{i=0}^r \in P^r(I)$; the time bases polynomials given in Definition 3.5, and sequence of coefficients $\{U_j\}_{j=0}^r$, $\{V_i\}_{i=0}^r \subset Z \times Z$. With

$$U_{DG,+}(t) = \sum_{j=0}^{r} \varphi_j^+(t) U_j \text{ and } V_+(t) = \sum_{i=0}^{r} \psi_i^+(t) V_i.$$

Definition 3.7
For $I = (t_0, t_1)$ and using time bases polynomials given in Definition 3.5, the following matrices are defined

$$\mathcal{A}_{ij} := \int_I \frac{d}{dt}\varphi_j \psi_i dt + \varphi_j^+(t_0)\psi_i^+(t_0), \text{ and } \mathcal{B}_{ij} := \int_I \varphi_i \psi_j dt, \text{ for } i, j = 0, \cdots, r, \text{ see [44, (4.5)]}$$

With using Definitions 3.5, 3.6, and 3.7, the time stepping formulation in (3.24) at a generic time step I reads:

$$\begin{aligned}
&\text{Find } \{U_j\}_{j=0}^r \subset Z \times Z \text{ such that} \\
&\sum_{i,j=0}^{r} \Big\{ \mathcal{A}_{ij}(U_j, V_i)_E + \mathcal{B}_{ij}\hat{a}(U_j, V_i) \Big\} = \sum_{i=0}^{r} \Big\{ \int_I \psi_i (f, v_{2,i})_H dt + \psi_i^+(t_0)(U_{DG,-}^0, V_i)_E \Big\}, \forall \{V_i\}_{i=0}^r \subset Z \times Z, \\
&\text{for given initial data } U_{DG,-}^0 \text{ at a given time step } I = (t_0, t_1), \text{ and forcing data } f \in H^1(J; H).
\end{aligned} \tag{3.26}$$

The resulting spatial problems in (3.26) can still be simplified using the transformation in the time bases, choosing the normalised Legendre polynomials, and then the diagonalisation in the complex arithmetic's. The coming lines will explain that in details.

Definition 3.8
With the affine transformation for the time bases $\{\varphi_j\}_{j=0}^r$, $\{\psi_i\}_{i=0}^r \in P^r(t_0, t_1)$ *given in Definition 3.5 to their variants* $\{\hat{\varphi}_j\}_{j=0}^r$, $\{\hat{\psi}_i\}_{i=0}^r \in P^r(-1,1)$*, then the matrices* $\mathcal{A}_{ij}$, $\mathcal{B}_{ij}$ *given in Definition 3.7, can be written as the following:*

$$\mathcal{A}_{ij} = \hat{\mathcal{A}}_{ij} := \int_{-1}^{1} \partial_t \hat{\varphi}_j \hat{\psi}_i d\hat{t} + \hat{\varphi^+}_j(-1)\hat{\psi}_i^+(-1), \text{ see [44, (4.6)]}, \tag{3.27}$$

$$\mathcal{B}_{ij} = \frac{k}{2}\hat{\mathcal{B}}_{ij} := \frac{k}{2}\int_{-1}^{1} \hat{\varphi}_j \hat{\psi}_i d\hat{t}, \text{ see [44, (4.7)]}. \tag{3.28}$$

Accordingly with the affine transformation done in Definition 3.8 now for the r.h.s of (3.26) will be written as

$$\begin{aligned} \int_I \psi_i (f, v_{2,i})_H dt &= \frac{k}{2}\int_{-1}^{1} \hat{\psi}_i (f \circ Q, v_{2,i})_H d\hat{t}, \text{ and} \\ \psi_i^+(t_0)(U_{DG,-}^0, V_i)_E &= \hat{\psi}_i^+(-1)(U_{DG,-}^0, V_i)_E, \qquad \text{for } i = 0, \cdots, r. \end{aligned} \tag{3.29}$$

Then, the formulation in (3.26) now reads:

$$\sum_{i,j=0}^{r} \Big\{ \hat{\mathcal{A}}_{ij}(U_j, V_i)_E + \frac{k}{2}\hat{\mathcal{B}}_{ij}\hat{a}(U_j, V_i) \Big\} = \sum_{i=0}^{r} \Big\{ \frac{k}{2}\int_{-1}^{1} \hat{\psi}_i (f \circ Q, v_{2,i})_H d\hat{t} + \hat{\psi}_i^+(-1)(U_{DG,-}^0, V_i)_E \Big\}, \tag{3.30}$$

Definition 3.9 (Normalised Legendre polynomials)
Let $\{\hat{\varphi}_j\}_{j=0}^r$, $\{\hat{\psi}_i\}_{i=0}^r \in P^r(-1,1)$ *be the time basis functions. The time basis functions are chosen to be the normalised Legendre polynomials which are defined on* $(-1,1)$*, such that*

$$\hat{\varphi}_i = \hat{\psi}_i = \sqrt{(i + 1/2)} L_i, \quad i = 0, \cdots, r, \text{ see [44, (4.17)]}$$

and have the following properties:

$$L_i(1) = 1, \text{ and } L_i(-1) = (-1)^i.$$

With choosing the time basis functions to be the normalised Legendre polynomials, then the matrices $\hat{\mathcal{A}}_{ij}$, and $\hat{\mathcal{B}}_{ij}$ in (3.30) become:

$$\hat{\mathcal{A}}_{ij} = \mathcal{A}_{ij}^L := \sqrt{(i+1/2)(j+1/2)}\Big[\int_{-1}^{1} \partial_t L_i(t) L_j(t) dt + (-1)^{i+j}\Big], \tag{3.31}$$

and with the use of the orthogonality of the normalised Legendre polynomials

$$\begin{aligned} \hat{\mathcal{B}}_{ij} &= \sqrt{(i+1/2)(j+1/2)} \int_{-1}^{1} L_i L_j d\hat{t} \\ &= \delta_{ij}, \text{ which is the identity matrix, for } i,j = 0, \cdots, r, \text{ see [44, (4.18)]}. \end{aligned} \tag{3.32}$$

Repeating the same for the r.h.s of (3.30) analogously, then the formulation in (3.30) with defining

$$x_i^1 := \sqrt{(i+1/2)}, \text{ and } x_i^2 := \sqrt{(i+1/2)}(-1)^i, \text{ for } i = 0, \cdots, r, \tag{3.33}$$

would become

$$\sum_{i,j=0}^{r}\left\{\mathcal{A}^L_{ij}(U_j,V_i)_E+\frac{k}{2}\delta_{ij}\hat{a}(U_j,V_i)\right\}=\sum_{i=0}^{r}\left\{\frac{k}{2}x_i^1\int_{-1}^{1}L_i(\hat{t})(f\circ Q,v_{2,i})_H d\hat{t}+x_i^2(U^0_{DG,-},V_i)_E\right\}, \tag{3.34}$$

Remark 3.4 (Diagonalisation in $\mathbb{C}$)
Following the practical experiments in [44], the resulting matrix; $\mathcal{A}^L$ in (3.31), after the temporal basis transformation can be diagonalised in $\mathbb{C}$ (for $0 \le r \le 100$, where practical interests do not go beyond that number of polynomial degrees in time):
There exists a matrix $Y \in \mathbb{C}^{(r+1)\times(r+1)}$ such that

$Y^{-1}\mathcal{A}^L Y = \operatorname{diag}(\lambda_1,\cdots,\lambda_{r+1})$ *or similarly* $\mathcal{A}^L = Y\operatorname{diag}(\lambda_1,\cdots,\lambda_{r+1})Y^{-1}$, *see [44, 4.3 Decoupling].*

The transformed matrix Y discussed in [44] is not unitary (i.e. $YY^{-1} \neq I$) and becomes strongly ill-conditioned (i.e. it has a high condition number; if $\operatorname{Cond}(Y) = \| Y \| \cdot \| Y^{-1} \| \to \infty$) for large r, but in practice, r varies between 0 and 10, and according to the practical observations in [44], with ranges for $0 \le r \le 100$, it is not a big disadvantage.

From Remark 3.4, unique complex functions $\mathfrak{u}_{1,q}$, and $\mathfrak{u}_{2,q}$ are defined such that

$$u_{1,j}=\sum_{q=0}^{r}Y_{jq}\mathfrak{u}_{1,q},\qquad u_{2,j}=\sum_{q=0}^{r}Y_{jq}\mathfrak{u}_{2,q},$$

and similarly for $\mathfrak{v}_{1,p}$, and $\mathfrak{v}_{2,p}$ such that

$$v_{1,i}=\sum_{p=0}^{r}(Y^{-1})_{pi}\mathfrak{v}_{1,p},\ \text{ and }\quad v_{2,i}=\sum_{p=0}^{r}(Y^{-1})_{pi}\mathfrak{v}_{2,p},$$

and defining the linear functional

$$f_i(\mathfrak{v}_{2,p}):=\int_{-1}^{1}L_i(\hat{t})(f\circ Q,\mathfrak{v}_{2,p})_H d\hat{t},\ \text{ for } i=0,\cdots,r$$

then via the linearity of the formulation in (3.34) on p. 31 it can be rewritten in terms of these unique complex ansatz and test functions as follows

Find $\{\mathfrak{U}_q\}_{q=0}^r \subset Z\times Z$ such that

$$\sum_{p,i,j,q=0}^{r}\left\{(Y^{-1})_{pi}\mathcal{A}^L_{ij}Y_{jq}(\mathfrak{U}_q,\mathfrak{V}_p)_E+\frac{k}{2}(Y^{-1})_{pi}\delta_{ij}Y_{jq}\hat{a}(\mathfrak{U}_q,\mathfrak{V}_p)\right\}=\sum_{p,i=0}^{r}(Y^{-1})_{pi}\Big\{\frac{k}{2}x_i^1f_i(\mathfrak{v}_{2,p})+x_i^2(U^0_{DG,-},\mathfrak{V}_p)_E\Big\},$$

$\forall\{\mathfrak{V}_p\}_{p=0}^r \subset Z\times Z,$ given initial data $U^0_{DG,-}$ at a given time step I, and forcing data $f \in H^1(J;H)$.

Since $Y^{-1}\mathcal{A}^L Y = \operatorname{diag}(\lambda_1,\cdots,\lambda_{r+1})$, then

$$\sum_{p,q=0}^{r}\left\{\lambda_p\delta_{pq}(\mathfrak{U}_q,\mathfrak{V}_p)_E+\frac{k}{2}\delta_{pq}\hat{a}(\mathfrak{U}_q,\mathfrak{V}_p)\right\}=\sum_{p,i=0}^{r}(Y^{-1})_{pi}\Big\{\frac{k}{2}x_i^1f_i(\mathfrak{v}_{2,p})+x_i^2(U^0_{DG,-},\mathfrak{V}_p)_E\Big\}, \tag{3.35}$$

and after rearranging the system with (3.33) on p. 30 and defining

$$\begin{aligned}(\beta_p^1)_q &:= (Y^{-1})_{pq}\, x_q^1 = (Y^{-1})_{pq}\sqrt{(q+1/2)}, \text{ and}\\ (\beta_p^2)_q &:= (Y^{-1})_{pq}\, x_q^2 = (Y^{-1})_{pq}\sqrt{(q+1/2)}(-1)^q,\end{aligned} \tag{3.36}$$

it would be given as follows

Problem 3.2.3 (Spatial problems)
Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let λ_p be the eigenvalues of the matrix $\mathcal{A}^L$ given in (3.31) on p. 30. Let $(\beta_p^1)_q$ and $(\beta_p^2)_q$ be the $(r+1)$ vectors given in (3.36). Let $\hat{a}(\,,\,)$ be the skew-symmetric form given in Definition 2.7 on p. 14. Let $(\,,\,)_E$ be the inner product given in Corollary 2.2 on p. 10 and

$$f_q(\mathfrak{v}_{2,p}) = \int_{-1}^{1} L_q(\hat{t})(f\circ Q, \mathfrak{v}_{2,p})_H d\hat{t}.$$

Find $\{\mathfrak{U}_p\}_{p=0}^r \subset Z\times Z$ such that

$$\begin{aligned}&\lambda_p(\mathfrak{U}_p,\mathfrak{V}_p)_E + \frac{k}{2}\hat{a}(\mathfrak{U}_p,\mathfrak{V}_p) = \frac{k}{2}\sum_{q=0}^{r}\Big\{(\beta_p^1)_q f_q(\mathfrak{v}_{2,p})\Big\} + \sum_{q=0}^{r}\Big\{(\beta_p^2)_q\Big\}(U_{DG,-}^0,\mathfrak{V}_p)_E,\\ &\quad \text{for each } p=0,\cdots,r, \forall\{\mathfrak{V}_p\}_{p=0}^r\subset Z\times Z, \text{ given initial data } U_{DG,-}^0 \text{ at a given time step } I,\\ &\quad \text{and forcing data } f\in H^1(J;H).\end{aligned} \tag{3.37}$$

Thus, at a given time step I, there are $2(r+1)$ systems to be solved where both $\mathfrak{U}$, $\mathfrak{V}$, the ansatz and test functions, respectively are complex.

3.2.1.2 Reduced spatial problems

The coupled formulation of the spatial problems in (3.37) in component-wise reads:

Find $\{\mathfrak{u}_{2,p}\}_{p=0}^r \subset Z$ and $\{\mathfrak{u}_{1,p}\}_{p=0}^r \subset Z$ such that

$$\lambda_p a(\mathfrak{u}_{1,p},\mathfrak{v}_{1,p}) = \frac{k}{2}a(\mathfrak{u}_{2,p},\mathfrak{v}_{1,p}) + \sum_{q=0}^{r}\Big\{(\beta_p^2)_q\Big\}a(u_{1,DG,-}^0,\mathfrak{v}_{1,p}), \tag{3.38}$$

$$\lambda_p(\mathfrak{u}_{2,p},\mathfrak{v}_{2,p})_H + \frac{k}{2}a(\mathfrak{u}_{1,p},\mathfrak{v}_{2,p}) = \frac{k}{2}\sum_{q=0}^{r}\Big\{(\beta_p^1)_q f_q(\mathfrak{v}_{2,p})\Big\} + \sum_{q=0}^{r}\Big\{(\beta_p^2)_q\Big\}(u_{2,DG,-}^0,\mathfrak{v}_{2,p})_H, \tag{3.39}$$

for each $p=0,\cdots,r$, for all $\{\mathfrak{v}_{1,p}\}_{p=0}^r\subset Z$, $\{\mathfrak{v}_{2,p}\}_{p=0}^r\subset Z$, with given data $f\in H^1(J;H)$, and $u_{1,DG,-}^0\in Z, u_{2,DG,-}^0\in Z$.

Now, the equation (3.38) is rearranged, by dividing both sides with λ_p (where $\lambda_p\neq 0$ and $0 < \operatorname{Re}\lambda_p \le \max(1,r)^2$, see [44, Lemma 4.1]) and then taking $\mathfrak{v}_{1,p}=\mathfrak{v}_{2,p}$ in (3.38):

$$a(\mathfrak{u}_{1,p},\mathfrak{v}_{2,p}) = \frac{k}{2\lambda_p}a(\mathfrak{u}_{2,p},\mathfrak{v}_{2,p}) + \frac{1}{\lambda_p}\sum_{q=0}^{r}\Big\{(\beta_p^2)_q\Big\}a(u_{1,DG,-}^0,\mathfrak{v}_{2,p}). \tag{3.40}$$

Then, replacing (3.40) back in (3.39) and multiplying the latter with $4\lambda_p$. Thus, it is enough to solve for $\{\mathfrak{u}_{2,p}\}_{p=0}^r \subset Z$ with defining

$$\mathfrak{f}(\mathfrak{v}_{2,p}) := 2k\lambda_p \sum_{q=0}^{r} \Big\{(\beta_p^1)_q f_q(\mathfrak{v}_{2,p})\Big\} = 2k\lambda_p \sum_{q=0}^{r} \Big\{(\beta_p^1)_q \int_{-1}^{1} L_q(\hat{t})(f\circ Q, \mathfrak{v}_{2,p})_H d\hat{t}\Big\},$$

such that with defining the following $b(\,,)$ form and $\mathfrak{f} \in L^2(J;Z)'$ functional,

$$b(\mathfrak{u}_{2,p}, \mathfrak{v}_{2,p}) := 4\lambda_p^2(\mathfrak{u}_{2,p}, \mathfrak{v}_{2,p})_H + k^2 a(\mathfrak{u}_{2,p}, \mathfrak{v}_{2,p}), \tag{3.41}$$

Thus the formulation is now reduced to read:

Problem 3.2.4 (Reduced spatial problems)
Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let $b(\,,)$ be the form and $\mathfrak{f}(\cdot)$ the linear functional given in (3.41).
Find $\{\mathfrak{u}_{2,p}\}_{p=0}^r \subset Z$:

$$b(\mathfrak{u}_{2,p}, \mathfrak{v}_{2,p}) = \mathfrak{f}(\mathfrak{v}_{2,p}) + \sum_{q=0}^{r} \Big\{(\beta_p^2)_q\Big\}\Big(4\lambda_p(u^0_{2,DG,-}, \mathfrak{v}_{2,p})_H - 2ka(u^0_{1,DG,-}, \mathfrak{v}_{2,p})\Big),$$

for each $p = 0, \cdots, r$, for all $\{\mathfrak{v}_{2,p}\}_{p=0}^r \subset Z$. (3.42)

After solving for unknowns $\{\mathfrak{u}_{2,p}\}_{p=0}^r \subset Z$ with given data $f \in H^1(J;H)$ and $u^0_{1,DG,-}$, $u^0_{2,DG,-} \in Z$, then it comes to update the values of $\{\mathfrak{u}_{1,p}\}_{p=0}^r \subset Z$:

$$\mathfrak{u}_{1,p} = \frac{k}{2\lambda_p}\mathfrak{u}_{2,p} + \frac{1}{\lambda_p}\sum_{q=0}^{r}\Big\{(\beta_p^2)_q\Big\}u^0_{1,DG,-}, \tag{3.43}$$

which need to be solved at any given time step I.

Now, comes to show that these reduced problems are solvable which also concludes that our real semi-discrete formulation is solvable too.

Lemma 3.4 (Inf-sup condition of reduced spatial problem's forms) *Let Z be the Hilbert space over the field $\mathbb{C}$ given in Definition 2.1 on p. 6. Let $c_1 \in \mathbb{C}$ with $Re\ c_1 \in \mathbb{R}$ and $c_2 \in \mathbb{R}$, then the form*

$$b(u, v) := c_1(u, v)_H + c_2 a(u, v), \tag{3.44}$$

satisfies

$$\inf_{u\in Z}\sup_{v\in Z} \frac{|b(u, v)|}{\| u \|_Z \cdot \| v \|_Z} \geq C = \alpha c_2 > 0. \tag{3.45}$$

The imaginary parts of $u, v \in Z$ may not be zero. $\alpha \in \mathbb{R}$ is a real constant which corresponds to the lower bound of the form $a(\,,)$ which satisfies the properties given in Theorem A.3 on p. 178:

$$\begin{aligned} a(u, v) &= a(v, u), && \textit{for real } u, v \in Z \textit{ or}\\ a(u, v) &= \overline{a(v, u)}, && \textit{for complex } u, v \in Z\\ |a(u, v)| &\leq M \| u \|_Z \| v \|_Z, && \forall\ \ u, v \in Z,\\ a(u, u) &\geq \alpha \| u \|_Z^2, && \forall\ \ u \in Z. \end{aligned}$$

Proof:
In the form $b(\, , \,)$ in (3.44), choose $v = u \in Z$, then

$$\inf_{u\in Z}\sup_{v\in Z}\frac{|b(u,v)|}{\parallel u \parallel_Z \cdot \parallel v \parallel_Z} = \inf_{u\in Z}\sup_{v\in Z}\frac{\left|c_1(u,v)_H + c_2 a(u,v)\right|}{\parallel u \parallel_Z \parallel v \parallel_Z} \geq \inf_{u\in Z}\frac{\left|c_1 \parallel u \parallel_H^2 + c_2 a(u,u)\right|}{\parallel u \parallel_Z^2}$$

Here, with knowing that

$$|x| \geq \text{ Re } x, \text{ for } x \in \mathbb{C},$$

using the *ellipticity* of the form $a(\, , \,)$, then

$$\begin{aligned}\left|c_1 \parallel u \parallel_H^2 + c_2 a(u,u)\right| \geq \text{Re }\left(c_1 \parallel u \parallel_H^2 + c_2 a(u,u)\right) &= \overbrace{(\text{ Re } c_1)\parallel u \parallel_H^2}^{>0} + c_2 a(u,u)\\ &\geq c_2 a(u,u)\\ &\geq c_2 \alpha \parallel u \parallel_Z^2,\end{aligned}$$

Thus

$$\inf_{u\in Z}\sup_{v\in Z}\frac{|b(u,v)|}{\parallel u \parallel_Z \cdot \parallel v \parallel_Z} \geq \inf_{u\in Z}\frac{c_2\alpha \parallel u \parallel_Z^2}{\parallel u \parallel_Z^2} = c_2\alpha > 0.$$

□

Lemma 3.5 (Continuity of reduced spatial problem's forms) *Let Z be the Hilbert space over the field $\mathbb{C}$ given in Definition 2.1 on p. 6. Let $c_1 \in \mathbb{C}$ and $c_2 \in \mathbb{R}$, then the form*

$$b(u,v) = c_1(u,v)_H + c_2 a(u,v), \tag{3.46}$$

satisfies

$$|b(u,v)| \leq \bar{C} \parallel u \parallel_Z \parallel v \parallel_Z, \qquad \forall u,\, v \in Z, \bar{C} > 0, \tag{3.47}$$

where the imaginary parts of $u, v \in Z$ may not be zero. $\alpha \in \mathbb{R}$ is a real constant which corresponds to the lower bound of the form $a(\, , \,)$ which satisfies the properties given in Theorem A.3 on p. 178:

$$\begin{aligned} a(u,v) &= a(v,u), && \textit{for real } u,v \in Z \textit{ or}\\ a(u,v) &= \overline{a(v,u)}, && \textit{for complex } u,v \in Z\\ |a(u,v)| &\leq M \parallel u \parallel_Z \parallel v \parallel_Z, && \forall \quad u,v \in Z,\\ a(u,u) &\geq \alpha \parallel u \parallel_Z^2, && \forall \quad u \in Z.\end{aligned}$$

Proof:
Using *Cauchy inequality*, the *continuity* of the form $a(,)$, and that $\parallel u \parallel_H \leq c \parallel u \parallel_Z, \forall\, u \in Z$, then

$$\begin{aligned} b(u,v) = c_1(u,v)_H + c_2 a(u,v) &\leq |c_1| \parallel u \parallel_H \parallel v \parallel_H + c_2 M \parallel u \parallel_Z \parallel v \parallel_Z\\ &\leq |c_1| c^2 \parallel u \parallel_Z \parallel v \parallel_Z + c_2 M \parallel u \parallel_Z \parallel v \parallel_Z\\ &= (|c_1|c^2 + c_2 M) \parallel u \parallel_Z \parallel v \parallel_Z\\ &= \bar{C} \parallel u \parallel_Z \parallel v \parallel_Z, \qquad \forall u, v \in Z, \bar{C} > 0,\end{aligned}$$

and that completes the proof. □

Lemma 3.6 (Existence and uniqueness of continuous reduced spatial problem's)
Let Z and H be the Hilbert spaces over the field $\mathbb{C}$ given in Definition 2.1 on p. 6. There exist unique complex functions $\{\mathfrak{u}_{2,p}\}_{p=0}^{r} \subset Z$ which solve the $(r+1)$ spatial problems in (3.42) on p. 33, as well as $\{\mathfrak{u}_{1,p}\}_{p=0}^{r} \subset Z$ given in (3.43) on p. 33, for given data $f \in H^1(J;H)$ and initial data $u^0_{1,DG,-}, u^0_{2,DG,-} \in Z$.

Proof:
Firstly, the form $b(\, ,)$ in (3.42) is continuous with using Lemma 3.5 after letting $c_1 = 4\lambda_p \in \mathbb{C}$ and $c_2 = k^2 \in \mathbb{R}$ where k is the time step size, such that

$$b(\mathfrak{u}_{2,p}, \mathfrak{v}_{2,p}) = 4\lambda_p^2 (\mathfrak{u}_{2,p}, \mathfrak{v}_{1,p})_H + k^2 a(\mathfrak{u}_{2,p}, \mathfrak{v}_{1,p}) \leq (4|\lambda_p| c^2 + k^2 M) \parallel \mathfrak{u}_{2,p} \parallel_Z \parallel \mathfrak{v}_{1,p} \parallel_Z, \text{ for } p = 0, \cdots, r.$$

Secondly, the form $b(\, ,)$ in (3.42) for $\mathfrak{u}_{2,p}, \mathfrak{v}_{2,p} \in Z$ satisfies the Inf-sup condition in Lemma 3.4 on p. 33 after letting $c_1 = 4\lambda_p$ and $c_2 = k^2$, where the Inf-sup constant is $k^2\alpha$.
Thirdly, for the continuous of the r.h.s of (3.42) with

$$|r.h.s| = |\mathfrak{f}(\mathfrak{v}_{2,p}) + \sum_{q=0}^{r} \Big\{ (\beta_p^2)_q \Big\} \Big(4\lambda_p (u^0_{2,DG,-}, \mathfrak{v}_{2,p})_H - 2ka(u^0_{1,DG,-}, \mathfrak{v}_{2,p}) \Big) |. \tag{3.48}$$

Here, with using *Cauchy inequality*

$$\begin{aligned} \mathfrak{f}(\mathfrak{v}_{2,p}) &= 2k\lambda_p \sum_{q=0}^{r} \Big\{ (\beta_p^1)_q \int_{-1}^{1} L_q(\hat{t})(f \circ Q, \mathfrak{v}_{2,p})_H d\hat{t} \Big\} \\ &\leq 2k|\lambda_p| \sum_{q=0}^{r} \Big\{ |(\beta_p^1)_q| \parallel f \circ Q \parallel_{L^2((-1,1);H)} \parallel L_q(\hat{t}) \mathfrak{v}_{2,p} \parallel_{L^2((-1,1);H)} \Big\}. \end{aligned}$$

The test functions $\{\mathfrak{v}_{2,p}\}_{p=0}^{r}$ are constants in time, then with using the definition of the $L^2((-1,1);H)$ norm, the orthogonal property of the Legendre polynomials, and the fact that $\parallel \mathfrak{v}_{2,p} \parallel_H \leq c \parallel \mathfrak{v}_{2,p} \parallel_Z$, yield

$$\begin{aligned} \parallel L_q(\hat{t}) \mathfrak{v}_{2,p} \parallel_{L^2((-1,1);H)} = \Big(\int_{-1}^{1} L_q^2(\hat{t}) \parallel \mathfrak{v}_{2,p} \parallel_H^2 d\hat{t} \Big)^{1/2} &= \Big(\int_{-1}^{1} L_q^2(\hat{t}) d\hat{t} \Big)^{1/2} \parallel \mathfrak{v}_{2,p} \parallel_H \\ &= \Big(\frac{2}{2q+1} \Big)^{1/2} \parallel \mathfrak{v}_{2,p} \parallel_H \\ &\leq c \Big(\frac{2}{2q+1} \Big)^{1/2} \parallel \mathfrak{v}_{2,p} \parallel_Z = c\hat{x}_q \parallel \mathfrak{v}_{2,p} \parallel_Z, \end{aligned}$$

then using the temporal transformation back from $L^2((-1,1);H)$ to $L^2(I=(t_0,t_1);H)$ and based on the given relation between the time variable t and $\hat{t}$ in (3.25) on p. 29, yield

$$\parallel f \circ Q \parallel^2_{L^2((-1,1);H)} = \int_{-1}^{1} \parallel f \circ Q \parallel_H^2 d\hat{t} = \frac{2}{k} \int_I \parallel f \parallel_H^2 dt,$$

Thus

$$\mathfrak{f}(\mathfrak{v}_{2,p}) = 2k\lambda_p \sum_{q=0}^{r} \Big\{ (\beta_p^1)_q \int_{-1}^{1} L_q(\hat{t})(f \circ Q, \mathfrak{v}_{2,p})_H d\hat{t} \Big\} \leq c\sqrt{2^3 k} |\lambda_p| \sum_{q=0}^{r} (\beta_p^1)_q \hat{x}_q \parallel f \parallel_{L^2(I;H)} \parallel \mathfrak{v}_{2,p} \parallel_Z .$$

Secondly, for

$$\sum_{q=0}^{r}\left\{(\beta_p^2)_q\right\}\Big(4\lambda_p(u^0_{2,DG,-},\mathfrak{v}_{2,p})_H - 2ka(u^0_{1,DG,-},\mathfrak{v}_{2,p})\Big),$$

with using again *Cauchy inequality*, the *continuity* of the form $a(\,,\,)$,
and again the inequality $\|\,\mathfrak{v}_{2,p}\,\|_H \leq c\,\|\,\mathfrak{v}_{2,p}\,\|_Z$, then plugging everything back in (3.48) on p. 35 imply

$$|r.h.s| \leq \Bigg(c\sqrt{2^3k}|\lambda_p|\sum_{q=0}^{r}|(\beta_p^1)_q|\hat{x}_q\,\|\,f\,\|_{L^2(I;H)} + |\sum_{q=0}^{r}(\beta_p^2)_q|\Big(4c|\lambda_p|\,\|\,u^0_{2,DG,-}\,\|_H + 2kM\,\|\,u^0_{1,DG,-}\,\|_Z\,\Big)\Bigg)\,\|\,\mathfrak{v}_{2,p}\,\|_Z\,. \tag{3.49}$$

Now, going back to Theorem [Babuška] A.4 on p. 179 with having Lemmas 3.4 and 3.5, then there exist unique functions $\{\mathfrak{u}_{2,p}\}_{p=0}^{r} \in Z$ which solve (3.42). Moreover, from the identity (3.43) on p. 33 that is

$$\mathfrak{u}_{1,p} = \frac{k}{2\lambda_p}\mathfrak{u}_{2,p} + \frac{1}{\lambda_p}\sum_{q=0}^{r}(\beta_p^2)_q u^0_{1,DG,-}, \tag{3.50}$$

for each $p = 0,\cdots,r$, then from the given existence and uniqueness of $\{\mathfrak{u}_{2,p}\}_{p=0}^{r} \subset Z$ and the given forcing data $f \in H^1(J;H)$ and initial data $u^0_{1,DG,-}, u^0_{2,DG,-} \in Z$ which are well defined, that also concludes that $\{\mathfrak{u}_{1,p}\}_{p=0}^{r} \subset Z$ at a given time step I also exists and is unique. □

Theorem 3.2 (Existence and uniqueness of the semi-discrete solution) *Let Z and H be the Hilbert spaces over the field $\mathbb{C}$ given in Definition 2.1 on p. 6.*
Let $\mathcal{V}^{\underline{r}}(\mathcal{N};Z\times Z) = \{U : J \to Z\times Z : U|_{I_n} \in P^{r_n}(I_n;Z\times Z), 0 \leq n \leq N-1\}$ be the semi-discrete space given in Definition 3.1. There exists a unique real semi-discrete vector-valued function $U_{DG} \in \mathcal{V}^{\underline{r}}(\mathcal{N};Z\times Z)$ which solves the formulation (3.19) on p. 26.

Proof:
Since the real semi-discrete vector-valued function U_{DG} is uniquely written as

$$U_{DG} = \sum_{n=0}^{N-1} U_{DG}|_{I_n},$$

where

$$U_{DG}|_{I_n} = \sum_{j=0}^{r_n}\varphi_j(t)U_j, \text{ for } Y^{r_n} \in \mathbb{C}^{(r_n+1)\times(r_n+1)} \text{ and } U_j = \sum_{i=0}^{r_n} Y^{r_n}_{ji}\mathfrak{U}_i \subset Z\times Z.$$

Then the shown existence and uniqueness of $\{\mathfrak{U}_i\}_{i=0}^{r_n} \subset Z\times Z$ in Lemma 3.6 on p. 35 implies that $U_{DG} \in \mathcal{V}^{\underline{r}}(\mathcal{N};Z\times Z)$ does exist and is unique. □

3.2.2 The stability estimate with zero forcing data $f = 0$

This section includes the lemmas and theorems of the local and global stability estimates of the semi-discrete vector-valued function $U_{DG} \in \mathcal{V}^{\underline{r}}(\mathcal{N};Z\times Z)$ which solves the high-order in time DGFEM formulation given in (3.19) on p. 26 with forcing data $f = 0$. The left-sided limits estimate of the semi-discrete solution is shown in Lemma 3.7 on p. 37 with Corollary 3.3 on p. 38. The Inf-sup condition for the local semi-discrete form of the formulation (3.19) on p. 26 is given in Lemma 3.8 on p. 38. The local stability estimate of the semi-discrete solution is proven in Lemma 3.9 on p. 40 which uses the result of Lemma 3.8.

Lemma 3.7 (Identity estimate of left-sided limit of the semi-discrete solution with $f = 0$)

Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let $P^{r_n}(I_n; Z \times Z)$ be the semi-discrete space of r_nth order polynomials in time at I_n with coefficients in $Z \times Z$ given in Definition 3.1 on p. 19. The solution $U_{DG} \in P^{r_n}(I_n; Z \times Z)$ of (3.23) on p. 28 with $f = 0$ and given initial data $U_0 \in Z \times Z$ satisfies:

$$\| U^n_{DG,-} \|^2_E + \sum_{m=0}^{n-1} \Big(\| [U_{DG}]^m \|^2_E \Big) = \| U_0 \|^2_E, \; \text{for } n \le N, \tag{3.51}$$

where U_{DG} and U_0 are real

$$\| U \|^2_E = a(u_1, u_1) + \| u_2 \|^2_H, \; \forall\, U \in Z \times Z,$$

and $[U]^n = U^+_n - U^-_n$ is the jump at a time node t_n, for $n = 0, \cdots, N-1$, given in Definition 3.3 on p. 20.

Proof:
Firstly, with choosing $V = U_{DG} \in P^{r_m}(I_m; Z \times Z)$ in (3.23) on p. 28 for some $m = 0, \cdots, n-1$, letting $f = 0$, and since U_{DG} is real and thus the form $a(\,,\,)$ is real and symmetric, thus

$$\hat{a}(U_{DG}, U_{DG}) = -a(u_{2,DG}, u_{1,DG}) + a(u_{1,DG}, u_{2,DG}) = 0,$$

imply

$$\int_{I_m} (\partial_t U_{DG}, U_{DG})_E \, dt + \| U^m_{DG,+} \|^2_E - (U^m_{DG,-}, U^m_{DG,+})_E = 0,$$

for $I_m = (t_m, t_{m+1})$ and with the symmetry of the $E-$inner product with real vector-valued functions as shown in Corollary 2.2 on p. 10:

$$(U, V)_E = a(u_1, v_1) + (u_2, v_2)_H = a(v_1, u_1) + (v_2, u_2)_H = (V, U)_E,$$

and then taking the one-sided limits at the nodes $(t_{m+1} - s)$, and $(t_m + s)$ as $s \to 0$, yields

$$\begin{aligned} &\frac{1}{2} \int_{I_m} \frac{d}{dt} \| U_{DG} \|^2_E \, dt + \| U^m_{DG,+} \|^2_E - (U^m_{DG,-}, U^m_{DG,+})_E = 0 \\ \Longrightarrow\; & \| U^{m+1}_{DG,-} \|^2_E - \| U^m_{DG,+} \|^2_E + 2 \| U^m_{DG,+} \|^2_E - 2(U^m_{DG,-}, U^m_{DG,+})_E = 0. \end{aligned}$$

Now, with rearranging the terms with adding $\| U^m_{DG,-} \|^2_E - \| U^m_{DG,-} \|^2_E$, yields

$$\| U^{m+1}_{DG,-} \|^2_E + \overbrace{\| U^m_{DG,+} \|^2_E - 2(U^m_{DG,-}, U^m_{DG,+})_E + \| U^m_{DG,-} \|^2_E}^{=\|[U_{DG}]^m\|^2_E} - \| U^m_{DG,-} \|^2_E = 0,$$

and with summing over $m = 0, \cdots, n-1$, yields

$$\| U^n_{DG,-} \|^2_E + \sum_{m=0}^{n-1} \| [U_{DG}]^m \|^2_E = \| U^0_{DG,-} \|^2_E, \text{ for } n \leq N,$$

where $U^0_{DG,-} = U_0$ is the given initial data, and that completes the proof. □

Corollary 3.3 (Stability of left-sided limits of the semi-discrete solution with $f = 0$) *Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. The following estimate holds for $0 \leq n \leq N$,*

$$\| U^n_{DG,-} \|^2_E \leq \| U_0 \|^2_E . \tag{3.52}$$

Proof:
Now, going back to the resulting estimate in (3.51), which is

$$\| U^n_{DG,-} \|^2_E + \overbrace{\sum_{m=0}^{n-1} \| [U_{DG}]^m \|^2_E}^{\geq 0} = \| U_0 \|^2_E \Longrightarrow \| U^n_{DG,-} \|^2_E \leq \| U_0 \|^2_E,$$

and that completes the proof. □

Lemma 3.8 (Inf-sup condition of local form of the semi-discrete formulation) *Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let $P^r(I; Z \times Z)$ be the semi-discrete space of rth order polynomials in time at a generic time step $I = (t_0, t_1)$ with coefficients in $Z \times Z$ given in Definition 3.1 on p. 19. The form*

$$b_{DG}(U, V) = \int_I \{(\partial_t U, V)_E + \hat{a}(U, V)\} dt + (U_+(t_0), V_+(t_0))_E, \tag{3.53}$$

satisfies

$$\inf_{U \in P^r(I;Z\times Z)} \sup_{V \in P^r(I;Z\times Z)} \frac{|b_{DG}(U,V)|}{\| U \|_{L^2(I;E)} \cdot \| V \|_{L^2(I;E)}} \geq C = \frac{2 \min_p Re\, \lambda_p}{k \| Y \|^2_2} > 0, \tag{3.54}$$

for $p = 0, \cdots, r$, where r is the time polynomial degree at the given time step I, the matrix $Y \in \mathbb{C}^{(r+1)\times(r+1)}$, λ_p with $Re\, \lambda_p > 0$ the eigenvalues of the matrix $\mathcal{A}^L$ given in (3.31) on p. 30, k is the time step size, and the normed-product space $(P^r(I; Z \times Z), \| \cdot \|_{L^2(I;E)})$, such that

$$\| U \|^2_{L^2(I;E)} = \int_I \overbrace{\{a(u_1, u_1) + \| u_2 \|^2_H\}}^{\|U\|^2_E} dt, \text{ for a vector-valued functions } U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in P^r(I; Z \times Z).$$

Proof:
Firstly, in (3.53) choose $V = U$, then

$$\inf_{U \in P^r(I;Z\times Z)} \sup_{V \in P^r(I;Z\times Z)} \frac{|b_{DG}(U,V)|}{\| U \|_{L^2(I;E)} \cdot \| V \|_{L^2(I;E)}} \geq \inf_{U \in P^r(I;Z\times Z)} \frac{|b_{DG}(U,U)|}{\| U \|^2_{L^2(I;E)}}.$$

Secondly, with the use of the unique representation of $U \in P^r(I; Z \times Z)$ and choosing the temporal basis transformation as done at Section 3.2.1.1 starting on p. 28, and the choice of Normalised Legendre polynomials given in Definition 3.9 on p. 30 and their orthogonal properties in $P^r(-1,1)$:

$$
\begin{aligned}
\| U \|_{L^2(I;E)}^2 = \int_I \| U \|_E^2 \, dt &= \int_I \| \sum_{j=0}^{r} \varphi_j(t) U_j \|_E^2 \, dt \\
&= \frac{k}{2} \int_{-1}^{1} \| \sum_{j=0}^{r} \hat{\varphi}_j(\hat{t}) U_j \|_E^2 \, d\hat{t} \\
&= \frac{k}{2} \int_{-1}^{1} \| \sum_{j=0}^{r} \sqrt{(j+1/2)} L_j(\hat{t}) U_j \|_E^2 \, d\hat{t} \\
&= \frac{k}{2} \int_{-1}^{1} (\sum_{j=0}^{r} \sqrt{(j+1/2)} L_j(\hat{t}) U_j, \sum_{j=0}^{r} \sqrt{(j+1/2)} L_j(\hat{t}) U_j)_E d\hat{t} \\
&= \frac{k}{2} \sum_{j=0}^{r} \| U_j \|_E^2 \, (j+1/2) \int_{-1}^{1} L_j^2(\hat{t}) d\hat{t} = \frac{k}{2} \sum_{j=0}^{r} \| U_j \|_E^2,
\end{aligned}
\tag{3.55}
$$

the diagonalisation step such that there exist unique vector-valued functions $\{\mathfrak{U}_i\}_{i=0}^{r} \in Z \times Z$, and a matrix $Y \in \mathbb{C}^{(r+1)\times(r+1)}$, such that

$$
U_j = \sum_{i=0}^{r} Y_{ji} \mathfrak{U}_i, \text{ or in vector notation } \vec{U} = Y \cdot \vec{\mathfrak{U}}, \text{ for } \vec{\mathfrak{U}} \in (Z \times Z)^{r+1},
$$

and with using the matrix compatibility, then

$$
\begin{aligned}
\sum_{j=0}^{r} \| U_j \|_E^2 &= \sum_{j=0}^{r} \| \sum_{i=0}^{r} Y_{ji} \mathfrak{U}_i \|_E^2 \\
&\leq \| Y \|_2^2 \sum_{j=0}^{r} \| \mathfrak{U}_j \|_E^2, \text{ where } \| \cdot \|_2 \text{ is the spectral norm, see (A.6) on p. 173,}
\end{aligned}
$$

which imply that

$$
\| U \|_{L^2(I;E)}^2 = \frac{k}{2} \sum_{j=0}^{r} \| U_j \|_E^2 \leq \frac{k}{2} \| Y \|_2^2 \sum_{j=0}^{r} \| \mathfrak{U}_j \|_E^2 . \tag{3.56}
$$

Then, the form (3.53) on p. 38 now is rewritten as

$$
\begin{aligned}
b_{DG}(U,V) = \sum_{p=0}^{r} a_{\lambda_p}(\mathfrak{U}_p, \mathfrak{V}_p) &= \sum_{p=0}^{r} \lambda_p (\mathfrak{U}_p, \mathfrak{V}_p)_E + \frac{k}{2} \hat{a}(\mathfrak{U}_p, \mathfrak{V}_p) \\
&= \sum_{p=0}^{r} \lambda_p (\mathfrak{U}_p, \mathfrak{V}_p)_E + \frac{k}{2} \big(a(\mathfrak{u}_{1,p}, \mathfrak{v}_{2,p}) - a(\mathfrak{u}_{2,p}, \mathfrak{v}_{1,p}) \big).
\end{aligned}
$$

Secondly, from above time decoupling and diagonalisation process such that

$$
U = \sum_{j=0}^{r} \phi_j(t) \sum_{i=0}^{r} Y_{ji} \mathfrak{U}_i
$$

yield

$$\inf_{U\in P^r(I;Z\times Z)}\sup_{V\in P^r(I;Z\times Z)}\frac{\left|b_{DG}(U,V)\right|}{\| U \|_{L^2(I;E)}\cdot\| V \|_{L^2(I;E)}}\geq\inf_{U\in P^r(I;Z\times Z)}\frac{\left|b_{DG}(U,U)\right|}{\| U \|^2_{L^2(I;E)}}$$

$$=\inf_{U\in P^r(I;Z\times Z)}\frac{\left|\sum_{p=0}^r a_{\lambda_p}(\mathfrak{U}_p,\mathfrak{U}_p)\right|}{\| U \|^2_{L^2(I;E)}}$$

$$\geq\inf_{\vec{\mathfrak{U}}\in(Z\times Z)^{r+1}}\frac{\left|\sum_{p=0}^r a_{\lambda_p}(\mathfrak{U}_p,\mathfrak{U}_p)\right|}{\frac{k}{2}\| Y \|_2^2\sum_{p=0}^r\| \mathfrak{U}_p \|_E^2}.$$

Here and after switching to the complex arithmetic

$$\begin{aligned}\left|\sum_{p=0}^r a_{\lambda_p}(\mathfrak{U}_p,\mathfrak{U}_p)\right| &= \left|\sum_{p=0}^r\lambda_p\| \mathfrak{U}_p \|_E^2+\frac{k}{2}\big(a(\mathfrak{u}_{1,p},\mathfrak{u}_{2,p})-a(\mathfrak{u}_{2,p},\mathfrak{u}_{1,p})\big)\right|\\
&= \left|\sum_{p=0}^r\lambda_p\| \mathfrak{U}_p \|_E^2+\frac{k}{2}\left(a(\mathfrak{u}_{1,p},\mathfrak{u}_{2,p})-\overline{a(\mathfrak{u}_{1,p},\mathfrak{u}_{2,p})}\right)\right|\\
&= \left|\sum_{p=0}^r\lambda_p\| \mathfrak{U}_p \|_E^2+k\,i\,\mathrm{Im}\,\left(a(\mathfrak{u}_{1,p},\mathfrak{u}_{2,p})\right)\right|\\
&\geq \mathrm{Re}\left(\sum_{p=0}^r\lambda_p\| \mathfrak{U}_p \|_E^2+k\,i\,\mathrm{Im}\,\left(a(\mathfrak{u}_{1,p},\mathfrak{u}_{2,p})\right)\right)\\
&= \sum_{p=0}^r(\mathrm{Re}\,\lambda_p)\| \mathfrak{U}_p \|_E^2\geq\min_p(\mathrm{Re}\,\lambda_p)\sum_{p=0}^r\| \mathfrak{U}_p \|_E^2,\end{aligned}$$

which yields that

$$\begin{aligned}\inf_{U\in P^r(I;Z\times Z)}\sup_{V\in P^r(I;Z\times Z)}\frac{\left|b_{DG}(U,V)\right|}{\| U \|_{L^2(I;E)}\cdot\| V \|_{L^2(I;E)}} &\geq\inf_{\vec{\mathfrak{U}}\in(Z\times Z)^{r+1}}\frac{\left|\sum_{p=0}^r a_{\lambda_p}(\mathfrak{U}_p,\mathfrak{U}_p)\right|}{\frac{k}{2}\| Y \|_2^2\sum_{p=0}^r\| \mathfrak{U}_p \|_E^2}\\
&\geq\inf_{\vec{\mathfrak{U}}\in(Z\times Z)^{r+1}}\frac{\min_p(\mathrm{Re}\,\lambda_p)\sum_{p=0}^r\| \mathfrak{U}_p \|_E^2}{\frac{k}{2}\| Y \|_2^2\sum_{p=0}^r\| \mathfrak{U}_p \|_E^2}\\
&=\frac{2\min_p(\mathrm{Re}\,\lambda_p)}{k\| Y \|_2^2}>0.\end{aligned}$$

□

Lemma 3.9 (Local stability estimate of semi-discrete formulation with $f=0$) *Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let $P^r(I;Z\times Z)$ be the semi-discrete* *space of rth order polynomials in time at a generic time step* $I = (t_0, t_1)$ *with coefficients in* $Z \times Z$ *given in Definition 3.1 on p. 19. Let* $(\ ,\ )_E$ *be the inner product given in Corollary 2.2 on p. 10. Let* $\hat{a}(\ ,\ )$ *be the skew-symmetric form given in Definition 2.7 on p. 14:*

$$\hat{a}(U,V) = -a(u_2, v_1) + a(u_1, v_2), \text{ and } (U,V)_E = a(u_1, v_1) + (u_2, v_2)_H, \forall U, V \in Z \times Z, \text{ respectively.}$$

At a generic time step $I = (t_0, t_1)$ *with* $f = 0$ *and* $U^0_{DG,-}$ *the given initial data at* I*, the semi-discrete vector-valued function* $U_{DG} \in P^r(I; Z \times Z)$ *which solves*

$$\overbrace{\int_I \Big\{ (\partial_t U_{DG}, V)_E + \hat{a}(U_{DG}, V) \Big\} dt + (U_{DG,+}(t_0), V_+(t_0))_E}^{b_{DG}(U_{DG},V)=} = (U_{DG,-}(t_0), V_+(t_0))_E,$$

for all $V \in P^r(I; Z \times Z)$*, where* $U_{DG,-}(t_0) = U^0_{DG,-}$ *to be the initial value at the time step* I*.*

satisfies

$$\| U_{DG} \|_{L^2(I;E)} \leq \frac{k \| Y \|_2^2 (2r+1)}{2 \min_p \ Re\ \lambda_p} \| U^0_{DG,-} \|_E, \tag{3.57}$$

where k *is the time step size,* r *is the approximation order of polynomials in time,* $\| Y \|_2^2$ *is the square of the spectral norm (see (A.6) on p. 173) of the matrix* $Y \in \mathbb{C}^{(r+1)\times(r+1)}$*, and* λ_p *with* $Re\ \lambda_p > 0$ *for* $p = 0, \cdots, r$ *are the eigenvalues of the matrix* $\mathcal{A}^L$ *given in (3.31) on p. 30 and*

$$\| U \|^2_{L^2(I;E)} = \int_I \overbrace{\{a(u_1, u_1) + \| u_2 \|^2_H\}}^{\|U\|^2_E} dt, \text{ for a vector-valued functions } U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in P^r(I; Z \times Z).$$

Proof:
With using Lemma 3.8 on p. 38 and then *Cauchy inequality*, then

$$\begin{aligned} \| U_{DG} \|_{L^2(I;E)} &\leq \frac{k \| Y \|_2^2}{2(\min_p \ \mathrm{Re}\ \lambda_p)} \sup_{V \in P^r(I;Z\times Z)} \frac{\Big| b_{DG}(U_{DG}, V) \Big|}{\| V \|_{L^2(I;E)}} \\ &= \frac{k \| Y \|_2^2}{2(\min_p \ \mathrm{Re}\ \lambda_p)} \sup_{V \in P^r(I;Z\times Z)} \frac{\Big| (U^0_{DG,-}, V_+(t_0))_E \Big|}{\| V \|_{L^2(I;E)}} \\ &\leq \frac{k \| Y \|_2^2}{2(\min_p \ \mathrm{Re}\ \lambda_p)} \sup_{V \in P^r(I;Z\times Z)} \frac{\| U^0_{DG,-} \|_E \| V_+(t_0) \|_E}{\| V \|_{L^2(I;E)}}. \end{aligned} \tag{3.58}$$

Now, to bound the norm $\| V_+(t_0) \|_E$ from above with the use of time basis transformation and using Normalised Legendre polynomials given in Definition 3.9 on p. 30 such that

$$V_+(t_0) = \sum_{p=0}^{r} \phi_p^+(t_0) V_p = \sum_{p=0}^{r} \hat{\phi}_p^+(-1) V_p = \sum_{p=0}^{r} \sqrt{(p+1/2)} L_p(-1) V_p = \sum_{p=0}^{r} \sqrt{(p+1/2)} (-1)^p V_p,$$

then with applying the $r+1$ *Cauchy inequality* imply

$$\| V_+(t_0) \|_E^2 = \| \sum_{p=0}^{r} \sqrt{(p+1/2)}(-1)^p V_p \|_E^2 \le (r+1) \sum_{p=0}^{r} \| \sqrt{(p+1/2)}(-1)^p V_p \|_E^2$$
$$\le (r+1) \sum_{p=0}^{r} \| \sqrt{(r+1/2)}(-1)^r V_p \|_E^2$$
$$= (r+1)\frac{(2r+1)}{2} \sum_{p=0}^{r} \| V_p \|_E^2 \le (2r+1)^2 \sum_{p=0}^{r} \| V_p \|_E^2 . \tag{3.59}$$

In the proof of Lemma 3.8 in (3.55) on p. 39:

$$\| U \|_{L^2(I;E)}^2 = \frac{k}{2} \sum_{p=0}^{r} \| U_p \|_E^2 \Longleftrightarrow \sqrt{\sum_{p=0}^{r} \| U_p \|_E^2} = \sqrt{\frac{2}{k}} \| U \|_{L^2(I;E)}, \tag{3.60}$$

and then with going back to (3.58) with using (3.59) and (3.60), then

$$\| U_{DG} \|_{L^2(I;E)} \le \frac{k \| Y \|_2^2}{2(\min_p \ \mathrm{Re}\ \lambda_p)} \sup_{V \in P^r(I; Z \times Z)} \frac{\sqrt{2}(2r+1) \| U_{DG,-}^0 \|_E \| V \|_{L^2(I;E)}}{\sqrt{k} \| V \|_{L^2(I;E)}}$$
$$= \sqrt{\frac{k}{2}} \frac{\| Y \|_2^2 (2r+1)}{\min_p \ \mathrm{Re}\ \lambda_p} \| U_{DG,-}^0 \|_E$$
$$\le \frac{k}{2} \frac{\| Y \|_2^2 (2r+1)}{\min_p \ \mathrm{Re}\ \lambda_p} \| U_{DG,-}^0 \|_E . \qquad \square \tag{3.61}$$

Theorem 3.3 (Global stability estimate of the semi-discrete formulation with $f = 0$) *Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6.*
Let $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z \times Z) = \Big\{ U : J \to Z \times Z \ : \ U|_{I_n} \in P^{r_n}(I_n; Z \times Z), 0 \le n \le N-1 \Big\}$ be the semi-discrete space given in Definition 3.1 on p. 19, the solution $U_{DG} \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z \times Z)$ of (3.19) on p. 26 with $f = 0$ and initial data $U_0 \in Z \times Z$ satisfies

$$\| U_{DG} \|_{L^2(J;E)} \le \mathcal{C} \| U_0 \|_E, \ \textit{for}\ \mathcal{C} < \infty, \tag{3.62}$$

where $\mathcal{C} = \max(T \frac{\| Y^{r_n} \|_2^2 (2r_n+1)}{2 \min_p \ Re\ \lambda_p})$ for $T < \infty$ is a real constant depending on the maximum value of the approximation order r_n for $n = 0, \cdots, N-1$, the square of the spectral norm (see (A.6) on p. 173) of the matrix $Y^{r_n} \in \mathbb{C}^{(r_n+1)\times(r_n+1)}$, and the eigenvalues λ_p with $Re\ \lambda_p > 0$ of the matrix $\mathcal{A}^L$ given in (3.31) on p. 30 for $p = 0, \cdots, r_n$, and

$$\| U \|_{L^2(J;E)}^2 = \int_J \{ \overbrace{a(u_1, u_1) + \| u_2 \|_H^2}^{\|U\|_E^2} \} dt, \ \textit{for a vector-valued functions}\ U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z \times Z).$$

Proof:
Starting with using the estimate (3.57) in Lemma 3.9 on p. 40 with summing over all time steps $I_n = (t_n, t_{n+1})$ such that $J = \cup_{n=0}^{N-1} I_n$ and

$$\| U_{DG} \|_{L^2(J;E)} = \sum_{n=0}^{N-1} \| U_{DG} \|_{L^2(I_n;E)},$$

using the estimate (3.52) in Corollary 3.3 on p. 38 that is

$$\| U^n_{DG,-} \|^2_E \leq \| U_0 \|^2_E,$$

and considering $k = \frac{T}{N}$ such that for $p = 0, \cdots, r_n$, then

$$\begin{aligned}
\| U_{DG} \|_{L^2(J;E)} = \sum_{n=0}^{N-1} \| U_{DG} \|_{L^2(I_n;E)} &\leq \sum_{n=0}^{N-1} \left\{ \frac{T}{2N} \frac{\| Y^{r_n} \|_2^2 \, (2r_n + 1)}{(\min_p \ \mathrm{Re}\ \lambda_p} \| U^n_{DG,-} \|_E \right\} \\
&= \frac{T}{2N} \sum_{n=0}^{N-1} \left\{ \frac{\| Y^{r_n} \|_2^2 \, (2r_n + 1)}{\min_p \ \mathrm{Re}\ \lambda_p} \| U^n_{DG,-} \|_E \right\} \\
&\leq \frac{T}{2N} \max \Big(\frac{\| Y^{r_n} \|_2^2 \, (2r_n + 1)}{\min_p \ \mathrm{Re}\ \lambda_p} \Big) \sum_{n=0}^{N-1} \underbrace{\| U^n_{DG,-} \|_E}_{\leq \| U_0 \|_E} \\
&= \max \Big(T \frac{\| Y^{r_n} \|_2^2 \, (2r_n + 1)}{2 \min_p \ \mathrm{Re}\ \lambda_p} \Big) \| U_0 \|_E = \mathcal{C} \| U_0 \|_E,
\end{aligned}$$

and that completes the proof. □

3.3 High-order in time DGFEM with conformal spatial discretisation, fully-discrete formulation

This section discusses the approximation analysis of the fully-discrete formulation given in (3.66) on p. 44 after using the conformal finite dimensional spatial space given in Definition 3.10. In Section 3.3.1, existence and uniqueness of the fully-discrete solution is shown in Theorem 3.4 on p. 48. The latter uses the result of Lemma 3.12 on p. 47, where the same techniques used to show the existence and uniqueness of the semi-discrete solution are used here too. In Section 3.3.2 starting on p. 48 discusses the stability estimate of the fully-discrete solution.

Definition 3.10 (The conformal finite dimensional spatial space) *Let Z be the Hilbert space given in Definition 2.1 on p. 6. Let $Z^h \subset Z$ be a finite dimensional subspace of Z, such that*

$$(Z^h \times Z^h, \| \cdot \|_{Z \times Z}), \text{ is a finite-dimensional product space which is a subspace of } (Z \times Z, \| \cdot \|_{Z \times Z}). \tag{3.63}$$

Let $\hat{a}(\,,\,)$ be the skew-symmetric form given in Definition 2.7 on p. 14 and $(\,,\,)_E$ be the inner product given in Corollary 2.2 on p. 10:

$$\hat{a}(U, V) = -a(u_2, v_1) + a(u_1, v_2), \text{ and } (U, V)_E = a(u_1, v_1) + (u_2, v_2)_H, \forall U, V \in Z^h \times Z^h, \text{ respectively.}$$

For the form:

$$B_{DG}(U^h_{DG},V^h)=\sum_{n=0}^{N-1}\int_{I_n}\Big\{(\partial_t U^h_{DG},V^h)_E+\hat{a}(U^h_{DG},V^h)\Big\}dt+\sum_{n=1}^{N-1}([U_{DG}]^{h,n},V^{h,n}_+)_E+(U^{h,0}_{DG,+},V^{h,0}_+)_E, \tag{3.64}$$

and the linear functional:

$$F_{DG}(V^h)=\sum_{n=0}^{N-1}\int_{I_n}(f,v^h_2)_H dt, \tag{3.65}$$

given in Section 3.2 on p. 25, the fully-discrete formulation of Problem 2.2.2 on p. 15 reads:

Problem 3.3.1 (Fully-discrete variational formulation)
Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let $Z^h\times Z^h$ be the discrete finite-dimensional product subspace given in Definition 3.10 on p. 43. Let $\mathcal{V}^{\underline{r}}(\mathcal{N};Z^h\times Z^h)$ be the fully-discrete finite-dimensional product subspace of $\mathcal{V}^{\underline{r}}(\mathcal{N};Z\times Z)$ where the latter is given in Definition 3.1 on p. 19. Let $B_{DG}(\,,)$ be the form given in (3.64) and $F_{DG}(\cdot)$ the linear functional in (3.65).
Find $U^h_{DG}\in\mathcal{V}^{\underline{r}}(\mathcal{N};Z^h\times Z^h)$ such that

$$B_{DG}(U^h_{DG},V^h)=F_{DG}(V^h)+(U^h_0,V^{h,0}_+)_E,\quad \forall\ V^h\in\mathcal{V}^{\underline{r}}(\mathcal{N};Z^h\times Z^h), \tag{3.66}$$

with given initial data $U^h_0=\begin{pmatrix}u^h_{1,0}\\ u^h_{2,0}\end{pmatrix}=U^{h,0}_{DG,-}\in Z^h\times Z^h$ and forcing data $f\in H^1(J;H)$.

The time-stepping scheme of (3.66) would be

Problem 3.3.2 (Fully-discrete time-stepping formulation)
Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let $Z^h\times Z^h$ the discrete finite-dimensional product subspace given in Definition 3.10 on p. 43. Let $P^{r_n}(I_n;Z^h\times Z^h)$ be the fully-discrete finite-dimensional product subspace of $P^{r_n}(I_n;Z\times Z)$ where the latter is the semi-discrete space of r_nth order polynomials in time at I_n with coefficients in $Z\times Z$ given in Definition 3.1 on p. 19. Let $(\,,)_E$ be the inner product given in Corollary 2.2 on p. 10. Let $\hat{a}(\,,)$ be the skew-symmetric form given in Definition 2.7 on p. 14:

$\hat{a}(U,V)=-a(u_2,v_1)+a(u_1,v_2)$, and $(U,V)_E=a(u_1,v_1)+(u_2,v_2)_H, \forall U,V\in Z^h\times Z^h$, respectively.

Find $U^h_{DG}\in P^{r_n}(I_n;Z^h\times Z^h)$ such that

$$\begin{aligned}&\int_{I_n}\Big\{(\partial_t U^h_{DG},V^h)_E+\hat{a}(U^h_{DG},V^h)\Big\}dt+(U^{h,n}_{DG,+},V^{h,n}_+)_E=\int_{I_n}(f,v^h_2)_H dt+(U^{h,n}_{DG,-},V^{h,n}_+)_E,\\ &\text{for all } V^h\in P^{r_n}(I_n;Z^h\times Z^h), \text{ with } U^{h,n}_{DG,-} \text{ to be the initial value at the time step } I_n,\\ &\text{for } 0\le n\le N-1.\end{aligned} \tag{3.67}$$

Assumption 3.1 (The E-projection of the given initial data)
Let Z be the Hilbert space given in Definition 2.1 on p. 6. Let $Z^h\subset Z$ be the conformal finite dimensional subspace of Z. For the initial data $U_0\in Z\times Z$ of Problem 2.2.2 on p. 15 and U^h_0 to be the initial data of (3.66) and $\bar{\Pi}:=\begin{pmatrix}\bar{\Pi}_1 & 0\\ 0 & \bar{\Pi}_2\end{pmatrix}:Z\times Z\to Z^h\times Z^h$.
Find $U^h_0\in Z^h\times Z^h$:

$$(U^h_0-U_0,V^h)_E=0,\forall\, V^h\in Z^h\times Z^h, \tag{3.68}$$

i.e. $U^h_0:=\bar{\Pi}U_0$ is the $E-$projection of U_0 into $Z^h\times Z^h$.

Lemma 3.10 ((Discrete) Inf-sup condition of local form of the fully-discrete formulation) *Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6 and Z^h be the conformal finite dimensional subspace of the Hilbert space Z given in Definition 3.10 on p. 43. Let $P^r(I; Z^h \times Z^h)$ be the fully-discrete finite-dimensional product subspace of $P^r(I; Z \times Z)$ where the latter is the semi-discrete space of rth order polynomials in time at a generic time step $I = (t_0, t_1)$ with coefficients in $Z \times Z$ given in Definition 3.1 on p. 19. The form*

$$b_{DG}(U, V) = \int_I \{(\partial_t U, V)_E + \hat{a}(U, V)\}dt + (U_+(t_0), V_+(t_0))_E, \text{ for } U, V \in P^r(I; Z^h \times Z^h) \tag{3.69}$$

satisfies

$$\inf_{U \in P^r(I;Z^h \times Z^h)} \sup_{V \in P^r(I;Z^h \times Z^h)} \frac{|b_{DG}(U, V)|}{\| U \|_{L^2(I;E)} \cdot \| V \|_{L^2(I;E)}} \geq C = \frac{2 \min_p Re\, \lambda_p}{k \| Y \|_2^2} > 0. \tag{3.70}$$

For $p = 0, \cdots, r$, where r is the time polynomial degree at the given time step I, the matrix $Y \in \mathbb{C}^{(r+1)\times(r+1)}$. λ_p with $Re\, \lambda_p > 0$ are the eigenvalues of the matrix $\mathcal{A}^L$ given in (3.31) on p. 30, k is the time step size, and fully-discrete normed-product space $(P^r(I; Z^h \times Z^h), \| \cdot \|_{L^2(I;E)})$, such that

$$\| U \|^2_{L^2(I;E)} = \int_I \{\overbrace{a(u_1, u_1) + \| u_2 \|^2_H}^{\|U\|^2_E}\}dt, \text{ for vector-valued functions } U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in P^r(I; Z^h \times Z^h),$$

and

$$\hat{a}(U, V) = -a(u_2, v_1) + a(u_1, v_2), \text{ and } (U, V)_E = a(u_1, v_1) + (u_2, v_2)_H, \forall U, V \in Z^h \times Z^h, \text{ respectively,}$$

where the form $a(,)$ is given in Definition 2.2 on p. 6:

$$\begin{aligned} |a(u, v)| &\leq M \| u \|_Z \| v \|_Z, && \forall \quad u, v \in Z^h \subset Z, \\ a(u, u) &\geq \alpha \| u \|^2_Z, && \forall \quad u \in Z^h \subset Z. \end{aligned}$$

Proof:
The form $b_{DG}(,)$ in (3.69) is the same as the one in (3.53) in Lemma 3.8 on p. 38 for $U, V \in P^r(I; Z^h \times Z^h) \subset P^r(I; Z \times Z)$ and this is because Z^h is a conformal finite dimensional subspace of Z and thus the conformal finite dimensional space $(P^r(I; Z^h \times Z^h), \| \cdot \|_{L^2(I;E)})$ is a subspace of $P^r(I; Z \times Z)$. Thus all the steps in the proof of Lemma 3.8 on p. 38 still hold and that completes the proof. □

3.3.1 Existence and uniqueness, reduced discrete spatial problems

Let $P^r(I; Z^h \times Z^h)$ be the fully-discrete finite-dimensional product subspace of rth order polynomials in time at a generic time step $I = (t_0, t_1)$ with coefficients in $Z^h \times Z^h$. Let $U^h_{DG} P^r(I; Z^h \times Z^h)$ be the fully-discrete vector-valued function which solves (3.67):

$$U^h_{DG} = \sum_{j=0}^{r} \varphi_j(t) U^h_j, \text{ for time basis } \varphi_j(t) \in P^r(I), \text{ and unique coefficients } U^h_j \in Z^h \times Z^h, \tag{3.71}$$

which approximates $U_{DG} = \sum_{j=0}^{r} \varphi_j U_j$ the solution of (3.23) on p. 28. Now, with recalling the time decoupling process done previously in Section 3.2.1 and the diagonalisation i.e. for $\{U\}^r_{j=0} =$

$\left\{ \begin{pmatrix} u_{1,j} \\ u_{2,j} \end{pmatrix} \right\}_{j=0}^{r}$ such that $U_j = \sum_{i=0}^{r} Y_{ji} \mathfrak{U}_i$, where $Y \in \mathbb{C}^{(r+1)\times(r+1)}$ is the transformation matrix and $\{\mathfrak{U}_j\}_{j=0}^r \in Z \times Z$ is the solution of the formulation (3.37) on p. 32 which need to be solved at the generic time step I. Then the following hold for $\{U_j^h\}_{j=0}^r = \left\{ \begin{pmatrix} u_{1,j}^h \\ u_{2,j}^h \end{pmatrix} \right\}_{j=0}^{r}$, such that

$$U_j^h = \sum_{i=0}^{r} Y_{ji} \mathfrak{U}_i^h.$$

Then let $\{\mathfrak{U}_j^h\}_{j=0}^r \subset Z^h \times Z^h \subset Z \times Z$ be the conformal approximative solution of (3.37) on p. 32, which solves the following formulation:

Problem 3.3.3 (Discrete spatial problems)
Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let $Z^h \times Z^h$ the discrete finite-dimensional product subspace given in Definition 3.10 on p. 43. Let λ_p be the eigenvalues of the matrix $\mathcal{A}^L$ given in (3.31) on p. 30. Let $(\,,)_E$ be the inner product given in Corollary 2.2 on p. 10 and let $(\beta_p^1)_q$ and $(\beta_p^2)_q$ be elements given in (3.36):

$$(U,V)_E = a(u_1,v_1) + (u_2,v_2)_H, \forall U,V \in Z^h \times Z^h, \tag{3.72}$$

$$(\beta_p^1)_q = (Y^{-1})_{pq}\, x_q^1 = (Y^{-1})_{pq}\sqrt{(q+1/2)}, \text{ and} \tag{3.73}$$

$$(\beta_p^2)_q = (Y^{-1})_{pq}\, x_q^2 = (Y^{-1})_{pq}\sqrt{(q+1/2)}(-1)^q, \text{ respectively.} \tag{3.74}$$

Find $\{\mathfrak{U}_p^h\}_{p=0}^r \subset Z^h \times Z^h$:

$$\lambda_p(\mathfrak{U}_p^h, \mathfrak{V}_p^h)_E + \frac{k}{2}\Big(a(\mathfrak{u}_{1,p}^h, \mathfrak{v}_{2,p}) - a(\mathfrak{u}_{2,p}^h, \mathfrak{v}_{1,p}^h)\Big) = \frac{k}{2}\sum_{i=0}^{r}\Big\{(\beta_p^1)_i \int_{-1}^{1} L_i(\hat{t})(f\circ Q, \mathfrak{v}_{2,p}^h)_H d\hat{t}\Big\} + \sum_{i=0}^{r}\Big\{(\beta_p^2)_i\Big\}(U_{DG,-}^{h,0}, \mathfrak{V}_p^h)_E,$$

for each $p = 0, \cdots, r$, for all $\{\mathfrak{V}_p^h\}_{p=0}^r \subset Z^h \times Z^h$, given forcing data $f \in H^1(J;H)$, and $U_{DG,-}^{h,0} \in Z^h \times Z^h$ initial data at the given generic time step $I = (t_0, t_1)$, and k is the time step size. (3.75)

Problem 3.3.4 (Discrete reduced spatial problems)
Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let $Z^h \times Z^h$ be the discrete finite-dimensional product subspace given in Definition 3.10 on p. 43. Let λ_p be the eigenvalues of the matrix $\mathcal{A}^L$ given in (3.31) on p. 30. Let $(\beta_p^1)_q$ and $(\beta_p^2)_q$ be the $(r+1)$ row vectors defined in (3.36) and also given in (3.74).
Again with following the same steps of reduction done in Section 3.2.1.2 for (3.75), then the formulation in (3.75) with

$$b(\mathfrak{u}_{2,p}^h, \mathfrak{v}_{2,p}^h) = 4\lambda_p^2(\mathfrak{u}_{2,p}^h, \mathfrak{v}_{2,p}^h)_H + k^2 a(\mathfrak{u}_{2,p}^h, \mathfrak{v}_{2,p}^h),$$

$$\mathfrak{f}(\mathfrak{v}_{2,p}^h) = 2k\lambda_p \sum_{q=0}^{r}\Big\{(\beta_p^1)_q \int_{-1}^{1} L_q(\hat{t})(f\circ Q, \mathfrak{v}_{2,p}^h)_H d\hat{t}\Big\},$$

now reads:
Find $\{\mathfrak{u}_{2,p}^h\}_{p=0}^r \subset Z^h$*:*

$$b(\mathfrak{u}_{2,p}^h, \mathfrak{v}_{2,p}^h) = \mathfrak{f}(\mathfrak{v}_{2,p}^h) + \sum_{q=0}^{r} \Big\{(\beta_p^2)_q\Big\} \Big(4\lambda_p(u_{2,DG,-}^{h,0}, \mathfrak{v}_{2,p}^h)_H - 2ka(u_{1,DG,-}^{h,0}, \mathfrak{v}_{2,p}^h)\Big),$$
$$\text{for each } p = 0, \cdots, r, \text{ for all } \{\mathfrak{v}_{2,p}^h\}_{p=0}^r \in Z^h. \tag{3.76}$$

After solving for the unknowns $\{\mathfrak{u}_{2,p}^h\}_{p=0}^r \subset Z^h$ *with given data* $f \in H^1(J;H)$ *and* $U_{DG,-}^{h,0} \in Z^h \times Z^h$*, then comes to update the values of* $\{\mathfrak{u}_{1,p}^h\}_{p=0}^r \subset Z^h$*:*

$$\mathfrak{u}_{1,p}^h = \frac{k}{2\lambda_p}\mathfrak{u}_{2,p}^h + \frac{1}{\lambda_p}\sum_{i=0}^{r}\Big\{(\beta_p^2)_i\Big\}u_{1,DG,-}^{h,0}, \tag{3.77}$$

which need to be solved in every given time step I*.*

Lemma 3.11 ((Discrete) Inf-sup condition of discrete reduced spatial problem's forms) *Let* Z *and* H *be the Hilbert spaces given in Definition 2.1 on p. 6. Let* $(Z^h, \|\cdot\|_Z)$ *be the conformal finite dimensional subspace of the Hilbert space* $(Z, \|\cdot\|_Z)$ *over the field* $\mathbb{C}$ *given in Definition 3.10 on p. 43. Let* $c_1 \in \mathbb{C}$ *such that* $Re\ c_1 > 0$ *and* $c_2 \in \mathbb{R}$*, then the form*

$$b(u, v) = c_1(u, v)_H + c_2 a(u, v), \tag{3.78}$$

satisfies

$$\inf_{u\in Z^h}\sup_{v\in Z^h}\frac{|b(u,v)|}{\| u \|_Z \cdot \| v \|_Z} \geq C = \alpha c_2 > 0. \tag{3.79}$$

The imaginary parts of these functions may not be zero, and $\alpha \in \mathbb{R}$ *a real constant which corresponds to the lower bound of the form* $a(\,,)$ *given in Theorem A.3 on p. 178:*

$$\begin{aligned} |a(u,v)| &\leq M \| u \|_Z \| v \|_Z, && \forall \quad u, v \in Z^h \subset Z, \\ a(u,u) &\geq \alpha \| u \|_Z^2, && \forall \quad u \in Z^h \subset Z. \end{aligned}$$

Proof:
The form $b(\,,)$ in (3.78) is the same as the one in (3.44) in Lemma 3.4 on p. 33 with (u, $v \in Z^h \subset Z$) and this is based on the choice of the conformal finite dimensional space $(Z^h, \|\cdot\|_Z)$ which is a subspace of $(Z, \|\cdot\|_Z)$, then the same arguments in the proof of Lemma 3.4 on p. 33 hold and that complete the proof. □

Lemma 3.12 (Uniqueness and existence of discrete reduced spatial problem's) *Let* Z *and* H *be the Hilbert spaces given in Definition 2.1 on p. 6. Let* $(Z^h, \|\cdot\|_Z)$ *be the conformal finite dimensional subspace of the Hilbert space* $(Z, \|\cdot\|_Z)$ *over the field* $\mathbb{C}$ *given in Definition 3.10 on p. 43. There exist unique functions* $\{\mathfrak{u}_{2,p}^h\}_{p=0}^r \subset Z^h$ *which solve the* $(r+1)$ *spatial problems (3.76), as well as* $\{\mathfrak{u}_{1,p}^h\}_{p=0}^r \subset Z^h$ *in (3.77).*

Proof:
Firstly, the form in (3.76) on p. 47 is continuous with using Lemma 3.5 on p. 34 after letting $c_1 =$

$4\lambda_p \in \mathbb{C}$ and $c_2 = k^2 \in \mathbb{R}$ since λ_p with Re $\lambda_p > 0$ is the eigenvalues of the matrix $\mathcal{A}^L$ given in (3.31) on p. 30, k is the time step size, thus

$$b(\mathfrak{u}^h_{2,p}, \mathfrak{v}^h_{1,p}) = 4\lambda_p^2(\mathfrak{u}^h_{2,p}, \mathfrak{v}^h_{1,p})_H + k^2 a(\mathfrak{u}^h_{2,p}, \mathfrak{v}^h_{1,p}) \leq (4|\lambda_p|c^2 + k^2 M) \parallel \mathfrak{u}^h_{2,p} \|_Z \| \mathfrak{v}^h_{1,p} \|_Z, \text{ for } \mathfrak{u}^h_{2,p}, \mathfrak{v}^h_{1,p} \in Z^h \subset Z \text{ for } p = 0, \cdots, r.$$

Secondly, and again based on the choice of the conformal finite dimensional space $(Z^h, \|\cdot\|_Z)$ which is a subspace of $(Z, \|\cdot\|_Z)$, then the form $b(\,,\,)$ in (3.76) for each $p = 0, \cdots, r$ satisfies the discrete Inf-sup condition in Lemma 3.11 on p. 47 after letting $c_1 = 4\lambda_p$ and $c_2 = k^2$ then the Inf-sup constant would be $k^2\alpha$. Finally, with going back to (Babuška) Theorem A.4 on p. 179, and with having Lemmas 3.11 and 3.5, then there exist unique functions $\{\mathfrak{u}^h_{2,p}\}_{p=0}^r \subset Z^h \subset Z$ which solve (3.76) on p. 47. Moreover, from the identity (3.77) on p. 47 that is

$$\mathfrak{u}^h_{1,p} = \frac{k}{2\lambda_p}\mathfrak{u}^h_{2,p} + \frac{1}{\lambda_p}\sum_{q=0}^{r}(\beta_p^2)_q u^{h,0}_{1,DG,-}, \tag{3.80}$$

for each $p = 0, \cdots, r$, then from the given existence and uniqueness of $\{\mathfrak{u}_{2,p}\}_{p=0}^r \subset Z$ and the given forcing data $f \in H^1(J; H)$ and initial data $u^{h,0}_{1,DG,-}, u^0_{2,DG,-} \in Z$ which are well defined, that also concludes that $\{\mathfrak{u}^h_{1,p}\}_{p=0}^r \in Z^h \subset Z$ at a given time step I also exists and is unique. □

Theorem 3.4 (Existence and uniqueness of the fully-discrete solution)
Let Z and H be Hilbert spaces over the field $\mathbb{C}$ given in Definition 2.1 on p. 6. Let Z^h be the conformal finite dimensional subspace of the Hilbert space Z given in Definition 3.10 on p. 43. Let $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h \times Z^h) = \Big\{U : J \to Z^h \times Z^h \,:\, U|_{I_n} \in P^{r_n}(I_n; Z^h \times Z^h), 0 \leq n \leq N-1\Big\}$ be the fully-discrete finite-dimensional product subspace of $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z \times Z)$, where the latter space is given in Definition 3.1 on p. 19. There exists a unique fully-discrete vector-valued function $U^h_{DG} \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h \times Z^h)$ which solves the formulation (3.66) on p. 44.

Proof:
Since the fully-discrete vector-valued function U^h_{DG} is uniquely written as

$$U^h_{DG} = \sum_{n=0}^{N-1} U^h_{DG}|_{I_n},$$

such that

$$U^h_{DG}|_{I_n} = \sum_{j=0}^{r_n}\varphi_j(t)U^h_j, \text{ where for } Y^{r_n} \in \mathbb{C}^{(r_n+1)\times(r_n+1)},\ U^h_j = \sum_{i=0}^{r_n} Y^{r_n}_{ji}\mathfrak{U}^h_i \subset Z \times Z.$$

The shown existence and uniqueness of $\{\mathfrak{U}^h_i\}_{i=0}^{r_n} \subset Z^h \times Z^h$ (after letting $r = r_n$) in Lemma 3.12 on p. 47 implies that $U^h_{DG} \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h \times Z^h)$ does exist and is unique. □

3.3.2 Stability estimates of the fully-discrete formulation with $f = 0$

Lemma 3.13 (Identity estimate of left-sided limits of fully-discrete solution with $f = 0$)
Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let Z^h be the conformal finite dimensional subspace of the Hilbert space Z given in Definition 3.10 on p. 43. Let $P^{r_m}(I_m; Z^h \times Z^h)$ be

the fully-discrete finite-dimensional product subspace of $P^{r_m}(I_m; Z \times Z)$ *where the latter is the semi-discrete space of* r_m*th order polynomials in time at* I_m *with coefficients in* $Z \times Z$ *given in Definition 3.1 on p. 19. With* $f = 0$ *and given initial data* $U_0^h \in Z^h \times Z^h$*, the solution* $U_{DG}^h \in P^{r_m}(I_m; Z^h \times Z^h)$ *of (3.67) on p. 44 satisfies:*

$$\| U_{DG,-}^{h,n} \|_E^2 + \sum_{m=0}^{n-1} \Big(\| [U_{DG}^h]^m \|_E^2 \Big) = \| U_0^h \|_E^2, \; for \; n \leq N, \tag{3.81}$$

where U_{DG}^h *and* U_0^h *are real and*

$$\| U \|_E^2 = a(u_1, u_1) + \| u_2 \|_H^2, \; \forall \, U \in Z^h \times Z^h,$$

$[U]^n = U_n^+ - U_n^-$ *is the jump at a time node* t_n*, for* $n = 0, \cdots, N-1$*, given in Definition 3.3 on p. 20, and* $a(\, , \,)$ *is th form given in Definition 2.2 on p. 6 which has the following properties:*

$$\begin{aligned} |a(u,v)| &\leq M \| u \|_Z \| v \|_Z, \qquad &\forall \quad u, v \in Z^h \subset Z, \\ a(u,u) &\geq \alpha \| u \|_Z^2, \qquad &\forall \quad u \in Z^h \subset Z. \end{aligned}$$

Proof:
With repeating the same arguments in the proof of Lemma 3.7 on p. 37 which hold here since they don't depend on the discrete product spatial space, $Z^h \times Z^h$ which is a conformal finite dimensional subspace of $Z \times Z$, and that completes the proof. □

Lemma 3.14 (Local stability estimate of fully-discrete formulation with $f = 0$)
Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let Z^h be the conformal finite dimensional subspace of the Hilbert space Z given in Definition 3.10 on p. 43. Let $P^r(I; Z^h \times Z^h)$ be the fully-discrete finite-dimensional product subspace of $P^r(I; Z \times Z)$ where the latter is the semi-discrete space of rth order polynomials in the generic time step $I = (t_0, t_1)$ with coefficients in $Z \times Z$ given in Definition 3.1 on p. 19. Let $(\,,)_E$ be the inner product given in Corollary 2.2 on p. 10. Let $\hat{a}(\,,)$ be the skew-symmetric form given in Definition 2.7 on p. 14:

$$\hat{a}(U, V) = -a(u_2, v_1) + a(u_1, v_2), \text{ and } (U, V)_E = a(u_1, v_1) + (u_2, v_2)_H, \forall U, V \in Z^h \times Z^h, \text{ respectively,}$$

where $a(\,,)$ is th form given in Definition 2.2 on p. 6 which has the following properties:

$$\begin{aligned} |a(u, v)| &\leq M \parallel u \parallel_Z \parallel v \parallel_Z, && \forall \quad u, v \in Z^h \subset Z, \\ a(u, u) &\geq \alpha \parallel u \parallel_Z^2, && \forall \quad u \in Z^h \subset Z. \end{aligned}$$

At a generic time step $I = (t_0, t_1)$ with $f = 0$ and $U_{DG,-}^{h,0}$ the given initial data at I, the fully-discrete vector-valued function $U_{DG}^h \in P^r(I; Z^h \times Z^h)$ which solves

$$\overbrace{\int_I \Big\{(\partial_t U_{DG}^h, V^h)_E + \hat{a}(U_{DG}^h, V^h)\Big\} dt + (U_{DG,+}^h(t_0), V_+^h(t_0))_E}^{b_{DG}(U_{DG}^h, V^h)=} = (U_{DG,-}^h(t_0), V_+^h(t_0))_E,$$

for all $V^h \in P^r(I; Z^h \times Z^h)$, where $U_{DG,-}^h(t_0) = U_{DG,-}^{h,0}$ is the initial value at the generic time step I.

satisfies

$$\parallel U_{DG}^h \parallel_{L^2(I;E)} \leq \frac{k}{2} \frac{\parallel Y \parallel_2^2 (2r+1)}{\min\limits_p \; Re\ \lambda_p} \parallel U_{DG,-}^{h,0} \parallel_E, \tag{3.82}$$

where k is the time step size, r is the approximation order of polynomials in time, $\parallel Y \parallel_2^2$ is the square of the spectral norm (see (A.6) on p. 173) of the matrix $Y \in \mathbb{C}^{(r+1)\times(r+1)}$, and λ_p with $Re\ \lambda_p > 0$ for $p = 0, \cdots, r$ are the eigenvalues of the matrix $\mathcal{A}^L$ given in (3.31) on p. 30.

Proof:
Using Lemma 3.10 on p. 45 with the same arguments in the proof of Lemma 3.9 on p. 40, that completes the proof. □

Theorem 3.5 (Global stability estimate of the fully-discrete formulation with $f = 0$)
Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let Z^h be the conformal finite dimensional subspace of the Hilbert space Z given in Definition 3.10 on p. 43.
Let $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h \times Z^h) = \Big\{U : J \to Z^h \times Z^h : U|_{I_n} \in P^{r_n}(I_n; Z^h \times Z^h), 0 \leq n \leq N-1\Big\}$ be the fully-discrete finite-dimensional product subspace of $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z \times Z)$, where the latter space is given in Definition 3.1 on p. 19. The solution $U_{DG}^h \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h \times Z^h)$ of (3.66) on p. 44 with $f = 0$ and initial data $U_0^h \in Z^h \times Z^h$ satisfies

$$\parallel U_{DG}^h \parallel_{L^2(J;E)} \leq \mathcal{C} \parallel U_0^h \parallel_E, \text{ for } \mathcal{C} < \infty, \tag{3.83}$$

where $\mathcal{C} = \max_n(T\frac{\| Y^{r_n} \|_2^2 (2r_n+1)}{2\min_p \ Re\ \lambda_p})$ *is depending on the maximum value of the approximation order* r_n *for* $n = 0, \cdots, N-1$*, the square of the spectral norm (see* (A.6) *on p. 173) of the matrix* $Y^{r_n} \in \mathbb{C}^{(r_n+1)\times(r_n+1)}$*, and the eigenvalues* λ_p *with* $Re\ \lambda_p > 0$ *of the matrix* $\mathcal{A}^L$ *given in* (3.31) *on p. 30 for* $p = 0, \cdots, r_n$ *and*

$$\| U \|_{L^2(J;E)}^2 = \int_J \{\overbrace{a(u_1,u_1) + \| u_2 \|_H^2}^{\|U\|_E^2}\} dt, \ \text{for vector-valued functions } U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h \times Z^h).$$

Proof:
Firstly, from Lemma 3.13 on p. 48 we have

$$\begin{aligned} &\| U_{DG,-}^{h,n} \|_E^2 + \overbrace{\sum_{m=0}^{n-1} \Big(\| [U_{DG}^h]^m \|_E^2 \Big)}^{\geq 0} = \| U_0^h \|_E^2 \\ \Longrightarrow &\| U_{DG,-}^{h,n} \|_E^2 \leq \| U_0^h \|_E^2, \end{aligned}$$

With this estimate and Lemma 3.14 on p. 50:

$$\| U_{DG}^h \|_{L^2(J;E)} = \sum_{n=0}^{N-1} \| U_{DG}^h \|_{L^2(I_n;E)} \leq \sum_{n=0}^{N-1} \frac{k}{2} \frac{\| Y \|_2^2 (2r+1)}{\min_p \ \mathrm{Re}\ \lambda_p} \| U_{DG,-}^{h,n} \|_E,$$

then following the rest of the arguments in the proof of Theorem 3.3 on p. 42, that completes the proof. □

3.4 Implementation

At a given generic time step with the time basis transformation as done in Section 3.2.1.1, the fully-discrete time stepping scheme of our formulation reads

$$\begin{aligned} &\text{Find } \{U_j^h\}_{j=0}^r \subset Z^h \times Z^h \text{ such that} \\ &\sum_{i,j=0}^r \Big\{ \mathcal{A}_{ij}(U_j^h, V_i^h)_E + \mathcal{B}_{ij}\hat{a}(U_j^h, V_i^h) \Big\} = \sum_{i=0}^r \Big\{ \int_I \psi_i(f, v_{2,i}^h)_H dt + \psi_i^+(t_0)(U_{DG,-}^{h,0}, V_i^h)_E \Big\}, \\ &\forall \{V_i\}_{i=0}^r \subset Z^h \times Z^h, \\ &\text{for given initial data } U_{DG,-}^{h,0} \text{ at a given time step } I = (t_0, t_1), \text{ and forcing data } f \in H^1(J; H). \end{aligned} \tag{3.84}$$

With $\mathcal{M} = \dim Z^h$, the following matrices and vectors are introduced:

$$\begin{aligned} \mathcal{S}_{ij} &:= \hat{a}(U_j^h, V_i^h), \ \text{for } \mathcal{S} \in \mathbb{R}^{\mathcal{M}\times\mathcal{M}}, \\ \bar{\mathcal{M}}_{ij} &:= (U_j^h, V_i^h)_E, \ \text{for } \mathcal{E} \in \mathbb{R}^{\mathcal{M}\times\mathcal{M}}, \\ \text{with r.h.s } \bar{\mathcal{B}}_i(V) &= \sum_{i=0}^r \Big\{ \int_I \psi_i(f, v_{2,i}^h)_H dt + \psi_i^+(t_0)(U_{DG,-}^{h,0}, V_i^h)_E \Big\}. \end{aligned} \tag{3.85}$$

Then, with making use of matrix-vector notation, the formulation in (3.84) reads

$$\sum_{j=0}^{r}\Big(\mathcal{A}_{ij}\bar{\mathcal{M}}+\mathcal{B}_{ij}\mathcal{S}\Big)U_j^h=\vec{\bar{\mathcal{B}}}_i(V),\forall i=0,\cdots,r, \tag{3.86}$$

which has to be solved in every given time step for obtaining the fully-discrete solution $U\in(2\cdot\mathcal{M}\cdot(r+1))$, where $\vec{\bar{\mathcal{B}}}_i(V)=(\bar{\mathcal{B}}_i(V^1),\bar{\mathcal{B}}_i(V^2),\cdots,\bar{\mathcal{B}}_i(V^{\mathcal{M}}))$.

With higher polynomial degrees r and $2\mathcal{M}$ basis functions in space, the systems become difficult to solve. But since this system can be diagonalized as given in (3.75) on p. 46 where we now solve for $\mathfrak{U}_j^h\in\mathbb{C}^{2(r+1)}$:

$$\Big(\lambda_j\bar{\mathcal{M}}+\frac{k}{2}\mathcal{S}\Big)\mathfrak{U}_j^h=\vec{\bar{\mathcal{B}}}_i(\mathfrak{V}), \tag{3.87}$$

where the coefficients $\vec{U_j}$ are derived by the following inverse transformation

$$\vec{U_j}=\sum_{i=0}^{r}Y_{ji}\mathfrak{U}_i^h,\ \text{for } Y\in\mathbb{C}^{(r+1)\times(r+1)}. \tag{3.88}$$

The decoupling step with diagonalization reduces the computation from solving for $2\cdot\mathcal{M}\cdot(r+1)$ unknowns to solve only for $(r+1)$ systems with $2\mathcal{M}$ unknowns which can be computed in parallel.

3.5 The a priori error estimate

Similar projection arguments and techniques used by C Johnson, 1993, (see [30]) are used here to get the a priori error estimates, but this time with abstract spatial operator and spaces. Also the properties of the time projection given in [44, Definition 1.9] will be used with its approximation in [44, Lemma 1.16], where Definition 3.12 extended it to become a $2\times2-$ matrix spatio-time projection operator in the $a(\,,\,)$-inner product. In Theorem 3.6 on p. 61, the a priori error estimate of the last left-sided limit in the E-norm of the difference from the continuous to fully-discrete solution is given. Also, we show two global a priori error estimates. The first one is in the global $L^2(J;E)$-norm is shown in Lemma 3.23 on p. 71. The second one is the maximum over the local $L^2(I_n;E)$ norms for $n=0,\cdots,N-1$ is shown in Lemma 3.24 on p. 74. Both lemmas uses the resulting a priori error estimate in the Local $L^2(I;E)$-norm shown in Theorem 3.7 on p. 68 and then Theorem 3.6 on p. 61.

Problem 3.5.1; an auxiliary semi-discrete formulation (see [30, c.f. Backward problem (3.1)]), is introduced and will be shown later in Lemma 3.15 that it is equivalent to our original semi-discrete formulation with $f=0$. Then Lemma 3.16 on p. 55 shows that Problem 3.5.1 is solvable and its solution has the same stability as the semi-discrete solution of our scheme in (3.19) on p. 26.
Following the same techniques in [30] to show the a priori error estimates, the fully-discrete formulation of Problem 3.5.1 is given in Problem 3.5.2 on p. 55. The latter will be used in the proof of Theorem 3.6 to get the a priori error estimate from the continuous to fully-discrete solution in the E-norm. Also, in Lemma 3.23 on p. 71 the a priori error estimate is shown in the global $L^2(J;E)$-norm. The latter uses the local a priori error estimates shown in Theorem 3.7 on p. 68.

Problem 3.5.1 (The semi-discrete auxiliary problem)
Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6.
Let $\mathcal{V}^{\underline{r}}(\mathcal{N};Z\times Z)=\Big\{U:J\to Z\times Z\,:\,U|_{I_n}\in P^{r_n}(I_n;Z\times Z),0\le n\le N-1\Big\}$ *be the semi-discrete*

space given in Definition 3.1 on p. 19. Let $\hat{a}(\,,)$ *be the skew-symmetric form given in Definition 2.7 on p. 14,* $(\,,)_E$ *be the inner product given in Corollary 2.2 on p. 10, and* $B_{DG}(\,,)$ *be the form given in (3.17) on p. 26:*

$$\begin{aligned}\hat{a}(U_{DG}, V) &= -a(u_{2,DG}, v_1) + a(u_{1,DG}, v_2),\\ (U, V)_E &= a(u_1, v_1) + (u_2, v_2)_H, \quad \forall U, V \in Z \times Z, \text{ and}\\ B_{DG}(U, V) &= \sum_{n=0}^{N-1} \int_{I_n} \Big\{(\partial_t U, V)_E + \hat{a}(U, V)\Big\} dt + \sum_{n=1}^{N-1} ([U]^n, V_+^n)_E + (U_+^0, V_+^0)_E.\end{aligned}$$

Find a real vector-valued function $\Phi \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z \times Z)$*, which satisfies*

$$B_{DG}(V, \Phi) = (V_-^N, \Theta_-^N)_E, \forall V \in \mathcal{V}^r(\mathcal{N}; Z \times Z), \tag{3.89}$$

for given data $\Theta_-^N \in Z \times Z$*.*

Lemma 3.15 (Equivalence between Problem 3.5.1 and the semi-discrete formulation) *Let* Z *and* H *be the Hilbert spaces given in Definition 2.1 on p. 6.*
Let $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z \times Z) = \Big\{U : J \to Z \times Z \,:\, U|_{I_n} \in P^{r_n}(I_n; Z \times Z), 0 \le n \le N-1\Big\}$ *be the semi-discrete space given in Definition 3.1 on p. 19 where* $J = (0, T)$ *is the time interval. Let* $\Phi \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z \times Z)$ *be the solution of Problem 3.5.1 for given data* $\Theta_-^N \in Z \times Z$*, then* $\bar{\Phi}(t) = \begin{pmatrix}\Phi_1(T-t)\\ -\Phi_2(T-t)\end{pmatrix}$ *for* $0 < t < T < \infty$ *is a solution of the semi-discrete formulation (3.19) on p. 26 with* $f = 0$ *and given data* $\binom{\Theta_1}{-\Theta_2}_-^N \in Z \times Z$*. Moreover, let* $\bar{\Phi} \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z \times Z)$ *be a solution of the semi-discrete formulation (3.19) on p. 26 with* $f = 0$ *and given data* $\binom{\Theta_1}{-\Theta_2}_-^N \in Z \times Z$*, then* $\Phi(t) = \begin{pmatrix}\bar{\Phi}_1(T-t)\\ -\bar{\Phi}_2(T-t)\end{pmatrix}$ *is the solution of Problem 3.5.1 for given data* $\Theta_-^N \in Z \times Z$*.*

Proof:
Firstly, in Problem 3.5.1 and starting with the form $B_{DG}(V, \Phi)$ in (3.89) and using the result of Lemma 3.3 on p. 26, then

$$B_{DG}(V, \Phi) = \sum_{n=0}^{N-1} \int_{I_n} \Big\{ - (\partial_t \Phi, V)_E - \hat{a}(\Phi, V)\Big\} dt - \sum_{n=1}^{N-1} ([\Phi]^n, V_-^n)_E + (\Phi_-^N, V_-^N)_E = (\Theta_-^N, V_-^N)_E. \tag{3.90}$$

Secondly, with the following transformation on the time variable:

$$t = T - \tau \iff \tau = T - t, \text{ such that } dt = -d\tau, \tag{3.91}$$

and the time nodes become

$$t_n = \tau_{N-n} \text{ with } t_{n+1} = \tau_{N-(n+1)}, \text{ where now } \tau_N = T, \text{ and } \tau_0 = 0,$$

such that the following integral

$$\int_{t_n}^{t_{n+1}} f(t) dt$$

after the time variable transformation becomes

$$\int_{\tau_{N-n}}^{\tau_{N-(n+1)}} f(T-\tau)(-d\tau) = -\int_{\tau_{N-n-1}}^{\tau_{N-n}} f(T-\tau)(-d\tau) = \int_{\tau_{N-n-1}}^{\tau_{N-n}} f(T-\tau) d\tau,$$

also, letting

$$\bar{U}(\tau)=\begin{pmatrix}\bar{u}_1(\tau)\\ \bar{u}_2(\tau)\end{pmatrix}:=\begin{pmatrix}u_1(T-\tau)\\ -u_2(T-\tau)\end{pmatrix}\Longrightarrow U(t)=\begin{pmatrix}\bar{u}_1(T-t)\\ -\bar{u}_2(T-t)\end{pmatrix}.$$

For the left/right-sided limits at any time nodes given in Definition 3.3 on p. 20 would be e.g.

$$u_-^n=\lim_{s\to 0}u(t_n-s)=\lim_{s\to 0}u(T-(T-(t_n-s)))=\lim_{s\to 0}\bar{u}(T-t_n+s)=\lim_{s\to 0}\bar{u}(\tau_{N-n}+s)=\bar{u}_+^{N-n},$$

the same holds for the right-sided limits i.e.

$$u_+^n=\lim_{s\to 0}u(t_n+s)=\bar{u}_-^{N-n},$$

and that

$$u_-^N=\bar{u}_+^0,\ i.e.\text{ for the jump over a given time node :}$$

$$[U]^n=U_+^n-U_-^n=\begin{pmatrix}\bar{u}_1\\ -\bar{u}_2\end{pmatrix}_-^{N-n}-\begin{pmatrix}\bar{u}_1\\ -\bar{u}_2\end{pmatrix}_+^{N-n}=-\left[\begin{pmatrix}\bar{u}_1\\ -\bar{u}_2\end{pmatrix}\right]^{N-n},\ \text{ for } n=0,\cdots,N-1.$$

With all above transformation, Problem 3.5.1 in (3.90) on p. 52 becomes

$$\sum_{n=0}^{N-1}\int_{\tau_{N-n-1}}^{\tau_{N-n}}\left\{-\left(-\partial_\tau\begin{pmatrix}\bar{\Phi}_1(\tau)\\ -\bar{\Phi}_2(\tau)\end{pmatrix},\begin{pmatrix}\bar{v}_1(\tau)\\ -\bar{v}_2(\tau)\end{pmatrix}\right)_E-\hat{a}\left(\begin{pmatrix}\bar{\Phi}_1(\tau)\\ -\bar{\Phi}_2(\tau)\end{pmatrix},\begin{pmatrix}\bar{v}_1(\tau)\\ -\bar{v}_2(\tau)\end{pmatrix}\right)\right\}d\tau$$
$$-\sum_{n=1}^{N-1}\left(-\left[\begin{pmatrix}\bar{\Phi}_1(\tau)\\ -\bar{\Phi}_2(\tau)\end{pmatrix}\right]^{N-n},\begin{pmatrix}\bar{v}_1(\tau)\\ -\bar{v}_2(\tau)\end{pmatrix}_+^{N-n}\right)_E+\left(\begin{pmatrix}\bar{\Phi}_1(\tau)\\ -\bar{\Phi}_2(\tau)\end{pmatrix}_+^0,\begin{pmatrix}\bar{v}_1(\tau)\\ -\bar{v}_2(\tau)\end{pmatrix}_+^0\right)_E$$
$$=\left(\Theta_-^N,\begin{pmatrix}\bar{v}_1(\tau)\\ -\bar{v}_2(\tau)\end{pmatrix}_+^0\right)_E.$$

Rearranging the terms with the use of the skew-symmetry property of $\hat{a}(\,,\,)$ given in Remark 2.7 on p. 14:

$$-\hat{a}\left(\begin{pmatrix}\bar{\Phi}_1(\tau)\\ -\bar{\Phi}_2(\tau)\end{pmatrix},\begin{pmatrix}\bar{v}_1(\tau)\\ -\bar{v}_2(\tau)\end{pmatrix}\right)=-\big(-a(-\bar{\Phi}_2(\tau),\bar{v}_1(\tau))+a(\bar{\Phi}_1(\tau),-\bar{v}_2(\tau))\big)$$
$$=-\big(a(\bar{\Phi}_2(\tau),\bar{v}_1(\tau))-a(\bar{\Phi}_1(\tau),\bar{v}_2(\tau))\big)=-\hat{a}(\bar{V},\bar{\Phi})=\hat{a}(\bar{\Phi},\bar{V}),$$

yield

$$\overbrace{\sum_{n=0}^{N-1}\int_{\tau_{N-n-1}}^{\tau_{N-n}}\Big\{(\partial_\tau\bar{\Phi},\bar{V})_E+\hat{a}(\bar{\Phi},\bar{V})\Big\}d\tau+\sum_{n=1}^{N-1}([\bar{\Phi}]^{N-n},\bar{V}_+^{N-n})_E+(\bar{\Phi}_+^0,\bar{V}_+^0)_E}^{=B_{DG}(\bar{\Phi},\bar{V})}=\left(\Theta_-^N,\begin{pmatrix}\bar{v}_1(\tau)\\ -\bar{v}_2(\tau)\end{pmatrix}_+^0\right)_E$$
$$=\left(\begin{pmatrix}\Theta_1\\ -\Theta_2\end{pmatrix}_-^N,\bar{V}_+^0\right)_E. \tag{3.92}$$

With (3.92) it concludes that $\bar{\Phi}(\tau)=\begin{pmatrix}\Phi_1(T-\tau)\\ -\Phi_2(T-\tau)\end{pmatrix}$ is a solution of (3.19) on p. 26 with now $f=0$ and given data $\left(\begin{smallmatrix}\Theta_1\\ -\Theta_2\end{smallmatrix}\right)_-^N$ in the r.h.s that is

$$\left(\begin{pmatrix}\Theta_1\\ -\Theta_2\end{pmatrix}_-^N,\bar{V}_+^0\right)_E. \tag{3.93}$$

Finally, and via the given transformation on the time variable in (3.91) the other way holds and that completes the proof. $\square$

Lemma 3.16 (Solvability of the semi-discrete auxiliary problem) *Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6.*
Let $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z\times Z) = \Big\{U : J \to Z\times Z : U|_{I_n} \in P^{r_n}(I_n; Z\times Z), 0 \le n \le N-1\Big\}$ be the semi-discrete space given in Definition 3.1 on p. 19. Problem 3.5.1 on p. 52 is solvable and for given data $\Theta_-^N \in Z\times Z$ its real solution satisfies

$$\| \Phi \|_{L^2(J;E)} \le \mathcal{C} \| \Theta_-^N \|_E, \tag{3.94}$$

and $\mathcal{C} < \infty$ is the same real constant appearing in Theorem 3.3 on p. 42, where

$$\| U \|^2_{L^2(I;E)} = \int_I \{\overbrace{a(u_1,u_1) + \| u_2 \|^2_H}^{\|U\|^2_E}\} dt, \ \textit{for vector-valued functions } U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathcal{V}^r(\mathcal{N}; Z\times Z),$$

and the form $a(\,,\,)$ is given in Definition 2.2 on p. 6:

$$\begin{aligned} |a(u,v)| &\le M \| u \|_Z \| v \|_Z, \qquad \forall \quad u,v \in Z, \\ a(u,u) &\ge \alpha \| u \|^2_Z, \qquad \forall \quad u \in Z. \end{aligned}$$

Proof:
From Lemma 3.15 on p. 53, $\Phi(t)$ the solution of Problem 3.5.1 is transformed to be $\begin{pmatrix} \bar{\Phi}_1(T-t) \\ -\bar{\Phi}_2(T-t) \end{pmatrix}$, and accordingly with the test function such that it is equivalent to the solution of the semi-discrete formulation (3.19) on p. 26 with given data $\begin{pmatrix} \Theta_1 \\ -\Theta_2 \end{pmatrix}^N_-$ and $f = 0$. The equivalence then concludes that the following holds

$$\| \bar{\Phi} \|_{L^2(J;E)} \le \mathcal{C} \| \Theta_-^N \|_E,$$

and since $\bar{\Phi}(\tau) = \begin{pmatrix} \Phi_1(T-\tau) \\ -\Phi_2(T-\tau) \end{pmatrix}$ then also holds

$$\| \Phi \|_{L^2(J;E)} \le \mathcal{C} \| \Theta_-^N \|_E \,.$$

That concludes that Problem 3.5.1 with given data $\binom{\Theta_1}{\Theta_2}^N_- \in Z\times Z$ is solvable and its solution $\Phi \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z\times Z)$ has the same stability estimates of the semi-discrete solution of (3.19) on p. 26. $\square$

Problem 3.5.2 (The fully-discrete auxiliary problem)
Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let $Z^h \times Z^h$ the discrete finite-dimensional product subspace given in Definition 3.10 on p. 43. Let $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h\times Z^h)$ be the fully-discrete finite-dimensional product subspace of $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z\times Z)$ where the latter is given in Definition 3.1 on p. 19. Let $\hat{a}(\,,\,)$ be the skew-symmetric form given in Definition 2.7 on p. 14, $(\,,\,)_E$ be the inner product given in Corollary 2.2 on p. 10, and $B_{DG}(\,,\,)$ be the form given in (3.17) on p. 26:

$$\begin{aligned} \hat{a}(U_{DG}, V) &= -a(u_{2,DG}, v_1) + a(u_{1,DG}, v_2), \\ (U,V)_E &= a(u_1,v_1) + (u_2,v_2)_H, \quad \forall U,V \in Z^h \times Z^h \subset Z\times Z, \ \textit{and} \\ B_{DG}(U,V) &= \sum_{n=0}^{N-1} \int_{I_n} \Big\{(\partial_t U, V)_E + \hat{a}(U,V)\Big\} dt + \sum_{n=1}^{N-1} ([U]^n, V_+^n)_E + (U_+^0, V_+^0)_E, \end{aligned}$$

where $a(,)$ *is th form given in Definition 2.2 on p. 6 which has the following properties:*

$$\begin{aligned} |a(u,v)| &\leq M \parallel u \parallel_Z \parallel v \parallel_Z, && \forall \quad u,v \in Z^h \subset Z, \\ a(u,u) &\geq \alpha \parallel u \parallel_Z^2, && \forall \quad u \in Z^h \subset Z. \end{aligned}$$

Find $\Phi^h \in \mathcal{V}^r(\mathcal{N}; Z^h \times Z^h)$*, which is a real vector-valued function, satisfies*

$$B_{DG}(V^h, \Phi^h) = (V_-^{h,N}, \Theta_-^{h,N})_E, \forall V^h \in \mathcal{V}^r(\mathcal{N}; Z^h \times Z^h), \tag{3.95}$$

for given data $\Theta_-^{h,N} \in Z^h \times Z^h$*.*

Lemma 3.17 (Equivalence between the fully-discrete auxiliary problem and formulation) *Let* Z *and* H *be the Hilbert spaces given in Definition 2.1 on p. 6. Let* Z^h *be the conformal finite dimensional subspace of the Hilbert space* Z *given in Definition 3.10 on p. 43. Let* $\Phi^h \in \mathcal{V}^r_-(\mathcal{N}; Z^h \times Z^h)$ *be the solution of Problem 3.5.2 on p. 55 for given data* $\Theta_-^{h,N} \in Z^h \times Z^h$*, then* $\bar{\Phi}^h(t) = \begin{pmatrix} \Phi_1^h(T-t) \\ -\Phi_2^h(T-t) \end{pmatrix}$*, for* $0 < t < T < \infty$*, is a solution of the fully-discrete formulation (3.66) on p. 44 with* $f = 0$ *and given data* $\begin{pmatrix} \Theta_1^h \\ -\Theta_2^h \end{pmatrix}_-^N \in Z^h \times Z^h$*. Let* $\bar{\Phi}^h \in \mathcal{V}^r_-(\mathcal{N}; Z^h \times Z^h)$ *be a solution of the fully-discrete formulation (3.66) on p. 44 with* $f = 0$ *and given data* $\begin{pmatrix} \Theta_1^h \\ -\Theta_2^h \end{pmatrix}_-^N \in Z^h \times Z^h$*, then* $\Phi^h(t) = \begin{pmatrix} \bar{\Phi}_1^h(T-t) \\ -\bar{\Phi}_2^h(T-t) \end{pmatrix}$ *is the solution of Problem 3.5.2 for given data* $\Theta_-^{h,N} \in Z^h \times Z^h$*.*

Proof:
Knowing that the product space $Z^h \times Z^h$ is a conformal finite dimensional subspace of $Z \times Z$ and that renders having the same form and linear r.h.s used in the semi-discrete as well as the fully-discrete formulation. Also, since the transformation in Lemma 3.15 was done in time i.e. the same equivalence would also hold between Problem 3.5.2 and the fully-discrete formulation in (3.66) on p. 44 with $f = 0$ and given data $\begin{pmatrix} \Theta_1^h \\ -\Theta_2^h \end{pmatrix}_-^N \in Z^h \times Z^h$ and that completes the proof. □

Lemma 3.18 (Solvability of the fully-discrete auxiliary problem)
Let Z *and* H *be the Hilbert spaces given in Definition 2.1 on p. 6. Let* Z^h *be the conformal finite dimensional subspace of the Hilbert space* Z *given in Definition 3.10 on p. 43.*
Let $\mathcal{V}^r_-(\mathcal{N}; Z^h \times Z^h)$ *be the fully-discrete finite-dimensional product subspace of* $\mathcal{V}^r_-(\mathcal{N}; Z \times Z) = \Big\{ U : J \to Z \times Z \ : \ U|_{I_n} \in P^{r_n}(I_n; Z \times Z), 0 \leq n \leq N-1 \Big\}$ *which is the semi-discrete space given in Definition 3.1 on p. 19. Problem 3.5.2 on p. 55 is solvable and for given data* $\Theta_-^{h,N} \in Z^h \times Z^h$ *its solution satisfies*

$$\parallel \Phi^h \parallel_{L^2(J;E)} \leq \mathcal{C} \parallel \Theta_-^{h,N} \parallel_E, \tag{3.96}$$

and

$$\parallel \Phi_-^{h,n} \parallel_E^2 + \sum_{m=0}^{n-1} \Big(\parallel [\Phi^h]^m \parallel_E^2 \Big) = \parallel \Theta_-^{h,N} \parallel_E^2, \ \textit{for } n \leq N, \tag{3.97}$$

for given data $\Theta_-^{h,N} \in Z^h \times Z^h$, *and* $\mathcal{C} < \infty$ *is the same real constant appearing in Theorem 3.5 on p. 50, where*

$$\| U \|^2_{L^2(I;E)} = \int_I \{\overbrace{a(u_1,u_1) + \| u_2 \|^2_H}^{\|U\|^2_E}\} dt, \ \textit{for}\ U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathcal{V}^r(\mathcal{N}; Z^h \times Z^h) \subset \mathcal{V}^r(\mathcal{N}; Z \times Z),$$

and the form $a(\,,)$ *is given in Definition 2.2 on p. 6:*

$$\begin{aligned} |a(u,v)| \le M \| u \|_Z \| v \|_Z, \qquad & \forall \quad u,v \in Z^h \subset Z, \\ a(u,u) \ge \alpha \| u \|^2_Z, \qquad & \forall \quad u \in Z^h \subset Z. \end{aligned}$$

Proof:
From Lemma 3.17 on p. 56, $\Phi^h(t)$ the solution of Problem 3.5.2 on p. 55 can be transformed to be $\begin{pmatrix} \bar{\Phi}^h_1(T-t) \\ -\bar{\Phi}^h_2(T-t) \end{pmatrix}$, and accordingly with the test function, such that it is equivalent to the solution of the fully-discrete formulation (3.66) on p. 44 with $f = 0$ and given data $\begin{pmatrix} \Theta^h_1 \\ -\Theta^h_2 \end{pmatrix}^N_-$. The equivalence then concludes that the following hold

$$\| \bar{\Phi}^h \|_{L^2(J;E)} \le \mathcal{C} \| \Theta^{h,N}_- \|_E, \ \text{using Theorem 3.5 on p. 50},$$

and

$$\| \bar{\Phi}^{h,n}_- \|^2_E + \sum_{m=0}^{n-1} \Big(\| [\bar{\Phi}^h]^m \|^2_E \Big) = \| \Theta^{h,N}_- \|^2_E, \ \text{for } n \le N, \ \text{using Lemma 3.13 on p. 48}.$$

Since $\bar{\Phi}^h(\tau) = \begin{pmatrix} \Phi^h_1(T-\tau) \\ -\Phi^h_2(T-\tau) \end{pmatrix}$ then also hold

$$\| \Phi \|_{L^2(J;E)} \le \mathcal{C} \| \Theta^N_- \|_E,$$

and

$$\| \Phi^{h,n}_- \|^2_E + \sum_{m=0}^{n-1} \Big(\| [\Phi^h]^m \|^2_E \Big) = \| \Theta^{h,N}_- \|^2_E, \ \text{for } n \le N.$$

That concludes that Problem 3.5.2 with given data $\begin{pmatrix} \Theta^h_1 \\ \Theta^h_2 \end{pmatrix}^N_- \in Z^h \times Z^h$ is solvable and its solution $\Phi^h \in \mathcal{V}^r_-(\mathcal{N}; Z^h \times Z^h)$ has the same stability estimates of the fully-discrete solution of (3.66) on p. 44. □

Corollary 3.4 *From (3.97) in Lemma 3.18 on p. 56 there hold*

$$\| \Phi^{h,n}_- \|_E \le \| \Theta^{h,N}_- \|_E, \tag{3.98}$$

and

$$\sum_{m=0}^{n-1} \Big(\| [\Phi^h]^m \|^2_E \Big) \le \| \Theta^{h,N}_- \|^2_E, \tag{3.99}$$

for $n \le N$.

Proof:
Starting with (3.97) then

$$\| \Phi_-^{h,n} \|_E^2 + \overbrace{\sum_{m=0}^{n-1} \Big(\| [\Phi^h]^m \|_E^2 \Big)}^{\geq 0} = \| \Theta_-^{h,N} \|_E^2 \Longrightarrow \| \Phi_-^{h,n} \|_E^2 \leq \| \Theta_-^{h,N} \|_E^2,$$

similarly

$$\overbrace{\| \Phi_-^{h,n} \|_E^2}^{\geq 0} + \sum_{m=0}^{n-1} \Big(\| [\Phi^h]^m \|_E^2 \Big) = \| \Theta_-^{h,N} \|_E^2 \Longrightarrow \sum_{m=0}^{n-1} \Big(\| [\Phi^h]^m \|_E^2 \Big) \leq \| \Theta_-^{h,N} \|_E^2,$$

and that completes the proof. □

Definition 3.11 (The spatial matrix projection operator) *Let Z be the Hilbert space given in Definition 2.1 on p. 6. Let Z^h be the conformal finite dimensional subspace of the Hilbert space Z given in Definition 3.10 on p. 43. For $u_1, u_2 \in Z$, we define the spatial scalar projection operator $\hat{\Pi} : Z \to Z^h$:*

$$a(\hat{\Pi} u - u, v) = 0, \quad \forall\, v \in Z^h, \tag{3.100}$$

Lemma 3.19 *The spatial projection operator $\hat{\Pi}$ given in Definition 3.11 is well defined.*

Proof:
To show uniqueness and then existence for $\hat{\Pi} : Z \to Z^h$. For $u \in Z$, let u^1, $u^2 \in Z^h$ to be two functions which satisfy (3.100):

$$a(u^1 - u, v) = 0, \text{ and } a(u^2 - u, v) = 0, \quad \forall\, v \in Z^h. \tag{3.101}$$

with taking the difference of $u^1 - u^2 =: w$ in (3.101) which now reads
Find $w \in Z^h$:

$$a(w, v) = 0, \tag{3.102}$$

then with using the *continuity* of the form $a(\,,)$ and Cauchy inequality,

$$a(w, v) \leq M \| w \|_Z \| v \|_Z,$$

and with choosing $v = w \in Z^h$, and using *ellipticity* of the form $a(\,,)$, then

$$\alpha \| w \|_Z^2 \leq a(w, w) = 0, \Longrightarrow 0 = w = u_1 - u_2, \Longrightarrow u_1 = u_2.$$

Now, since Z is a Hilbert space (see Lemma 2.2 on p. 11) then with applying Lax-Milgram theorem (see Theorem A.3 on p. 178) it concludes the existence and the uniqueness, and that completes the proof. □

Definition 3.12 (The spatio-time projection operator) *Let $I = (-1, 1)$ be the given time step, Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let $P^r(I; Z^h \times Z^h)$ be the fully-discrete space of rth order polynomials in the time step $I = (-1, 1)$ with coefficients in $Z^h \times Z^h$ given in Definition 3.1 on p. 19. For $U = \binom{u_1}{u_2} \in L^2(I; Z \times Z)$ which is continuous at $t = 1$. The following two spatio-time projection operators are defined, via the $r + 1$, $r \geq 1$ conditions:*

$$\tilde{\Pi}^r = \begin{pmatrix} \tilde{\Pi}_1^r & 0 \\ 0 & \tilde{\Pi}_2^r \end{pmatrix} : L^2(J; Z \times Z) \to P^r(I; Z^h \times Z^h), \tag{3.103}$$

such that

$$\int_I a(\tilde{\Pi}_1^r u_1 - u_1, v_1)dt = 0, \forall\, v_1 \in P^{r-1}(I; Z^h), \text{ with } \tilde{\Pi}_1^r u_1(+1) = \hat{\Pi} u_1(+1) \in Z^h, \tag{3.104}$$

and

$$\int_I a(\tilde{\Pi}_2^r u_2 - u_2, v_2)dt = 0, \forall\, v_2 \in P^{r-1}(I; Z^h), \text{ with } \tilde{\Pi}_2^r u_2(+1) = \hat{\Pi} u_2(+1) \in Z^h. \tag{3.105}$$

Here, $a(\,,)$ is the form which is given in Definition 2.2 on p. 6 which satisfies the following properties:

$$\begin{aligned} |a(u, v)| &\leq M \parallel u \parallel_Z \parallel v \parallel_Z, \qquad && \forall \quad u, v \in Z, \\ a(u, u) &\geq \alpha \parallel u \parallel_Z^2, && \forall \quad u \in Z, \end{aligned}$$

Lemma 3.20 (Well-defined spatio-time matrix projection operator) *Let $\tilde{\Pi}^r = \begin{pmatrix} \tilde{\Pi}_1^r & 0 \\ 0 & \tilde{\Pi}_2^r \end{pmatrix}$ be the matrix time projection operator given in Definition 3.12 on p. 59. It is well defined.*

Proof:
To show uniqueness, and doing it in component wise:
Firstly, for given $u_1 \in L^2(I; Z)$, let $u_1^1 \in P^r(I; Z^h)$ and $u_1^2 \in P^r(I; Z^h)$ where both satisfy (3.104) such that

$$\int_I a(u_1^1 - u_1, v_1)dt = 0, \text{ and } \int_I a(u_1^2 - u_1, v_1)dt = 0, \forall\, v_1 \in P^{r-1}(I; Z^h), \tag{3.106}$$

with

$$u_1^1(+1) = \hat{\Pi} u_1(+1) \text{ and } u_1^2(+1) = \hat{\Pi} u_1(+1). \tag{3.107}$$

Then from taking the difference of $u_1^1 - u_1^2$ which can be written in the following finite series

$$u_1^1 - u_1^2 = \sum_{i=0}^{r} \bar{v}_i L_i, \text{ for } \bar{v}_i = \int_I (u_1^1 - u_1^2) L_i\, dt \in Z^h, (\text{ see [44, Lemma 1.10]}), \tag{3.108}$$

and for any $j = \{0, \cdots, r-1\}$ and choosing $v_1 = L_j v \in P^{r-1}(I, Z^h)$ and using the orthogonal property of Legendre polynomials, yield

$$\begin{aligned} 0 = \int_I a(u_1^1 - u_1^2, L_j v)dt = \sum_{i=0}^{r} a(\bar{v}_i, v) \int_I L_i L_j dt = \sum_{i=0}^{r} a(\bar{v}_i, v)\delta_{ij} \frac{2}{2j+1} = a(\bar{v}_j, v)\frac{2}{2j+1}, \\ \Longrightarrow 0 = a(\bar{v}_j, v), \text{ for } j = \{0, \cdots, r-1\}. \end{aligned} \tag{3.109}$$

Now, with choosing $v = \bar{v}_j \in Z^h$ and using the *ellipticity* of the form $a(\,,)$, imply

$$M \parallel \bar{v}_j \parallel_Z^2 \leq a(\bar{v}_j, \bar{v}_j) = 0, \tag{3.110}$$

and, from (3.110), implies that $\bar{v}_j = 0$, for $j = \{0, \cdots, r-1\}$, and with going back to (3.108), yields that

$$u_1^1 - u_1^2 = \overbrace{\sum_{j=0}^{r-1} \bar{v}_j L_j}^{=0} + \bar{v}_r L_r \Longrightarrow u_1^1 - u_1^2 = \bar{v}_r L_r,$$

and finally from (3.104) i.e.

$$0 = u_1^1(+1) - u_1^2(+1) = \bar{v}_r L_r(+1) = \bar{v}_r \Longrightarrow \bar{v}_r = 0,$$

which concludes that $u_1^1 = u_1^2$. Also the same holds for the other component. To show existence and with following the same argument in [44, Lemma 1.10] such that, from Lemma 2.2 on p. 11 $L^2(J; Z^h \times Z^h)$ is a separable Hilbert space i.e. it has orthonormal bases (see Theorem A.5 and Definition A.20 on p. 179) and with choosing the Legendre polynomials as the orthonormal basis in $L^2(I) = L^2(-1, 1)$, then $U = \sum_{i=0}^{\infty} L_i U_i$, is the Legendre expansion of $U \in L^2(I; Z \times Z)$ which exists and is continuous in $t = 1$, and setting

$$\tilde{\Pi}^r U = \sum_{i=0}^{r-1} L_i U_i + \big((\hat{\Pi} I)U(1) - \sum_{i=0}^{r-1} U_i\big) L_r,$$

then that also concludes that $\tilde{\Pi}^r U \in P^r(I; Z^h \times Z^h)$, which satisfies (3.104) and (3.105) on p. 59, does exist and that completes the proof. □

Remark 3.5 *For an arbitrary interval (a, b) with $k := b - a > 0$, then $\Pi^r_{(a,b)}$ correspondingly via the linear map $Q : (-1, 1) \to (a, b)$, $\bar{t} \mapsto t = \frac{1}{2}(a + b + k\bar{t})$ is defined as*

$$\Pi^r_{(a,b)} U = [\Pi^r(U \circ Q)] \circ Q^{-1}, \text{ see [44, c.f. (1.41)]} \tag{3.111}$$

Definition 3.13 (New non-empty normed spaces) *Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let $A \in L(Z, Z')$ be the operator given in Definition 2.2 on p. 6. From Definition A.19 with Corollaries A.7 and A.8 on p. 176, a non-empty spaces D_A and $\bar{D}_A$ over the field $\mathbb{C}$ which are a subspace of Z is defined as*

$$\begin{aligned} &D_A := \{u \in Z : Au \in H\}, \text{ with a norm } \| u \|^2_{D_A} := \| u \|^2_Z + \| Au \|^2_H, \forall u \in D_A, \text{ and} \\ &\bar{D}_A := \{u \in Z : Au \in Z\}, \text{ with a norm } \| u \|^2_{\bar{D}_A} := \| u \|^2_Z + \| Au \|^2_Z, \forall u \in \bar{D}_A \end{aligned} \tag{3.112}$$

Assumption 3.2 (Regularity of the solution after discretising in time) *Given the higher regularity in Corollary 2.1 in (2.11) on p. 8 for the first component on the solution of the variational formulation, such that for $f \in H^1(J; H)$ and $\partial_t u_2 \in L^2(J; H)$, then from (2.20) on p. 11 that is*

$$\overbrace{\partial_t u_2}^{\in L^2(J;H)} + \overbrace{A\, u_1}^{\in L^2(J;H)} = \overbrace{f}^{\in L^2(J;H)},$$

then it means from Definition 3.13 that $u_1 \in L^2(J; D_A)$. The same is also assumed for the corresponding initial data $u_{1,0} := u_0 \in D_A \subset Z$ which does not affect the given existence and uniqueness theorem in Chapter 2. Moreover, at a generic time step $I = (t_0, t_1)$ the first component of the spatio-time projected solution is assumed to be in H i.e. $A\tilde{\Pi}^r_1 u_1 \in P^r(I; H)$.

Lemma 3.21 (Galerkin orthogonality) *Let Z be the Hilbert space given in Definition 2.1 on p. 6. Let Z^h be the conformal finite dimensional subspace of the Hilbert space Z given in Definition 3.10 on p. 43. Let $B_{DG}(\,,)$ be the form given in (3.17) on p. 26:*

$$B_{DG}(U_{DG},V) := \sum_{n=0}^{N-1}\int_{I_n}\Big\{(\partial_t U_{DG},V)_E + \hat{a}(U_{DG},V)\Big\}dt + \sum_{n=1}^{N-1}([U_{DG}]^n, V_+^n)_E + (U^0_{DG,+},V^0_+)_E.$$

For $U \in L^2(J;Z\times Z)$ which is the continuous solution of Problem 2.2.2 on p. 15, and $U^h_{DG} \in \mathcal{V}^{\underline{r}}(\mathcal{N};Z^h\times Z^h)$ the solution of the fully-discrete formulation (3.66) on p. 44, there holds

$$B_{DG}(U-U^h_{DG},V^h) = 0,\ \ \forall\, V^h \in \mathcal{V}^{\underline{r}}(\mathcal{N};Z^h\times Z^h) \subset \mathcal{V}^r(\mathcal{N};Z\times Z). \tag{3.113}$$

Proof:
Firstly, the continuous solution of Problem 2.2.2 on p. 15 which is a solution of the equivalent variational formulation in (3.12) on p. 23 which satisfies Theorem 3.1 on p. 23:

$$B_{DG}(U,V) = F_{DG}(V) + (U_0,V^0_+)_E = \sum_{n=0}^{N-1}\int_{I_n}(f,v_2)_H dt + (U_0,V^0_+)_E, \forall\, V \in \mathcal{V}^{\underline{r}}(\mathcal{N};Z\times Z), \tag{3.114}$$

with taking the difference between U and U^h_{DG} in (3.114) and with $V = V^h \in \mathcal{V}^{\underline{r}}(\mathcal{N};Z^h\times Z^h) \subset \mathcal{V}^{\underline{r}}(\mathcal{N};Z\times Z)$, then

$$B_{DG}(U-U^h_{DG},V^h) = (U_0-U^h_0, V^{h,0}_+)_E, \forall\, V^h \in \mathcal{V}^{\underline{r}}(\mathcal{N};Z^h\times Z^h).$$

Secondly, with the use of Assumption 3.1 on p. 44 such that $U^h_0 := \bar{\Pi}U_0$ is the $E-$projection of given initial data U_0 into $Z^h\times Z^h$, thus

$$B_{DG}(U-U^h_{DG},V^h) = 0, \forall\, V^h \in \mathcal{V}^{\underline{r}}(\mathcal{N};Z^h\times Z^h),$$

and that completes the proof. □

Theorem 3.6 (A priori error estimate in the last left sided limit)
Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let Z^h be the conformal finite dimensional subspace of Z. Let D_A be the non-empty normed space given in Definition 3.13 on p. 60. Let $\mathcal{V}^{\underline{r}}(\mathcal{N};Z^h\times Z^h) = \Big\{U : J \to Z^h\times Z^h : U|_{I_n} \in P^{r_n}(I_n;Z^h\times Z^h), 0 \le n \le N-1\Big\}$ be the fully-discrete finite-dimensional product subspace of $\mathcal{V}^{\underline{r}}(\mathcal{N};Z\times Z)$, where the latter space is given in Definition 3.1 on p. 19. For $U \in L^2(J;Z\times Z)$ the solution of the continuous formulation Problem 2.2.2 on p. 15 and $U^h_{DG} \in \mathcal{V}^{\underline{r}}(\mathcal{N};Z^h\times Z^h)$ the solution of the fully-discrete formulation (3.66) on p. 44. We assume the additional regularity $U \in H^1(J;Z\times Z)$. Then there holds

$$\begin{aligned}\|\,(U-U^h_{DG})^N_-\,\|_E \le&\ \|\,(U-\tilde{\Pi}^r U)^N_-\,\|_E + \frac{2\sqrt{\mathcal{P}_{r_n}}\mathcal{C}}{k}\|\,\tilde{\Pi}^r_2 u_2 - u_2\,\|_{L^2(J;H)} + \sum_{n=1}^{N-1}\|\,(\hat{\Pi}u_2-u_2)^n_-\,\|_H \\ &+ \sqrt{2}\,\|\,(\hat{\Pi}u_2-u_2)^N_-\,\|_H + \frac{\max(1,M)\mathcal{C}}{\sqrt{\min(\alpha,1)}}\big(\,\|\,A(\tilde{\Pi}^r_1 u_1 - u_1)\,\|_{L^2(J;H)} + \|\,\tilde{\Pi}^r_2 u_2 - u_2\,\|_{L^2(J;Z)}\,\big),\end{aligned} \tag{3.115}$$

where $\hat{\Pi}$ is the spatial projection operator given in Definition 3.10 on p. 43, and $\tilde{\Pi}^r = \begin{pmatrix}\tilde{\Pi}^r_1 & 0\\ 0 & \tilde{\Pi}^r_2\end{pmatrix}$ is the spatio-time projection operator given in Definition 3.12 on p. 59. k is the time step size, $M < \infty$ and

$\alpha \in \mathbb{R}$, the upper and lower bounds of the form $a(\,,\,)$, respectively which satisfies the properties given in Theorem A.3 on p. 178:

$$|a(u,v)| \leq M \parallel u \parallel_Z \parallel v \parallel_Z, \qquad \forall \quad u,v \in Z^h \subset Z,$$
$$a(u,u) \geq \alpha \parallel u \parallel_Z^2, \qquad \forall \quad u \in Z^h \subset Z.$$

and

$$\mathcal{C} = \max_n (T \frac{\parallel Y^{r_n} \parallel_2^2 (2r_n+1)}{2 \min_p \; Re\; \lambda_p}),$$

is depending on the maximum value of the approximation order r_n for $n = 0, \cdots, N-1$, the square of the spectral norm (see (A.6) on p. 173) of the matrix $Y^{r_n} \in \mathbb{C}^{(r_n+1)\times(r_n+1)}$, and λ_p with $Re\;\lambda_p > 0$ the eigenvalues of the matrix $\mathcal{A}^L$ given in (3.31) on p. 30 for $p = 0, \cdots, r_n$. $\mathcal{P}_{r_n}$ is a real constant which also depends on the maximum approximation order r_n.

Proof:
The additional regularity $U \in H^1(J; Z \times Z)$ ensures that the point evaluation of U in all time mesh nodes is well defined. Firstly, with the *triangle inequality*

$$\parallel (U - U_{DG}^h)_-^N \parallel_E \leq \parallel (U - \tilde{\Pi}^r U)_-^N \parallel_E + \parallel (\tilde{\Pi}^r U - U_{DG}^h)_-^N \parallel_E .$$

Now, to find the upper bound of $\parallel (\tilde{\Pi}\Pi U - U_{DG}^h)_-^N \parallel_E$, Problem 3.5.2 on p. 55; the fully-discrete auxiliary problem on p. 55, and it stability will be used:
Let $\Theta^h = \tilde{\Pi}^r U - U_{DG}^h$ s.t. $\Theta_-^{h,N} = (\tilde{\Pi}^r U - U_{DG}^h)_-^N \in Z^h \times Z^h$, for U the solution of continuous formulation Problem 2.2.2 on p. 15, and U_{DG}^h the solution of the fully-discrete formulation in (3.66) on p. 44.
Problem 3.5.2 in (3.95) states:
Find a real vector-valued function $\Phi^h \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h \times Z^h)$, which satisfies

$$B_{DG}(V^h, \Phi^h) = \mathcal{F}_{DG}(V^h) := (V_-^{h,N}, \Theta_-^{h,N})_E, \forall V^h \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h \times Z^h), \tag{3.116}$$

for given data $\Theta_-^{h,N} \in Z^h \times Z^h$. Now, with choosing $V^h = \Theta^h \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h \times Z^h)$ in (3.116).
Then

$$\begin{aligned} \parallel \Theta_-^{h,N} \parallel_E^2 = B_{DG}(\Theta^h, \Phi^h) = B_{DG}(\tilde{\Pi}^r U - U_{DG}^h, \Phi^h) &= B_{DG}(\tilde{\Pi}^r U - U + U - U_{DG}^h, \Phi^h) \\ &= B_{DG}(\tilde{\Pi}^r U - U, \Phi^h) + B_{DG}(U - U_{DG}^h, \Phi^h). \end{aligned}$$

With having

$B_{DG}(U - U_{DG}^h, \Phi^h) = 0$, from Lemma 3.21 on p. 61, for $V^h = \Phi^h \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h \times Z^h) \subset \mathcal{V}^{\underline{r}}(\mathcal{N}; Z \times Z)$.

Then

$$\parallel \Theta_-^{h,N} \parallel_E^2 = B_{DG}(\tilde{\Pi}^r U - U, \Phi^h). \tag{3.117}$$

In $B_{DG}(\tilde{\Pi}^r U - U, \Phi^h)$ with using Lemma 3.3 on p. 26, yields

$$
\begin{aligned}
B_{DG}(\tilde{\Pi}^r U - U, \Phi^h) = &- \sum_{n=0}^{N-1} \int_{I_n} a(\tilde{\Pi}_1^{r_n} u_1 - u_1, \partial_t \Phi_1^h) dt - \sum_{n=0}^{N-1} \int_{I_n} (\tilde{\Pi}_2^{r_n} u_2 - u_2, \partial_t \Phi_2^h)_H dt \\
&- \sum_{n=1}^{N-1} ((\tilde{\Pi}^{r_n} U - U)_-^n, [\Phi^h]^n)_E + ((\tilde{\Pi}^{r_N} U - U)_-^N, \Phi_-^{h,N})_E \\
&+ \sum_{n=0}^{N-1} \int_{I_n} a(\tilde{\Pi}_1^{r_n} u_1 - u_1, \Phi_2^h) dt - \sum_{n=0}^{N-1} \int_{I_n} a(\tilde{\Pi}_2^{r_n} u_2 - u_2, \Phi_1^h) dt \\
&= \sum_{j=1}^{6} \mathcal{E}_j .
\end{aligned}
\tag{3.118}
$$

Now, with using the spatio-time projection result in (3.104) on p. 59, implies

$$
\mathcal{E}_1 = - \sum_{n=0}^{N-1} \int_{I_n} a(\tilde{\Pi}_1^{r_n} u_1 - u_1, \partial_t \Phi_1^h) d = 0.
$$

In $\mathcal{E}_2$ in (3.118) on p. 63 and with first using *Cauchy inequality* yields

$$
\mathcal{E}_2 = \int_J (\tilde{\Pi}_2^r u_2 - u_2, \partial_t \Phi_2^h)_H dt \leq \| \tilde{\Pi}_2^r u_2 - \Pi_2 u_2 \|_{L^2(J;H)} \| \partial_t \Phi_2^h \|_{L^2(J;H)}, \tag{3.119}
$$

With Lemma 3.22 there holds

$$
\| \partial_t \Phi^h \|_{L^2(J;H)}^2 \leq \frac{4}{k^2} \mathcal{P}_{r_n} \| \Phi^h \|_{L^2(J;H)}^2 \tag{3.120}
$$

Now after taking the square root of both sides of (3.120) and then plugging it back into (3.119) yields

$$
\mathcal{E}_2 \leq \frac{2\sqrt{\mathcal{P}_{r_n}} \mathcal{C}}{k} \| \tilde{\Pi}_2^r u_2 - u_2 \|_{L^2(J;H)} \| \Theta_-^{h,N} \|_E .
$$

For $\mathcal{E}_3$ and $\mathcal{E}_4$. Starting with, $\mathcal{E}_3$ and from the identities (3.104) and (3.105) on p. 59 s.t. $\tilde{\Pi}_1^r u_1(+1) = \hat{\Pi} u_1(+1)$ and $\tilde{\Pi}_2^r u_2(+1) = \hat{\Pi} u_2(+1)$ imply

$$
\begin{aligned}
\mathcal{E}_3 = - \sum_{n=1}^{N-1} ((\tilde{\Pi}^{r_n} U - U)_-^n, [\Phi^h]^n)_E &= - \sum_{n=1}^{N-1} \{ a(\hat{\Pi} u_1 - u_1)_-^n, [\Phi_1^h]^n) + (\hat{\Pi} u_2 - u_2)_-^n, [\Phi_2^h]^n)_H \} \\
&= - \sum_{n=1}^{N-1} (\hat{\Pi} u_2 - u_2)_-^n, [\Phi_2^h]^n)_H,
\end{aligned}
\tag{3.121}
$$

now with using *Cauchy inequality*, the estimate in (3.99) in Corollary 3.4 on p. 57, and at the end Lemma A.3 on p. 181, then

$$
\begin{aligned}
\mathcal{E}_3 &\leq \sqrt{\sum_{n=1}^{N-1} \| (\hat{\Pi} u_2 - u_2)_-^n \|_H^2} \sqrt{\sum_{n=1}^{N-1} \underbrace{\| [\Phi_2^h]^n \|_H^2}_{\leq a([\Phi_1^h]^n, [\Phi_1^h]^n) + \| [\Phi_2^h]^n \|_H^2}} \\
&= \sqrt{\sum_{n=1}^{N-1} \| (\hat{\Pi} u_2 - u_2)_-^n \|_H^2} \sqrt{\sum_{n=1}^{N-1} \| [\Phi^h]^n \|_E^2} \\
&\leq \sqrt{\sum_{n=1}^{N-1} \| (\hat{\Pi} u_2 - u_2)_-^n \|_H^2} \| \Theta_-^{h,N} \|_E \leq \sum_{n=1}^{N-1} \| (\hat{\Pi} u_2 - u_2)_-^n \|_H \| \Theta_-^{h,N} \|_E .
\end{aligned}
\tag{3.122}
$$

Also, for $\mathcal{E}_4$ i.e.

$$\begin{aligned}\mathcal{E}_4 = ((\tilde{\Pi}^{r_N} U - U)_-^N, \Phi_-^{h,N})_E &= a((\hat{\Pi} u_1 - u_1)_-^N, \Phi_{1,-}^{h,N}) + ((\hat{\Pi} u_2 - u_2)_-^N, \Phi_{2,-}^{h,N})_H \\ &= ((\hat{\Pi} u_2 - u_2)_-^N, \Phi_{2,-}^{h,N})_H \\ &\leq \| (\hat{\Pi} u_2 - u_2)_-^N \|_H \| \Phi_{2,-}^{h,N} \|_H \\ &\leq \| (\hat{\Pi} u_2 - u_2)_-^N \|_H \left(\sqrt{a(\Phi_{1,-}^{h,N}, \Phi_{1,-}^{h,N})} + \| \Phi_{2,-}^{h,N} \|_H \right) \\ &\leq \| (\hat{\Pi} u_2 - u_2)_-^N \|_H \sqrt{2} \left(a(\Phi_{1,-}^{h,N}, \Phi_{1,-}^{h,N}) + \| \Phi_{2,-}^{h,N} \|_H^2 \right)^{1/2} \\ &\leq \sqrt{2} \| (\hat{\Pi} u_2 - u_2)_-^N \|_H \| \Phi_-^{h,N} \|_E \\ &\leq \sqrt{2} \| (\hat{\Pi} u_2 - u_2)_-^N \|_H \| \Theta_-^{h,N} \|_E .\end{aligned} \tag{3.123}$$

Now, for $\mathcal{E}_5 + \mathcal{E}_6$ and with referring to Assumption 3.2 on p. 60 for the given higher regularity of the function u_1 and the fully-projected u_1:

$$a(\tilde{\Pi}_1 u_1 - u_1, \Phi_2^h) := (A(\tilde{\Pi}_1 u_1 - u_1), \Phi_2^h)_H, \text{ for , and } \Phi_2^h \in Z^h \subset Z.$$

Thus, after using *Cauchy inequality*, the *continuity* of the form $a(,)$, the inequality in Lemma 2.1 on p. 9, and (3.96) on p. 56 then

$$\begin{aligned}\mathcal{E}_5 + \mathcal{E}_6 &= \int_J (A(\tilde{\Pi}_1^r u_1 - u_1), \Phi_2^h)_H dt - \int_J a(\tilde{\Pi}_2^r u_2 - u_2, \Phi_1^h) dt \\ &\leq \| A(\tilde{\Pi}_1^r u_1 - u_1) \|_{L^2(J;H)} \| \Phi_2^h \|_{L^2(J;H)} + M \| \tilde{\Pi}_2^r u_2 - u_2 \|_{L^2(J;Z)} \| \Phi_1^h \|_{L^2(J;Z)} \\ &\leq \max(1, M) \left(\| A(\tilde{\Pi}_1^r u_1 - u_1) \|_{L^2(J;H)}^2 + \| \tilde{\Pi}_2^r u_2 - u_2 \|_{L^2(J;Z)}^2 \right)^{1/2} \\ &\qquad \underbrace{\left(\| \Phi_2^h \|_{L^2(J;H)}^2 + \| \Phi_1^h \|_{L^2(J;Z)}^2 \right)^{1/2}}_{= \| \Phi^h \|_{L^2(J;X)}} \\ &\leq \frac{\max(1, M)\mathcal{C}}{\sqrt{\min(\alpha, 1)}} \left(\| A(\tilde{\Pi}_1^r u_1 - u_1) \|_{L^2(J;H)} + \| \tilde{\Pi}_2^r u_2 - u_2 \|_{L^2(J;Z)} \right) \| \Theta_-^{h,N} \|_E,\end{aligned} \tag{3.124}$$

and with plugging the resulting estimates in back into (3.118) on p. 63, yield

$$\begin{aligned}B_{DG}(\tilde{\Pi}^r U - U, \Phi^h) &= \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4 + \mathcal{E}_5 + \mathcal{E}_6 \\ &\leq \Bigg(\frac{\sqrt{2\mathcal{P}_{r_n}}\mathcal{C}}{k} \| \tilde{\Pi}_2^r u_2 - u_2 \|_{L^2(J;H)} \\ &+ \sum_{n=1}^{N-1} \| (\hat{\Pi} u_2 - u_2)_-^n \|_H + \sqrt{2} \| (\hat{\Pi} u_2 - u_2)_-^N \|_H \\ &+ \frac{\max(1, M)\mathcal{C}}{\sqrt{\min(\alpha, 1)}} \left(\| A(\tilde{\Pi}_1^r u_1 - u_1) \|_{L^2(J;H)} + \| \tilde{\Pi}_2^r u_2 - u_2 \|_{L^2(J;Z)} \right) \Bigg) \| \Theta_-^{h,N} \|_E\end{aligned} \tag{3.125}$$

Finally,

$$\begin{aligned}\| (U - U_{DG}^h)_-^N \|_E \leq \| (U - \tilde{\Pi}^r U)_-^N \|_E + \frac{\sqrt{2\mathcal{P}_{r_n}}\mathcal{C}}{k} \| \tilde{\Pi}_2^r u_2 - u_2 \|_{L^2(J;H)} + \sum_{n=1}^{N-1} \| (\hat{\Pi} u_2 - u_2)_-^n \|_H \\ + \sqrt{2} \| (\hat{\Pi} u_2 - u_2)_-^N \|_H + \frac{\max(1, M)\mathcal{C}}{\sqrt{\min(\alpha, 1)}} \left(\| A(\tilde{\Pi}_1^r u_1 - u_1) \|_{L^2(J;H)} + \| \tilde{\Pi}_2^r u_2 - u_2 \|_{L^2(J;Z)} \right),\end{aligned} \tag{3.126}$$

and that completes the proof. □

Corollary 3.5 *Let $U \in L^2(J; Z \times Z)$ be the solution of the continuous formulation the problem in (2.33) and $U_{DG}^h \in \mathcal{V}_-^r(\mathcal{N}; Z^h \times Z^h)$ be the solution of the fully-discrete formulation (3.66). We assume the additional regularity $U \in H^2(J; Z \times Z), Au_1 \in H^2(J; H)$. Then there holds*

$$
\begin{aligned}
\|(U - U_{DG}^h)_-^N\|_E \leq\ & \|(U - \hat{\Pi}U)_-^N\|_E + \sqrt{2}\sum_{n=1}^{N} \|(u_2 - \hat{\Pi}u_2)_-^n\|_H \\
& + \left(\frac{2\sqrt{\mathcal{P}_{r_n}}\mathcal{C}}{k} + \frac{\max(1, M)\mathcal{C}}{\sqrt{\min(\alpha, 1)}}\right)\Big(C(\frac{k}{2})^2 \frac{1}{\max\{1, r\}^2}\|u_2\|_{H^2(J;Z)} \\
& + C\max\{1, r\}^{1/2}\|u_2 - \hat{\Pi}u_2\|_{L^2(J;Z)} \\
& + C\max\{1, r\}^{-1/2}(\frac{k}{2})^{1/2}\sum_{n=1}^{N}\|(u_2 - \hat{\Pi}u_2)_-^n\|_Z\Big) \\
& + \frac{\max(1, M)\mathcal{C}}{\sqrt{\min(\alpha, 1)}}\Big(C(\frac{k}{2})^2\frac{1}{\max\{1, r\}^2}\|Au_1\|_{H^2(J;H)} \\
& + C\max\{1, r\}^{1/2}\|A(u_1 - \hat{\Pi}u_1)\|_{L^2(J;H)} \\
& + C\max\{1, r\}^{-1/2}(\frac{k}{2})^{1/2}\sum_{n=1}^{N}\|A(u_1 - \hat{\Pi}u_1)_-^n\|_H\Big),
\end{aligned}
\tag{3.127}
$$

where $M < \infty$ and $\alpha \in \mathbb{R}$, the upper and lower bounds of $a(\, , \,)$ and

$$\mathcal{C} = \max_n(T\frac{\| \ Y^{r_n} \ \|_2^2 \ (2r_n + 1)}{2\min_p \ \ Re \ \lambda_p}),$$

which is depending on the maximum value of the approximation order r_n for $n = 0, \cdots, N-1$.

Proof:
In Theorem 3.6 we split all terms $U - \tilde{\Pi}^r U$ in two parts $U - \tilde{\Pi}U = U - \Pi^r U + \Pi^r U - \tilde{\Pi}^r U$ and then apply the approximation result in time [44, Theorem 1.19] with $s = s_0 = 1, r \geq 1, I = (0, k)$ using the identity $\tilde{\Pi}^r = \Pi^r\hat{\Pi}$ with the time interpolation operator Π^r:

$$\|u - \Pi^r u\|_{L^2(I;\cdot)} \leq C\left(\frac{k}{2}\right)^2 \frac{1}{\max\{1, r\}^2}\|u''\|_{L^2(I;\cdot)}. \tag{3.128}$$

We also need the stability result [44, Lemma 1.16]

$$\|\Pi^r u\|_{L^2(I;\cdot)}^2 \leq C\max\{1, r\}\|u\|_{L^2(I;\cdot)}^2 + \frac{C}{\max\{1, r\}}\frac{k}{2}\|(u)_-\|_\cdot^2. \tag{3.129}$$

We now bound the 2 relevant terms in Theorem 3.6 involving the time interpolation operator:

$$
\begin{aligned}
\|A(u_1 - \tilde{\Pi}^r u_1)\|_{L^2(J;H)} &\leq \|A(u_1 - \Pi^r u_1 + \Pi^r u_1 - \tilde{\Pi}^r u_1)\|_{L^2(J;H)} \\
&\leq \|A(u_1 - \Pi^r u_1)\|_{L^2(J;H)} + \|\Pi^r A(u_1 - \hat{\Pi}u_1)\|_{L^2(J;H)} \\
&\leq C(\frac{k}{2})^2\frac{1}{\max\{1, r\}^2}\|Au_1\|_{H^2(J;H)} \\
&\quad + C\max\{1, r\}^{1/2}\|A(u_1 - \hat{\Pi}u_1)\|_{L^2(J;H)} \\
&\quad + C\max\{1, r\}^{-1/2}(\frac{k}{2})^{1/2}\sum_{n=1}^{N}\|A(u_1 - \hat{\Pi}u_1)_-^n\|_H.
\end{aligned}
$$

Analogously

$$
\begin{aligned}
\|u_2 - \tilde{\Pi}^r u_2\|_{L^2(J;Z)} &\leq \|u_2 - \Pi^r u_2 + \Pi^r u_2 - \tilde{\Pi}^r u_2\|_{L^2(J;Z)} \\
&\leq \|u_2 - \Pi^r u_2\|_{L^2(J;Z)} + \|\Pi^r(u_2 - \hat{\Pi} u_2)\|_{L^2(J;Z)} \\
&\leq C(\frac{k}{2})^2 \frac{1}{\max\{1,r\}^2} \|u_2\|_{H^2(J;Z)} \\
&+ C \max\{1,r\}^{1/2} \|u_2 - \hat{\Pi} u_2\|_{L^2(J;Z)} \\
&+ C \max\{1,r\}^{-1/2} (\frac{k}{2})^{1/2} \sum_{n=1}^{N} \|(u_2 - \hat{\Pi} u_2)_-^n\|_Z.
\end{aligned}
$$

Combining now the individual terms with Theorem 3.6 we obtain

$$
\begin{aligned}
\|(U - U_{DG}^h)_-^N\|_E &\leq \|(U - \hat{\Pi} U)_-^N\|_E + \frac{2\sqrt{\mathcal{P}_{r_n}}\mathcal{C}}{k} \|u_2 - \tilde{\Pi}^r u_2\|_{L^2(J;H)} \\
&+ \frac{\max(1,M)\mathcal{C}}{\sqrt{\min(\alpha,1)}} \big(\|A(u_1 - \tilde{\Pi}^r u_1)\|_{L^2(J;H)} + \|u_2 - \tilde{\Pi}^r u_2\|_{L^2(J;Z)} \big) \\
&+ \sum_{n=1}^{N-1} \|(u_2 - \hat{\Pi} u_2)_-^n\|_H + \sqrt{2} \|(u_2 - \hat{\Pi} u_2)_-^N\|_H \\
&\leq \|(U - \hat{\Pi} U)_-^N\|_E + \sqrt{2} \sum_{n=1}^{N} \|(u_2 - \hat{\Pi} u_2)_-^n\|_H \\
&+ \left(\frac{2\sqrt{\mathcal{P}_{r_n}}\mathcal{C}}{k} + \frac{\max(1,M)\mathcal{C}}{\sqrt{\min(\alpha,1)}} \right) \|u_2 - \tilde{\Pi}^r u_2\|_{L^2(J;Z)} \\
&+ \frac{\max(1,M)\mathcal{C}}{\sqrt{\min(\alpha,1)}} \|A(u_1 - \tilde{\Pi}^r u_1)\|_{L^2(J;H)} \\
&\leq \|(U - \hat{\Pi} U)_-^N\|_E + \sqrt{2} \sum_{n=1}^{N} \|(u_2 - \hat{\Pi} u_2)_-^n\|_H \\
&+ \left(\frac{2\sqrt{\mathcal{P}_{r_n}}\mathcal{C}}{k} + \frac{\max(1,M)\mathcal{C}}{\sqrt{\min(\alpha,1)}} \right) \Big(C(\frac{k}{2})^2 \frac{1}{\max\{1,r\}^2} \|u_2\|_{H^2(J;Z)} \\
&+ C \max\{1,r\}^{1/2} \|u_2 - \hat{\Pi} u_2\|_{L^2(J;Z)} \\
&+ C \max\{1,r\}^{-1/2} (\frac{k}{2})^{1/2} \sum_{n=1}^{N} \|(u_2 - \hat{\Pi} u_2)_-^n\|_Z \Big) \\
&+ \frac{\max(1,M)\mathcal{C}}{\sqrt{\min(\alpha,1)}} \Big(C(\frac{k}{2})^2 \frac{1}{\max\{1,r\}^2} \|Au_1\|_{H^2(J;H)} \\
&+ C \max\{1,r\}^{1/2} \|A(u_1 - \hat{\Pi} u_1)\|_{L^2(J;H)} \\
&+ C \max\{1,r\}^{-1/2} (\frac{k}{2})^{1/2} \sum_{n=1}^{N} \|A(u_1 - \hat{\Pi} u_1)_-^n\|_H \Big)
\end{aligned}
$$

□

Corollary 3.6 *Let $U \in L^2(J; Z \times Z)$ be the solution of the continuous formulation the problem in (2.33) and $U_{DG}^h \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h \times Z^h)$ be the solution of the fully-discrete formulation (3.66). We* *assume the additional regularity* $U \in H^2(J; Z \times Z), Au_1 \in H^2(J; H)$ *and we assume the following approximation bound in space for all* $1 \le n \le N$

$$\begin{aligned} \|(U - \hat{\Pi}U)_-^N\|_E + \|(u_2 - \hat{\Pi}u_2)_-^n\|_H + \|(u_2 - \hat{\Pi}u_2)_-^n\|_Z + \|A(u_1 - \hat{\Pi}u_1)_-^n\|_H \\ + \|u_2 - \hat{\Pi}u_2\|_{L^2(J;Z)} + \|A(u_1 - \hat{\Pi}u_1)\|_{L^2(J;H)} \le C_U h^2 \end{aligned} \tag{3.130}$$

with the constant C_U *only depending on* U *and* J*. Then we obtain the convergence result in* h *and* k *:*

$$\|(U - U_{DG}^h)_-^N\|_E = \mathcal{O}(h^2 + k^{-1}h^2 + k + k^2 + k^{-1/2}h^2) \tag{3.131}$$

Proof:
Using Corollary 3.5 we obtain

$$\begin{aligned} \|(U - U_{DG}^h)_-^N\|_E &=\le C(h^2 + k^{-1}h^2 + k + k^2 + h^2 + k^{1/2}k^{-1}h^2 + k^2 + h^2 + k^{1/2}k^{-1}h^2) \\ &= \mathcal{O}(h^2 + k^{-1}h^2 + k + k^2 + k^{-1/2}h^2) \end{aligned}$$

□

Lemma 3.22 *Let* $\Phi^h \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z)$*. Then there holds*

$$\| \partial_t \Phi^h \|_{L^2(J;H)}^2 \le \frac{4}{k^2} \mathcal{P}_{r_n} \| \Phi^h \|_{L^2(J;H)}^2 \tag{3.132}$$

with $\mathcal{P}_{r_n} = \max_n (r_n + 1)^4$.

Proof:
we can write $\Phi^h|_{I_n} = \sum_{j=0}^{r_n} \phi_j(t) \Phi_j^h$ with $\Phi_j^h \in Z^h$. Due to the normalisation of the $\phi_j(t)$ we have

$$\| \Phi^h \|_{L^2(I_n;H)}^2 = \frac{k}{2} \sum_{j=0}^{r} \| \Phi_j^h \|_H^2 . \tag{3.133}$$

For the derivative we obtain $\partial_t \Phi^h = \sum_{j=0}^{r_n} \partial_t \phi_j(t) \Phi_j^h$, with $\partial_t \phi_j(t) \in P^{r_n - 1}(I_n)$,. Applying the *Cauchy inequality* for the $(r_n + 1)$ sums we obtain

$$\| \partial_t \Phi^h \|_{L^2(J;H)}^2 \le \sum_{n=0}^{N-1} (r_n + 1) \sum_{j=0}^{r_n} \int_{I_n} (\partial_t \phi_j(t))^2 dt \| \Phi_j^h \|_H^2,$$

Here, with letting $I_n \to I = (t_0, t_1)$, recalling the affine transformation i.e. for $t = \frac{1}{2}(t_0 + t_1 + k\hat{t})$, and choosing the time basis's transported variant to be the normalised Legendre polynomials s.t.

$$\partial_t \phi_j(t) = \frac{2}{k} \sqrt{j + 1/2} \partial_{\hat{t}} L_j(\hat{t}) \Longrightarrow \int_{I_n} (\partial_t \phi_j(t))^2 dt = \frac{(2j+1)}{k} \int_{-1}^{1} (\partial_{\hat{t}} L_j(\hat{t}))^2 d\hat{t}$$

For Legendre polynomials it can be easily show that

$$\int_{-1}^{1} (\partial_{\hat{t}} L_j(\hat{t}))^2 d\hat{t} = j(j+1), \tag{3.134}$$

i.e. we obtain

$$\int_{I_n} (\partial_t \phi_j(t))^2 dt = \frac{(2j+1)}{k} j(j+1) \leq \frac{2}{k}(j+1)^3.$$

Using the norm identity (3.133) and the estimate in (3.96) we finally obtain

$$\| \partial_t \Phi^h \|^2_{L^2(J;H)} \leq \underbrace{\frac{2}{k} \max_n (r_n+1)^4}_{=:\mathcal{P}_{r_n}} \sum_{n=0}^{N-1} \underbrace{\sum_{j=0}^{r_n} \| \Phi^h_j \|^2_H}_{=\frac{2}{k}\|\Phi^h\|^2_{L^2(I_n;H)}} = \frac{4}{k^2} \mathcal{P}_{r_n} \| \Phi^h \|^2_{L^2(J;H)} . \tag{3.135}$$

□

Theorem 3.7 (Local a priori error estimate in the $L^2(I;E)$-norm)
Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let Z^h be the conformal finite dimensional subspace of Z. Let D_A be the non-empty normed space given in Definition 3.13 on p. 60. For $U \in L^2(J; Z \times Z)$ the solution of Problem 2.2.2 on p. 15 and $U^h_{DG} \in \mathcal{V}^r_-(\mathcal{N}; Z^h \times Z^h)$ the solution of the fully-discrete formulation (3.66) on p. 44, then at a generic time step $I = (t_0, t_1)$. We assume the additional regularity $U \in H^2(J; Z \times Z)$, $Au_1 \in H^2(J, H)$. Then there holds

$$\begin{aligned}
\| U - U^h_{DG} \|_{L^2(I;E)} \leq & \| U - \tilde{\Pi}^r U \|_{L^2(I;E)} \\
& + (\frac{k \| Y \|^2_2}{2 \min_p Re\, \lambda_p}) \Big(C(\frac{\sqrt{2\mathcal{P}_r}}{k} + \frac{\max(1,M)}{\sqrt{\min(\alpha,1)}}) \Big((\frac{k}{2})^2 \frac{1}{\max\{1,r\}^2} \|u_2\|_{H^2(J;Z)} \\
& + \max\{1,r\}^{1/2} \|u_2 - \hat{\Pi} u_2\|_{L^2(J;Z)} + \max\{1,r\}^{-1/2} (\frac{k}{2})^{1/2} \sum_{n=1}^{N} \|(u_2 - \hat{\Pi} u_2)^n_- \|_Z \Big) \\
& + \frac{2}{\sqrt{k}} (2r+1) \| (\hat{\Pi}_2 u_2 - u_2)_- \|_H + \sqrt{\frac{2}{k}} (2r+1) \| (U - U^h_{DG})^0_- \|_E \\
& + \frac{\max(1,M) C}{\sqrt{\min(\alpha,1)}} \Big((\frac{k}{2})^2 \frac{1}{\max\{1,r\}^2} \|Au_1\|_{H^2(J;H)} + \max\{1,r\}^{1/2} \|A(u_1 - \hat{\Pi} u_1)\|_{L^2(J;H)} \\
& + \max\{1,r\}^{-1/2} (\frac{k}{2})^{1/2} \sum_{n=1}^{N} \|A(u_1 - \hat{\Pi} u_1)^n_- \|_H \Big) \Big),
\end{aligned} \tag{3.136}$$

where $\hat{\Pi}$ is the spatial projection operator given in Definition 3.10 on p. 43, and $\tilde{\Pi}^r = \begin{pmatrix} \tilde{\Pi}^r_1 & 0 \\ 0 & \tilde{\Pi}^r_2 \end{pmatrix}$ is the spatio-time projection operator given in Definition 3.12 on p. 59. k is the time step size, $M < \infty$ and $\alpha \in \mathbb{R}$, the upper and lower bounds of the form $a(\,,)$, respectively given in Theorem A.3 on p. 178 and $\mathcal{P}_r$ is a real constant which also depends on the maximum approximation order r at the generic time step I. k is the time step size, $\| Y \|^2_2$ the square of the spectral norm (see (A.6) on p. 173) of the matrix $Y^r \in \mathbb{C}^{(r+1)\times(r+1)}$, and the eigenvalues λ_p of the matrix $\mathcal{A}^L$ given in (3.31) on p. 30 for $p = 0, \cdots, r$.

Proof:
Starting with the triangle inequality

$$\| U - U^h_{DG} \|_{L^2(I;E)} \leq \| U - \tilde{\Pi}^r U \|_{L^2(I;E)} + \| \tilde{\Pi}^r U - U^h_{DG} \|_{L^2(I;E)} . \tag{3.137}$$

Now, since $\tilde{\Pi}^r U - U_{DG}^h \in P^r(I; Z^h \times Z^h)$, then the discrete inf-sup condition in Lemma 3.10 on p. 45 will be used to bound $\| \tilde{\Pi}^r U - U_{DG}^h \|_{L^2(I;E)}$ where the difference between the continuous and fully discrete solution at a local generic time stepping scheme is

$$b_{DG}(U - U_{DG}^h, V^h) = ((U - U_{DG}^h)_-^0, V_+^{h,0})_E,$$

and with knowing that

$$\sup(A + B) \leq \sup(A) + \sup(B), \text{ see [23, 1.2 The Set } \mathbb{R}],$$

then we have

$$\begin{aligned}
\| \tilde{\Pi}^r U - U_{DG}^h \|_{L^2(I;E)} &\leq \Big(\frac{k \| Y \|_2^2}{2 \min_p \operatorname{Re} \lambda_p}\Big) \sup_{V^h \in P^r(I;Z^h \times Z^h)} \frac{b_{DG}(\tilde{\Pi}^r U - U_{DG}^h, V^h)}{\| V^h \|_{L^2(I;E)}} \\
&\leq \Big(\frac{k \| Y \|_2^2}{2 \min_p \operatorname{Re} \lambda_p}\Big) \Big(\sup_{V^h \in P^r(I;Z^h \times Z^h)} \frac{b_{DG}(\tilde{\Pi}^r U - U, V^h)}{\| V^h \|_{L^2(I;E)}} \\
&\qquad + \sup_{V^h \in P^r(I;Z^h \times Z^h)} \frac{b_{DG}(U - U_{DG}^h, V^h)}{\| V^h \|_{L^2(I;E)}} \Big) \\
&= \Big(\frac{k \| Y \|_2^2}{2 \min_p \operatorname{Re} \lambda_p}\Big) \Big(\sup_{V^h \in P^r(I;Z^h \times Z^h)} \frac{b_{DG}(\tilde{\Pi}^r U - U, V^h)}{\| V^h \|_{L^2(I;E)}} \\
&\qquad + \sup_{V^h \in P^r(I;Z^h \times Z^h)} \frac{((U - U_{DG}^h)_-^0, V_+^{h,0})_E}{\| V^h \|_{L^2(I;E)}} \Big).
\end{aligned} \tag{3.138}$$

Now, with using *Cauchy inequality* and then recalling the steps in (3.59) and (3.60) in the proof of Lemma 3.9 on p. 40:

$$\| V_+^{h,0} \|_E \leq \sqrt{\frac{2}{k}} (2r + 1) \| V^h \|_{L^2(I;E)},$$

then

$$\begin{aligned}
\sup_{V^h \in P^r(I;Z^h \times Z^h)} \frac{((U - U_{DG}^h)_-^0, V_+^{h,0})_E}{\| V^h \|_{L^2(I;E)}} &\leq \sup_{V^h \in P^r(I;Z^h \times Z^h)} \frac{\| (U - U_{DG}^h)_-^0 \|_E \| V_+^{h,0} \|_E}{\| V^h \|_{L^2(I;E)}} \\
&\leq \sqrt{\frac{2}{k}} (2r + 1) \| (U - U_{DG}^h)_-^0 \|_E \,.
\end{aligned} \tag{3.139}$$

With recalling Corollary 3.2 on p. 28, then

$$\begin{aligned}
b_{DG}(\tilde{\Pi}^r U - U, V^h) &= \int_I \Big\{ - (\tilde{\Pi}^r U - U, \partial_t V^h)_E + \hat{a}(\tilde{\Pi}^r U - U, V^h) \Big\} dt + ((\tilde{\Pi}^r U - U)_-, V_-^h)_E \\
&= \int_I -a(\tilde{\Pi}_1^r u_1 - u_1, \partial_t v_1^h) dt - \int_I (\tilde{\Pi}_2^r u_2 - u_2, \partial_t v_2^h)_H dt \\
&+ ((\tilde{\Pi}^r U - U)_-, V_-^h)_E + \int_I a(\tilde{\Pi}_1^r u_1 - u_1, v_2^h) dt - \int_I a(\tilde{\Pi}_2^r u_2 - u_2, v_1^h) dt \\
&= \sum_{j=0}^{5} \mathcal{E}_j.
\end{aligned} \tag{3.140}$$

With recalling the estimates of $\mathcal{E}_1$ and $\mathcal{E}_2$, in (3.118) on p. 63 in the proof of Theorem 3.6 on p. 61 but this time without the sum over all time steps and just being at a generic time step $I = (t_0, t_1)$, then now we have

$$\mathcal{E}_1 = 0, \text{ with using the spatio-time projection property,} \tag{3.141}$$

$$\mathcal{E}_2 \leq \frac{\sqrt{2\mathcal{P}_r}}{k} \parallel \tilde{\Pi}_2^r u_2 - u_2 \parallel_{L^2(I;H)} \parallel V^h \parallel_{L^2(I;E)}, \tag{3.142}$$

and with recalling the steps in (3.59) and (3.60) in the proof of Lemma 3.9 on p. 40 but this time with V_-^h:

$$\parallel V_-^h \parallel_E = \parallel V_-^h(t_1) \parallel_E = \sqrt{\parallel \sum_{p=0}^{r} \sqrt{(p+1/2)} V_p^h \parallel_E^2} \leq \sqrt{(2r+1)^2 \sum_{p=0}^{r} \parallel V_p^h \parallel_E^2} \leq \sqrt{\frac{2}{k}} (2r+1) \parallel V^h \parallel_{L^2(I;E)},$$

and the identity in (3.104) on p. 59 s.t. $\tilde{\Pi}_1^r u_1(+1) = \hat{\Pi}_1 u_1(+1)$, then

$$\begin{aligned} \mathcal{E}_3 = ((\tilde{\Pi}^r U - U)_-, V_-^h)_E &= a(\hat{\Pi}_1 u_1 - u_1)_-, v_{1,-}^h) + (\hat{\Pi}_2 u_2 - u_2)_-, v_{2,-}^h)_H \\ &= (\hat{\Pi}_2 u_2 - u_2)_-, v_{2,-}^h)_H \\ &\leq \parallel (\hat{\Pi} u_2 - u_2)_- \parallel_H \underbrace{\parallel v_{2,-}^h \parallel_H}_{\leq \sqrt{2} \parallel V_-^h \parallel_E} \\ &\leq \frac{2}{\sqrt{k}} (2r+1) \parallel (\hat{\Pi}_2 u_2 - u_2)_- \parallel_H \parallel V^h \parallel_{L^2(I;E)}, \end{aligned} \tag{3.143}$$

Again recalling the estimates of in the proof of Theorem 3.6 on p. 61 then

$$\mathcal{E}_4 + \mathcal{E}_5 \leq \frac{\max(1, M)}{\sqrt{\min(\alpha, 1)}} \left(\parallel A(\tilde{\Pi}_1^r u_1 - u_1) \parallel_{L^2(I;H)} + \parallel \tilde{\Pi}_2^r u_2 - u_2 \parallel_{L^2(I;Z)} \right) \parallel V^h \parallel_{L^2(I;E)} . \tag{3.144}$$

Now, with recalling the approximation steps done in Corollary 3.5 then we have

$$\begin{aligned} &\parallel A(\tilde{\Pi}_1^r u_1 - u_1) \parallel_{L^2(I;H)} + \parallel \tilde{\Pi}_2^r u_2 - u_2 \parallel_{L^2(I;Z)} \\ &+ \parallel \tilde{\Pi}_2^r u_2 - u_2 \parallel_{L^2(I;H)} \leq C(\frac{k}{2})^2 \frac{1}{\max\{1,r\}^2} \|Au_1\|_{H^2(J;H)} + C \max\{1,r\}^{1/2} \|A(u_1 - \hat{\Pi} u_1)\|_{L^2(J;H)} \\ &+ C \max\{1,r\}^{-1/2} (\frac{k}{2})^{1/2} \sum_{n=1}^{N} \|A(u_1 - \hat{\Pi} u_1)_-^n\|_H + C(\frac{k}{2})^2 \frac{1}{\max\{1,r\}^2} \|u_2\|_{H^2(J;Z)} \\ &+ C \max\{1,r\}^{1/2} \|u_2 - \hat{\Pi} u_2\|_{L^2(J;Z)} \\ &+ C \max\{1,r\}^{-1/2} (\frac{k}{2})^{1/2} \sum_{n=1}^{N} \|(u_2 - \hat{\Pi} u_2)_-^n\|_Z. \end{aligned} \tag{3.145}$$

Finally we have

$$
\begin{aligned}
\| U - U_{DG}^h \|_{L^2(I;E)} \leq & \| U - \tilde{\Pi}^r U \|_{L^2(I;E)} \\
& + (\frac{k \parallel Y \parallel_2^2}{2 \min_p \operatorname{Re} \lambda_p}) \Big(C (\frac{\sqrt{2\mathcal{P}_r}}{k} + \frac{\max(1,M)}{\sqrt{\min(\alpha,1)}}) \Big((\frac{k}{2})^2 \frac{1}{\max\{1,r\}^2} \|u_2\|_{H^2(J;Z)} \\
& + \max\{1,r\}^{1/2} \|u_2 - \hat{\Pi} u_2\|_{L^2(J;Z)} + \max\{1,r\}^{-1/2} (\frac{k}{2})^{1/2} \sum_{n=1}^{N} \|(u_2 - \hat{\Pi} u_2)_-^n\|_Z \Big) \\
& + \frac{2}{\sqrt{k}} (2r+1) \parallel (\hat{\Pi}_2 u_2 - u_2)_- \parallel_H + \sqrt{\frac{2}{k}} (2r+1) \parallel (U - U_{DG}^h)_-^0 \parallel_E \\
& + \frac{\max(1,M) C}{\sqrt{\min(\alpha,1)}} ((\frac{k}{2})^2 \frac{1}{\max\{1,r\}^2} \|Au_1\|_{H^2(J;H)} + \max\{1,r\}^{1/2} \|A(u_1 - \hat{\Pi} u_1)\|_{L^2(J;H)} \\
& + \max\{1,r\}^{-1/2} (\frac{k}{2})^{1/2} \sum_{n=1}^{N} \|A(u_1 - \hat{\Pi} u_1)_-^n\|_H) \Big),
\end{aligned}
$$

and that completes the proof. □

Lemma 3.23 (First global a priori error estimate in the $L^2(J;E)$-norm)
Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let Z^h be the conformal finite dimensional subspace of Z. For $U \in L^2(J; Z \times Z)$ the solution of the continuous formulation Problem 2.2.2 on p. 15 and $U_{DG}^h \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h \times Z^h)$ the solution of the fully-discrete formulation (3.66) on p. 44. We assume the additional regularity $U \in H^2(J; Z \times Z)$, $Au_1 \in H^2(J, H)$. Then there holds

$$
\begin{aligned}
\| U - U_{DG}^h \|_{L^2(J;E)} \leq & \| U - \tilde{\Pi}^r U \|_{L^2(J;E)} + C\mathcal{C}(\frac{2\max(\sqrt{\mathcal{P}_{r_n}})}{T} + \frac{\max(1,M)}{\sqrt{\min(\alpha,1)}})((\frac{k}{2})^2 \frac{1}{\max\{1,r\}^2} \|u_2\|_{H^2(J;Z)} \\
& + \max\{1,r\}^{1/2} \|u_2 - \hat{\Pi} u_2\|_{L^2(J;Z)} + \max\{1,r\}^{-1/2} (\frac{k}{2})^{1/2} \sum_{n=1}^{N} \|(u_2 - \hat{\Pi} u_2)_-^n\|_Z) \\
& + \frac{\mathcal{C} \max(2r_n + 1)}{\sqrt{TN}} \sum_{n=0}^{N-1} \parallel (\hat{\Pi}_2 u_2 - u_2)_-^{n+1} \parallel_H + \frac{C\mathcal{C} \max(1,M)}{\sqrt{\min(\alpha,1)}} ((\frac{k}{2})^2 \frac{1}{\max\{1,r\}^2} \|Au_1\|_{H^2(J;H)} \\
& + \max\{1,r\}^{1/2} \|A(u_1 - \hat{\Pi} u_1)\|_{L^2(J;H)} + \max\{1,r\}^{-1/2} (\frac{k}{2})^{1/2} \sum_{n=1}^{N} \|A(u_1 - \hat{\Pi} u_1)_-^n\|_H) \\
& + \frac{\sqrt{2N} \max(2r_n+1)}{\sqrt{T}} \Big(\sum_{n=0}^{N-1} \parallel (U - \tilde{\Pi}^{r_n} U)_-^n \parallel_E + \sqrt{2} \sum_{n=1}^{N} \sum_{n=1}^{N} \|(u_2 - \hat{\Pi} u_2)_-^n\|_H \\
& + \sum_{n=1}^{N} \Big(\frac{2\sqrt{\mathcal{P}_{r_n}} \mathcal{C}}{k} + C \frac{\max(1,M)\mathcal{C}}{\sqrt{\min(\alpha,1)}} \Big) \Big((\frac{k}{2})^2 \frac{1}{\max\{1,r\}^2} \|u_2\|_{H^2(J;Z)} \\
& + \max\{1,r\}^{1/2} \|u_2 - \hat{\Pi} u_2\|_{L^2(J;Z)} + \max\{1,r\}^{-1/2} (\frac{k}{2})^{1/2} \sum_{n=1}^{N} \|(u_2 - \hat{\Pi} u_2)_-^n\|_Z \Big) \\
& + \frac{C \max(1,M)\mathcal{C}}{\sqrt{\min(\alpha,1)}} \sum_{n=1}^{N} \Big((\frac{k}{2})^2 \frac{1}{\max\{1,r\}^2} \|Au_1\|_{H^2(J;H)} \\
& + \max\{1,r\}^{1/2} \|A(u_1 - \hat{\Pi} u_1)\|_{L^2(J;H)} + \max\{1,r\}^{-1/2} (\frac{k}{2})^{1/2} \sum_{n=1}^{N} \|A(u_1 - \hat{\Pi} u_1)_-^n\|_H \Big) \Big)
\end{aligned}
\tag{3.146}
$$

where $\hat{\Pi}$ *is the spatial projection operator given in Definition 3.10 on p. 43, and* $\tilde{\Pi}^r = \binom{\tilde{\Pi}_1^r\ 0}{0\ \tilde{\Pi}_2^r}$ *is the spatio-time projection operator given in Definition 3.12 on p. 59.* $M < \infty$ *and* $\alpha \in \mathbb{R}$*, the upper and lower bounds of the form* $a(\ ,\ )$*, respectively given in Theorem A.3 on p. 178 and* $\mathcal{P}_r$ *is a real constant which also depends on the maximum approximation order* r *at the generic time step* I*.* k *is the time step size,* $\parallel Y \parallel_2^2$ *the square of the spectral norm (see (A.6) on p. 173) of the matrix* $Y^r \in \mathbb{C}^{(r+1)\times(r+1)}$*, and the eigenvalues* λ_p *of the matrix* $\mathcal{A}^L$ *given in (3.31) on p. 30 for* $p = 0, \cdots, r$ *with*

$$\mathcal{C} = \max_n\left(\frac{T \parallel Y^{r_n} \parallel_2^2}{2 \min_p Re\ \lambda_p}\right),$$

Proof:
Starting with summing over all time steps and then using the resulting estimate in Theorem 3.7 on p. 68 with $k = \frac{T}{N}$ and

$$\mathcal{C} = \max_n\left(\frac{T \parallel Y^{r_n} \parallel_2^2}{2 \min_p \mathrm{Re}\ \lambda_p}\right),$$

then

$$\begin{aligned}
\parallel U - U_{DG}^h \parallel_{L^2(J;E)} = \sum_{n=0}^{N-1} \parallel U - U_{DG}^h \parallel_{L^2(I_n;E)} \leq \parallel U - \tilde{\Pi}^r U \parallel_{L^2(J;E)} \\
+ C\Big(\frac{\sqrt{2}\mathcal{C} \max(\sqrt{\mathcal{P}_{r_n}})}{T} + \frac{\mathcal{C}\max(1,M)}{\sqrt{\min(\alpha,1)}}\Big)\Big((\frac{k}{2})^2 \frac{1}{\max\{1,r\}^2} \|u_2\|_{H^2(J;Z)} \\
+ \max\{1,r\}^{1/2} \|u_2 - \hat{\Pi} u_2\|_{L^2(J;Z)} \\
+ \max\{1,r\}^{-1/2} (\frac{k}{2})^{1/2} \sum_{n=1}^{N} \|(u_2 - \hat{\Pi} u_2)_-^n\|_Z\Big) \\
+ \frac{\mathcal{C}\max(2r_n+1)}{\sqrt{TN}} \sum_{n=0}^{N-1} \parallel (\hat{\Pi}_2 u_2 - u_2)_-^{n+1} \parallel_H \\
+ \frac{C\mathcal{C}\max(1,M)}{\sqrt{\min(\alpha,1)}}\Big((\frac{k}{2})^2 \frac{1}{\max\{1,r\}^2} \|Au_1\|_{H^2(J;H)} \\
+ \max\{1,r\}^{1/2} \|A(u_1 - \hat{\Pi} u_1)\|_{L^2(J;H)} \\
+ \max\{1,r\}^{-1/2} (\frac{k}{2})^{1/2} \sum_{n=1}^{N} \|A(u_1 - \hat{\Pi} u_1)_-^n\|_H\Big) \\
+ \frac{\sqrt{2N}\max(2r_n+1)}{\sqrt{T}} \sum_{n=0}^{N-1} \parallel (U - U_{DG}^h)_-^n \parallel_E .
\end{aligned}$$

Now, from the resulting estimate in Corollary 3.5 on p 65:

$$
\begin{aligned}
\|(U-U_{DG}^h)_-^N\|_E \le \|(U-\hat{\Pi}U)_-^N\|_E + \sqrt{2}\sum_{n=1}^{N}\|(u_2-\hat{\Pi}u_2)_-^n\|_H \\
+\left(\frac{2\sqrt{\mathcal{P}_{r_n}}\mathcal{C}}{k}+C\frac{\max(1,M)\mathcal{C}}{\sqrt{\min(\alpha,1)}}\right)\Big((\frac{k}{2})^2\frac{1}{\max\{1,r\}^2}\|u_2\|_{H^2(J;Z)} \\
+\max\{1,r\}^{1/2}\|u_2-\hat{\Pi}u_2\|_{L^2(J;Z)} \\
+\max\{1,r\}^{-1/2}(\frac{k}{2})^{1/2}\sum_{n=1}^{N}\|(u_2-\hat{\Pi}u_2)_-^n\|_Z\Big) \\
+\frac{C\max(1,M)\mathcal{C}}{\sqrt{\min(\alpha,1)}}\Big((\frac{k}{2})^2\frac{1}{\max\{1,r\}^2}\|Au_1\|_{H^2(J;H)} \\
+\max\{1,r\}^{1/2}\|A(u_1-\hat{\Pi}u_1)\|_{L^2(J;H)} \\
+\max\{1,r\}^{-1/2}(\frac{k}{2})^{1/2}\sum_{n=1}^{N}\|A(u_1-\hat{\Pi}u_1)_-^n\|_H\Big),
\end{aligned}
$$

then we have

$$
\begin{aligned}
\sum_{n=0}^{N-1}\|(U-U_{DG}^h)_-^n\|_E \le \sum_{n=0}^{N-1}\|(U-\hat{\Pi}U)_-^n\|_E + \sqrt{2}\sum_{n=1}^{N}\sum_{n=1}^{N}\|(u_2-\hat{\Pi}u_2)_-^n\|_H \\
+\sum_{n=1}^{N}\left(\frac{2\sqrt{\mathcal{P}_{r_n}}\mathcal{C}}{k}+C\frac{\max(1,M)\mathcal{C}}{\sqrt{\min(\alpha,1)}}\right)\Big((\frac{k}{2})^2\frac{1}{\max\{1,r\}^2}\|u_2\|_{H^2(J;Z)} \\
+\max\{1,r\}^{1/2}\|u_2-\hat{\Pi}u_2\|_{L^2(J;Z)} \\
+\max\{1,r\}^{-1/2}(\frac{k}{2})^{1/2}\sum_{n=1}^{N}\|(u_2-\hat{\Pi}u_2)_-^n\|_Z\Big) \\
+\frac{C\max(1,M)\mathcal{C}}{\sqrt{\min(\alpha,1)}}\sum_{n=1}^{N}\Big((\frac{k}{2})^2\frac{1}{\max\{1,r\}^2}\|Au_1\|_{H^2(J;H)} \\
+\max\{1,r\}^{1/2}\|A(u_1-\hat{\Pi}u_1)\|_{L^2(J;H)} \\
+\max\{1,r\}^{-1/2}(\frac{k}{2})^{1/2}\sum_{n=1}^{N}\|A(u_1-\hat{\Pi}u_1)_-^n\|_H\Big).
\end{aligned}
\tag{3.147}
$$

Finally, with plugging all the terms together

$$
\begin{aligned}
\| U-U_{DG}^{h} \|_{L^2(J;E)} \leq & \| U - \tilde{\Pi}^r U \|_{L^2(J;E)} + C\mathcal{C}\Big(\frac{2\max(\sqrt{\mathcal{P}_{r_n}})}{T} + \frac{\max(1,M)}{\sqrt{\min(\alpha,1)}}\Big)\Big((\frac{k}{2})^2 \frac{1}{\max\{1,r\}^2}\|u_2\|_{H^2(J;Z)} \\
& + \max\{1,r\}^{1/2}\|u_2 - \hat{\Pi}u_2\|_{L^2(J;Z)} + \max\{1,r\}^{-1/2}(\frac{k}{2})^{1/2}\sum_{n=1}^{N}\|(u_2-\hat{\Pi}u_2)_{-}^{n}\|_Z\Big) \\
& + \frac{\mathcal{C}\max(2r_n+1)}{\sqrt{TN}}\sum_{n=0}^{N-1} \| (\hat{\Pi}_2 u_2 - u_2)_{-}^{n+1} \|_H + \frac{C\mathcal{C}\max(1,M)}{\sqrt{\min(\alpha,1)}}\Big((\frac{k}{2})^2\frac{1}{\max\{1,r\}^2}\|Au_1\|_{H^2(J;H)} \\
& + \max\{1,r\}^{1/2}\|A(u_1-\hat{\Pi}u_1)\|_{L^2(J;H)} + \max\{1,r\}^{-1/2}(\frac{k}{2})^{1/2}\sum_{n=1}^{N}\|A(u_1-\hat{\Pi}u_1)_{-}^{n}\|_H\Big) \\
& + \frac{\sqrt{2N}\max(2r_n+1)}{\sqrt{T}}\Bigg(\sum_{n=0}^{N-1} \| (U-\hat{\Pi}U)_{-}^{n} \|_E + \sqrt{2}\sum_{n=1}^{N}\sum_{n=1}^{N}\|(u_2-\hat{\Pi}u_2)_{-}^{n}\|_H \\
& + \sum_{n=1}^{N}\Bigg(\frac{2\sqrt{\mathcal{P}_{r_n}}\mathcal{C}}{k} + C\frac{\max(1,M)\mathcal{C}}{\sqrt{\min(\alpha,1)}}\Bigg)\Big((\frac{k}{2})^2\frac{1}{\max\{1,r\}^2}\|u_2\|_{H^2(J;Z)} \\
& + \max\{1,r\}^{1/2}\|u_2-\hat{\Pi}u_2\|_{L^2(J;Z)} + \max\{1,r\}^{-1/2}(\frac{k}{2})^{1/2}\sum_{n=1}^{N}\|(u_2-\hat{\Pi}u_2)_{-}^{n}\|_Z\Big) \\
& + \frac{C\max(1,M)\mathcal{C}}{\sqrt{\min(\alpha,1)}}\sum_{n=1}^{N}\Big((\frac{k}{2})^2\frac{1}{\max\{1,r\}^2}\|Au_1\|_{H^2(J;H)} \\
& + \max\{1,r\}^{1/2}\|A(u_1-\hat{\Pi}u_1)\|_{L^2(J;H)} + \max\{1,r\}^{-1/2}(\frac{k}{2})^{1/2}\sum_{n=1}^{N}\|A(u_1-\hat{\Pi}u_1)_{-}^{n}\|_H\Big)\Bigg),
\end{aligned}
$$

and that completes the proof. □

Lemma 3.24 (Second global a priori error estimate in the $L^2 - E$-norm)
Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let Z^h be the conformal finite dimensional subspace of Z. For $U \in L^2(J; Z \times Z)$ the solution of the continuous formulation Problem 2.2.2 on p. 15 and $U_{DG}^h \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h \times Z^h)$ the solution of the fully-discrete formulation (3.66) on

p. 44. We assume the additional regularity $U \in H^2(J; Z \times Z)$, $Au_1 \in H^2(J, H)$. Then there holds

$$
\begin{aligned}
\max_m \| U - U^h_{DG} \|_{L^2(I_m;E)} \leq \max_m \Bigg(& \| U - \tilde{\Pi}^r U \|_{L^2(I_m;E)} \\
& + C\Big(\frac{\sqrt{2}\mathcal{C}\max(\sqrt{\mathcal{P}_{r_m}})}{T} + \frac{\mathcal{C}\max(1,M)}{\sqrt{\min(\alpha,1)}}\Big)\Big((\frac{k}{2})^2 \frac{1}{\max\{1,r_m\}^2}\|u_2\|_{H^2(J;Z)} \\
& + \max\{1,r_m\}^{1/2}\|u_2 - \hat{\Pi}u_2\|_{L^2(J;Z)} \\
& + \max\{1,r_m\}^{-1/2}(\frac{k}{2})^{1/2}\sum_{m=1}^{N}\|(u_2 - \hat{\Pi}u_2)^m_-\|_Z\Big) \\
& + \frac{\mathcal{C}\max(2r_m+1)}{\sqrt{TN}}\sum_{m=0}^{N-1}\| (\hat{\Pi}_2 u_2 - u_2)^{m+1}_- \|_H \\
& + \frac{C\mathcal{C}\max(1,M)}{\sqrt{\min(\alpha,1)}}\Big((\frac{k}{2})^2\frac{1}{\max\{1,r_m\}^2}\|Au_1\|_{H^2(J;H)} \\
& + \max\{1,r_m\}^{1/2}\|A(u_1 - \hat{\Pi}u_1)\|_{L^2(J;H)} \\
& + \max\{1,r_m\}^{-1/2}(\frac{k}{2})^{1/2}\sum_{m=1}^{N}\|A(u_1 - \hat{\Pi}u_1)^m_-\|_H\Big) \\
& + \frac{\sqrt{2N}\max(2r_m+1)}{\sqrt{T}}\sum_{m=0}^{N-1}\Bigg(\|(U - \hat{\Pi}U)^m_-\|_E + \sqrt{2}\sum_{m=1}^{N}\|(u_2 - \hat{\Pi}u_2)^m_-\|_H \\
& + \Bigg(\frac{2\sqrt{\mathcal{P}_{r_m}}\mathcal{C}}{k} + C\frac{\max(1,M)\mathcal{C}}{\sqrt{\min(\alpha,1)}}\Bigg)\Big((\frac{k}{2})^2\frac{1}{\max\{1,r_m\}^2}\|u_2\|_{H^2(J;Z)} \\
& + \max\{1,r_m\}^{1/2}\|u_2 - \hat{\Pi}u_2\|_{L^2(J;Z)} \\
& + \max\{1,r_m\}^{-1/2}(\frac{k}{2})^{1/2}\sum_{m=1}^{N}\|(u_2 - \hat{\Pi}u_2)^m_-\|_Z\Big) \\
& + \frac{C\max(1,M)\mathcal{C}}{\sqrt{\min(\alpha,1)}}\Big((\frac{k}{2})^2\frac{1}{\max\{1,r_m\}^2}\|Au_1\|_{H^2(J;H)} \\
& + \max\{1,r_m\}^{1/2}\|A(u_1 - \hat{\Pi}u_1)\|_{L^2(J;H)} \\
& + \max\{1,r_m\}^{-1/2}(\frac{k}{2})^{1/2}\sum_{m=1}^{N}\|A(u_1 - \hat{\Pi}u_1)^m_-\|_H\Big)\Bigg)\Bigg)
\end{aligned}
\tag{3.148}
$$

where Π is the interpolant in time related to the projection operator given in Definition 3.12 on p. 59 i.e. on each time interval I_n $\Pi U|_{I_n} = \Pi^{r_n}_{I_n}(U|_{I_n}) = \begin{pmatrix} \Pi^{r_n}_1 & 0 \\ 0 & \Pi^{r_n}_2 \end{pmatrix}(U|_{I_n})$, for $n = 0, \cdots, N-1$. $\tilde{\Pi} = \begin{pmatrix} \hat{\Pi} & 0 \\ 0 & \hat{\Pi} \end{pmatrix}$ is the spatial matrix projection operator given in Definition 3.11 on p. 58 and

$$
\mathcal{C} = \max_n\Big(T\frac{\| Y^{r_m} \|_2^2 \, (2r_m+1)}{2\min_p \ Re\ \lambda_p}\Big),
$$

is depending on the maximum value of the approximation order r_m for $m = 0, \cdots, N-1$, the square of the spectral norm (see (A.6) on p. 173) of the matrix $Y^{r_m} \in \mathbb{C}^{(r_m+1)\times(r_m+1)}$, and the eigenvalues λ_p of the matrix $\mathcal{A}^L$ given in (3.31) on p. 30 for $p = 0, \cdots, r_m$. $\mathcal{P}_{r_m}$ is a real constant which also depends on the maximum approximation order r_m and C is a generic constant (see [44, Lemma 1.16]).

$M < \infty$ and $\alpha \in \mathbb{R}$, the upper and lower bounds of the form $a(\,,\,)$, respectively given in Theorem A.3 on p. 178.

Proof:
Starting with the resulting estimate in Theorem 3.7 on p. 68 with taking the maximum at the given $m = 0, \cdots, N-1$ with

$$\mathcal{C} = \max_m \left(\frac{T \parallel Y^{r_m} \parallel_2^2}{2 \min_p \mathrm{Re}\ \lambda_p}\right),$$

$$\begin{aligned}
\max_m \parallel U - U_{DG}^h \parallel_{L^2(I_m;E)} \leq \max_m \Bigg(& \parallel U - \tilde{\Pi}^r U \parallel_{L^2(I_m;E)} \\
& + C\Big(\frac{\sqrt{2}\mathcal{C}\max(\sqrt{\mathcal{P}_{r_m}})}{T} + \frac{\mathcal{C}\max(1,M)}{\sqrt{\min(\alpha,1)}}\Big)\Big((\frac{k}{2})^2 \frac{1}{\max\{1,r_m\}^2}\|u_2\|_{H^2(J;Z)} \\
& + \max\{1,r_m\}^{1/2}\|u_2 - \hat{\Pi}u_2\|_{L^2(J;Z)} \\
& + \max\{1,r_m\}^{-1/2}(\frac{k}{2})^{1/2}\sum_{m=1}^{N}\|(u_2 - \hat{\Pi}u_2)_-^m\|_Z\Big) \\
& + \frac{\mathcal{C}\max(2r_m+1)}{\sqrt{TN}}\sum_{m=0}^{N-1} \parallel (\hat{\Pi}_2 u_2 - u_2)_-^{m+1} \parallel_H \\
& + \frac{C\mathcal{C}\max(1,M)}{\sqrt{\min(\alpha,1)}}\Big((\frac{k}{2})^2\frac{1}{\max\{1,r_m\}^2}\|Au_1\|_{H^2(J;H)} \\
& + \max\{1,r_m\}^{1/2}\|A(u_1 - \hat{\Pi}u_1)\|_{L^2(J;H)} \\
& + \max\{1,r_m\}^{-1/2}(\frac{k}{2})^{1/2}\sum_{n=1}^{N}\|A(u_1 - \hat{\Pi}u_1)_-^m\|_H\Big) \\
& + \frac{\sqrt{2N}\max(2r_m+1)}{\sqrt{T}}\sum_{m=0}^{N-1}\Bigg(\|(U - \hat{\Pi}U)_-^m\|_E + \sqrt{2}\sum_{m=1}^{N}\|(u_2 - \hat{\Pi}u_2)_-^m\|_H \\
& + \left(\frac{2\sqrt{\mathcal{P}_{r_m}}\mathcal{C}}{k} + C\frac{\max(1,M)\mathcal{C}}{\sqrt{\min(\alpha,1)}}\right)\Big((\frac{k}{2})^2\frac{1}{\max\{1,r_m\}^2}\|u_2\|_{H^2(J;Z)} \\
& + \max\{1,r_m\}^{1/2}\|u_2 - \hat{\Pi}u_2\|_{L^2(J;Z)} \\
& + \max\{1,r\}^{-1/2}(\frac{k}{2})^{1/2}\sum_{m=1}^{N}\|(u_2 - \hat{\Pi}u_2)_-^m\|_Z\Big) \\
& + \frac{C\max(1,M)\mathcal{C}}{\sqrt{\min(\alpha,1)}}\Big((\frac{k}{2})^2\frac{1}{\max\{1,r_m\}^2}\|Au_1\|_{H^2(J;H)} \\
& + \max\{1,r_m\}^{1/2}\|A(u_1 - \hat{\Pi}u_1)\|_{L^2(J;H)} \\
& + \max\{1,r_m\}^{-1/2}(\frac{k}{2})^{1/2}\sum_{m=1}^{N}\|A(u_1 - \hat{\Pi}u_1)_-^m\|_H\Big)\Bigg)\Bigg)
\end{aligned} \tag{3.149}$$

and that completes the proof. □

Chapter 4

A variational formulation of a first-order system in time with a general vector-valued functional

In this chapter, the same methods used in Chapter 3 will be used here again to approximate a variational formulation of a first-order system in time with a functional of a more general forcing data. The general theorems provided in this chapter will be very useful when it comes to show existence, uniqueness, and stability estimate of the variational formulation of first-order system in time with given linear constrains in Part II of this thesis.

4.1 The continuous variational formulation of a first-order system in time with a general vector-valued functional

This section will discuss the general stability, existence, and uniqueness of a continuous variational formulation of first-order system in time with a general vector-valued functional. All theorems to be shown here are going to be very important and useful to show existence and uniqueness of semi-discrete formulation in Part II of this thesis. The stability estimate of Problem 4.1.1 on p. 77 is shown in Theorem 4.1 on p. 78. Existence and uniqueness is shown in Theorem 4.2 on p. 80.

Problem 4.1.1 (Variational formulation of first-order system in time with general functional)

Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let $A \in L(Z, Z')$ be the spatial operator given in Definition 2.2 on p. 6. Let $a : Z \times Z \to \mathbb{C}$ be a Hermitian form which satisfies the properties given in Theorem A.3 on p. 178:

$$
\begin{aligned}
|a(u,v)| &\leq M \parallel u \parallel_Z \parallel v \parallel_Z, && \forall \quad u, v \in Z,\\
a(u,u) &\geq \alpha \parallel u \parallel_Z^2, && \forall \quad u \in Z,
\end{aligned}
$$

and

$$
\begin{aligned}
a(u,v) &= a(v,u), && \textit{for real functions } u, v \in Z, \ \textit{(Symmetric)},\\
a(u,v) &= \overline{a(v,u)}, && \textit{for complex functions } u, v \in Z, \ \textit{(Conjugate symmetric)},
\end{aligned}
$$

$$
\langle Au, v \rangle_{Z' \times Z} = a(u,v), \forall\, u, v \in Z.
$$

Let D_A be the subspace of the Hilbert space Z given in Definition 3.13 on p. 60:

$$D_A := \{u \in Z : \ Au \in H\}, \ \textit{with a norm} \ \| u \|_{D_A}^2 := \| u \|_Z^2 + \| Au \|_H^2, \forall u \in D_A. \tag{4.1}$$

Let

$$F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in L^2(J; Z \times H) \ \textit{be a general vector-valued function where} \ f_1 \ \textit{is in} \ L^2(J; D_A) \subset L^2(J; Z). \tag{4.2}$$

A general linear functional (see Definition A.18 on p. 176) $\mathcal{F} : L^2(J; Z \times H) \to \mathbb{C}$ and since L^2 is a pivot space i.e. $(L^2)' = L^2$ (see [3, 5.3 Normal Subspaces of a pivot space]) and $H \equiv H'$ from the Gelfand triple $(Z \subset H \subset Z')$, $\mathcal{F} \in (L^2(J; Z \times H))' \equiv L^2(J; Z' \times H)$:

$$\mathcal{F}(V) := \int_J \{ \overbrace{a(f_1, v_1) + (f_2, v_2)_H}^{(F,V)_E=} \} dt, \ \forall V \in L^2(J; Z \times H), \tag{4.3}$$

where all vector-valued functions here are real. Here, the regularity of the vector-valued function F in (4.2) is chosen that way so that the existence can be shown as well as to have well defined norms for the approximated initial data when it comes to show a priori error estimates in Part II of this theses. Find $U \in L^2(J; Z \times Z)$ with $\partial_t U \in L^2(J; Z \times H) \subset L^2(J; Z \times Z')$:

$$B(U, V) = \mathcal{F}(V), \ \forall V \in L^2(J; Z \times Z), \tag{4.4}$$

for given vector-valued initial data $U_0 = \begin{pmatrix} u_{1,0} \\ u_{2,0} \end{pmatrix} \in Z \times Z$.

Theorem 4.1 (Stability of continuous variational formulation with general vector-valued func

Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let D_A be the subspace of the Hilbert space Z given in Definition 3.13 on p. 60. In the normed product space $(L^2(J; Z \times Z), \| \cdot \|_{L^2(J;E)})$, the solution of the variational formulation in (4.4) with a functional $\mathcal{F} \in L^2(J; Z' \times H)$ with general forcing data $F = \binom{f_1}{f_2} \in L^2(J; Z \times H)$ and initial data $U_0 = \binom{u_{1,0}}{u_{2,0}} \in Z \times Z$, where $f_1 \in L^2(J; D_A) \subset L^2(J; Z)$ and $u_{1,0} \in D_A \subset Z$, satisfies

$$\| U \|_{L^2(J;E)} \leq T\sqrt{C} (\| F \|_{L^2(J;E)} + \| U_0 \|_E), \tag{4.5}$$

for

$$\| U \|_{L^2(J;E)}^2 = \int_J \Big\{ \overbrace{a(u_1, u_1) + \| u_2 \|_H^2}^{\|U\|_E^2=} \Big\} dt, \ \forall U \in L^2(J; Z \times Z), \tag{4.6}$$

where C does not depend on $T < \infty$.

Proof:
Starting with the general variational formulation (4.4) with (4.3) on p. 78 and considering the vector-valued test function to be

$$V = \begin{cases} \bar{V}(\tau), & \bar{V}(\tau) \in L^2((0,t); Z \times Z), \ \text{for } t < T < \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

Then it becomes

$$\int_0^t \big\{a(\partial_\tau u_1, \bar{v}_1) + \langle \partial_\tau u_2, \bar{v}_2\rangle_{Z'\times Z} + \hat{a}(U,\bar{V})\big\}d\tau = \mathcal{F}(\bar{V}) = \int_0^t (F,\bar{V})_E d\tau. \tag{4.7}$$

With choosing $\bar{V} = U \in L^2((0,t); Z\times Z)$, and then repeating the same steps done to proof Lemma 2.5:

$$a(\partial_\tau u_1, u_1) = \frac{1}{2}\frac{d}{d\tau}a(u_1,u_1),\ \text{by the symmetry of } a(\,,), \text{ and}$$
$$\langle \partial_\tau u_2, u_2\rangle_{Z'\times Z} = \frac{1}{2}\frac{d}{d\tau}\parallel u_2 \parallel_H^2,\ \text{respectively,}$$

then using the *Cauchy inequality* with the *Bochner integral* norm in Definition A.22 on p. 181 with $p = 2$ for $F \in L^2((0,t); Z\times H)$ with the Hilbert space $(L^2((0,t); Z\times H), \|\cdot\|_E)$, and *Young's inequality* (see Lemma A.2 on p. 181) in the r.h.s of (4.7), imply

$$\int_0^t \{a(\partial_\tau u_1, u_1) + \langle \partial_\tau u_2, u_2\rangle_{Z'\times Z} \overbrace{-a(u_2,u_1) + a(u_1,u_2)}^{=0}\}d\tau = \int_0^t (F,U)_E d\tau$$

$$\begin{aligned}\Longrightarrow \frac{1}{2}\int_0^t \frac{d}{d\tau}\{\overbrace{a(u_1,u_1) + \parallel u_2 \parallel_H^2}^{=\|U\|_E^2}\}d\tau &= \int_0^t (F,U)_E d\tau \\ &\le \sqrt{T}\parallel F \parallel_{L^2((0,t);E)} \frac{1}{\sqrt{T}} \parallel U \parallel_{L^2((0,t);E)} \\ &\le \frac{1}{2}\big(T \parallel F \parallel^2_{L^2((0,t);E)} + \frac{1}{T}\parallel U \parallel^2_{L^2((0,t);E)}\big),\end{aligned}$$

with multiplying both sides with 2 and then using the *Gronwall's inequality* given in Theorem A.8 on p. 182 imply

$$\parallel U(t) \parallel_E^2 \le C\Big(T \parallel F \parallel^2_{L^2((0,t);E)} + \parallel U_0 \parallel_E^2\Big), \tag{4.8}$$

where C does not depend on T. Now, taking the integral over $J = (0,T)$ of both sides in (2.38) and using the inequality

$$\int_0^t u^2 d\tau \le \int_0^t u^2 d\tau + \int_t^T u^2 d\tau = \int_J u^2 dt,\ \forall t \in J = (0,T),\ 0 < t < T < \infty.$$

Knowing that $\parallel U_0 \parallel_E^2$ and $\displaystyle\int_J \parallel F \parallel_E^2 dt$ are constants in time, then

$$\begin{aligned}\parallel U \parallel^2_{L^2(J;E)} &\le C\big(T \underbrace{\int_J\int_0^t \parallel F \parallel_E^2 d\tau dt}_{\le \int_J\int_J \parallel F \parallel_E^2 dtdt} + T \parallel U_0 \parallel_E^2\big) \\ &= TC\big(T \parallel F \parallel^2_{L^2(J;E)} + \parallel U_0 \parallel_E^2\big)\end{aligned}$$

$\Longrightarrow \parallel U \parallel_{L^2(J;E)} \le T\sqrt{C}\big(\parallel F \parallel_{L^2(J;E)} + \parallel U_0 \parallel_E\big)$, after using Lemma A.3 on p. 181. □

Theorem 4.2 (Existence, uniqueness of variational formulation with general vector-valued fun
Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let $A \in L(Z, Z')$ be the spatial operator given in Definition 2.2 on p. 6 such for a Hermitian form $a : Z \times Z \to \mathbb{C}$ which satisfies the properties given in Theorem A.3 on p. 178:

$$\langle Au, v\rangle_{Z' \times Z} = a(u, v), \forall\, u, v \in Z.$$

Let D_A be the subspace of the Hilbert space Z given in Definition 3.13 on p. 60. There exists unique vector-valued function $U \in L^2(J; Z \times Z)$ which solves (4.4) in Problem 4.1.1.

Proof:
Firstly, to show existence for the solution of Problem 4.1.1 starts with writing it as a strong form as follows

$$\begin{aligned} \partial_t A u_1 - A u_2 &= A f_1, \\ \partial_t u_2 + A\, u_1 &= f_2, \end{aligned} \tag{4.9}$$

with given initial data $u_1|_{t=0} = u_{1,0} \in D_A \subset Z$, and $u_2|_{t=0} = u_{2,0} := \bar{u}_0 \in Z$ and from (4.3) on p. 78:

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in L^2(J; D_A \times H) \subset L^2(J; Z \times H). \tag{4.10}$$

Now, in order to show existence of u_1 and u_2 with their first time derivatives we will show that they are defined by existing functions. To do that we recall the existing system in (2.20) on p. 11, transform (4.9) to an existence system of the form of (2.20) on p. 11.
First, with the invertibility of the operator $A \in L(Z, Z')$ then with recalling the system (2.20) on p. 11 which now reads to avoid any confusion with notations

$$\begin{aligned} \partial_t A \bar{u}_1 - A \bar{u}_2 &= 0, \\ \partial_t \bar{u}_2 + A \bar{u}_1 &= f, \end{aligned} \tag{4.11}$$

with given initial data $\bar{u}_1|_{t=0} = \bar{u}_{1,0} := u_0 \in Z$, $\bar{u}_2|_{t=0} = \bar{u}_{2,0} := \bar{u}_0 \in Z_0$ and forcing data $f \in H^1(J; H)$. Theorem 2.3 on p. 17 shows that $\bar{u}_1 \in L^2(J; Z)$, $\bar{u}_2 \in L^2(J; Z)$, and $\partial_t \bar{u}_1 \in L^2(J; Z)$ with $\partial_t \bar{u}_2 \in L^2(J; H) \subset L^2(J; Z')$ which solve (4.11) do exist, with the regularity of $\bar{u}_1$ and the initial data $\bar{u}_{1,0}$ given in Assumption 3.2 on p. 60.
Then, with defining

$$\delta_1 := \bar{u}_1 - u_1 \text{ and } \bar{u}_2 - u_2 = 0, \text{ with } \delta_{1,0} = \bar{u}_{1,0} - u_{1,0}, \tag{4.12}$$

and after taking the difference of the two systems; (4.11) and (4.9):

$$\begin{aligned} \partial_t A \delta_1 - A(-f_1) &= 0, \\ f_2 + A\, \delta_1 &= f, \end{aligned} \tag{4.13}$$

and knowing that $f_1 \in L^2(J; D_A) \subset L^2(J; Z)$ from (4.10) and thus the operator $A \in L(Z, Z')$ is invertible and that

$$\delta_{1,0} = \bar{u}_{1,0} - u_{1,0} \in D_A \text{ which is constant in time}, \tag{4.14}$$

then

$$\partial_t A\delta_1 = -Af_1 \Longrightarrow \delta_1 = \delta_{1,0} - \int_0^t f_1 d\tau, \text{ for } t < T < \infty, \text{ i.e. } \delta_1 \in L^2(J; D_A) \subset L^2(J; Z), \tag{4.15}$$

and also from (4.10) $f_2 \in L^2(J;H) \subset L^2(J;Z')$ and $\delta_1 \in L^2(J;D_A) \subset L^2(J;Z)$ from (4.15), which are well defined then

$$\Longrightarrow f = f_2 + \underbrace{A\,\delta_1}_{\in L^2(J;H)} \in L^2(J;H) \subset L^2(J;Z'). \tag{4.16}$$

Secondly, let

$$\begin{aligned} \delta_1 &:= \delta_{1,0} - \int_0^t f_1 d\tau \in L^2(J; D_A) \subset L^2(J; Z), \\ f &:= f_2 + A\,\delta_1 \in L^2(J;H) \subset L^2(J;Z'), \end{aligned} \tag{4.17}$$

and with the existence of $\bar{u}_1$ and $\bar{u}_2$ in (4.11) shown in Theorem 2.3 on p. 17 with considering the regularity of $\bar{u}_1$ given in Assumption 3.2 on p. 60 such that $\bar{u}_1 \in L^2(J;D_A)$, then there exists

$$u_1 := \bar{u}_1 - \delta_1 \in L^2(J;D_A) \subset L^2(J;Z), u_2 := \bar{u}_2 \in L^2(J;Z), \text{ and } u_{1,0} := \bar{u}_{1,0} - \delta_{1,0} \in D_A \subset Z. \tag{4.18}$$

Finally, with going back to (4.9) on p. 80 with using the identities in (4.18), (4.17), and the system (4.11), then

$$\begin{aligned} \partial_t Au_1 - Au_2 &= \partial_t A(\bar{u}_1 - \delta_1) - Au_2 = \overbrace{\partial_t A\bar{u}_1 - A\bar{u}_2}^{=0} - \partial_t A\delta_1 = -\partial_t A(\delta_{1,0} - \int_0^t f_1 d\tau) = Af_1, \\ \partial_t u_2 + Au_1 &= \partial_t \bar{u} + A(\bar{u}_1 - \delta_1) = \overbrace{\partial_t \bar{u} + A\bar{u}_1}^{=f} - A\delta_1 = f_2 + A\,\delta_1 - A\delta_1 = f_2, \end{aligned} \tag{4.19}$$

and this concludes that $u_1 \in L^2(J;D_A) \subset L^2(J;Z)$ and $u_2 \in L^2(J;Z)$ do exist. Moreover for well defined $f_1 \in L^2(J;Z)$ and $f_2 \in L^2(J;H)$, the existence of $u_1 \in L^2(J;Z)$ and $u_2 \in L^2(J;Z)$, the invertibility of the operator $A \in L(Z, Z')$, and from (4.19), then

$$\begin{aligned} \partial_t Au_1 - Au_2 = Af_1, &\Longrightarrow \partial_t u_1 = u_2 + f_1 L^2(J;Z), \\ \partial_t u_2 + Au_1 = f_2, &\Longrightarrow \partial_t u_2 = f_2 - Au_1 \in L^2(J;H), \end{aligned} \tag{4.20}$$

which also conclude that

$$\partial_t u_1 \in L^2(J;Z) \text{ and } \partial_t u_2 \in L^2(J;H) \subset L^2(J;Z'), \tag{4.21}$$

exist. Thus, with the stability estimate shown in Theorem 4.1 on p. 78, then $U \in L^2(J; Z \times Z)$ with $\partial_t U \in L^2(J; Z \times H)$ with well defined forcing data

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in L^2(J; D_A \times H), \tag{4.22}$$

and given initial data $\bar{U} \in D_A \times Z \subset Z \times Z$, the existence of the vector-valued function U which solves Problem 4.1.1 on p. 77 is shown.

Secondly, to show uniqueness, let $U^1 = \binom{u_1^1}{u_2^1}$ and $U^2 = \binom{u_1^2}{u_2^2}$ be two functions which solve the formulation in (4.4) on p. 78 with same forcing and initial data, then when letting $\bar{W} := U^1 - U^2 \in L^2(J; Z\times Z)$ with $\partial_t \bar{W} \in L^2(J; Z \times H)$ which solves

$$\int_J \{a(\partial_t \bar{w}_1, v_1) + \langle \partial_t \bar{w}_2, v_2\rangle_{Z'\times Z} + \hat{a}(\bar{W}, V)\}dt = \mathbf{0},$$

for all $V \in L^2(J; Z \times Z)$ with initial data $\bar{W}_0 = \mathbf{0}$. Now, with choosing $V = \bar{W} \in L^2(J; Z \times Z)$ and then following that same steps done at the beginning of this proof to show stability but this time with zero vector-valued forcing data and initial data i.e.

$$\| \bar{W} \|^2_{L^2(J;E)} = 0, \Longrightarrow 0 = \bar{W} = U^1 - U^2,$$

which implies that

$$U^1 = U^2. \tag{4.23}$$

Finally, there exists unique solution $U \in L^2(J; Z \times Z)$ of Problem 4.1.1 on p. 77. □

4.2 High-order in time DGFEM, semi-discrete formulation with a functional of general forcing data F

The section includes the general theorem of existence and uniqueness of the semi-discrete vector-valued function which solves Problem 4.2.1 on p. 82. The latter is having a general structure with general vector-valued functional. All theorems to be shown here are going to be very important and useful to show existence and uniqueness of semi-discrete formulation in Part II of this thesis. In this Section, Problem 4.1.1 on p. 77 is discretised in time then in space using the same method in Chapter 3. The semi-discrete formulation is given in (4.26) on p. 83. In Theorem 4.4 on p. 86 it will be shown that this formulation is solvable.

Problem 4.2.1 (Semi-discrete formulation with a functional of general forcing data) *Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let $A \in L(Z, Z')$ be the spatial operator given in Definition 2.2 on p. 6 for a Hermitian form $a : Z \times Z \to \mathbb{C}$ which satisfies the properties given in Theorem A.3 on p. 178:*

$$\langle Au, v\rangle_{Z'\times Z} = a(u, v), \forall\, u, v \in Z.$$

Let D_A be the subspace of the Hilbert space Z given in Definition 3.13 on p. 60. Let $(\,,)_E$ be the inner product given in Corollary 2.2 on p. 10. Let $B_{DG}(\,,)$ be the form given in (3.17) on p. 26:

$$(U, V)_E = a(u_1, v_1) + (u_2, v_2)_H \text{ and } \hat{a}(U, V) = -a(u_2, v_1) + a(u_1, v_2),$$

and

$$B_{DG}(U_{DG}, V) = \sum_{n=0}^{N-1} \int_{I_n} \Big\{(\partial_t U_{DG}, V)_E + \hat{a}(U_{DG}, V)\Big\}dt + \sum_{n=1}^{N-1} ([U_{DG}]^n, V_+^n)_E + (U^0_{DG,+}, V^0_+)_E.$$

Let $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z \times Z) \subset L^2(J; Z \times Z)$ be the semi-discrete space given in Definition 3.1 on p. 19:

$$\mathcal{V}^{\underline{r}}(\mathcal{N}; Z \times Z) = \{U : J \to Z \times Z \,:\, U|_{I_n} \in P^{r_n}(I_n; Z \times Z), 0 \le n \le N-1\}\,.$$

$\mathcal{V}^{\underline{r}}(\mathcal{N}; Z\times Z)$ is the linear space consisting of piecewise polynomials in time with coefficients in $Z\times Z$. $P^{r_n}(I_n; Z\times Z)$ denotes the space of r_nth order polynomials in time at I_n with coefficients in $Z\times Z$. Here $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z\times Z)\subset L^2(J; Z\times Z)$.
For vector-valued function

$$F=\begin{pmatrix} f_1\\ f_2\end{pmatrix}\in L^2(J; Z\times H) \text{ be a general vector-valued function where } f_1 \text{ is in } L^2(J; D_A)\subset L^2(J; Z). \tag{4.24}$$

A general linear functional (see Definition A.18 on p. 176) $\mathcal{F}_{DG}: L^2(J; Z\times H)\to\mathbb{C}$ i.e. $\mathcal{F}_{DG}\in (L^2(J; Z\times H))'$

$$\mathcal{F}_{DG}(V)=\sum_{n=0}^{N-1}\int_{I_n}\{\overbrace{a(f_1,v_1)+(f_2,v_2)_H}^{(F,V)_E=}\}dt=\int_J\{\overbrace{a(f_1,v_1)+(f_2,v_2)_H}^{(F,V)_E=}\}dt,\ \forall V\in L^2(J; Z\times H), \tag{4.25}$$

and here all vector-valued functions in semi-discrete space $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z\times Z)$ are also included in this linear map since $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z\times Z)\subset L^2(J; Z\times H)$.
Find $U_{DG}\in\mathcal{V}^{\underline{r}}(\mathcal{N}; Z\times Z)$ such that

$$B_{DG}(U_{DG},V)=\mathcal{F}_{DG}(V)+(U^0_{DG,-},V^0_+)_E,\quad \forall\ V\in\mathcal{V}^{\underline{r}}(\mathcal{N}; Z\times Z), \tag{4.26}$$

with given initial data $U^0_{DG,-}=U_0=\begin{pmatrix} u_{1,0}\\ u_{2,0}\end{pmatrix}\in Z\times Z$, where $u_{1,0}\in D_A\subset Z$.

The time-stepping formulation of (4.26) would read:

Problem 4.2.2 (Semi-discrete time-stepping scheme with general forcing data)
Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let D_A be the subspace given in Definition 3.13 on p. 60. Let $(\,,)_E$ be the inner product given in Corollary 2.2 on p. 10 and $\hat{a}(\,,)$ be the skew-symmetric form given in Definition 2.7 on p. 14:

$$(U,V)_E=a(u_1,v_1)+(u_2,v_2)_H,\ \text{and}\ \hat{a}(U,V)=-a(u_2,v_1)+a(u_1,v_2).$$

Let $P^{r_n}(I_n; Z\times Z)$ be the semi-discrete space of r_nth order polynomials in time at I_n with coefficients in $Z\times Z$ given in Definition 3.1 on p. 19. For $F=\binom{f_1}{f_2}\in L^2(I_n; Z\times H)$ where $f_1\in L^2(I_n; D_A)$, let $\mathcal{F}^n_{DG}\in(L^2(I_n; Z\times H))'$:

$$\mathcal{F}^n_{DG}(V):=\int_{I_n}(F,V)_E dt,\ \text{for all}\ V\in L^2(I_n; Z\times H), 0\le n\le N-1. \tag{4.27}$$

Find $U_{DG}\in P^{r_n}(I_n; Z\times Z)$:

$$\overbrace{\int_{I_n}\Big\{(\partial_t U_{DG},V)_E+\hat{a}(U_{DG},V)\Big\}dt+(U^n_{DG,+},V^n_+)_E}^{b^n_{DG}(U_{DG},V)=}=\overbrace{\int_{I_n}(F,V)_E dt}^{\mathcal{F}^n_{DG}(V):=}+(U^n_{DG,-},V^n_+)_E,$$
$$\text{for all } V\in P^r(I_n; Z\times Z),\ \text{with } U^n_{DG,-} \text{ to be the initial data at a given time step } I_n=(t_n,t_{n+1}), \tag{4.28}$$

4.2.1 Existence and uniqueness

Theorem 4.3 (Existence, uniqueness of the local semi-discrete solution with general $r.h.s$)

Let Z and H be the Hilbert spaces over the field $\mathbb{C}$ given in Definition 2.1 on p. 6. At a generic time step $I = (t_0, t_1)$ there exists a unique real semi-discrete vector-valued function $U_{DG} \in P^r(I; Z \times Z)$ which solves Problem 4.2.2.

Proof:
Firstly, with repeating the same time decoupling and the diagonalisation process done in Section 3.2.1.1 starting on p. 28, then the formulation (4.28) at a generic time step I would read

$$\begin{aligned}&\text{Find } \{\mathfrak{U}_p\}_{p=0}^r \subset Z \times Z \text{ such that}\\ &\lambda_p(\mathfrak{U}_p, \mathfrak{V}_p)_E + \frac{k}{2}\hat{a}(\mathfrak{U}_p, \mathfrak{V}_p) = \frac{k}{2}\sum_{q=0}^{r}\Big\{(\beta_p^1)_q \int_{-1}^{1} L_q(\hat{t})(F \circ Q, \mathfrak{V}_p)_E d\hat{t}\Big\} + \sum_{q=0}^{r}\Big\{(\beta_p^2)_q\Big\}(U_{DG,-}^0, \mathfrak{V}_p)_E,\\ &\text{for each } p = 0, \cdots, r, \forall \{\mathfrak{V}_p\}_{p=0}^r \subset Z \times Z, \text{ given initial data } U_{DG,-}^0 \text{ at a given time step } I = (t_0, t_1),\\ &\text{and forcing data } F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in L^2(J; Z \times H).\end{aligned} \tag{4.29}$$

and in component-wise (4.29) is written as:

$$\lambda_p a(\mathfrak{u}_{1,p}, \mathfrak{v}_{1,p}) = a(\mathfrak{u}_{2,p}, \mathfrak{v}_{1,p}) + \frac{k}{2}\sum_{q=0}^{r}\Big\{(\beta_p^1)_q \int_{-1}^{1} L_q(\hat{t}) a(f_1 \circ Q, \mathfrak{v}_{1,p}) d\hat{t}\Big\} + \sum_{q=0}^{r}\Big\{(\beta_p^2)_q\Big\} a(u_{1,DG,-}^0, \mathfrak{v}_{1,p})$$

$$\lambda_p(\mathfrak{u}_{2,p}, \mathfrak{v}_{2,p})_H + \frac{k}{2} a(\mathfrak{u}_{1,p}, \mathfrak{v}_{2,p}) = \frac{k}{2}\sum_{q=0}^{r}\Big\{(\beta_p^1)_q \int_{-1}^{1} L_q(\hat{t})(f_2 \circ Q, \mathfrak{v}_{2,p})_H d\hat{t}\Big\} + \sum_{q=0}^{r}\Big\{(\beta_p^2)_q\Big\}(u_{2,DG,-}^0, \mathfrak{v}_{2,p})_H.$$

Secondly, the $2(r + 1)$ spatial problems in (4.29) can be reduced with again recalling the same steps done in Section 3.2.1.2 starting on p. 32 such that with

$$\begin{aligned}\mathfrak{f}^1(\mathfrak{v}_{2,p}) &:= -k^2\sum_{q=0}^{r}\Big\{(\beta_p^1)_q \int_{-1}^{1} L_q(\hat{t}) a(f_1 \circ Q, \mathfrak{v}_{2,p}) d\hat{t}\Big\}\\ \mathfrak{f}^2(\mathfrak{v}_{2,p}) &:= 2k\lambda_p\sum_{q=0}^{r}\Big\{(\beta_p^1)_q \int_{-1}^{1} L_q(\hat{t})(f_2 \circ Q, \mathfrak{v}_{2,p})_H d\hat{t}\Big\},\end{aligned} \tag{4.30}$$

and with a form and a linear functional $\mathfrak{f} \in (Z \times H)' \equiv Z' \times H$ (since $H \equiv H'$ is a pivot space):

$$\begin{aligned}b(\mathfrak{u}_{2,p}, \mathfrak{v}_{2,p}) &:= 4\lambda_p^2(\mathfrak{u}_{2,p}, \mathfrak{v}_{2,p})_H + k^2 a(\mathfrak{u}_{2,p}, \mathfrak{v}_{2,p})\\ \mathfrak{f}(\mathfrak{v}_{2,p}) &:= \mathfrak{f}^1(\mathfrak{v}_{2,p}) + \mathfrak{f}^2(\mathfrak{v}_{2,p}), \forall v_2 \in Z,\end{aligned} \tag{4.31}$$

it would now read:
Find $\{\mathfrak{u}_{2,p}\}_{p=0}^r \subset Z$:

$$b(\mathfrak{u}_{2,p}, \mathfrak{v}_{2,p}) = \mathfrak{f}(\mathfrak{v}_{2,p}) + \sum_{q=0}^{r}\Big\{(\beta_p^2)_q\Big\}\Big(4\lambda_p(u_{2,DG,-}^0, \mathfrak{v}_{2,p})_H - 2ka(u_{1,DG,-}^0, \mathfrak{v}_{2,p})\Big), \tag{4.32}$$

for each $p = 0, \cdots, r$, for all $\{\mathfrak{v}_{2,p}\}_{p=0}^r \subset Z$.

After solving for unknowns $\{\mathfrak{u}_{2,p}\}_{p=0}^{r} \subset Z$ with given data $F = \binom{f_1}{f_2} \in L^2(J; Z \times H)$ and $u^0_{1,DG,-}$, $u^0_{2,DG,-} \in Z$, then it comes to update the values of $\{\mathfrak{u}_{1,p}\}_{p=0}^{r} \subset Z$:

$$\mathfrak{u}_{1,p} = \frac{k}{2\lambda_p}\mathfrak{u}_{2,p} + \frac{1}{\lambda_p}\sum_{q=0}^{r}\Big\{(\beta_p^2)_q\Big\}u^0_{1,DG,-} + \frac{k}{2\lambda_p}\sum_{q=0}^{r}\Big\{(\beta_p^1)_q\int_{-1}^{1}L_q(\hat{t})f_1\circ Q d\hat{t}\Big\}, \tag{4.33}$$

which need to be solved at any given time step I.
Now, the form $b(\mathfrak{u}_{2,p}, \mathfrak{v}_{2,p})$ given in (4.32) is continuous with using Lemma 3.5 on p. 34 after letting $c_1 = 4\lambda_p \in \mathbb{C}$ and $c_2 = k^2 \in \mathbb{R}$:

$$b(\mathfrak{u}_{2,p}, \mathfrak{v}_{2,p}) = 4\lambda_p^2(\mathfrak{u}_{2,p}, \mathfrak{v}_{1,p})_H + k^2 a(\mathfrak{u}_{2,p}, \mathfrak{v}_{1,p}) \le (4|\lambda_p|c^2 + k^2M) \parallel \mathfrak{u}_{2,p} \parallel_Z \parallel \mathfrak{v}_{1,p} \parallel_Z, \text{ for } p = 0, \cdots, r,$$

and satisfies the Inf-sup condition in Lemma 3.4 on p. 33 where the Inf-sup constant is $k^2\alpha$. Also, the r.h.s of (4.32) we have

$$|r.h.s| = |\mathfrak{f}^1(\mathfrak{v}_{2,p}) + \mathfrak{f}^2(\mathfrak{v}_{2,p}) + \sum_{q=0}^{r}\Big\{(\beta_p^2)_q\Big\}\Big(4\lambda_p(u^0_{2,DG,-}, \mathfrak{v}_{2,p})_H - 2ka(u^0_{1,DG,-}, \mathfrak{v}_{2,p})\Big)|. \tag{4.34}$$

The same steps done in the proof of Lemma 3.6 on p. 35 are recalled in order to show boundedness of the r.h.s, where now there is one additional term in the functional in (4.34) that is $\mathfrak{f}^1(\mathfrak{v}_{2,p})$ and the other two terms are the same as the ones in (3.48) on p. 35 after letting $f \to f_2$, and from (3.49) on p. 36, then

$$\begin{aligned}|\mathfrak{f}^2(\mathfrak{v}_{2,p}) &+ \sum_{q=0}^{r}\Big\{(\beta_p^2)_q\Big\}\Big(4\lambda_p(u^0_{2,DG,-}, \mathfrak{v}_{2,p})_H - 2ka(u^0_{1,DG,-}, \mathfrak{v}_{2,p})\Big)| \\ &\le \Bigg(c\sqrt{2^3k}|\lambda_p|\sum_{q=0}^{r}|(\beta_p^1)_q|\hat{x}_q \parallel f_2 \parallel_{L^2(I;H)} + |\sum_{q=0}^{r}(\beta_p^2)_q|\Big(4c|\lambda_p| \parallel u^0_{2,DG,-} \parallel_H \\ &\qquad + 2kM \parallel u^0_{1,DG,-} \parallel_Z\Big)\Bigg) \parallel \mathfrak{v}_{2,p} \parallel_Z .\end{aligned} \tag{4.35}$$

Here and with using the *continuity* of the form $a(\,,\,)$ and then following the rest of steps done to get the upper bound of the term $\mathfrak{f}(\mathfrak{v}_{2,p})$ in the proof of Lemma 3.6 on p. 35 yield

$$|\mathfrak{f}^1(\mathfrak{v}_{2,p})| = |k^2\sum_{q=0}^{r_n}\Big\{(\beta_p^1)_q\int_{-1}^{1}L_q(\hat{t})a(f_1\circ Q, \mathfrak{v}_{2,p})d\hat{t}\Big\}| \le \sqrt{2k^3}M\sum_{q=0}^{r_n}|(\beta_p^1)_q|\hat{x}_q \parallel f_1 \parallel_{L^2(I;Z)} \parallel \mathfrak{v}_{2,p} \parallel_Z .$$

Then

$$\begin{aligned}|r.h.s| \le \Bigg(&c\sqrt{2^3k}|\lambda_p|\sum_{q=0}^{r}|(\beta_p^1)_q|\hat{x}_q \parallel f_2 \parallel_{L^2(I;H)} + |\sum_{q=0}^{r}(\beta_p^2)_q|\Big(4c|\lambda_p| \parallel u^0_{2,DG,-} \parallel_H \\ &+ 2kM \parallel u^0_{1,DG,-} \parallel_Z\Big) + \sqrt{2k^3}M\sum_{q=0}^{r_n}|(\beta_p^1)_q|\hat{x}_q \parallel f_1 \parallel_{L^2(I;Z)}\Bigg) \parallel \mathfrak{v}_{2,p} \parallel_Z .\end{aligned} \tag{4.36}$$

Thus, going back to (Babuška) Theorem A.4 on p. 179 with having Lemmas 3.4 and 3.5 and (4.36), then there exist unique functions $\{\mathfrak{u}_{2,p}\}_{p=0}^{r_n} \subset Z$ which solve (4.32).
For the identity (4.33) on p. 85:

$$\mathfrak{u}_{1,p} = \frac{k}{2\lambda_p}\mathfrak{u}_{2,p} + \frac{1}{\lambda_p}\sum_{q=0}^{r}\Big\{(\beta_p^2)_q\Big\}u_{1,DG,-}^0 + \frac{k}{2\lambda_p}\sum_{q=0}^{r}\Big\{(\beta_p^1)_q\int_{-1}^{1}L_q(\hat{t})f_1\circ Q d\hat{t}\Big\},$$

where

$$\parallel \frac{k}{2\lambda_p}\sum_{q=0}^{r}\Big\{(\beta_p^1)_q\int_{-1}^{1}L_q(\hat{t})f_1\circ Q d\hat{t}\Big\}\parallel_Z \leq \frac{k}{2|\lambda_p|}\sum_{q=0}^{r}\Big\{|(\beta_p^1)_q|\int_{-1}^{1}L_q(\hat{t})\parallel f_1\circ Q\parallel_Z d\hat{t}\Big\} < \infty,$$

for each $p = 0, \cdots, r$, then from the given existence and uniqueness of $\{\mathfrak{u}_{2,p}\}_{p=0}^{r} \subset Z$ and the given forcing data $f_1 \in L^2(J;Z)$, $f_2 \in L^2(J;H)$ and initial data $u_{1,DG,-}^0, u_{2,DG,-}^0 \in Z$ which are well defined, that also concludes that $\{\mathfrak{u}_{1,p}\}_{p=0}^{r} \subset Z$ at a given time step I also exists and is unique.
Finally, for given forcing data $F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in L^2(J; Z\times H)$ and initial data $U_{DG,-}^0 \in Z\times Z$ at given generic time step I there exist unique functions $\{\mathfrak{U}_p\}_{p=0}^{r} \subset Z\times Z$ which solve the $2(r+1)$ spatial problems (4.29) on p. 84. Now since

$$U_{DG}|_I = \sum_{j=0}^{r}\varphi_j(t)U_j, \text{ for } Y \in \mathbb{C}^{(r+1)\times(r+1)}: \ U_j = \sum_{i=0}^{r} Y_{ji}\mathfrak{U}_i \in Z\times Z.$$

From the existence and uniqueness of $\{\mathfrak{U}_i\}_{i=0}^{r} \subset Z\times Z$, then there exists unique vector-valued function $U_{DG} \in P^r(I; Z\times Z)$ which solves Problem 4.2.2. □

Theorem 4.4 (Existence and uniqueness of the semi-discrete solution with general $r.h.s$)

Let Z and H be the Hilbert spaces over the field $\mathbb{C}$ given in Definition 2.1 on p. 6. There exists a unique real semi-discrete vector-valued function $U_{DG} \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z\times Z)$ which solves the formulation (4.26) on p. 83.

Proof:
Firstly, the semi-discrete vector-valued function U_{DG} is uniquely written as

$$U_{DG} = \sum_{n=0}^{N-1} U_{DG}|_{I_n},$$

where

$$U_{DG}|_{I_n} = \sum_{j=0}^{r_n}\varphi_j(t)U_j, \text{ for } Y^{r_n} \in \mathbb{C}^{(r_n+1)\times(r_n+1)}: \ U_j = \sum_{i=0}^{r_n} Y_{ji}^{r_n}\mathfrak{U}_i \in Z\times Z.$$

Then, from the existence and uniqueness of $U_{DG}|_{I_n} \in P^{r_n}(I_n; Z\times Z)$ for $n = 0, \cdots, N-1$ shown in Theorem 4.3 on p. 84 implies that $U_{DG} \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z\times Z)$ does exist and is unique. □

4.2.2 Stability estimates

This section includes the general lemmas and theorems of the local and global stability estimates of the semi-discrete vector-valued function which solves Problem 4.2.1 on p. 82. The latter is having a general structure with general vector-valued functional. The same techniques used in Section 3.2.2 starting on p. 36 will be used here with treating the additional terms in the functional which are bounded as well as we will see in the coming sections.
All lemmas and theorems to be shown here are going to be very important and useful to get the complete proofs of existence, uniqueness, and stability estimates in Part II of this thesis.

Lemma 4.1 (Local stability estimate of semi-discrete formulation with general $r.h.s$) *Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. The solution of the time-stepping scheme in (4.28) on p. 83 and at a generic time step $I = (t_0, t_1)$ that is*

$$\overbrace{\int_I \Big\{(\partial_t U_{DG}, V)_E + \hat{a}(U_{DG}, V)\Big\}dt + (U^0_{DG,+}, V^0_+)_E}^{b_{DG}(U_{DG},V)=} = \overbrace{\int_I (F, V)_E dt}^{=\mathcal{F}_{DG}(V)} + (U^0_{DG,-}, V^0_+)_E, \tag{4.37}$$

for all $V \in P^r(I; Z \times Z)$, with $U^0_{DG,-}$ to be the initial data at a given time step $I = (t_0, t_1)$,

where $V^0_+ = V_+(t_0)$, satisfies

$$\| U_{DG} \|_{L^2(I;E)} \leq \frac{k \parallel Y \parallel_2^2 (2r+1)}{2 \min\limits_p \ Re\ \lambda_p}\Big(\| F \|_{L^2(I;E)} + \| U^0_{DG,-} \|_E \Big), \tag{4.38}$$

where k is the time step size at I. λ_p with $Re\ \lambda_p > 0$ are the eigenvalues of the matrix $\mathcal{A}^L$ given in (3.31) on p. 30 for $p = 0, \cdots, r$, $\| Y^r \|_2^2$ is the square of the spectral norm (see (A.6) on p. 173) of the transformation matrix $Y \in \mathbb{C}^{(r+1)\times(r+1)}$ and r is the approximation order at the given generic time step I.

Proof:
Firstly the form $b_{DG}(\,,)$ in (4.37) satisfies the local Inf-sup condition in Lemma 3.8 on p. 38 with recalling the first steps in the proof of Lemma 3.9 more precisely in (3.58) on p. 41, but this time with one additional term in the $r.h.s$:

$$\begin{aligned} \| U_{DG} \|_{L^2(I;E)} &\leq \frac{k \parallel Y \parallel_2^2}{2(\min\limits_p \ \text{Re}\ \lambda_p)} \sup_{V \in P^r(I;Z\times Z)} \frac{\Big|b_{DG}(U_{DG}, V)\Big|}{\| V \|_{L^2(I;E)}} \\ &= \frac{k \parallel Y \parallel_2^2}{2(\min\limits_p \ \text{Re}\ \lambda_p)} \sup_{V \in P^r(I;Z\times Z)} \frac{\Big| \overbrace{\int_I (F, V)_E dt}^{\text{the additional term}} + (U^0_{DG,-}, V^0_+)_E \Big|}{\| V \|_{L^2(I;E)}} \end{aligned} \tag{4.39}$$

Now, getting the upper bound for the additional term with using *Cauchy inequality* would be

$$\Big| \int_I (F, V)_E dt \Big| \leq \| F \|_{L^2(I;E)} \| V \|_{L^2(I;E)} \tag{4.40}$$

Then with adding (4.40) to the resulting estimate in (3.61) on p. 42 and plugging them back to (4.39) and since the time step size $0 < k < 1$, yield

$$\Longrightarrow \| U_{DG} \|_{L^2(I;E)} \le \frac{k \| Y \|_2^2}{2 \min\limits_p \ \mathrm{Re}\ \lambda_p} \Big(\| F \|_{L^2(I;E)} + \frac{\sqrt{2}(2r+1)}{\sqrt{k}} \| U^0_{DG,-} \|_E \Big)$$

$$\le \frac{k}{2} \frac{\| Y \|_2^2}{\min\limits_p \ \mathrm{Re}\ \lambda_p} \| F \|_{L^2(I;E)} + \sqrt{\frac{k}{2}} \| Y \|_2^2 \frac{(2r+1)}{\min\limits_p \ \mathrm{Re}\ \lambda_p} \| U^0_{DG,-} \|_E \Big)$$

$$\le \frac{k}{2} \frac{\| Y \|_2^2 \, (2r+1)}{\min\limits_p \ \mathrm{Re}\ \lambda_p} \Big(\| F \|_{L^2(I;E)} + \| U^0_{DG,-} \|_E \Big). \qquad \square$$

Lemma 4.2 (Estimate of a special recursion relation) *Let $c > 0$ with assuming*

$$\delta^{j+1} \le c(d^j + \delta^j), \tag{4.41}$$

and

$$\delta^j \le \sum_{i=0}^{j-1} c^{j-i} d^i + c^j \delta^0, \tag{4.42}$$

then there holds

$$\delta^{j+1} \le \sum_{i=0}^{j} c^{j+1-i} d^i + c^{j+1} \delta^0,$$

Proof:
Starting with (4.41) for $j = 0$, then

$$\delta^{0+1} \le c d^0 + c \delta^0, \tag{4.43}$$

and for $j = 1$

$$\Longrightarrow \delta^{1+1} \le c\, d^1 + c\, \delta^1,$$

and with using the inequality in (4.43), implies

$$\delta^{1+1} \le c\, d^1 + c\, \delta^1 \le c\, d^1 + c\,(c\, d^0 + c\, \delta^0) = c\, d^1 + c^2\, d^0 + c^2 \delta^0, \tag{4.44}$$

and for $j = 2$

$$\Longrightarrow \delta^{2+1} \le c\, d^2 + c\, \delta^2, \tag{4.45}$$

and again substituting (4.44) in (4.45), yields

$$\begin{aligned} \delta^{2+1} &\le c\, d^2 + c\, \delta^2 \le c\, d^2 + c\,(c\, d^1 + c^2\, d^0 + c^2 \delta^0) = c\, d^2 + c^2\, d^1 + c^{2+1} d^0 + c^{2+1} \delta^0 \\ &= \sum_{i=0}^{2} c^{2+1-i} d^i + c^{2+1} \delta^0, \end{aligned} \tag{4.46}$$

Then with having (4.42) and from (4.46) for general j then by induction we have

$$\delta^{j+1} \le \sum_{i=0}^{j} c^{j+1-i} d^i + c^{j+1} \delta^0,$$

and that completes the proof. $\square$

Lemma 4.3 (Left-sided limits estimate of semi-discrete solution with general $r.h.s$)
Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let $P^{r_m}(I_m; Z \times Z)$ be the semi-discrete space of r_mth order polynomials in time at I_m with coefficients in $Z \times Z$ given in Definition 3.1 on p. 19. The semi-discrete vector-valued function $U_{DG} \in P^{r_m}(I_m; Z \times Z)$ which solves the following formulation

$$\int_{I_m} \Big\{(\partial_t U_{DG}, V)_E + \hat{a}(U_{DG}, V)\}dt + (U^m_{DG,+}, V^m_+)_E = \int_{I_m} (F, V)_E \, dt + (U^m_{DG,-}, V^m_+)_E, \quad \text{for all } V \in P^{r_m}(I_m; Z \times Z), \tag{4.47}$$

for $U^m_{DG,-} \in Z \times Z$ to be the initial data at a given time step $I_m = (t_m, t_{m+1})$ for $m = 0, \cdots, n-1$, ($n \leq N$) and $F = \binom{f_1}{f_2} \in L^2(J; Z \times H)$ to be a general forcing data, and $U_0 \in Z \times Z$ to be the given initial data, satisfies

$$\| U^n_{DG,-} \|_E \leq \Big(1 + \frac{\mathcal{C}}{N}\Big)^n \Big(\| F \|_{L^2(J;E)} + \| U_0 \|_E \Big), \tag{4.48}$$

where

$$\| U \|^2_E = a(u_1, u_1) + \| u_2 \|^2_H, \forall \, U \in Z \times Z, \tag{4.49}$$

and

$$\mathcal{C} = \max_m \Big(\frac{T \| Y^{r_m} \|^2_2 \, (2r_m + 1)}{2 \min\limits_p \; Re \; \lambda_p} \Big),$$

for $T < \infty$, $\| Y^{r_m} \|^2_2$ is the square of spectral norm (see (A.6) on p. 173) of the transformation matrix $Y^{r_m} \in \mathbb{C}^{(r_m+1)\times(r_m+1)}$ and λ_p with $Re \; \lambda_p > 0$ for $p = 0, \cdots, r_m$ is the corresponding eigenvalues of the diagonalised matrix A^L as defined in (3.31) on p. 30.

Proof:
Starting with choosing $V = U_{DG} \in P^{r_m}(I_m; Z \times Z)$ in (4.47) and then repeating the same first steps done in the l.h.s of (3.23) on p. 28 in the proof of Lemma 3.7 and before summing up over m and with applying the *Cauchy* and then *Young's inequality*, (see Lemma A.2 on p. 181), in the r.h.s of (4.47), imply

$$\begin{aligned} \| U^{m+1}_{DG,-} \|^2_E + \overbrace{\| [U_{DG}]^m \|^2_E}^{\geq 0} - \| U^m_{DG,-} \|^2_E &= 2 \int_{I_m} (F, U_{DG})_E dt \\ \Longrightarrow \| U^{m+1}_{DG,-} \|^2_E - \| U^m_{DG,-} \|^2_E \leq 2 \int_{I_m} (F, U_{DG})_E \, dt &\leq 2 \| F \|_{L^2(I_m;E)} \| U_{DG} \|_{L^2(I_m;E)} \\ &\leq \| F \|^2_{L^2(I_m;E)} + \| U_{DG} \|^2_{L^2(I_m;E)}, \end{aligned}$$

now with going back to the resulting estimate in Lemma 4.1 on p. 87 with letting $I \to I_m$ which means that $U^0_{DG,-} \to U^m_{DG,-}$ with considering a conformal time step size i.e. $k = \frac{T}{N}$ and using Lemma A.3 on

p. 181 three times after taking the square root of both sides, then

$$\| U^{m+1}_{DG,-} \|^2_E - \| U^m_{DG,-} \|^2_E \leq \| F \|^2_{L^2(I_m;E)} + \frac{k^2}{4}\frac{(\| Y^{r_m} \|^2_2)^2(2r_m+1)^2}{(\min_p \ \text{Re } \lambda_p)^2}\Big(\| F \|^2_{L^2(I_m;E)} + \| U^m_{DG,-} \|^2_E \Big)$$

$$= \| F \|^2_{L^2(I_m;E)} + \frac{T^2(\| Y^{r_m} \|^2_2)^2(2r_m+1)^2}{4N^2(\min_p \ \text{Re } \lambda_p)^2}\Big(\| F \|^2_{L^2(I_m;E)} + \| U^m_{DG,-} \|^2_E \Big)$$

$$\Longrightarrow \| U^{m+1}_{DG,-} \|_E = \Bigg(\|F \|^2_{L^2(I_m;E)} + \frac{T^2(\| Y^{r_m} \|^2_2)^2(2r_m+1)^2}{4N^2(\min_p \ \text{Re } \lambda_p)^2}\Big(\| F \|^2_{L^2(I_m;E)} + \| U^m_{DG,-} \|^2_E \Big) + \| U^m_{DG,-} \|^2_E \Bigg)^{1/2}$$

$$\leq \Big(\|F \|^2_{L^2(I_m;E)} + \| U^m_{DG,-} \|^2_E \Big)^{1/2} + \frac{T \| Y^{r_m} \|^2_2 (2r_m+1)}{4N \min_p \ \text{Re } \lambda_p}\Big(\| F \|^2_{L^2(I_m;E)} + \| U^m_{DG,-} \|^2_E \Big)^{1/2}$$

$$\Longrightarrow \| U^{m+1}_{DG,-} \|_E \leq \Big(\frac{T \| Y^{r_m} \|^2_2 (2r_m+1)}{4N \min_p \ \text{Re } \lambda_p} + 1\Big)\Big(\| F \|_{L^2(I_m;E)} + \| U^m_{DG,-} \|_E \Big).$$

Moreover

$$\Longrightarrow \| U^{m+1}_{DG,-} \|_E \leq \Big(\frac{T \| Y^{r_m} \|^2_2 (2r_m+1)}{4N \min_p \ \text{Re } \lambda_p} + 1\Big)\Big(\| F \|_{L^2(I_m;E)} + \| U^m_{DG,-} \|_E \Big)$$

$$\leq \Big(1 + \frac{1}{N} \max\big(\frac{T \| Y^{r_m} \|^2_2 (2r_m+1)}{2 \min_p \ \text{Re } \lambda_p}\big)\Big)\Big(\| F \|_{L^2(I_m;E)} + \| U^m_{DG,-} \|_E \Big) \tag{4.50}$$

$$= \Big(1 + \frac{\mathcal{C}}{N}\Big)\Big(\| F \|_{L^2(I_m;E)} + \| U^m_{DG,-} \|_E \Big).$$

Now,let

$$\delta^{m+1} := \| U^{m+1}_{DG,-} \|_E, \quad \delta^m := \| U^m_{DG,-} \|_E, \text{ and } d^m := \| F \|_{L^2(I_m;E)}, \tag{4.51}$$

then (4.50) can be written as

$$\delta^{m+1} \leq \Big(1 + \frac{\mathcal{C}}{N}\Big)(d^m + \delta^m), \text{ for } m = 0, \cdots, n-1,$$

and with using the recurrence relation in Lemma 4.2 on p. 88:

$$\delta^n \leq \sum_{m=0}^{n-1} \Big(1 + \frac{\mathcal{C}}{N}\Big)^{(n-m)} d^m + \Big(1 + \frac{\mathcal{C}}{N}\Big)^n \delta^0,$$

and with using the defined terms in (4.51) with $U^0_{DG,-} = U_0$ yields

$$\| U^n_{DG,-} \|_E \leq \sum_{m=0}^{n-1} \Big(1 + \frac{\mathcal{C}}{N}\Big)^{n-m} \| F \|_{L^2(I_m;E)} + \Big(1 + \frac{\mathcal{C}}{N}\Big)^n \| U_0 \|_E, \text{ for } n \leq N. \tag{4.52}$$

In the r.h.s of (4.52)

$$\sum_{m=0}^{n-1}(1+\frac{\mathcal{C}}{N})^{(n-m)} \parallel F \parallel_{L^2(I_m;E)} = \Big(1+\frac{\mathcal{C}}{N}\Big)^n \parallel F \parallel_{L^2(I_0;E)} + \cdots + \Big(1+\frac{\mathcal{C}}{N}\Big) \parallel F \parallel_{L^2(I_{n-1};E)}$$
$$\leq \Big(1+\frac{\mathcal{C}}{N}\Big)^n \sum_{m=0}^{n-1} \parallel F \parallel_{L^2(I_m;E)} = \Big(1+\frac{\mathcal{C}}{N}\Big)^n \parallel F \parallel_{L^2(J;E)} .$$

Then

$$\parallel U^n_{DG,-} \parallel_E \leq \Big(1+\frac{\mathcal{C}}{N}\Big)^n \Big(\parallel F \parallel_{L^2(J;E)} + \parallel U_0 \parallel_E \Big),$$

and that completes the proof. □

Theorem 4.5 (Global stability estimate of semi-discrete formulation with general *r.h.s*)
Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. With a general forcing data $F \in L^2(J; Z \times H)$ and initial data $U_0 \in Z \times Z$ the solution $U_{DG} \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z \times Z)$ of (4.26) in Problem 4.2.1 on p. 82 satisfies

$$\parallel U_{DG} \parallel_{L^2(J;E)} \leq (e^{\mathcal{C}} - 1)(\parallel F \parallel_{L^2(J;E)} + \parallel U_0 \parallel_E).$$

For bounded real constant

$$\mathcal{C} = \max_n(T\frac{\parallel Y^{r_n} \parallel_2^2 (2r_n+1)}{2\min\limits_p \ Re\ \lambda_p}),\ \ for\ p = 0, \cdots, r_n,$$

for $T < \infty$, $\parallel Y^{r_n} \parallel_2^2$ is the square of spectral norm (see (A.6) on p. 173) of the transformation matrix $Y^{r_m} \in \mathbb{C}^{(r_n+1)\times(r_n+1)}$ and λ_p with $Re\ \lambda_p > 0$ for $p = 0, \cdots, r_n$ is the corresponding eigenvalues of the diagonalised matrix A^L as defined in (3.31) on p. 30.

Proof:
Summing over all time steps I_n such that $J = \cup_{n=0}^{N-1} I_n$, then

$$\parallel U_{DG} \parallel_{L^2(J;E)} = \sum_{n=0}^{N-1} \parallel U_{DG} \parallel_{L^2(I_n;E)} .$$

Now, with using the resulting estimate in Lemma 4.1 on p. 87 after letting $I \to I_n$ and the correspondence initial data accordingly and after that Lemma 4.3 on p. 89, then

$$\begin{aligned}
\| U_{DG} \|_{L^2(J;E)} &\leq \sum_{n=0}^{N-1} \| U_{DG} \|_{L^2(I_n;E)} \\
&\leq \sum_{n=0}^{N-1} \left\{ k \frac{\| Y^{r_n} \|_2^2 \, (2r_n+1)}{2 \min\limits_p \ \mathrm{Re}\ \lambda_p} \Bigg(\| F \|_{L^2(I_n;E)} + \| U^n_{DG,-} \|_E \Bigg) \right\} \\
&= \sum_{n=0}^{N-1} \left\{ \frac{T \| Y^{r_n} \|_2^2 \, (2r_n+1)}{2N \min\limits_p \ \mathrm{Re}\ \lambda_p} \Bigg(\| F \|_{L^2(I_n;E)} + \| U^n_{DG,-} \|_E \Bigg) \right\} \\
&\leq \frac{1}{N} \max_n \Big(\frac{T \| Y^{r_n} \|_2^2 \, (2r_n+1)}{2 \min\limits_p \ \mathrm{Re}\ \lambda_p} \Big) \sum_{n=0}^{N-1} \left\{ \Bigg(\| F \|_{L^2(I_n;E)} + \| U^n_{DG,-} \|_E \Bigg) \right\} \\
&= \frac{\mathcal{C}}{N} \underbrace{\sum_{n=0}^{N-1} \| F \|_{L^2(I_n;E)}}_{=\|F\|_{L^2(J;E)}} + \frac{\mathcal{C}}{N} \sum_{n=0}^{N-1} \| U^n_{DG,-} \|_E \\
&\leq \frac{\mathcal{C}}{N} \| F \|_{L^2(J;E)} + \frac{\mathcal{C}}{N} \sum_{n=0}^{N-1} \left\{ \Big(1 + \frac{\mathcal{C}}{N}\Big)^n \right\} \Bigg(\| F \|_{L^2(J;E)} + \| U_0 \|_E \Bigg) \\
&= \Bigg(\frac{\mathcal{C}}{N} + \frac{\mathcal{C}}{N} \sum_{n=0}^{N-1} \left\{ \Big(\frac{\mathcal{C}}{N} + 1 \Big)^n \right\} \Bigg) \| F \|_{L^2(J;E)} + \frac{\mathcal{C}}{N} \sum_{n=0}^{N-1} \left\{ \Big(\frac{\mathcal{C}}{N} + 1 \Big)^n \right\} \| U_0 \|_E \, .
\end{aligned}$$

Here, with the use of the geometric series formula with $a \neq 1$:

$$\sum_{i=0}^{n-1} a^i = \frac{a^n - 1}{a - 1}, \ \text{(see [31, (7.6)])}$$

then

$$\sum_{n=0}^{N-1} \Big(\frac{\mathcal{C}}{N} + 1 \Big)^n = \frac{\Big(\frac{\mathcal{C}}{N} + 1 \Big)^N - 1}{\Big(\frac{\mathcal{C}}{N} + 1 \Big) - 1} = \frac{N}{\mathcal{C}} \Bigg(\Big(\frac{\mathcal{C}}{N} + 1 \Big)^N - 1 \Bigg).$$

Then

$$\| U_{DG} \|_{L^2(J;E)} \leq \Bigg(\frac{\mathcal{C}}{N} + \Bigg(\Big(\frac{\mathcal{C}}{N} + 1 \Big)^N - 1 \Bigg) \Bigg) \| F \|_{L^2(J;E)} + \Bigg(\Big(\frac{\mathcal{C}}{N} + 1 \Big)^N - 1 \Bigg) \| U_0 \|_E \, .$$

For the boundedness of the latter estimate:
First, let $x = N$ such that $0 < N < \infty$, and given fixed $0 < \mathcal{C} < \infty$,

$$f(x) := \Bigg(\Big(\frac{\mathcal{C}}{x} + 1 \Big)^x - 1 \Bigg). \tag{4.53}$$

Now, it is necessary to check if the latter function is increasing or decreasing. This is done by calculating the first derivative with respect to x and check its sign.
With knowing that

$$1 - \frac{1}{x} \leq \log(x) \leq x - 1, \ \text{for } x > 0, \ \text{(see [9, (9)])}$$

then

$$f'(x) = \left(\frac{\mathcal{C}}{x}+1\right)^x \left(\log\left(\frac{\mathcal{C}}{x}+1\right) - \frac{\mathcal{C}}{\mathcal{C}+x}\right) \geq \left(\frac{\mathcal{C}}{x}+1\right)^x \underbrace{\left(\left(1-\frac{1}{\frac{\mathcal{C}}{x}+1}\right) - \frac{\mathcal{C}}{\mathcal{C}+x}\right)}_{=0}.$$

Secondly, with using this result, it concludes that the function $f(x)$ given in (4.53) is bounded by its limit as $x \to \infty$ with $\mathcal{C} < \infty$. For $\frac{\mathcal{C}}{x}$ which is bounded from above by $\mathcal{C}$, then

$$\lim_{x\to+\infty} \left(\frac{\mathcal{C}}{x} + \overbrace{\left(\left(\frac{\mathcal{C}}{x}+1\right)^x - 1\right)}^{f(x)}\right) \leq e^{\mathcal{C}} - 1.$$

Thus,

$$\| U_{DG} \|_{L^2(J;E)} \leq (e^{\mathcal{C}} - 1)(\| F \|_{L^2(J;E)} + \| U_0 \|_E),$$

and that completes the proof. □

4.3 High-order in time DGFEM with conformal spatial discretisation, fully-discrete formulation with a functional of general forcing data F

The section includes the general theorem of existence, uniqueness, and stability estimates of fully-discrete vector-valued function which solves Problem 4.3.1 on p. 93 which is having a structure with general vector-valued functional. The same conformal spatial discretisation method used in Chapter 3 is also going to be used here in order to get the fully-discrete formulation.
All theorems to be shown here are going to be very important and useful to show existence, uniqueness, and stability estimates of fully-discrete formulation in Part II of this thesis. In Theorem 4.7 on p. 96 it will be shown that Problem 4.2.1 on p. 82 is solvable. Theorem 4.8 on p. 98 shows the global stability estimate of fully-discrete solution of Problem 4.2.1.

Problem 4.3.1 (Fully-discrete formulation with general forcing data)
Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let $Z^h \times Z^h$ be the discrete finite-dimensional product subspace of $Z \times Z$ given in Definition 3.10 on p. 43. Let $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h \times Z^h)$ be the fully-discrete finite-dimensional product subspace of $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z \times Z)$ where the latter is given in Definition 3.1 on p. 19. Here $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h \times Z^h)$ is a subspace of $L^2(J; Z \times Z)$. Let $(\,,)_E$ be the inner product given in Corollary 2.2 on p. 10. Let $\hat{a}(\,,)$ be the skew-symmetric form given in Definition 2.7 on p. 14. Let $B_{DG}(\,,)$ be the form given in (3.64):

$$(U, V)_E = a(u_1, v_1) + (u_2, v_2)_H, \text{ and } \hat{a}(U, V) = -a(u_2, v_1) + a(u_1, v_2),$$

and

$$B_{DG}(U_{DG}, V) = \sum_{n=0}^{N-1} \int_{I_n} \left\{(\partial_t U_{DG}, V)_E + \hat{a}(U_{DG}, V)\right\} dt + \sum_{n=1}^{N-1} ([U_{DG}]^n, V_+^n)_E + (U^0_{DG,+}, V^0_+)_E.$$

Let $\mathcal{F}_{DG} \in (L^2(J; Z \times H))'$ be the linear functional:

$$\begin{aligned}\mathcal{F}_{DG}(V^h) &:= \sum_{n=0}^{N-1} \int_{I_n} \{a(f_1, v_1^h) + \langle f_2, v_2^h \rangle_{Z' \times Z}\} dt \\ &= \sum_{n=0}^{N-1} \int_{I_n} \{\overbrace{a(f_1, v_1^h) + (f_2, v_2^h)_H}^{(F, V^h)_E =}\} dt, \; \forall V^h \in L^2(J; Z \times H),\end{aligned} \tag{4.54}$$

where all vector-valued functions in semi-discrete space $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h \times Z^h)$ are also included in this linear map since $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h \times Z^h) \subset L^2(J; Z \times H)$.
Find $U_{DG}^h \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h \times Z^h)$ such that

$$B_{DG}(U_{DG}^h, V^h) = \mathcal{F}_{DG}(V^h) + (U_{DG,-}^{h,0}, V_+^{h,0})_E, \quad \forall \; V^h \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h \times Z^h), \tag{4.55}$$

with given initial condition $U_{DG,-}^{h,0} = U_0^h = \begin{pmatrix} u_{1,0}^h \\ u_{2,0}^h \end{pmatrix} \in Z^h \times Z^h$.

The time-stepping formulation of (4.55) would read:

Problem 4.3.2 (Fully-discrete time-stepping scheme with general forcing data)
Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let $Z^h \times Z^h$ be the discrete finite-dimensional product subspace of $Z \times Z$ given in Definition 3.10 on p. 43. Let $P^{r_n}(I_n; Z^h \times Z^h)$ be the fully-discrete finite-dimensional product subspace of $P^{r_n}(I_n; Z \times Z)$ where the latter is given in Definition 3.1 on p. 19. Let $(\,,)_E$ be the inner product given in Corollary 2.2 on p. 10. Let $\hat{a}(\,,)$ be the skew-symmetric form given in Definition 2.7 on p. 14:

$$\hat{a}(U, V) = -a(u_2, v_1) + a(u_1, v_2), \;\; (U, V)_E = a(u_1, v_1) + (u_2, v_2)_H,$$

Find $U_{DG}^h \in P^{r_n}(I_n; Z^h \times Z^h)$:

$$\overbrace{\int_{I_n} \Big\{(\partial_t U_{DG}^h, V^h)_E + \hat{a}(U_{DG}^h, V^h)\Big\} dt + (U_{DG,+}^{h,n}, V_+^{h,n})_E}^{b_{DG}^n(U_{DG}^h, V^h) =} = \overbrace{\int_{I_n} (F, V^h)_E dt}^{\mathcal{F}_{DG}^n(V^h) :=} + (U_{DG,-}^{h,n}, V_+^{h,n})_E,$$

for all $V^h \in P^{r_n}(I_n; Z^h \times Z^h)$, with $U_{DG,-}^{h,0}$ to be the initial data at a given time step $I_n = (t_n, t_{n+1})$,
(4.56)

4.3.1 Existence and uniqueness

With repeating the same time decoupling process done in Section 3.2.1.1 starting on p. 28, then the formulation (4.56) at a generic time step $I = (t_0, t_1)$ would read

Problem 4.3.3 (Discrete spatial problems with a general forcing data)
Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let $Z^h \times Z^h$ be the discrete finite-dimensional product subspace of $Z \times Z$ given in Definition 3.10 on p. 43. Let $(\,,)_E$ be the inner product given in Corollary 2.2 on p. 10. Let λ_p be the eigenvalues of the matrix $\mathcal{A}^L$ given in (3.31) on p. 30. Let $(\beta_p^1)_q$ and $(\beta_p^2)_q$ be the $(r+1)$ vectors given in (3.36):

$$\begin{aligned}(U, V)_E &= a(u_1, v_1) + (u_2, v_2)_H, \forall U, V \in Z^h \times Z^h, \\ (\beta_p^1)_q &= (Y^{-1})_{pq}\, x_q^1 = (Y^{-1})_{pq} \sqrt{(q + 1/2)}, \;\; \textit{and} \\ (\beta_p^2)_q &= (Y^{-1})_{pq}\, x_q^2 = (Y^{-1})_{pq} \sqrt{(q + 1/2)}(-1)^q, \;\; \textit{respectively.}\end{aligned} \tag{4.57}$$

Find $\{\mathfrak{U}_p^h\}_{p=0}^r \subset Z^h \times Z^h$ *such that*

$$\lambda_p(\mathfrak{U}_p^h, \mathfrak{V}_p^h)_E + \frac{k}{2}\Big(a(\mathfrak{u}_{1,p}^h, \mathfrak{v}_{2,p}) - a(\mathfrak{u}_{2,p}^h, \mathfrak{v}_{1,p}^h)\Big) = \frac{k}{2}\sum_{q=0}^{r}\Big\{(\beta_p^1)_q \int_{-1}^{1} L_q(\hat{t})(F\circ Q, \mathfrak{V}_p^h)_E d\hat{t}\Big\}$$
$$+ \sum_{i=0}^{r}\Big\{(\beta_p^2)_q\Big\}(U_{DG,-}^{h,0}, \mathfrak{V}_p^h)_E,$$

for each $p = 0, \cdots, r, \forall\{\mathfrak{V}_p^h\}_{p=0}^r \subset Z^h \times Z^h$, *given initial data* $U_{DG,-}^{h,0}$ *at a given time step* I,
and forcing data $F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in L^2(J; Z \times H)$,

(4.58)

The $2(r+1)$ spatial problems in (4.58) can be reduced.

Problem 4.3.4 (Discrete reduced spatial problems with a general forcing data) *Let* Z *and* H *be the Hilbert spaces given in Definition 2.1 on p. 6. Let* Z^h *the discrete finite-dimensional subspace of* Z *given in Definition 3.10 on p. 43. Let* λ_p *be the eigenvalues of the matrix* $\mathcal{A}^L$ *given in (3.31) on p. 30. Let* $(\beta_p^1)_q$ *and* $(\beta_p^2)_q$ *be elements given in (4.57).*
With again recalling the same steps done in sections 3.2.1.2 and 4.2 starting on p. 32 and 82, respectively, such that with

$$\mathfrak{f}^1(\mathfrak{v}_{2,p}^h) = 2\lambda_p k \sum_{q=0}^{r}\Big\{(\beta_p^1)_q \int_{-1}^{1} L_q(\hat{t})(f\circ Q, \mathfrak{v}_{2,p}^h)_H d\hat{t}\Big\},$$
$$\mathfrak{f}^2(\mathfrak{v}_{2,p}^h) = -k^2 \sum_{q=0}^{r}\Big\{(\beta_p^1)_q \int_{-1}^{1} L_q(\hat{t}) a(f_1\circ Q, \mathfrak{v}_{2,p}^h) d\hat{t}\Big\},$$

and with a form and a linear functional $\mathfrak{f} \in (Z\times H)'$:

$$\begin{aligned} b(\mathfrak{u}_{2,p}^h, \mathfrak{v}_{2,p}^h) &= 4\lambda_p^2(\mathfrak{u}_{2,p}, \mathfrak{v}_{2,p}^h)_H + k^2 a(\mathfrak{u}_{2,p}^h, \mathfrak{v}_{2,p}^h) \\ \mathfrak{f}(\mathfrak{v}_{2,p}^h) &= \mathfrak{f}^1(\mathfrak{v}_{2,p}^h) + \mathfrak{f}^2(\mathfrak{v}_{2,p}^h), \forall v_2 \in Z, \end{aligned} \tag{4.59}$$

where all functions in the conformal finite dimensional subspace Z^h *are also included in this linear map since* $Z^h \subset Z$*. Then (4.58) is reduced and would now read:*
Find $\{\mathfrak{u}_{2,p}^h\}_{p=0}^r \subset Z^h$:

$$b(\mathfrak{u}_{2,p}^h, \mathfrak{v}_{2,p}) = \mathfrak{f}(\mathfrak{v}_{2,p}^h) + \sum_{q=0}^{r}\Big\{(\beta_p^2)_q\Big\}\Big(4\lambda_p(u_{2,DG,-}^{h,0}, \mathfrak{v}_{2,p}^h)_H - 2ka(u_{1,DG,-}^{h,0}, \mathfrak{v}_{2,p}^h)\Big),$$

for each $p = 0, \cdots, r$, *for all* $\{\mathfrak{v}_{2,p}^h\}_{p=0}^r \subset Z^h$.

(4.60)

After solving for unknowns $\{\mathfrak{u}_{2,p}^h\}_{p=0}^r \subset Z^h$ *with given data* $F = \binom{f_1}{f_2} \in L^2(J; Z\times H)$ *and* $u_{1,DG,-}^{h,0}$, $u_{2,DG,-}^{h,0} \in Z^h$, *then it comes to update the values of* $\{\mathfrak{u}_{1,p}^h\}_{p=0}^r \subset Z^h$:

$$\mathfrak{u}_{1,p}^h = \frac{k}{2\lambda_p}\mathfrak{u}_{2,p}^h + \frac{1}{\lambda_p}\sum_{q=0}^{r}\Big\{(\beta_p^2)_q\Big\}u_{1,DG,-}^{h,0} + \frac{k}{2\lambda_p}\sum_{q=0}^{r}\Big\{(\beta_p^1)_q \int_{-1}^{1} L_q(\hat{t}) f_1\circ Q d\hat{t}\Big\}, \tag{4.61}$$

which need to be solved at a given generic time step I.

Theorem 4.6 (Existence, uniqueness of discrete reduced problems of with general $r.h.s$**)**

Let Z and H be the Hilbert spaces over the field $\mathbb{C}$ given in Definition 2.1 on p. 6. Let Z^h be the conformal finite dimensional subspace of the Hilbert space Z given in Definition 3.10 on p. 43. There exist unique discrete functions $\{\mathfrak{u}^h_{2,p}\}_{p=0}^r \subset Z^h$ which solve (4.60) and $\{\mathfrak{u}^h_{1,p}\}_{p=0}^r \subset Z^h$ in (4.61).

Proof:
With using conformal spatial discetisation and since Z^h is a subspace of Z, the form $b(\, , \,)$ given in (4.60) is continuous with using Lemma 3.5 on p. 34 after letting $c_1 = 4\lambda_p \in \mathbb{C}$ and $c_2 = k^2 \in \mathbb{R}$:

$$b(\mathfrak{u}^h_{2,p}, \mathfrak{v}^h_{2,p}) = 4\lambda_p^2(\mathfrak{u}^h_{2,p}, \mathfrak{v}^h_{1,p})_H + k^2 a(\mathfrak{u}^h_{2,p}, \mathfrak{v}^h_{1,p}) \leq (4|\lambda_p|c^2 + k^2 M) \parallel \mathfrak{u}^h_{2,p} \parallel_Z \parallel \mathfrak{v}^h_{1,p} \parallel_Z, \text{ for } p = 0, \cdots, r,$$

and satisfies the discrete Inf-sup condition in Lemma 3.11 on p. 47 where the Inf-sup constant is $k^2\alpha$. For the r.h.s of (4.60) and with again using the same steps in the proof of Theorem 4.3, then

$$|r.h.s| \leq \Bigg(c\sqrt{2^3 k}|\lambda_p| \sum_{q=0}^{r} |(\beta^1_p)_q| \hat{x}_q \parallel f_2 \parallel_{L^2(I;H)} + \Big|\sum_{q=0}^{r}(\beta^2_p)_q\Big| \Big(4c|\lambda_p| \parallel u^{h,0}_{2,DG,-} \parallel_H + 2kM \parallel u^{h,0}_{1,DG,-} \parallel_Z \Big) + \sqrt{2k^3} M \sum_{q=0}^{r_n} |(\beta^1_p)_q| \hat{x}_q \parallel f_1 \parallel_{L^2(I;Z)} \Bigg) \parallel \mathfrak{v}^h_{2,p} \parallel_Z . \tag{4.62}$$

Thus, going back to (Babuška) Theorem A.4 on p. 179 again with having Lemmas 3.11 and 3.5 and (4.62), then there exist unique functions $\{\mathfrak{u}^h_{2,p}\}_{p=0}^r \subset Z^h$ which solve (4.60) on p. 95.
For the identity (4.61) on p. 95:

$$\mathfrak{u}^h_{1,p} = \frac{k}{2\lambda_p}\mathfrak{u}^h_{2,p} + \frac{1}{\lambda_p}\sum_{q=0}^{r}\Big\{(\beta^2_p)_q\Big\}u^{h,0}_{1,DG,-} + \frac{k}{2\lambda_p}\sum_{q=0}^{r}\Big\{(\beta^1_p)_q \int_{-1}^{1} L_q(\hat{t}) f_1 \circ Q d\hat{t}\Big\},$$

where

$$\parallel \frac{k}{2\lambda_p}\sum_{q=0}^{r}\Big\{(\beta^1_p)_q \int_{-1}^{1} L_q(\hat{t}) f_1 \circ Q d\hat{t}\Big\} \parallel_Z \leq \frac{k}{2|\lambda_p|}\sum_{q=0}^{r}\Big\{|(\beta^1_p)_q| \int_{-1}^{1} L_q(\hat{t}) \parallel f_1 \circ Q \parallel_Z d\hat{t}\Big\} < \infty,$$

for each $p = 0, \cdots, r$, then from the given existence and uniqueness of $\{\mathfrak{u}^h_{2,p}\}_{p=0}^r \subset Z^h$ and the given forcing data $f_1 \in L^2(J;Z)$, $f_2 \in L^2(J;H)$ and initial data $u^{h,0}_{1,DG,-}, u^{h,0}_{2,DG,-} \in Z$ which are well defined, that also concludes that $\{\mathfrak{u}^h_{1,p}\}_{p=0}^r \subset Z^h$ at a given time step I also exists and is unique. □

Theorem 4.7 (Existence and uniqueness of fully-discrete solution with general $r.h.s$**)**
Let Z and H be the Hilbert spaces over the field $\mathbb{C}$ given in Definition 2.1 on p. 6. Let $Z^h \times Z^h$ be the conformal finite dimensional subspace of the product-Hilbert space $Z \times Z$ given in Definition 3.10 on p. 43. There exists unique fully-discrete vector-valued function $U^h_{DG} \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h \times Z^h)$ which solves the formulation (4.55).

Proof:
Since the fully-discrete vector-valued function U^h_{DG} is uniquely written as

$$U^h_{DG} = \sum_{n=0}^{N-1} U^h_{DG}|_{I_n},$$

where

$$U_{DG}^h|_{I_n} = \sum_{j=0}^{r_n} \varphi_j(t) U_j^h, \text{ for } Y^{r_n} \in \mathbb{C}^{(r_n+1)\times(r_n+1)} : \ U_j^h = \sum_{i=0}^{r_n} Y_{ji}^{r_n} \mathfrak{U}_i^h \in Z^h \times Z^h.$$

Then, the shown existence and uniqueness of $\{\mathfrak{U}_i^h\}_{i=0}^{r_n} \subset Z^h \times Z^h$ in Theorem 4.6 on p. 96 implies that $U_{DG}^h \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h \times Z^h)$ does exist and is unique. □

4.3.2 The stability estimates

This section will provide local and global stability estimates of fully-discrete formulation with general forcing data F.

Lemma 4.4 (Local stability estimate for fully-discrete formulation with general $r.h.s$**)** *Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let $Z^h \times Z^h$ be the conformal finite dimensional subspace of the product-Hilbert space $Z \times Z$ given in Definition 3.10 on p. 43. At a generic time step $I = (t_0, t_1)$ with general vector-valued functional $\mathcal{F} \in (L^2(J; Z \times H))'$, a vector-valued function $U_{DG}^h \in P^r(I; Z^h \times Z^h)$ which solves*

$$\overbrace{\int_I \Big\{(\partial_t U_{DG}^h, V^h)_E + \hat{a}(U_{DG}^h, V^h)\Big\} dt + (U_{DG,+}^{h,0}, V_+^{h,0})_E}^{b_{DG}(U_{DG}^h, V^h)=} = \overbrace{\int_I (F, V^h)_E dt}^{\mathcal{F}_{DG}(V^h)=} + (U_{DG,-}^{h,0}, V_+^{h,0})_E,$$
$$\text{for all } V^h \in P^r(I; Z^h \times Z^h), \text{ with } U_{DG,-}^{h,0} \text{ to be the initial data at a given time step } I = (t_0, t_1), \tag{4.63}$$

satisfies

$$\| U_{DG}^h \|_{L^2(I;E)} \leq \frac{k \| Y \|_2^2 (2r+1)}{2 \min\limits_p \ Re\ \lambda_p} \Big(\| F \|_{L^2(I;E)} + \| U_{DG,-}^{h,0} \|_E \Big), \tag{4.64}$$

where k is the time step at I. λ_p with $Re\ \lambda > 0$ are the eigenvalues of the matrix $\mathcal{A}^L$ given in (3.31) on p. 30 for $p = 0, \cdots, r$, $\| Y^r \|_2^2$ is the square of the spectral norm (see (A.6) on p. 173) of the transformation matrix $Y \in \mathbb{C}^{(r+1)\times(r+1)}$ and r is the approximation order.

Proof:
Now, since the form $b_{DG}(\, ,)$ in (4.63) satisfies the discrete local Inf-sup condition in Lemma 3.10 on p. 45, then

$$\begin{aligned} \| U_{DG}^h \|_{L^2(I;E)} &\leq \frac{k \| Y \|_2^2}{2(\min\limits_p \ \mathrm{Re}\ \lambda_p)} \sup_{V^h \in P^r(I; Z^h \times Z^h)} \frac{\Big| b_{DG}(U_{DG}^h, V^h) \Big|}{\| V^h \|_{L^2(I;E)}} \\ &= \frac{k \| Y \|_2^2}{2(\min\limits_p \ \mathrm{Re}\ \lambda_p)} \sup_{V^h \in P^r(I; Z^h \times Z^h)} \frac{\Big| \int_I (F, V^h)_E dt + (U_{DG,-}^{h,0}, V_+^h)_E \Big|}{\| V^h \|_{L^2(I;E)}}, \end{aligned} \tag{4.65}$$

From here, and based on the choice of $Z^h \times Z^h$ which is the conformal finite dimensional subspaces of $Z \times Z$ the rest of the proof follows the same steps of proving the estimate in Lemma 4.1 on p. 87 since in this case the steps don't depend on the spatial spaces and that completes the proof. □

Lemma 4.5 (Left-sided limits estimate of fully-discrete solution with general $r.h.s$**)**
Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let $Z^h \times Z^h$ be the conformal finite dimensional subspace of the product-Hilbert space $Z \times Z$ given in Definition 3.10 on p. 43. The fully-discrete vector-valued function $U^h_{DG} \in P^{r_m}(I_m; Z^h \times Z^h)$ which solves the formulation

$$\int_{I_m} \Big\{ (\partial_t U^h_{DG}, V^h)_E + \hat{a}(U^h_{DG}, V^h) \} dt + (U^{m,h}_{DG,+}, V^{h,m}_+)_E = \overbrace{\int_{I_m} (F, V)_E \, dt + (U^{h,m}_{DG,-}, V^{h,m}_+)_E}^{\mathcal{F}_{DG}(V^h)=}, \quad (4.66)$$

for all $V^h \in P^{r_m}(I_m; Z^h \times Z^h)$,

where $U^{h,m}_{DG,-}$ is the initial data at a given time step $I_m = (t_m, t_{m+1})$ for $m = 0, \cdots, n-1$, ($n \leq N$) and $U^h_0 \in Z^h \times Z^h$ to be the given initial data, satisfies

$$\| U^{h,n}_{DG,-} \|_E \leq \Big(1 + \frac{\mathcal{C}}{N}\Big)^n \Big(\| F \|_{L^2(J;E)} + \| U^h_0 \|_E \Big), \quad (4.67)$$

where

$$\| U^h \|^2_E = a(u^h_1, u^h_1) + \| u^h_2 \|^2_H, \forall \, U^h \in Z^h \times Z^h, \quad (4.68)$$

and

$$\mathcal{C} = \max_m \Big(\frac{T \| Y^{r_m} \|^2_2 \, (2r_m + 1)}{2 \min_p \; Re \, \lambda_p} \Big),$$

for $T < \infty$, $\| Y^{r_m} \|^2_2$ is the square of spectral norm (see (A.6) on p. 173) of the transformation matrix $Y^{r_m} \in \mathbb{C}^{(r_m+1)\times(r_m+1)}$ and λ_p with $Re \, \lambda_m$ are the corresponding eigenvalues of the diagonalised matrix A^L as defined in (3.31) for $p = 0, \cdots, r_m$.

Proof:
Firstly, choosing $V^h = U^h_{DG} \in P^{r_m}(I_m; Z^h \times Z^h)$ in (4.66), then

$$\int_{I_m} \Big\{ (\partial_t U^h_{DG}, U^h_{DG})_E + \overbrace{\hat{a}(U^h_{DG}, U^h_{DG})}^{=0} \} dt + \| U^{m,h}_{DG,+} \|^2_E = \int_{I_m} (F, U^h_{DG})_E \, dt + (U^{h,m}_{DG,-}, U^{h,m}_{DG,+})_E.$$

Secondly, since the formulation in (4.66) satisfies same properties as the one in (4.37) on p. 87 and this is based on the choice of the space $Z^h \times Z^h$ which is the conformal finite dimensional subspaces of $Z \times Z$. Thus, the rest of the steps follow the same arguments in the proof of Lemma 4.3 on p. 89 which does not depend on the spatial spaces and that completes the proof. □

Theorem 4.8 (The global stability estimate of fully-discrete formulation with general $r.h.s$**)**
Let Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let $Z^h \times Z^h$ be the conformal finite dimensional subspace of the product-Hilbert space $Z \times Z$ given in Definition 3.10 on p. 43. With general forcing data $F \in L^2(J; Z \times H)$ and initial data $U^h_0 \in Z^h \times Z^h$, then the solution $U^h_{DG} \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h \times Z^h)$ of the formulation (4.66) satisfies

$$\| U^h_{DG} \|_{L^2(J;E)} \leq (\mathcal{C} + e^{\mathcal{C}} - 1) \| F \|_{L^2(J;E)} + (e^{\mathcal{C}} - 1) \| U^h_0 \|_E \, .$$

For bounded real constant

$$\mathcal{C} = \max_n \Big(T \frac{\| Y^{r_n} \|^2_2 \, (2r_n + 1)}{2 \min_p \; Re \, \lambda_p} \Big), \; \textit{for } p = 0, \cdots, r_n,$$

which appeared in Lemma 3.3 on p. 42.

Proof:
Firstly, with summing over all time steps I_n such that $J = \cup_{n=0}^{N-1} I_n$, then

$$\| U_{DG}^h \|_{L^2(J;E)} = \sum_{n=0}^{N-1} \| U_{DG}^h \|_{L^2(I_n;E)} .$$

Secondly, with using the resulting estimate in Lemma 4.4 on p. 97 and then using the estimate in Lemma 4.5 on p. 98, with following the same arguments in the proof of Theorem 4.5 on p. 91, then that completes the proof. □

Part II: Abstract theory analysis of High-order in time DGFEM for abstract linear wave equation with linear constrains

Chapter 5

Linear second-order in time evolution equation with linear constraints

In this chapter, a variational formulation of the first-order system in time considered in Chapter 3 of Part I of this thesis is now going to be discussed with the case if there are linear constraints e.g. in-homogeneous boundary data. These data are considered to be continuous in time and have high regularity in space. The first-order system in time variational formulation is analysed in abstract subset spaces.

In order to show existence, uniqueness, and stability estimates for first-order system in time formulation with linear constraints, the extension methodology is used. This approach is about splitting the solution of the formulation with linear constraints into two solutions. One is an existing function which solves an auxiliary elliptic problem which is depending on the given constraints in the trace space. The other is a solution of a transformed formulation where its analysis fits the given general abstract theory in Chapter 4 in Part I.

For the auxiliary problem, the space and the regularity of its solution are chosen very carefully in order to make sure all theory and regularity results from Part I are still applicable for the resulting transformed formulation in this chapter as well as in the coming ones when it comes to discretise in time and space.

5.1 Important Definitions and Lemmas

Definition 5.1 *Let:*

- *Z and H be separable, (see Definition A.5 on p. 172) Hilbert spaces over the field $\mathbb{C}$ such that Z is a subspace of H and dense in H, with Gelfand triple i.e. ($Z \subset H \subset Z'$).*

- *$A \in L(Z, Z')$ be a linear bounded operator, where Z' is the dual space of Z.*

- *With Definition A.19, Corollaries A.7 and A.8 on p. 176, a non-empty space D^A which is a subspace of Z is defined as*

$$D^A := \{u \in Z : \, Au \in H\}, \text{ with a norm } \| u \|_{D^A}^2 := \| u \|_Z^2 + \| Au \|_H^2, \forall \, u \in D^A.$$

Assumption 5.1 *$(D^A, \| \cdot \|_{D^A})$ in Definition 5.1 is Hilbert space.*

Corollary 5.1 (Separability of the Hilbert space D^A) *The Hilbert space D^A given in Definition 5.1 is separable.*

Proof:
Assumption 5.1 implies D^A is complete and that means it is a closed subspace of the separable Hilbert space Z and from Theorem A.2 on p. 178 it concludes that D^A is also separable. □

Definition 5.2 *Let Z and D^A be the Hilbert spaces given in Definition 5.1 with Assumption 5.1.*

- *Let Z^{tr} be a normed vector space.*
  Let $\gamma : Z \to Z^{tr}$ which is linear continuous surjective map, (see Definitions A.11 and A.13 on p. 174) i.e. $Z^{tr} = \gamma[Z]$. The norm in Z^{tr} is defined as

$$\| v \|_{Z^{tr}} := \inf_{\substack{u \in Z \\ \gamma u = v}} \| u \|_Z, \qquad \forall v \in Z^{tr}.$$

- *$D^{A,tr} := \gamma[D^A]$ with the norm*

$$\| v \|_{D^{A,tr}} := \inf_{\substack{u \in D^A \\ \gamma u = v}} \| u \|_{D^A}, \qquad \forall v \in D^{A,tr}.$$

- *For any $g \in Z^{tr}$ and $g \in D^{A,tr}$ the following subsets are defined over the field $\mathbb{C}$:*

$$Z_g := \{v \in Z : \gamma v = g\} \text{ and } D_g := \{v \in D^A : \gamma v = g\}.$$

- *Also the following subsets over the field $\mathbb{C}$:*

$$Z_0 := \{v \in Z : \gamma v = 0\} \text{ and } D_0 := \{v \in D^A : \gamma v = 0\}.$$

Corollary 5.2 (Separability of Z^{tr} and $D^{A,tr}$) *Z^{tr} and $D^{A,tr}$ given in Definition 5.2 are separable.*

Proof:
With using Corollary A.5 on p. 175 i.e. from Definition 5.2 on p. 102:

$$Z^{tr} = \gamma[Z], \text{ and } D^{A,tr} = \gamma[D^A],$$

and since Z and D^A are separable Hilbert spaces as given in Definition 5.1 with Assumption 5.1 and Corollary 5.1 on p. 101 and γ is linear and bounded operator, then that completes the proof. □

Corollary 5.3 *Let Z and D^A be the Hilbert spaces given in Definition 5.1 with Assumption 5.1 on p. 101. Then the subsets Z_g and D_g given in Definition 5.2 on p. 102 are not subspaces of Z and D^A, respectively. Moreover, the subsets Z_0 and D_0 given in Definition 5.2 are subspaces of $(Z, \| \cdot \|_Z)$ and $(D^A, \| \cdot \|_{D^A})$, respectively with $(Z_0, \| \cdot \|_Z)$ and $(D_0, \| \cdot \|_{D^A})$.*

Proof:
With recalling the three conditions given in the third point in Definition A.1 on p. 171:
Firstly, Let u_1, $u_2 \in Z_g$ and based on the definition of Z_g then

$$\gamma u_1 + \gamma u_2 = \gamma(u_1 + u_2) = 2g \rightarrow u_1 + u_2 \notin Z_g,$$

and the same thing holds for the subset D_g.
Secondly, for example if u_1 and u_2 are in Z_0 and $\alpha \in \mathbb{C}$, then

$$\alpha\gamma u_1 + \alpha\gamma u_2 = 0 \in Z_0,$$

with knowing that from the definition of Z_0 the zero vector is in Z_0. Also, the same thing holds for the subset D_0, then that completes the proof. □

Lemma 5.1 *Let Z and H be the Hilbert spaces given in Definition 5.1 on p. 101. Then $(Z_0, \|\cdot\|_Z)$ is a separable Hilbert space which is a subspace of H and dense in H i.e. have the Gelfand triple $(Z_0 \subset H \subset Z_0')$. Also*

$$Z_0 \subset Z \subset H \subset Z' \subset Z_0'. \tag{5.1}$$

Proof:
Firstly, for $\gamma \in L(Z, Z^{tr})$ which is a linear bounded surjective operator and the definition of the subspace Z_0 given in Definition 5.2 on p. 102:

$$Z_0 = \{v \in Z : \gamma v = 0\} \Longleftrightarrow \ker(\gamma) = \{v \in Z : \gamma v = 0\},$$

and from Corollary A.3 on p. 174 then Z_0 is closed. Also, with knowing that a closed subspace of a Hilbert space is a Hilbert space,(see [13, Orthogonal Complements and Projections]) and from Theorem A.2 on p. 178, Consequently, Z_0 is separable Hilbert space of Z. Secondly, from the given properties of the Hilbert spaces Z and H in Definition 5.1 on p. 101:

$$Z \text{ and } H \text{ are separable, } Z \text{ is subspace of } H \text{ and dense in } H, \text{ and have Gelfand triple } Z \subset H \subset Z',$$

and from the separability of Z_0 i.e. it has countable dense subsets, then it means that it is also dense in H, and for Z_0' the dual space of Z_0 the Gelfand triple holds that is $(Z_0 \subset H \subset Z_0')$ with the following spacial structure holds

$$Z_0 \subset Z \subset H \subset Z' \subset Z_0',$$

and that completes the proof. □

Corollary 5.4 *Let Z and D^A be the Hilbert spaces given in Definition 5.1 with Assumption 5.1 on p. 101. Let D_g and Z_g be the subsets and Z_0 and D_0 be the subspaces given in Definition 5.2 on p. 102. Then D_g and $D_{g'}$ are subsets of Z_g and $Z_{g'}$, respectively, and D_0 is a subspace of Z_0.*

Proof:
Since D^A is a Hilbert subspace of Z, and from the definitions of the subsets D_g and Z_g:

$$Z_g = \{v \in Z : \gamma v = g\} \text{ and } D_g = \{v \in D^A : \gamma v = g\},$$

they conclude that D_g is a subset of Z_g. Also, the same would hold for $D_{g'}$ and $Z_{g'}$. Similarly, from the definitions of the subspaces D_0 and Z_0 of D^A and Z, respectively:

$$Z_0 = \{v \in Z : \gamma v = 0\} \text{ and } D_0 = \{v \in D^A : \gamma v = 0\},$$

they imply that D_0 is a subspace of Z_0, and that concludes the proof. $\square$

Definition 5.3 *Let Z and H be the Hilbert spaces given in Definition 5.1 on p. 101. Let $A \in L(Z, Z')$. Let Z_0 be the Hilbert subspace of Z given in Definition 5.2 on p. 102. $a : Z \times Z \to \mathbb{C}$ is a Hermitian form such that*

$$a(u,v) := \langle Au, v\rangle_{Z' \times Z}, \text{ for } u, v \in Z,$$

and

$$\begin{aligned} |a(u,v)| &\leq M \parallel u \parallel_Z \parallel v \parallel_Z, &\quad \forall\, u, v \in Z, \\ a(u,u) &\geq \alpha \parallel u \parallel_Z^2, &\quad \forall u \in Z_0, \end{aligned}$$

which is real and symmetric if both u, v are real functions and if they are complex then

$$a(u,v) = \overline{a(v,u)}, \text{ (conjugate-symmetry)}.$$

Definition 5.4 *Considering Definitions 5.1 and 5.2 on p. 101, the following is defined*

$$W := H^1(J; D^A \times D^A),$$

with the norm

$$\parallel U \parallel_W^2 := \parallel U \parallel_{H^1(J;D^A \times D^A)}^2 = \parallel u_1 \parallel_{H^1(J;D^A)}^2 + \parallel u_2 \parallel_{H^1(J;D^A)}^2 .$$

Lemma 5.2 (New inner product) *Let Z and D^A be the Hilbert spaces given in Definition 5.1 with Assumption 5.1 on p. 101. Let W be the normed-product space given in Definition 5.4 on p. 104. The following is an inner product*

$$(U,V)_W := \int_J \{(\partial_t U, \partial_t V)_{D^A \times D^A} + (U,V)_{D^A \times D^A}\} dt = (U,V)_{H^1(J;D^A \times D^A)}, \ \forall\, U,\, V \in W.$$

Proof:

1. Bi-linearity:
   For any real vector-valued functions U^1, $U^2, V \in W$ and α_1, $\alpha_2 \in \mathbb{R}$, with the bi-linearity of the inner products $(\,,)_{D^A}$ and $(\,,)_Z$ of the Hilbert spaces D^A and Z, respectively then

$$\begin{aligned} ((\alpha_1 U^1 + \alpha_2 U^2), V)_W &= ((\alpha_1 U^1 + \alpha_2 U^2), V)_{H^1(J;D^A \times D^A)} \\ &= \alpha_1 (U^1, V)_{H^1(J;D^A \times D^A)} + \alpha_2 (U^2, V)_{H^1(J;D^A \times D^A)} \\ &= \alpha_1 (U^1, V)_W + \alpha_2 (U^2, V)_W = (U^1, \alpha_1 V)_W + (U^2, \alpha_2 V)_W. \end{aligned}$$

2. Symmetry:
   By definition and the symmetries of its individual terms with given real vector-valued functions, then

$$(U,V)_W = (U,V)_{H^1(J;D^A \times D^A)} = (V,U)_{H^1(J;D^A \times D^A)} = (V,U)_W.$$

3. Positive definiteness:
   For real vector-valued function $U \neq 0$ and by definition

$$(U,U)_W = (U,U)_{H^1(J;D^A\times D^A)} = \| U \|^2_{H^1(J;D^A\times D^A)} > 0,$$

   and this concludes that $(U,U)_W = 0$ if and only if $U \equiv 0$.

Moreover, since $H^1(J;D^A)$ is a Hilbert space and with using Definition A.22 on p. 181 such that

$$\| u_1 \|^2_{H^1(J;D^A)} < \infty, \ \forall u_1 \in H^1(J;D^A),$$

then

$$\| U \|^2_W = \| U \|^2_{H^1(J;D^A\times D^A)} < \infty \Longrightarrow \| U \|_W \ \text{ is defined on the whole } W.$$

Thus, the space W given in Definition 5.4 is a normed-product space with an inner product space i.e. $(W,(\,,\,)_W)$ is a pre-Hilbert space. □

Remark 5.1 *The inner product in Lemma 5.2 on p. 104 can also be written as*

$$\begin{aligned}(U,V)_W :&= \int_J \{(\partial_t u_1, \partial_t v_1)_{D^A} + (u_1,v_1)_{D^A} + (\partial_t u_2, \partial_t v_2)_{D^A} + (u_2,v_2)_{D^A}\}dt \\ &= (u_1,v_1)_{H^1(J;D^A)} + (u_2,v_2)_{H^1(J;D^A)}.\end{aligned} \tag{5.2}$$

Lemma 5.3 (New Hilbert space) *Let D^A be the Hilbert space given in Definition 5.1 with Assumption 5.1 on p. 101. Then W given in Definition 5.4 on p. 104 is a Hilbert space, i.e. $(W, \| \cdot \|_W)$ is complete, (see Definitions A.4 and A.16 on p. 172 and 176, respectively) and separable.*

Proof:
$(W,(\cdot,\cdot)_W)$ is a pre-Hilbert space as shown in Lemma 5.2. It remains to show that it is a Hilbert space. This is shown if the space W induced with the norm $\| \cdot \|_W$ is complete i.e. if every *Cauchy sequence* $\{U_n\}$ in W, (see Definition A.3 on p. 172), converges in W.
Let $\{U_n\}$ be a *Cauchy sequence* in $W = H^1(J;D^A \times D^A)$ where

$$W = H^1(J;D^A) \times H^1(J;D^A).$$

Since a product of two Hilbert spaces is a Hilbert space, i.e. since $H^1(J;D^A)$; with using Assumption 5.1 on p. 101 thus

$$\exists\, u_1 \in H^1(J;D^A) \text{ such that } u_{1,n} \to u_1 \text{ in } H^1(J;D^A), \tag{5.3}$$

and

$$\exists\, u_2 \in H^1(J;D^A) \text{ such that } u_{2,n} \to u_2 \text{ in } H^1(J;D^A), \tag{5.4}$$

Thus, from (5.3) and (5.4)

$$\text{the limit } U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \text{ is in } W, \tag{5.5}$$

which concludes that the space $(W,(\cdot,\cdot)_W)$ is a Hilbert space. Also since D^A is separable as well as the product $D^A \times D^A$, which also implies that $W = H^1(J;D^A \times D^A)$ is separable, (see [39, Remark 10.1.10]), and that completes the proof. □

Lemma 5.4 (Surjective time-extension operator) *Let D^A be the Hilbert space given in Definition 5.1 with Assumption 5.1 on p. 101. Let $D^{A,tr}$ be the separable normed vector space in Definition 5.2 on p. 102. Then $\gamma_{H^1} \in L(H^1(J; D^A), H^1(J; D^{A,tr}))$ is continuous (bounded) and surjective.*

Proof:
From Lemma 5.3 on p. 105 $H^1(J; D^A)$ is a separable Hilbert space i.e. from Theorem A.5 on p. 179 it has a countable orthonormal basis. Thus, $v \in H^1(J; D^A)$ is written as

$$v = \sum_n \psi_n(t) v_n, \text{ for orthonormal basis } \psi_n(t) \in H^1(J), \text{ and } v_n = (v, \psi_n)_{H^1(J)} \in D^A.$$

Now, with using Definition A.20 on p. 179 and knowing that $\gamma \in L(D^A, D^{A,tr})$ is bounded and time independent:

$$\gamma_{H^1} v := \sum_n \psi_n(t) \gamma v_n, \tag{5.6}$$

then

$$\| \gamma_{H^1} v \|^2_{H^1(J;D^{A,tr})} = \sum_n \| \gamma v_n \|^2_{D^{tr}} \leq \| \gamma \|^2_{D^A, D^{A,tr}} \sum_n \| v_n \|^2_{D^A} = \mathcal{C}_\gamma \| v \|^2_{H^1(J;D^A)},$$

and with the way the mapping is defined and the surjectivity of $\gamma \in L(D^A, D^{A,tr})$, then
$\gamma_{H^1} \in L(L^2(J; D^A), L^2(J; D^{A,tr}))$ is continuous and surjective. □

Definition 5.5 (The norm in $H^1(J; D^{A,tr})$) *Let D^A be the Hilbert space given in Definition 5.1 with Assumption 5.1 on p. 101. Let $D^{A,tr}$ be the separable normed vector space in Definition 5.2 on p. 102. Let $\gamma_{H^1} \in L(H^1(J; D^A), H^1(J; D^{A,tr}))$ be the continuous (bounded) and surjective operator given in Lemma 5.4. The norm in $H^1(J; D^{A,tr})$ is defined as*

$$\| g \|^2_{H^1(J;D^{A,tr})} = \inf_{\substack{u \in H^1(J;D^A) \\ \gamma_{H^1} u = g}} \| u \|^2_{H^1(J;D^A)}, \text{ for } g \in H^1(J; D^{A,tr}). \tag{5.7}$$

Lemma 5.5 (Matrix surjective time-extension operator)
Let $D^{A,tr}$ be the normed space given in Definition 5.2 on p. 102. Let W be the given Hilbert space in Definition 5.4 on p. 104. For vector-valued function $U \in W = H^1(J; D^A) \times H^1(J; D^A)$ and with $W^{tr} := H^1(J; D^{A,tr}) \times H^1(J; D^{A,tr})$, the time-extension of the continuous surjective map $\bar{\gamma} := \begin{pmatrix} \gamma_{H^1} & 0 \\ 0 & \gamma_{H^1} \end{pmatrix} \in L(W, W^{tr})$ is continuous and surjective.

Proof:
From Definition 5.4 on p. 104, then the vector-valued function $U \in W = H^1(J; D^A \times D^A)$ in component wise would be as

$$u_1 : J \to D^A, \text{ is in } H^1(J; D^A) \text{ and } u_2 : J \to D^A, \text{ which is in } H^1(J; D^A).$$

With recalling the proof of Lemma 5.4 on p. 106, respectively, then

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \sum_n \phi_n(t) \begin{pmatrix} u_{1,n} \\ u_{2,n} \end{pmatrix}, \text{ for orthonormal basis } \phi_n(t) \in H^1(J), \text{ and}$$

$$\begin{pmatrix} u_{1,n} \\ u_{2,n} \end{pmatrix} = (U, \phi_n(t) I)_{H^1(J) \times H^1(J)} \in D^A \times D^A :$$

$$\bar{\gamma} U = \begin{pmatrix} \gamma_{H^1} & 0 \\ 0 & \gamma_{H^1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \sum_n \phi_n(t) \overbrace{\begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} u_{1,n} \\ u_{2,n} \end{pmatrix}}^{\in D^{A,tr} \times D^{A,tr}},$$

and since the operator γ_{H^1} is linear, bounded, and surjective, then $\bar{\gamma} \in L(W, W^{tr})$ is bounded and surjective. □

Corollary 5.5 (The norm on the trace space W^{tr}) *Let $D^{A,tr}$ be the normed space given in Definition 5.2 on p. 102. Let W be the Hilbert space given in Definition 5.4 on p. 104 and $W^{tr} = H^1(J; D^{A,tr} \times D^{A,tr})$. Let $\bar{\gamma} : W \to W^{tr}$ be the time-extension continuous surjective map given in Lemma 5.5 on p. 106, then a norm in W^{tr} is*

$$\| G \|_{W^{tr}} := \inf_{\substack{U \in W \\ \bar{\gamma} U = G}} \| U \|_W, \quad \forall G \in W^{tr}. \tag{5.8}$$

Proof:
Firstly, Lemma 5.5 implies that $W^{tr} = \bar{\gamma}[W]$.
Secondly, for the norm of the Hilbert space W given in Definition 5.4 on p. 104, with the norm of the space $D^{A,tr}$ in Definition 5.2 on p. 102, and for $G = \binom{g_1}{g_2} = \binom{g}{g'} \in W^{tr}$ (where g' indicates the partial time derivative of g) and with taking the infimum of W where $\bar{\gamma} U = G$, then

$$\begin{aligned}\inf_{\substack{U \in W \\ \bar{\gamma} U = G}} \| U \|_W^2 &= \inf_{\substack{u_1 \in H^1(J;D^A) \\ \gamma_{H^1} u_1 = g_1}} \| u_1 \|_{H^1(J;D^A)}^2 + \inf_{\substack{u_1 \in H^1(J;D^A) \\ \gamma_{H^1} u_2 = g_2}} \| u_2 \|_{H^1(J;D^A)}^2 \\ &= \| g_1 \|_{H^1(J;D^{A,tr})}^2 + \| g_2 \|_{H^1(J;D^{A,tr})}^2 = \| G \|_{H^1(J;D^{A,tr} \times D^{A,tr})}^2 = \| G \|_{W^{tr}},\end{aligned}$$

and that completes the proof. □

Definition 5.6 (Subset and subspace of the Hilbert space W) *Let W be the Hilbert space given in Definition 5.4 on p. 104 and $W^{tr} = H^1(J; D^{A,tr} \times D^{A,tr})$. Let $\bar{\gamma} : W \to W^{tr}$ be the time-extension continuous surjective map given in Lemma 5.5 on p. 106. For any $G = \binom{g}{g'} \in W^{tr}$ the following set is defined*

$$W_G := \left\{ U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} | \, U \in W \;\; with \;\; \bar{\gamma} U = \begin{pmatrix} g \\ g' \end{pmatrix}, \, \partial_t u_1 \in L^2(J; D_{g'}) \;\; and \;\; \partial_t u_2 \in L^2(J; D^A) \right\}.$$

Also the subset:

$$W_{\mathbf{0}} = \left\{ U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} | \, U \in W \;\; with \;\; \bar{\gamma} U = \mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

Corollary 5.6 *$(W_{\mathbf{0}}, \| \cdot \|_W)$ is a subspace of $W = H^1(J; D^A \times D^A)$ and is a separable Hilbert space.*

Proof:
Firstly, from the definition of the subset $W_{\mathbf{0}}$, the zero vector is in $W_{\mathbf{0}}$. Also for any $U^1, U^2 \in W_{\mathbf{0}}$ and $\alpha \in \mathbb{C}$,

$$\alpha \bar{\gamma} U^1 + \alpha \bar{\gamma} U^1 = \mathbf{0}.$$

Secondly, with the use of Lemma 5.4 on p. 106 such that $\bar{\gamma} \in L(W, W^{tr})$ is a linear bounded surjective operator and also from the definition of the subspace $W_{\mathbf{0}}$:

$$W_{\mathbf{0}} := \left\{ U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} | \, U \in W \text{ with } \bar{\gamma} U = \mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \Longleftrightarrow \ker(\bar{\gamma}) = \{ V \in W : \bar{\gamma} V = 0 \},$$

then from Corollary A.3 on p. 174 then $W_{\mathbf{0}}$ is closed. Also, with knowing that a closed subspace of a Hilbert space is a Hilbert space,(see [13, Orthogonal Complements and Projections]) and from Theorem A.2 on p. 178, Consequently, $W_{\mathbf{0}}$ is separable Hilbert space of W, and that complete the proof. □

Lemma 5.6 (New inequalities) *Let Z, H, and D^A be the Hilbert spaces given in Definition 5.1 with Assumption 5.1 on p. 101. Let $(Z \times Z, \| \cdot \|_X)$ be the normed-product space with the equivalent E-norm in Definition 2.4 on p. 9 and in Lemma 2.1 on p. 9, the following hold*

$$\| V \|_E^2 \leq \max(c^2, M) \| V \|_{D^A \times D^A}^2, \text{ for } V \in Z \times Z, \tag{5.9}$$

and

$$\| V \|_X^2 \leq \max(c^2, 1) \| V \|_{D^A \times D^A}^2, \text{ for } V \in Z \times Z. \tag{5.10}$$

Proof:
Starting with the E-norm for $V \in Z \times Z$, using the *continuity* of the form $a(\,,)$, the inequality $\| v_2 \|_H \leq c \| v_2 \|_Z$, and the D_A-norm imply

$$\begin{aligned}
\| V \|_E^2 = a(v_1, v_1) + \| v_2 \|_H^2 &\leq M \| v_1 \|_Z^2 + c^2 \| v_2 \|_Z^2 \\
&\leq \max(M, c^2)\left(\| v_1 \|_Z^2 + \| v_2 \|_Z^2 \right) \\
&\leq \max(M, c^2)\left(\| v_1 \|_Z^2 + \| Av_1 \|_H^2 + \| v_2 \|_Z^2 + \| Av_2 \|_H^2 \right) \\
&= \max(M, c^2)\left(\| v_1 \|_{D^A}^2 + \| v_2 \|_{D^A}^2 \right) \\
&= \max(M, c^2) \| V \|_{D^A \times D^A}^2,
\end{aligned}$$

and for the X-norm

$$\begin{aligned}
\| V \|_X^2 = \| v_1 \|_Z^2 + \| v_2 \|_H^2 &\leq \| v_1 \|_Z^2 + c^2 \| v_2 \|_Z^2 \\
&\leq \max(1, c^2)\left(\| v_1 \|_Z^2 + \| v_2 \|_Z^2 \right) \\
&\leq \max(1, c^2)\left(\| v_1 \|_Z^2 + \| Av_1 \|_H^2 + \| v_2 \|_Z^2 + \| Av_2 \|_H^2 \right) \\
&= \max(1, c^2)\left(\| v_1 \|_{D^A}^2 + \| v_2 \|_{D^A}^2 \right) \\
&= \max(1, c^2) \| V \|_{D^A \times D^A}^2,
\end{aligned}$$

□

5.2 The variational formulation of second-order system in time evolution problem with linear constraints

Given Definitions 5.1 and 5.3 on p. 101 - 104 and suppose that

$f \in H^1(J; H)$ to be the forcing data,

$u_0 \in D_g \subset Z_g \subset Z$ and $\bar{u}_0 \in Z_{g'} \subset Z$ to be the compatible initial data

with, $g \in H^1(J; D^{A,tr})$, and $g' \in H^1(J; D^{A,tr})$, to be the smooth linear constraints on the trace space. (5.11)

The following formulation reads:

Problem 5.2.1 (The second-order in time formulation with linear constraints)
Let Z and H be the Hilbert spaces given in Definition 5.1 on p. 101. Let Z_g, $Z_{g'}$, and Z_0 be the subsets and subspace given in Definition 5.2 on p. 102. Let $a(\,,)$ be the form given in Definition 5.3 on p. 104. Find $u \in L^2(J; Z_g)$ with $\partial_t u \in L^2(J; Z_{g'})$ and $\partial_t^2 u \in L^2(J; H)$:

$$\int_J \{ \langle \partial_t^2 u, v \rangle_{Z' \times Z} + a(u, v) \} dt = F(v) = \int_J (f, v)_H dt, \ \ \forall v \in L^2(J; Z_0), \tag{5.12}$$

for given forcing data $f \in H^1(J; H)$, compatible initial data $u_0 \in Z_g \subset Z$, $\bar{u}_0 \in Z_{g'} \subset Z$, and $g \in H^1(J; D^{A,tr})$, $\partial_t g = g' \in H^1(J; D^{A,tr})$ to be smooth enough data at the trace space.

5.3 The continuous auxiliary (elliptic) problem

This section will introduce an existing continuous auxiliary (elliptic) problem which is going to be used later as part of the splitting technique to complete the theoretical analysis involved in this part i.e. showing the existence, uniqueness, and stability estimates of solution of the continuous variational formulation with given linear constraints.
With recalling Definitions 5.1 and 5.3 on p. 101 - 104, and Lemma 5.2 on p. 104, the following problem is introduced

Problem 5.3.1
Let D^A be the Hilbert space given in Definition 5.1 with Assumption 5.1 on p. 101. Let D_g be the subset of D^A given in Definition 5.2. Let W_G and $W_{\mathbf{0}}$ be the subset and the subspace of W, respectively, given in Definition 5.6 on p. 107. Let $(\,,)_W$ be the inner product given in Lemma 5.2 on p. 104 with $J = (0, T)$:

$$(U, V)_W = (U, V)_{H^1(J;D^A \times D^A)}, \text{ for } U, V \in W = H^1(J; D^A \times D^A).$$

Find $\hat{G} \in W_G = \left\{ U = \binom{u_1}{u_2} | U \in W \text{ with } \bar{\gamma} U = \binom{g}{g'},\ \partial_t u_1 \in L^2(J; D_{g'}) \text{ and } \partial_t u_2 \in L^2(J; D^A) \right\} \subset W \subset C([0, T]; D^A \times D^A)$:

$$(\hat{G}, V)_W = 0, \tag{5.13}$$

for all $V \in W_{\mathbf{0}} \subset W$ with $G = \binom{g_1}{g_2} := \binom{g}{g'} \in W^{tr} = H^1(J; D^{A,tr} \times D^{A,tr})$ where $\bar{\gamma} \in L(W, W^{tr})$ is the linear surjective matrix operator given in Lemma 5.5 on p. 106, and $G \in W^{tr} \subset C([0, T]; D^{A,tr} \times D^{A,tr})$.

Now, since $\hat{G} \in W_G \subset W \subset C([0, T]; D^A \times D^A)$, then

$$\partial_t \hat{g}_1 \in L^2(J; D_{g'}) \text{ and } \hat{g}_2 \in L^2(J; D_{g'}). \tag{5.14}$$

5.3.1 Existence, uniqueness, and stability estimate of the continuous auxiliary (elliptic) problem

Theorem 5.1 (Existence and uniqueness of continuous auxiliary (elliptic) problem)
Let D^A be the Hilbert space given in Definition 5.1 with Assumption 5.1 on p. 101. Let $D^{A,tr}$ be the normed space given in Definition 5.2 on p. 102. Let D_g be the subset of D^A given in Definition 5.2. Let W be the Hilbert space given in Definition 5.4 on p. 104 and $W^{tr} = H^1(J; D^{A,tr} \times D^{A,tr})$. Let W_G and $W_{\mathbf{0}}$ be the subset and the subspace of W, respectively, given in Definition 5.6 on p. 107. There exists a unique vector-valued function $\hat{G} \in W_G$ which solves the formulation in (5.13).

Proof:
To show existence, since $G \in W^{tr} = H^1(J; D^{A,tr} \times D^{A,tr})$ is in the trace space, therefore at least one vector-valued function $\bar{G} \in W = H^1(J; D^A \times D^A)$ exists which satisfies the data at the trace space i.e. $\bar{\gamma}\bar{G} = G$, then let

$$\hat{G} := \bar{G} + \tilde{G}, \text{ where } \tilde{G} \in W_{\mathbf{0}} \subset W, \tag{5.15}$$

then the formulation in (5.13) in terms of $\tilde{G} \in W_{\mathbf{0}} \subset W$ and $\bar{G} \in W$ now reads
Find $\tilde{G} \in W_{\mathbf{0}}$:

$$(\tilde{G}, V)_W = -(\bar{G}, V)_W, \qquad \forall V \in W_{\mathbf{0}} \subset W. \tag{5.16}$$

Firstly, since $(\,,\,)_W$ is the inner product in Hilbert space W as shown in Lemma 5.2 on p. 104, the following hold
1. Continuity:
Using *Cauchy inequality* yields

$$(\tilde{G}, V)_W \leq \| \tilde{G} \|_W \| V \|_W, \ \forall \tilde{G}, V \in W_{\mathbf{0}} \subset W.$$

2. Positive definite:

$$(\tilde{G}, \tilde{G})_W = \| \tilde{G} \|_W^2 \geq 0, \qquad\qquad \forall \tilde{G} \in W_{\mathbf{0}} \subset W.$$

Secondly, for the *r.h.s* of (5.16) with the use of *Cauchy inequality* again yields:

$$(\bar{G}, V)_W \leq \| \bar{G} \|_W \| V \|_W, \qquad\qquad \forall \bar{G} \in W, \ V \in W_{\mathbf{0}} \subset W.$$

Thus, with knowing that $W_{\mathbf{0}}$ is a Hilbert space (see Lemma 5.6 on p. 107), then by Lax-Milgram theorem (see Theorem A.3 on p. 178) there exists a unique solution $\tilde{G} \in W_{\mathbf{0}}$ which solves the formulation in (5.16). From the identity in (5.15) on p. 109 and since $\tilde{G} \in W_{\mathbf{0}}$ exists, it concludes that $\hat{G} \in W_G$ exists.
Finally, for the uniqueness, let $\hat{G}^1 = \begin{pmatrix} \hat{g}_1^1 \\ \hat{g}_2^1 \end{pmatrix} \in W_G$ and $\hat{G}^2 = \begin{pmatrix} \hat{g}_1^2 \\ \hat{g}_2^2 \end{pmatrix} \in W_G$ be two solutions which solve (5.13) on p. 109, and with taking the difference of the two solutions i.e. letting $\bar{W} := \hat{G}^1 - \hat{G}^2 \in W_{\mathbf{0}}$, then we have
Find $\bar{W} \in W_{\mathbf{0}}$:

$$(\bar{W}, V)_W = 0, \forall V \in W_{\mathbf{0}},$$

and with choosing $V = \bar{W} \in W_{\mathbf{0}}$, then

$$(\bar{W}, \bar{W})_W = \| \bar{W} \|_W^2 \geq 0 \Longrightarrow \bar{W} := \hat{G}^1 - \hat{G}^2 = 0,$$

which means $\hat{G}^1 = \hat{G}^2$, and that completes the proof. □

Lemma 5.7 (Stability estimate of continuous auxiliary (elliptic) problem)
Let D^A be the Hilbert space given in Definition 5.1 with Assumption 5.1 on p. 101. Let $D^{A,tr}$ be the normed space given in Definition 5.2 on p. 102. Let D_g be the subset of D^A given in Definition 5.2. Let W be the Hilbert space given in Definition 5.4 on p. 104 and $W^{tr} = H^1(J; D^{A,tr} \times D^{A,tr})$. Let W_G and $W_{\mathbf{0}}$ be the subset and the subspace of W, respectively, given in Definition 5.6 on p. 107. The vector-valued function $\hat{G} \in W_G$ which solves the formulation in (5.13) on p. 109 satisfies

$$\| \hat{G} \|_W \leq 2 \| G \|_{W^{tr}} . \tag{5.17}$$

For $G \in W^{tr}$, where $\| \cdot \|_{W^{tr}}$ is the norm on W^{tr} given in Corollary 5.5 on p. 107:

$$\| G \|_{W^{tr}} := \inf_{\substack{U \in W \\ \bar{\gamma} U = G}} \| U \|_W, \quad \forall G \in W^{tr}. \tag{5.18}$$

Proof:
Starting with the identity given in (5.15) on p. 109 and the use of the *triangle inequality*, then

$$\| \hat{G} \|_W = \| \bar{G} + \tilde{G} \|_W \leq \| \bar{G} \|_W + \| \tilde{G} \|_W . \tag{5.19}$$

Now, with choosing $V = \tilde{G} \in W_{\mathbf{0}}$, in the formulation (5.16) implies

$$\| \tilde{G} \|_W^2 \leq \| \bar{G} \|_W \| \tilde{G} \|_W ,$$

and dividing both sides with $\| \tilde{G} \|_W \neq 0$, yields

$$\| \tilde{G} \|_W \leq \| \bar{G} \|_W ,$$

then substituting the resulting estimate back in (5.19) implies

$$\| \hat{G} \|_W \leq 2 \| \bar{G} \|_W .$$

Now, with taking $\inf\limits_{\substack{\bar{G} \in W \\ \bar{\gamma}\bar{G} = G}}$ in both sides and using Corollary 5.5 on 107, imply

$$\| \hat{G} \|_W \leq 2 \inf_{\substack{\bar{G} \in W \\ \bar{\gamma}\bar{G} = G}} \| \bar{G} \|_W = 2 \| G \|_{W^{tr}} , \tag{5.20}$$

and that completes the proof. □

5.4 The variational formulation of the first-order system in time with linear constraints

In this Section, the first-order system in time variational formulation of (5.12) on p. 108 is given in Problem 5.4.1 on p. 112. The equivalence proof between the two problems is shown in Lemma 5.8 on p. 112. Uniqueness and existence of Problem 5.4.1 is proven in Theorem 5.2 on p. 115, where the proof of it uses the results from Theorem 5.1 on p. 109 and Lemma 4.2 on p. 80.

Definition 5.7 (New functions)
From Section 5.3 the following (linear constraints) smooth data are considered at the trace space:

$$\begin{aligned} &g_1 = g, \ \textit{the same function at the trace space of problem } (5.12), \ \textit{with} \\ &g_2 = \partial_t g = g', \end{aligned} \tag{5.21}$$

such that

$$G = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in W^{tr} = H^1(J; D^{A,tr} \times D^{A,tr}).$$

Then to rewrite the variational formulation in (5.12) as a first-order system in time the following functions are introduced

$$\begin{aligned} &u_1 := u, \ \textit{the solution of } (5.12), \ \textit{and} \\ &u_2 := \partial_t u. \end{aligned} \tag{5.22}$$

Now, with (5.22) in Definition 5.7:

$$\partial_t u_1 = u_2 \text{ and } \partial_t^2 u = \partial_t u_2,$$

then the formulation in (5.12) on p. 108 is rewritten in terms of the two functions in (5.22) would read

Find $u_1 \in L^2(J; Z_g)$, $u_2 \in L^2(J; Z_{g'})$ with $\partial_t u_2 \in L^2(J; H)$:

$$\int_J \{a(\partial_t u_1, v_1) - a(u_2, v_1)\} dt = 0$$
$$\int_J \{\langle \partial_t u_2, v_2 \rangle_{Z' \times Z} + a(u_1, v_2)\} dt = \int_J \langle f, v_2 \rangle_{Z' \times Z} dt, \ \forall v_1, v_2 \in Z_0,$$

for given initial data $u_{1,0} := u_0 \in D_g \subset Z_g$, $u_{2,0} := \bar{u} \in Z_{g'}$, forcing data $f \in H^1(J; H)$, and linear constraints $G \in W^{tr}$.

Definition 5.8
Let Z be the Hilbert space given in Definition 5.1 on p. 101. Let $Z_{g'}$ be the subset of Z given in Definition 5.2. Let Z_g and $Z_{g'}$ be the subsets given in Definition 5.1 on p. 101. Let u_1 and u_2 be the given functions in Definition 5.7:

$$U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in L^2(J; Z_g \times Z_{g'}), \ U_0 = \begin{pmatrix} u_{1,0} \\ u_{2,0} \end{pmatrix} \in Z_g \times Z_{g'}, \ F = \begin{pmatrix} 0 \\ f \end{pmatrix}, \tag{5.23}$$

for f, $u_{1,0}$, and $u_{2,0}$ to be the same data given in (5.12) on p.108 accordingly, and the following form $\hat{a}(U, V)$:

$$\int_J \hat{a}(U, V) dt = \int_J \{-a(u_2, v_1) + a(u_1, v_2)\} dt, \quad \forall V \in L^2(J; Z_0 \times Z_0), \tag{5.24}$$

where $a(\,,)$ is given in Definition 5.3 on p. 104.

Using the defined vector-valued functions and the form $\hat{a}(U, V)$ in Definition 5.8 then Problem 5.2.1 on p. 108 can be rewritten in terms of the new functions u_1 and u_2 given in Definition 5.7 in a vector form as follows

Problem 5.4.1 (First-order system in time variational formulation with linear constraints)
Let Z and H be the Hilbert spaces given in Definition 5.1 on p. 101. Let Z_g, $Z_{g'}$, and Z_0 be the subsets and subspace given in Definition 5.2 on p. 102. Let $D^{A,tr}$ be the normed space given in Definition 5.2 on p. 102. Let $\hat{a}(\,,)$ be the skew-symmetric form given in Definition 5.8.
Find $U \in L^2(J; Z_g \times Z_{g'})$ with $\partial_t U \in L^2(J; Z_{g'} \times H)$:

$$\int_J \{a(\partial_t u_1, v_1) + \langle \partial_t u_2, v_2 \rangle_{Z' \times Z} + \hat{a}(U, V)\} dt = \int_J (f, v_2)_H dt, \ \forall V \in L^2(J; Z_0 \times Z_0), \tag{5.25}$$

for given $f \in H^1(J; H)$, initial data $U|_{t=0} = \begin{pmatrix} u_{1,0} \\ u_{2,0} \end{pmatrix} \in D_g \times Z_{g'} \subset Z \times Z$, and smooth enough data $G = \binom{g_1}{g_2} \in W^{tr} = H^1(J; D^{A,tr} \times D^{A,tr})$.

Lemma 5.8 (Equivalence between variational formulations with linear constraints)
Problem 5.4.1 on p. 112 and Problem 5.2.1 on p. 108 are equivalent.

Proof:
Firstly, starting with Problem 5.2.1, knowing that Z_g, and $Z_{g'}$ are subsets of Z, and using the introduced functions in (5.22) in Definition 5.7:

$$u_1 := u \in L^2(J;Z_g) \subset L^2(J;Z), \partial_t u_1 = \partial_t u \in L^2(J;Z_{g'}) \subset L^2(J;Z), \text{ and}$$
$$u_2 := \partial_t u_1 \in L^2(J;Z_{g'}) \subset L^2(J;Z) \text{ with } \partial_t u_2 = \partial_t^2 u \in L^2(J;H) \subset L^2(J;Z').$$

Letting $v := v_2 \in L^2(J;Z_0)$ in (5.12) on p. 108 and for $v_1 \in L^2(J;Z_0)$, with using the Hermitian form $a : Z \times Z \to \mathbb{C}$ given in Definition 5.3 on p. 104, then

$$\int_J \{a(\partial_t u_1, v_1) - a(u_2, v_1)\}dt = 0$$
$$\int_J \{\langle \overbrace{\partial_t^2 u}^{\partial_t u_2 =}, v_2\rangle_{Z'\times Z} + a(\overbrace{u}^{u_1 =}, v_2)\}dt = \int_J \langle f, v_2\rangle_{Z'\times Z} dt, \ \forall v_1, v_2 \in L^2(J;Z_0),$$

for $u_{1,0} = u_0 \in D_g \subset Z$ and $u_{2,0} = \bar{u}_0 \in Z_{g'} \subset Z$ to be the compatible initial data with $G = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} := \begin{pmatrix} g \\ \partial_t g \end{pmatrix} \in W^{tr} = H^1(J; D^{A,tr} \times D^{A,tr})$ to be the smooth data at the trace space. Coupling the two equation with using the vector-valued functions in (5.23) and the skew-symmetric form $\hat{a}(\,,)$ in (5.24), respectively in Definition 5.8 on p. 112, conclude that

$$\text{Problem 5.2.1} \Longrightarrow \text{Problem 5.4.1}.$$

Secondly, starting now with Problem 5.4.1 that is

$$\int_J \{a(\partial_t u_1, v_1) + \langle \partial_t u_2, v_2\rangle_{Z'\times Z} + \hat{a}(U,V)\}dt = \int_J \langle f, v_2\rangle_{Z'\times Z} dt, \ \forall V \in L^2(J; Z_0 \times Z_0),$$

where

$$\int_J \hat{a}(U,V)dt = \int_J \{-a(u_2,v_1) + a(u_1,v_2)\}dt, \quad \forall V \in L^2(J;Z_0 \times Z_0),$$

and writing it in component-wise:

$$\begin{aligned} &\int_J a(\partial_t u_1 - u_2, v_1)dt = 0, \\ &\int_J \{\langle \partial_t u_2, v_2\rangle_{Z'\times Z} + a(u_1, v_2)\}dt = \int_J \langle f, v_2\rangle_{Z'\times Z} dt, \ \forall v_2 \in L^2(J;Z_0). \end{aligned} \tag{5.26}$$

Now, the first equation in the system in (5.26), since $\partial_t u_1 - u_2 \in L^2(J;Z_0)$ and from referring to Corollary A.10 on p. 179 for all $v_1 \in L^2(J;Z_0)$ and from Definition 5.3 on p. 104 which states that $a(\,,)$ is elliptic in Z_0 i.e. the operator A is invertible, then

$$0 = \int_J a(\partial_t u_1 - u_2, v_1)dt := \int_J \langle A(\partial_t u_1 - u_2), v_1\rangle_{Z'\times Z} dt \Longrightarrow \partial_t u_1 = u_2. \tag{5.27}$$

With $u_2 = \partial_t u_1 \in L^2(J;Z_{g'}) \subset L^2(J;Z)$, then for $u_1 = u \in L^2(J;Z_g) \subset L^2(J;Z)$ and $v_2 = v \in L^2(J;Z_0)$ conclude

$$\int_J \{\langle \partial_t^2 u, v\rangle_{Z'\times Z} + a(u,v)\}dt = \int_J \langle f, v\rangle_{Z'\times Z} dt, \ \forall v \in L^2(J;Z_0).$$

Thus

$$\text{Problem 5.4.1} \Longrightarrow \text{Problem 5.2.1},$$

and that completes the proof. □

5.4.1 Existence and uniqueness

With the use of the splitting technique i.e. the solution of Problem 5.4.1 on p. 112 is defined by two functions and we should show that these two functions do exist and are unique. Then $U \in L^2(J; Z_g \times Z_{g'})$ the solution of Problem 5.4.1 on p. 112 is defined by,

$$U := \hat{G} + \bar{U}. \tag{5.28}$$

Here, in (5.28) $\hat{G} \in W_G$ is the uniquely existing vector-valued function which solves the auxiliary problem (5.13) on p. 109 as shown in Lemma 5.1 on p. 109 the real vector-valued function $\bar{U}$ such that $\bar{U} \in L^2(J; Z_0 \times Z_0)$ with $\partial_t \bar{U} \in L^2(J; Z_0 \times H)$. Then the formulation in (5.25) on p. 112 can be rewritten in a vector-valued form in terms of $\bar{U}$ and $\hat{G}$ as follows

Find $\bar{U} \in L^2(J; Z_0 \times Z_0)$ with $\partial_t \bar{U} \in L^2(J; Z_0 \times H)$:

$$\begin{aligned}\int_J \{a(\partial_t \bar{u}_1, v_1) + \langle \partial_t \bar{u}_2, v_2 \rangle_{Z' \times Z} + \hat{a}(\bar{U}, V)\} dt = \int_J \big\{ & \langle f, v_2 \rangle_{Z' \times Z} - a(\partial_t \hat{g}_1, v_1) - \langle \partial_t \hat{g}_2, v_2 \rangle_{Z' \times Z} \\ & - \hat{a}(\hat{G}, V) \big\} dt, \qquad \forall V \in L^2(J; Z_0 \times Z_0),\end{aligned} \tag{5.29}$$

for given initial data

$$\bar{U}|_0 = \begin{pmatrix} \bar{u}_{1,0} \\ \bar{u}_{2,0} \end{pmatrix} = \bar{U}|_{t=0} = \begin{pmatrix} u_{1,0} \\ u_{2,0} \end{pmatrix} - \begin{pmatrix} \hat{g}_{1,0} \\ \hat{g}_{2,0} \end{pmatrix} \in D_0 \times Z_0 \subset Z_0 \times Z_0, \tag{5.30}$$

and a forcing data $f \in H^1(J; H)$ where $\hat{G}_0 = \left(\begin{smallmatrix} \hat{g}_{1,0} \\ \hat{g}_{2,0} \end{smallmatrix}\right) \in D_g \times D_{g'} \subset Z_g \times Z_{g'}$ is well defined initial data of the elliptic auxiliary problem in (5.13) on p. 109.
Here, with using Definition 5.3 on p. 104 and (5.24) on p. 112:

$$\begin{aligned} a(u, v) &= \langle Au, v \rangle_{Z' \times Z}, \\ \int_J \hat{a}(U, V) dt &= \int_J \{-a(u_2, v_1) + a(u_1, v_2)\} dt. \end{aligned}$$

Now, from Remark 5.1 on p. 103 such that D_0 is a subspace of Z_0, then with the regularity of the existing vector-valued function $\hat{G}$ shown in Theorem 5.1 on p. 109:

$$\partial_t \hat{g}_1 \in L^2(J; D_{g'}) \text{ and } \hat{g}_2 \in L^2(J; D_{g'}) \Longrightarrow \partial_t \hat{g}_1 - \hat{g}_2 \in L^2(J; D_0) \subset L^2(J; Z_0), \tag{5.31}$$

also with $(Z_0 \subset Z \subset H \subset Z' \subset Z_0')$:

$$\begin{aligned} & \partial_t \hat{g}_2 \in L^2(J; H) \text{ and } \hat{g}_1 \in L^2(J; D_g) \subset L^2(J; D^A) \\ \Longrightarrow & - \partial_t \hat{g}_2 - A\hat{g}_1 \in L^2(J; H) \subset L^2(J; Z_0'), \end{aligned} \tag{5.32}$$

then in the $r.h.s$ of (5.29) we have

$$\begin{aligned} r.h.s &= \int_J \{(f, v_2)_H - a(\partial_t \hat{g}_1, v_1) - \langle \partial_t \hat{g}_2, v_2 \rangle_{Z' \times Z} + a(\hat{g}_2, v_1) - \langle A\hat{g}_1, v_2 \rangle_{Z' \times Z}\} dt \\ &= \int_J a(-\partial_t \hat{g}_1 + \hat{g}_2, v_1) + (f - \partial_t \hat{g}_2 - A\hat{g}_1, v_2)_H dt \end{aligned} \tag{5.33}$$

where from (5.31) and (5.32) with the regularity of $f \in L^2(J; H)$, then

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} -\partial_t \hat{g}_1 + \hat{g}_2 \\ f - \partial_t \hat{g}_2 - A\hat{g}_1 \end{pmatrix} \in L^2(J; D_0 \times H) \subset L^2(J; Z_0 \times H), \tag{5.34}$$

and with defining the linear functional $\mathcal{F} : L^2(J; Z_0 \times H) \to \mathbb{C}$ such that $\mathcal{F} \in (L^2(J; Z_0 \times H))' \equiv L^2(J; Z_0' \times H)$ where $H \equiv H'$ from the (Gelfand triple) and $(L^2)' \equiv L^2$ since it is pivot:

$$\mathcal{F}(V) := \int_J \{\overbrace{a(f_1, v_1) + (f_2, v_2)_H}^{(F,V)_E}\} dt,\ \forall V \in L^2(J; Z_0 \times H). \tag{5.35}$$

Then with (5.35), the formulation of (5.29) on p. 114 would now read

Problem 5.4.2 (The transformed formulation)
Let Z and H be the Hilbert spaces given in Definition 5.1 on p. 101. Let Z_0 be the Hilbert subspace of Z given in Definition 5.2 on p. 102 with Remark 5.1 on p. 103. Let $\hat{a}(\,,\,)$ be the skew-symmetric form given in Definition 2.7 on p. 14 and $(\,,\,)_E$ be the inner product given in Corollary 2.2 on p. 10:

$$(U, V)_E = a(u_1, v_1) + (u_2, v_2)_E,$$
$$\int_J \hat{a}(U, V) dt = \int_J \{-a(u_2, v_1) + a(u_1, v_2)\} dt,$$

and a linear functional $\mathcal{F} \in L^2(J; Z_0' \times H)$ given in (5.35).
Find $\bar{U} \in L^2(J; Z_0 \times Z_0)$ with $\partial_t \bar{U} \in L^2(J; Z_0 \times H)$:

$$\overbrace{\int_J \{a(\partial_t \bar{u}_1, v_1) + \langle \partial_t \bar{u}_2, v_2 \rangle_{Z' \times Z} + \hat{a}(\bar{U}, V)\} \, dt}^{B(\bar{U},V)=} = \mathcal{F}(V),\ \ \forall V \in L^2(J; Z_0 \times Z_0), \tag{5.36}$$

for given initial data $\bar{U}|_0 = \begin{pmatrix} \bar{u}_{1,0} \\ \bar{u}_{2,0} \end{pmatrix} \in D_0 \times Z_0 \subset Z_0 \times Z_0$.

Theorem 5.2 (Existence, uniqueness of variational formulation with linear constraints)
Let Z, H, and D^A be the Hilbert spaces given in Definition 5.1 on p. 101 with Assumption 5.1 on p. 101. Let Z_g and Z_0 be the subset and the subspace of the Hilbert space Z, respectively, given in Definition 5.2 on p. 102 with Remark 5.1 on p. 103. Let W_G be the subset and the subspace of $W = H^1(J; D^A \times D^A)$, respectively, given in Definition 5.6 on p. 107. There exists unique real vector-valued solution $U \in L^2(J; Z_g \times Z_{g'})$ for Problem 5.4.1 on p. 112.

Proof:
Starting with the identity (5.28) on p. 114 that is

$$U = \hat{G} + \bar{U}.$$

Firstly, from Theorem 5.1 on p. 109 the vector-valued function $\hat{G} \in W_G$ exists and is unique.
Secondly, since the separable Hilbert space Z_0 satisfies all the properties of the considered space in the shown theory in Chapter 4 as shown in Lemma 5.1 on p. 103, also D_0 satisfy the given property given in Definition 3.13 on p. 60 where D_0 is a subspace of Z_0, and the forcing data in (5.34) defined by the functional in (5.35) on p. 115 are then considered as a particular choice of the general one given in Chapter 4 i.e. Problem 5.4.2 satisfies all proven properties of Problem 4.1.1 on p. 77 where now

$$F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} -\partial_t \hat{g}_1 + \hat{g}_2 \\ f - \partial_t \hat{g}_2 - A\hat{g}_1 \end{pmatrix} \in L^2(J; D_0 \times H) \subset L^2(J; Z_0 \times H), \tag{5.37}$$

and Lemma 4.2 on p. 80, implies $\bar{U} \in L^2(J; Z_0 \times Z_0)$; the solution of Problem 5.4.2, does exist and is unique. Thus, the vector-valued function $U \in L^2(J; Z_g \times Z_{g'}) \subset L^2(J; Z \times Z)$ which solves Problem 5.4.1 also exists and is unique. □

5.4.2 Stability estimate of first-order in time variational formulation with linear constraints

Theorem 5.3 (Global stability estimate of continuous formulation with linear constraints) *Let Z and H be the Hilbert spaces given in Definition 5.1 on p. 101. Let Z_g be the subset of the Hilbert space Z given in Definition 5.2 on p. 102. Let W be the Hilbert space given in Definition 5.4 on p. 104 and $W^{tr} = H^1(J; D^{A,tr} \times D^{A,tr})$ the normed-product space. With given forcing data $f \in H^1(J;H)$, compatible initial data $U_0 \in Z_g \times Z_{g'}$ and linear constraints $G \in W^{tr}$, then the real vector-valued function $U \in L^2(J; Z_g \times Z_{g'})$ which solves Problem 5.4.1 on p. 112 for $X = Z \times H$ satisfies*

$$\| U \|_{L^2(J;X)} \leq 2(4T\sqrt{C \max(M, c^2, 1)} + \sqrt{\max(1, c^2)}) \| G \|_{W^{tr}} + \frac{T\sqrt{2C}}{\sqrt{\min(\alpha, 1)}} \| f \|_{L^2(J;H)} + \frac{T\sqrt{C}}{\sqrt{\min(\alpha, 1)}} (\| U_0 \|_E + \sqrt{\max(c^2, M)} \| G_0 \|_{D^{A,tr} \times D^{A,tr}})), \tag{5.38}$$

for $T < \infty$ and C is a generic constant which does not depend on T. $M < \infty$ is the lower bound of the form $a(\,,\,)$ which satisfies the properties given in Definition 5.3 on p. 104. c is the constant in the inclusion inequality $\| u \|_H \leq c \| u \|_Z, \forall u \in Z$, and the norm $\| \cdot \|_{W^{tr}}$ given in Corollary 5.5 on p. 107.

Proof:
Starting with the identity in (5.28) on p. 114, using *triangle inequality*

$$\| U \|_{L^2(J;X)} \leq \| \bar{U} \|_{L^2(J;X)} + \| \hat{G} \|_{L^2(J;X)},$$

and substituting the resulting stability estimate in (4.5) on p. 78 in the proof of Lemma 4.2 and using Lemma 5.6 on p. 108 and the inequality between the E and the X norms:

$$\| \bar{U} \|_{L^2(J;X)} \leq \frac{1}{\sqrt{\min(\alpha, 1)}} \| \bar{U} \|_{L^2(J;E)} \leq \frac{T\sqrt{C}}{\sqrt{\min(\alpha, 1)}} (\| F \|_{L^2(J;E)} + \| \bar{U}_0 \|_E),$$

and

$$\| \hat{G} \|_{L^2(J;X)} \leq \sqrt{\max(1, c^2)} \| \hat{G} \|_{L^2(J;D^A \times D^A)},$$

which yield

$$\| U \|_{L^2(J;X)} \leq \frac{T\sqrt{C}}{\sqrt{\min(\alpha, 1)}} (\| F \|_{L^2(J;E)} + \| \bar{U}_0 \|_E) + \sqrt{\max(1, c^2)} \underbrace{\| \hat{G} \|_{L^2(J;D^A \times D^A)}}_{\leq \|\hat{G}\|_W}. \tag{5.39}$$

Here, with the use of the *Bochner integral* E-norm definition, the definitions in (5.35) on p. 115, *Cauchy* and *Young's* inequalities, the inclusion inequality $\| u \|_H \leq c \| u \|_Z, \forall u \in Z$, the norm of the

Hilbert space $(D^A, \|\cdot\|_{D^A})$, and Lemma A.3 on p. 181 twice, then

$$\begin{aligned}
\| F \|^2_{L^2(J;E)} = \| \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \|^2_{L^2(J;E)} &= \int_J \{a(\partial_t \hat{g}_1 + \hat{g}_2, \partial_t \hat{g}_1 + \hat{g}_2) + \| f + \partial_t \hat{g}_2 + A\hat{g}_1 \|^2_H\} dt \\
&\leq 2M \| \partial_t \hat{g}_1 \|^2_{L^2(J;Z)} + 2M \| \hat{g}_2 \|^2_{L^2(J;Z)} + 2 \| f \|^2_{L^2(J;H)} \\
&\qquad + 4c^2 \| \partial_t \hat{g}_2 \|^2_{L^2(J;Z)} + 4 \| A\hat{g}_1 \|^2_{L^2(J;H)} \\
&\leq 2M\big(\| \partial_t \hat{g}_1 \|^2_{L^2(J;Z)} + \| A\partial_t \hat{g}_1 \|^2_{L^2(J;H)} \big) + 4\max(M, c^2) \| \hat{g}_2 \|^2_{H^1(J;Z)} \\
&\qquad + 4 \| A\hat{g}_1 \|^2_{L^2(J;H)} + 4 \| \hat{g}_1 \|^2_{L^2(J;Z)} + 2 \| f \|^2_{L^2(J;H)} \\
&= 2M \| \partial_t \hat{g}_1 \|^2_{L^2(J;D^A)} + 4\max(M, c^2) \| \hat{g}_2 \|^2_{H^1(J;Z)} + 4 \| \hat{g}_1 \|^2_{L^2(J;D^A)} \\
&\qquad + 2 \| f \|^2_{L^2(J;H)} \\
&\leq 4\max(M, 1) \| \hat{g}_1 \|^2_{H^1(J;D^A)} + 4\max(M, c^2) \| \hat{g}_2 \|^2_{H^1(J;D^A)} \\
&\qquad + 2 \| f \|^2_{L^2(J;H)} \\
&\leq 4\max(M, c^2, 1) \| \hat{G} \|^2_{H^1(J;D^A \times D^A)} + 2 \| f \|^2_{L^2(J;H)} \\
&= 2\Big(2\max(M, c^2, 1) \| \hat{G} \|^2_W + \| f \|^2_{L^2(J;H)} \Big),
\end{aligned} \tag{5.40}$$

then plugging this estimate back in (5.39) and with the use of the stability of the $\hat{G}$ function given in Lemma 5.7 on p. 110 yields

$$\begin{aligned}
\| U \|_{L^2(J;X)} \leq 2(4T\sqrt{C\max(M, c^2, 1)} + \sqrt{\max(1, c^2)}) \| G \|_{W^{tr}} + \frac{T\sqrt{2C}}{\sqrt{\min(\alpha, 1)}} \| f \|_{L^2(J;H)} \\
+ \frac{T\sqrt{C}}{\sqrt{\min(\alpha, 1)}} \| \bar{U}_0 \|_E .
\end{aligned}$$

Here, from (5.30) on p. 114 for compatible initial data at the trace, the *triangle inequality*, and using the estimate in Lemma 5.6 on p. 108 but again this time with considering the correspondent norm of the Hilbert space D^A in Definition 5.1 on p. 101, then

$$\| \bar{U}_0 \|_E = \| U_0 - \hat{G}_0 \|_E \leq \| U_0 \|_E + \| \hat{G}_0 \|_E \leq \| U_0 \|_E + \sqrt{\max(c^2, M)} \| \hat{G}_0 \|_{D^A \times D^A},$$

and with using the norm of the space $D^{A,tr}$ given in Definition 5.2 on p. 102 after taking the infimum of $\hat{G}_0 \in D^A \times D^A$ in both sides where $\left(\begin{smallmatrix} \gamma & 0 \\ 0 & \gamma \end{smallmatrix}\right)\hat{G}_0 = G_0$, imply

$$\inf_{\substack{\hat{G}_0 \in D^A \times D^A \\ \left(\begin{smallmatrix} \gamma & 0 \\ 0 & \gamma \end{smallmatrix}\right)\hat{G}_0 = G_0}} \| \hat{G}_0 \|_{D^A \times D^A} = \| G_0 \|_{D^{A,tr} \times D^{A,tr}},$$

and this means

$$\| \bar{U}_0 \|_E \leq \| U_0 \|_E + \sqrt{\max(c^2, M)} \| G_0 \|_{D^{A,tr} \times D^{A,tr}}, \tag{5.41}$$

and that completes the proof. □

Lemma 5.9 (Last left-sided limit of the continuous transformed formulation) *Let Z and H be the Hilbert spaces given in Definition 5.1 on p. 101. Let Z_0 be the subspace of the Hilbert space Z given in Definition 5.2 on p. 102 with Remark 5.1 on p. 103. Let W be the Hilbert space given in Definition 5.4 on p. 104 and $W^{tr} = H^1(J; D^{A,tr} \times D^{A,tr})$ the normed-product space. With given forcing data $f \in H^1(J; H)$, compatible initial data $U_0 \in Z_g \times Z_{g'}$, and linear constraints $G \in W^{tr}$, then $\bar{U} \in L^2(J; Z_0 \times Z_0)$ the real solution of Problem 5.4.2 on p. 115 for $X = Z \times H$ satisfies*

$$\| \bar{U}_-^N \|_E \leq \sqrt{TC}\big(4\sqrt{\max(M, c^2, 1)} \| G \|_{W^{tr}} + \sqrt{2} \| f \|^2_{L^2(J;H)} + \| U_0 \|_E + \sqrt{\max(c^2, M)} \| G_0 \|_{D^{A,tr} \times D^{A,tr}} \big), \tag{5.42}$$

where C is a generic constant which does not depend on T. $M < \infty$ is the lower bound of the form $a(\,,\,)$ which satisfies the properties given in Definition 5.3 on p. 104. c is the constant in the inclusion inequality $\| u \|_H \leq c \| u \|_Z, \forall u \in Z$, and the norm $\| \cdot \|_{W^{tr}}$ is given in Corollary 5.5 on p. 107.

Proof:
Starting with choosing $V = \bar{U} \in L^2(J; Z_0 \times Z_0)$, which is real, in Problem 5.4.2 on p. 115 and recalling the proof of Lemma 2.5 on p. 16 but this time with $V \in L^2(J; Z_0 \times Z_0)$:

$$\begin{aligned} \int_J \{a(\partial_t \bar{u}_1, \bar{u}_1) + \langle \partial_t \bar{u}_2, \bar{u}_2 \rangle_{Z' \times Z} + \overbrace{\hat{a}(\bar{U}, \bar{U})}^{=0}\}\, dt &= \int_J (F, \bar{U})_E dt, \\ \Longrightarrow \frac{1}{2} \int_0^T \frac{d}{dt} \| \bar{U} \|^2_E \, dt &\leq \sqrt{T} \| F \|_{L^2(J;E)} \frac{1}{\sqrt{T}} \| \bar{U} \|_{L^2(J;E)} \\ &\leq \frac{1}{2}\big(T \| F \|^2_{L^2(J;E)} + \frac{1}{T} \| \bar{U} \|^2_{L^2(J;E)} \big). \end{aligned} \tag{5.43}$$

Now, with considering that the time interval $J = (0, T)$ can be partitioned into a time mesh of N time steps so that $J = \cup_{n=0}^{N-1} I_n$, where

$$I_n = (t_n, t_{n+1}), \quad 0 \leq n \leq N-1,$$

with nodes $0 =: t_0 < t_1 \ldots < t_N := T$. With recalling the left-right sided limits at such nodes in Definition 3.3 on p. 20 so that the vector-valued function $\bar{U}$ at the initial and last time nodes i.e. at $0 := t_0$ and $T := t_N$, respectively is given by

$$U_0 := U_-^0 = \lim_{s \to 0, s>0} U(t_0 - s) =: \lim_{s \to 0, s>0} U(0 - s), \text{ and}$$

$$U(T) := U_-^N = \lim_{s \to 0, s>0} U(t_N - s) =: \lim_{s \to 0, s>0} U(T - s), \text{ respectively.}$$

Thus, with also using the *Gronwall's inequality* given in Theorem A.8 on p. 182 imply

$$\| \bar{U}_-^N \|_E \leq \sqrt{TC}\big(\| F \|_{L^2(J;E)} + \| \bar{U}_0 \|_E \big).$$

Here, with recalling the estimates done in the proof of Theorem 5.3 on p. 116 to get the upper bound of the $\| F \|^2_{L^2(J;E)}$ norm in terms of the given data, using the stability of the $\hat{G}$ function given in Lemma 5.7 on p. 110, and the estimate in (5.41) on p. 117, then

$$\| \bar{U}_-^N \|_E \, dt \leq \sqrt{TC}\big(4\sqrt{\max(M, c^2, 1)} \| G \|_{W^{tr}} + \sqrt{2} \| f \|_{L^2(J;H)} + \| U_0 \|_E + \sqrt{\max(c^2, M)} \| G_0 \|_{D^{A,tr} \times D^{A,tr}} \big),$$

and that completes the proof. □

Theorem 5.4 (Last left-sided limit of the continuous formulation with linear constraints) *Let Z and H be the Hilbert spaces given in Definition 5.1 on p. 101. Let Z_g be the subset of the Hilbert space Z given in Definition 5.2 on p. 102. Let W be the Hilbert space and $W^{tr} = H^1(J; D^{A,tr} \times D^{A,tr})$ the normed-product space given in Definition 5.4 on p. 104. With given forcing data $f \in H^1(J; H)$, compatible initial data $U_0 \in Z_g \times Z_{g'}$ and $G \in W^{tr}$, then $U \in L^2(J; Z_g \times Z_{g'})$ the solution of Problem 5.4.1 on p. 112 for $X = Z \times H$ satisfies*

$$\| U_-^N \|_X \leq \frac{\sqrt{TC}}{\sqrt{\min(\alpha,1)}} \big(4\sqrt{\max(M,c^2,1)} \| G \|_{W^{tr}} + \sqrt{2} \| f \|_{L^2(J;H)} + \| U_0 \|_E + \sqrt{\max(c^2,M)} \| G_0 \|_{D^{A,tr}\times D^{A,tr}} \big) + \sqrt{\max(M,c^2)} \| G_-^N \|_{D^{A,tr}\times D^{A,tr}},$$

where C is a generic constant which does not depend on T. $M < \infty$ the lower bounds of the form $a(,)$ which satisfies the properties given in Definition 5.3 on p. 104. c the constant in the inclusion inequality $\| u \|_H \leq c \| u \|_Z, \forall u \in Z$, and the norm $\| \cdot \|_{W^{tr}}$ given in Corollary 5.5 on p. 107.

Proof:
Starting with the identity in (5.28) on p. 114, using *triangle* then the inequality in Lemma 5.6 for the corresponding D^A Hilbert space given in Definition 6.3 on p. 133 would be

$$\| U_-^N \|_X = \| \bar{U}_-^N + \hat{G}_-^N \|_X \leq \| \bar{U}_-^N \|_X + \| \hat{G}_-^N \|_X \leq \frac{1}{\sqrt{\min(\alpha,1)}} \| \bar{U}_-^N \|_E + \sqrt{\max(1,c^2)} \| \hat{G}_-^N \|_{D^A\times D^A},$$

now with using the estimates in Lemma 5.9 on p. 118 and the estimate in (5.41) on p. 117, then

$$\| U_-^N \|_E \leq \frac{\sqrt{TC}}{\sqrt{\min(\alpha,1)}} \big(4\sqrt{\max(M,c^2,1)} \| G \|_{W^{tr}} + \sqrt{2} \| f \|_{L^2(J;H)} + \| U_0 \|_E + \sqrt{\max(c^2,M)} \| G_0 \|_{D^{A,tr}\times D^{A,tr}} \big) + \sqrt{\max(1,c^2)} \| \hat{G}_-^N \|_{D^A\times D^A},$$

here, with using the norms of the spaces $D^{A,tr}$, and Z^{tr} in Definition 5.2 on p. 102 after taking the infimum of $\hat{G}_-^N \in D^A \times Z$ in both sides where $\left(\begin{smallmatrix}\gamma & 0\\ 0 & \gamma\end{smallmatrix}\right)\hat{G}_-^N = G_-^N$, imply

$$\inf_{\substack{\hat{G}_-^N \in D^A\times D^A \\ \left(\begin{smallmatrix}\gamma & 0\\ 0 & \gamma\end{smallmatrix}\right)\hat{G}_-^N = G_-^N}} \| \hat{G}_-^N \|_{D^A\times D^A} = \| G_-^N \|_{D^{A,tr}\times D^{A,tr}}, \tag{5.44}$$

and that completes the proof. □

Chapter 6

High-order in time DGFEM for first-order in time variational formulation with linear constraints

In this chapter, Problem 5.4.1 on p. 112 in chapter 5 is going to be discretised in time using High-order in time DGFEM with the same notations used in Part I, particularly in Chapter 3. Then, a conformal spatial discretisation will be used to get the fully-discrete formulation. The same splitting technique used in Chapter 5 is going to be used again here to show existence, uniqueness, and stability estimates. The a priori error estimates will be proven with again using the projection method used in Chapter 3 but this time with a particular spatial projection operator.

6.1 The semi-discrete continuous auxiliary (elliptic) problem

In Section 6.1.1, existence and uniqueness of the solution of the semi-discrete auxiliary (elliptic) problem (6.2) on p.121 will be shown in Theorem 6.1 on p. 122. The stability estimate of the solution of (6.2) is shown in Lemma 6.1 on p. 122. Theorem 6.1 and Lemma 6.1 will be used in order to show existence, uniqueness and stability estimate of the solution of the semi-discrete variational formulation with given linear constraints.

Definition 6.1 *Let Z and D^A be the Hilbert spaces given in Definition 5.1 with Assumption 5.1 on p. 101. Let $D^{A,tr}$ be the normed-space given in Definition 5.2 on p. 102.*
Let $W^{tr} = \bar{\gamma}[W] = H^1(J; D^{A,tr} \times D^{A,tr})$ be the normed-product space with the norm $\| \cdot \|_{W^{tr}}$ given in Corollary 5.5 on p. 107, where $\bar{\gamma} : H^1(J; D^A \times D^A) \to H^1(J; D^{A,tr} \times D^{A,tr})$ is time independent surjective matrix operator. Let W_G, and $W_{\mathbf{0}}$ be the subset and subspace given in Definitions 5.4 and 5.6 on p. 104 and 107. Let $J = \cup_{n=0}^{N-1} I_n$:

$$I_n = (t_n, t_{n+1}), \quad 0 \le n \le N-1$$

fff with nodes $0 =: t_0 < t_1 \ldots < t_N := T$, and associating with each time interval I_n an approximation order $r_n \ge 0$, with storing these temporal orders in the vector $\underline{r} := \{r_n\}_{n=0}^{N-1}$. The following conformal semi-discrete finite dimensional spaces are defined

$$W^{\underline{r}} := \Big\{ U | U \in H^1(J; D^A \times D^A) : U|_{I_n} \in P^{r_n}(I_n; D^A \times D^A), 0 \le n \le N-1 \Big\},$$

which is the conformal semi-discrete in time finite-dimensional subspace of $W = H^1(J; D^A \times D^A)$*, and*

$$W^{\underline{r},tr} := \Big\{U | U \in W^{tr} = H^1(J; D^{A,tr} \times D^{A,tr}) : U|_{I_n} \in P^{r_n}(I_n; D^{A,tr} \times D^{A,tr}), 0 \leq n \leq N-1\Big\},$$

with the defined norm $\| \cdot \|_{W^{tr}}$ *for* $\bar{\gamma} : W^{\underline{r}} \to W^{\underline{r},tr}$*:*

$$\| G_{CG} \|_{W^{tr}} = \inf_{\substack{U \in W^{\underline{r}} \\ \bar{\gamma} U = G_{CG}}} \| U \|_W, \quad \forall G_{CG} \in W^{\underline{r},tr},$$

which is a subspace of the normed-space $(W^{tr}, \| \cdot \|_{W^{tr}})$*. Moreover, for semi-discrete vector-valued function* $G_{CG} = \binom{g_{1,CG}}{g_{2,CG}} = \binom{g_{CG}}{g'_{CG}} \in W^{\underline{r},tr}$ *with*

$$\partial_t g_{CG} = g'_{CG}. \tag{6.1}$$

$$W^{\underline{r}}_{G_{CG}} := \Big\{U | U \in W^{\underline{r}} \text{ with } \bar{\gamma} U = G_{CG} \text{ and } \partial_t u_1 \in L^2(J; D_{g'_{CG}}), \partial_t u_2 \in L^2(J; D^A)\Big\},$$

which is the conformal semi-discrete in time finite-dimensional subset of W_G*, and*

$$W^{\underline{r}}_{\mathbf{0}} := \Big\{U | U \in W^{\underline{r}} \text{ with } \bar{\gamma} U = \mathbf{0}\Big\},$$

which is the conformal semi-discrete in time finite-dimensional subspace of the Hilbert space $(W_{\mathbf{0}}, \| \cdot \|_W)$.

Now, since $\hat{G} \in W = H^1(J; D^A \times D^A)$ (the continuous solution of the formulation of the auxiliary (elliptic) problem (5.13) given in Section 5.3 starting on p. 109) i.e.

$$\hat{G} \in C([0,T]; D^A \times D^A).$$

Thus, it is then considered to be still continuous at every time node $0 =: t_0 < t_1 \ldots < t_N := T$, then the semi-discrete formulation in time for (5.13) on p. 109 would read

Problem 6.1.1 (Semi-discrete auxiliary (elliptic) problem)
Let $W^{\underline{r}}_{G_{CG}}$ *be the semi-discrete finite dimensional subset given in Definition 6.1 on p. 120. Let* $(\,,)_W$ *be the inner product given in Lemma 5.2 on p. 104:*

$$(U,V)_W := \int_J \{(\partial_t U, \partial_t V)_{D^A \times D^A} + (U,V)_{D^A \times D^A}\} dt = (U,V)_{H^1(J; D^A \times D^A)}, \ \forall U, V \in W.$$

Find $\hat{G}_{CG} \in W^{\underline{r}}_{G_{CG}}$*:*

$$(\hat{G}_{CG}, V)_W = 0, \quad \forall V \in W^{\underline{r}}_{\mathbf{0}}, \tag{6.2}$$

where $\hat{G}_{CG} \in W \subset C([0,T]; D^A \times D^A)$ *and*

$$\partial_t \hat{g}_1 \in L^2(J; D_{g'_{CG}}), \text{ and } \hat{g}_2 \in L^2(J; D_{g'_{CG}}). \tag{6.3}$$

6.1.1 Existence, uniqueness and stability estimate of the semi-discrete auxiliary (elliptic) problem

Theorem 6.1 (Existence, uniqueness of semi-discrete auxiliary (elliptic) problem) *Let D^A and Z be the Hilbert spaces given in Definition 5.1 on p. 101. Let $W^{\underline{r}}_{\hat{G}_{CG}}$ be the finite dimensional subspace of the Hilbert space $W = H^1(J; D^A \times D^A)$ given in Definition 6.1 on p. 120. There exists unique solution $\hat{G}_{CG} \in W^{\underline{r}}_{\hat{G}_{CG}}$ for the formulation in (6.2) on p. 121.*

Proof:
Firstly, to show existence, let

$$\hat{G}_{CG} := \bar{G}_{CG} + \tilde{G}_{CG},$$

where $\bar{G}_{CG} \in W^{\underline{r}}$ is an existing vector-valued function which satisfies the data at the trace space and

$$\tilde{G}_{CG} \in W^{\underline{r}}_{\mathbf{0}}.$$

Then the formulation in (6.2) in terms of $\bar{G}_{CG}$ and $\tilde{G}_{CG}$ reads
Find $\tilde{G}_{CG} \in W^{\underline{r}}_{\mathbf{0}}$:

$$(\tilde{G}_{CG}, V)_W = -(\bar{G}_{CG}, V)_W, \qquad \forall V \in W^{\underline{r}}_{\mathbf{0}} \subset W_{\mathbf{0}}. \tag{6.4}$$

Now, since $W^{\underline{r}}_{\mathbf{0}}$ is a conformal finite-dimensional subspace of the Hilbert space $(W_{\mathbf{0}}, \| \cdot \|_W)$ i.e. it is a Hilbert space (see [51, Proof of Theorem 2.C.]) and with recalling the proof of Theorem 5.1 on p. 109 for $\tilde{G}_{CG} \in W^{\underline{r}}_{\mathbf{0}}$, thus, by Lax-Milligram theorem (see Theorem A.3 on p. 178), there exists unique solution $\tilde{G}_{CG} \in W^{\underline{r}}_{\mathbf{0}}$ which solves the formulation in (6.4).
Secondly, with the fact that

$$\hat{G}_{CG} := \bar{G}_{CG} + \tilde{G}_{CG} \tag{6.5}$$

and since $\tilde{G}_{CG}$ exists, the identity concludes that $\hat{G}_{CG} \in W^{\underline{r}}_{\hat{G}_{CG}}$ exists.
Finally, for the uniqueness, let $\hat{G}^1_{CG} = \begin{pmatrix} \hat{g}^1_1 \\ \hat{g}^1_2 \end{pmatrix} \in W^{\underline{r}}_{\hat{G}_{CG}}$ and $\hat{G}^2 = \begin{pmatrix} \hat{g}^2_1 \\ \hat{g}^2_2 \end{pmatrix} \in W^{\underline{r}}_{\hat{G}_{CG}}$ be two solution of (6.2), then let $\bar{W} := \hat{G}^1_{CG} - \hat{G}^2_{CG} \in W^{\underline{r}}_{\mathbf{0}}$ be the solution which solves:

$$(\bar{W}, V)_W = 0, \forall V \in W^{\underline{r}}_{\mathbf{0}},$$

and with choosing $V = \bar{W} \in W^{\underline{r}}_{\mathbf{0}}$, then

$$(\bar{W}, \bar{W})_W = \| \bar{W} \|^2_W \geq 0 \Longrightarrow \bar{W} = \hat{G}^1_{CG} - \hat{G}^2_{CG} = \mathbf{0},$$

which means that $\hat{G}^1_{CG} = \hat{G}^2_{CG}$, and that completes the proof. □

Lemma 6.1 (Stability estimate of the semi-discrete auxiliary (elliptic) problem) *Let $W^{\underline{r}}_{\hat{G}_{CG}}$ be the finite-dimensional subset of W given in Definition 6.1 on p. 120. Let $W^{\underline{r}}_{\mathbf{0}}$ be the finite-dimensional subspace of W given in Definition 6.1. Let $W^{\underline{r},tr}$ be the space given in Definition 6.1 with the norm*

$$\| G_{CG} \|_{W^{tr}} = \inf_{\substack{U \in W^{\underline{r}} \\ \bar{\gamma} U = G_{CG}}} \| U \|_W, \quad \forall G_{CG} \in W^{\underline{r},tr}.$$

The solution $\hat{G}_{CG} \in W^{\underline{r}}_{\hat{G}_{CG}}$ which solves the formulation in (6.2) on p. 121 satisfies

$$\| \hat{G}_{CG} \|_W \leq 2 \| G_{CG} \|_{W^{tr}} . \tag{6.6}$$

Proof:
Starting with the identity given in (6.5) on p. 122 and the use of the *triangle inequality* implies

$$\| \hat{G}_{CG} \|_W = \| \bar{G}_{CG} + \tilde{G}_{CG} \|_W \leq \| \bar{G}_{CG} \|_W + \| \tilde{G}_{CG} \|_W . \tag{6.7}$$

Now, to bound $\| \tilde{G}_{CG} \|_W$ is by finding the stability estimate of the solution the formulation (6.4). This starts with choosing $V = \tilde{G}_{DG} \in W_{\mathbf{0}}^{\underline{r}}$, in the formulation (6.4) and using the same steps in the proof of Lemma 5.7 on p. 110 then

$$\| \hat{G}_{CG} \|_W \leq 2 \inf_{\substack{\bar{G}_{CG} \in W^{\underline{r}} \\ \bar{\gamma}\bar{G}_{CG} = G_{CG}}} \| \bar{G}_{CG} \|_W = 2 \| G_{CG} \|_{W^{tr}}, \tag{6.8}$$

and that completes the proof. □

6.2 High-order in time DGFEM, semi-discrete formulation with linear constraints

Definition 6.2 *Let $\mathcal{N} = \{I_n\}_{n=0}^{N-1}$ be the partition of $J = (0, T)$ and associating with each time interval I_n an approximation order $r_n \geq 0$ with storing these temporal orders in the vector $\underline{r} := \{r_n\}_{n=0}^{N-1}$. With using the semi-discrete spaces given in Definition 3.1 on p. 19 and the finite dimensional subsets given in Definition 6.1 on p. 120 such that for any semi-discrete vector-valued function $G_{CG} = \binom{g_{1,CG}}{g_{2,CG}} = \binom{g_{CG}}{g'_{CG}} \in W^{\underline{r},tr}$ with*

$$\partial_t g_{CG} = g'_{CG}, \tag{6.9}$$

the new semi-discrete in time subset and subspace would be defined as

$$\mathcal{V}^{\underline{r}}(\mathcal{N}; Z_{g_{CG}} \times Z_{g'_{CG}}) = \Big\{ U : J \to Z_{g_{CG}} \times Z_{g'_{CG}} : U|_{I_n} \in P^{r_n}(I_n; Z_{g_{CG}} \times Z_{g'_{CG}}), 0 \leq n \leq N-1 \Big\},$$

and

$$\mathcal{V}^{\underline{r}}(\mathcal{N}; Z_0 \times Z_0) = \Big\{ U : J \to Z_0 \times Z_0 : U|_{I_n} \in P^{r_n}(I_n; Z_0 \times Z_0),\ 0 \leq n \leq N-1 \Big\}, \textit{respectively.}$$

The high-order in time DGFEM for Problem 5.4.1 on p. 112 reads:

Problem 6.2.1 (Semi-discrete formulation with linear constraints)
Let Z and H be the Hilbert spaces given in Definition 5.1 on p. 101. Let Z_g, $Z_{g'}$, and Z_0 be the subsets and subspace given in Definition 5.2 on p. 102. Let $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z_{g_{CG}} \times Z_{g'_{CG}})$ be the semi-discrete finite dimensional subset given in Definition 6.2 on p. 123. Let $\hat{a}(\,,)$ be the skew-symmetric form given in Definition 2.7 on p. 14 and $(\,,)_E$ be the inner product given in Corollary 2.2 on p. 10:

$$\hat{a}(U, V) = -a(u_2, v_1) + a(u_1, v_2), \text{ and } (U, V)_E = a(u_1, v_1) + (u_2, v_2)_E.$$

with the given form

$$B_{DG}(U_{DG}, V) = \sum_{n=0}^{N-1} \int_{I_n} \{(\partial_t U_{DG}, V)_E + \hat{a}(U_{DG}, V)\}\, dt + \sum_{n=1}^{N-1} ([U_{DG}]^n, V_+^n)_E + (U_{DG,+}^0, V_+^0)_E, \tag{6.10}$$

and with given forcing data $f \in H^1(J;H)$, the linear functional $F_{DG} \in (L^2(J;Z \times H))'$:

$$F_{DG}(V) = \sum_{n=0}^{N-1} \int_{I_n} (f, v_2)_H dt, \forall v_2 \in L^2(J;H), \tag{6.11}$$

where all semi-discrete test vector-valued functions in semi-discrete space $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z_0 \times Z_0)$ are also included in this linear map since $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z_0 \times Z_0) \subset L^2(J; Z \times H)$.
Find $U_{DG} \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z_{g_{CG}} \times Z_{g'_{CG}})$ such that

$$B_{DG}(U_{DG}, V) = F_{DG}(V) + (U^0_{DG,-}, V^0_+)_E, \quad \forall \ V \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z_0 \times Z_0). \tag{6.12}$$

For $U^n_{DG,-}$ to be the initial data at a given time step I_n, with the given compatible initial data $U^0_{DG,-} = U_0 \in D_{g_{CG}} \times Z_{g'_{CG}} \subset Z_{g_{CG}} \times Z_{g'_{CG}}$, and the semi-discrete data $G_{CG} \in W^{\underline{r},tr} \subset W^{tr} = H^1(J; D^{A,tr} \times D^{A,tr})$.

The time-stepping scheme for (6.12) would read:

Problem 6.2.2 (Semi-discrete time-stepping scheme with linear constraints)
Let Z and H be the Hilbert spaces given in Definition 5.1 on p. 101. Let Z_g, $Z_{g'}$, and Z_0 be the subsets and subspace given in Definition 5.2 on p. 102. Let $P^{r_n}(I_n; Z_{g_{CG}} \times Z_{g'_{CG}})$ be the semi-discrete finite dimensional subset given in Definition 6.2. Let $\hat{a}(\, ,)$ be the skew-symmetric form given in Definition 2.7 on p. 14 and $(\, ,)_E$ be the inner product given in Corollary 2.2 on p. 10:

$$\hat{a}(U,V) = -a(u_2, v_1) + a(u_1, v_2), \text{ and } (U,V)_E = a(u_1, v_1) + (u_2, v_2)_E.$$

Find $U_{DG} \in P^{r_n}(I_n; Z_{g_{CG}} \times Z_{g'_{CG}})$ such that

$$\overbrace{\int_{I_n} \Big\{ (\partial_t U_{DG}, V)_E + \hat{a}(U_{DG}, V) \Big\} dt + (U^n_{DG,+}, V^n_+)_E}^{b^n_{DG}(U_{DG},V)=} = \overbrace{\int_{I_n} (f, v_2)_H dt}^{F^n_{DG}(V)=} + (U^n_{DG,-}, V^n_+)_E,$$

for all $V \in P^{r_n}(I_n; Z_0 \times Z_0)$, with $U^n_{DG,-}$ to be the initial data at a given time step $I_n = (t_n, t_{n+1})$, (6.13)

the forcing data $f \in H^1(J;H)$, and the semi-discrete data $G_{CG} \in W^{\underline{r},tr} \subset W^{tr} = H^1(J; D^{A,tr} \times D^{A,tr})$.

6.2.1 Decoupling, existence and uniqueness

In this Section, to show the solvability of the semi-discrete formulation in (6.12), the splitting technique is used again as done in the previous chapter for the continuous problem i.e. the solution of (6.12) is defined by two functions which do exist and are unique. The first part of the split exists and is unique as shown in Theorem 6.1 on p. 122. The second part of the split which is a solution of (6.33) on p. 130, also exists and is unique and this is shown by recalling the same techniques used in Chapter 4 for formulations with general functional since the formulation in (6.33) is considered as a particular choice of the general structure in Chapter 4.

6.2.1.1 The transformed semi-discrete formulation

Let $\hat{G}_{CG} \in W^{\underline{r}}_{G_{CG}} \subset \mathcal{V}^{\underline{r}}(\mathcal{N}; Z_{g_{CG}} \times Z_{g'_{CG}})$ be the existing semi-discrete continuous real vector-valued function which solves the formulation in (6.2) on p. 121 i.e. at the nodes of I_n

$$[\hat{G}_{CG}]^n = 0, \text{ for } 0 \leq n \leq N-1,$$

then with defining

$$U_{DG} := \hat{G}_{CG} + \bar{U}_{DG}. \tag{6.14}$$

for real vector-valued function $\bar{U}_{DG} \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z_0 \times Z_0)$. With recalling (5.35) on p. 115, the form $a(\, ,)$ given in Definition 5.3 on p. 104, and for real vector-valued function for

$$F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} -\partial_t \hat{g}_{1,CG} + \hat{g}_{2,CG} \\ f - \partial_t \hat{g}_{2,CG} - A\hat{g}_{1,CG} \end{pmatrix} \in L^2(J; D_0 \times H) \in L^2(J; Z_0 \times H), \tag{6.15}$$

the functional $\mathcal{F}_{DG} \in (L^2(J; Z_0 \times H))'$ would be given by

$$\mathcal{F}_{DG}(V) := \sum_{n=0}^{N-1} \int_{I_n} \{\overbrace{a(f_1, v_1) + (f_2, v_2)_H}^{(F,V)_E}\} dt = \int_J (F, V)_E dt, \, \forall V \in L^2(J; Z_0 \times H), \tag{6.16}$$

and here all vector-valued functions in semi-discrete space $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z_0 \times Z_0)$ are also included in this linear map since $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z_0 \times Z_0) \subset L^2(J; Z \times H)$.
Then the formulation in (6.12) in terms of the two vector-valued functions $\hat{G}_{CG}$ and $\bar{U}_{DG}$ in (6.14) would now read

Problem 6.2.3 (Semi-discrete transformed formulation)
Let Z and H be the Hilbert spaces given in Definition 5.1 on p. 101. Let Z_0 be the Hilbert subspace given in Definition 5.2 on p. 102. Let $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z_0 \times Z_0)$ be the semi-discrete finite dimensional subspace given in Definition 6.2. Let $B_{DG}(\, ,)$ be the form given in (6.10) on p. 123 and the functional $\mathcal{F}_{DG}$ in (6.16). Let $(\, ,)_E$ be the inner product given in Corollary 2.2 on p. 10.
Find $\bar{U}_{DG} \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z_0 \times Z_0)$ such that

$$B_{DG}(\bar{U}_{DG}, V) = \mathcal{F}_{DG}(V) + (\bar{U}^0_{DG,-}, V^0_+)_E, \tag{6.17}$$

for all $V \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z_0 \times Z_0)$, and for given initial data $\bar{U}^0_{DG,-} \in Z_0 \times Z_0$.

Problem 6.2.4 (The semi-discrete time-stepping transformed formulation)
Let Z and H be the Hilbert spaces given in Definition 5.1 on p. 101. Let Z_0 be the Hilbert subspace given in Definition 5.2 on p. 102. Let $P^{r_n}(I_n; Z_0 \times Z_0)$ be the semi-discrete finite dimensional subspace given in Definition 6.2. Let $\hat{a}(\, ,)$ be the skew-symmetric form given in Definition 2.7 on p. 14 and $(\, ,)_E$ be the inner product given in Corollary 2.2 on p. 10:

$$\hat{a}(U, V) = -a(u_2, v_1) + a(u_1, v_2), \text{ and } (U, V)_E = a(u_1, v_1) + (u_2, v_2)_E.$$

For $F = \binom{f_1}{f_2} \in L^2(I_n; Z_0 \times H)$ let $\mathcal{F}^n_{DG} \in (L^2(I_n; Z_0 \times H))'$:

$$\mathcal{F}^n_{DG}(V) := \int_{I_n} (F, V)_E dt, \text{ for all } V \in L^2(I_n; Z_0 \times H), 0 \leq n \leq N-1, \tag{6.18}$$

where all semi-discrete test vector-valued functions in semi-discrete space $P^{r_n}(I_n; Z_0 \times Z_0)$ are also included in this linear map since $P^{r_n}(I_n; Z_0 \times Z_0) \subset L^2(J; Z_0 \times H)$.
Find $\bar{U}_{DG} \in P^{r_n}(I_n; Z_0 \times Z_0)$ such that

$$\overbrace{\int_{I_n} \Big\{(\partial_t \bar{U}_{DG}, V)_E + \hat{a}(\bar{U}_{DG}, V)\Big\}dt + (\bar{U}^n_{DG,+}, V^n_+)_E}^{b^n_{DG}(\bar{U}_{DG},V)=} = \overbrace{\int_{I_n} (F, V)_E dt}^{\mathcal{F}^n_{DG}(V)=} + (\bar{U}^n_{DG,-}, V^n_+)_E, \tag{6.19}$$

for all $V \in P^{r_n}(I_n; Z_0 \times Z_0)$, with $\bar{U}^n_{DG,-}$ to be the initial data at a given time step I_n.

Here, Z_0 is a separable Hilbert space as shown in Lemma 5.1 on p. 103 which satisfies all the properties of the Hilbert space considered in the general theory of Chapter 4. Also, D_0 is subspace of Z_0 and satisfies the properties of the given subspace in Definition 3.13 on p. 60. The functional in (6.19) is in $(L^2(I_n; Z_0 \times H))'$. Then the transformed formulation in (6.19) is a particular example of the general formulation in (4.28) on p. 83, where now with (6.3) on p. 121

$$-\partial_t \hat{g}_{1,CG} + \hat{g}_{2,CG}, \in L^2(J; D_0),$$

and

$$\partial_t \hat{g}_{2,CG} - A\hat{g}_{1,CG} \in L^2(J; H),$$

the terms in forcing data $F = \binom{f_1}{f_2}$ are

$$f_1 = -\partial_t \hat{g}_{1,CG} + \hat{g}_{2,CG}, \in L^2(J; D_0) \subset L^2(J; Z_0) \text{ and } f_2 = f - \partial_t \hat{g}_{2,CG} - A\hat{g}_{1,CG} \in L^2(J; H).$$

Now, since the transformed formulation (6.19) is a particular example of (4.28) on p. 83, then existence and uniqueness of the solution of (6.19) can be proven by following the same steps done in Section 4.2.

6.2.1.2 Time decoupling and the continuous reduced spatial problems of semi-discrete transformed formulation

With repeating the same steps in Section 4.2 starting on p. 82 and considering the formulation (4.28) on p. 83 with its equivalent $(r + 1)$ reduced spatial problems in (4.32) in Theorem 4.4 on p. 86, the formulation in (6.19) reads the same as (4.28), and accordingly, the $(r + 1)$ reduced spatial problems of (6.19) at a generic time step I with referring to (4.31) on p. 84 would read

Problem 6.2.5 (Continuous reduced spatial problems of the transformed formulation)
Let Z and H be the Hilbert spaces given in Definition 5.1 on p. 101. Let Z_0 be the Hilbert subspace given in Definition 5.2 on p. 102. Let $(\,,)_E$ be the inner product given in Corollary 2.2 on p. 10. Let λ_p be the eigenvalues of the matrix $\mathcal{A}^L$ given in (3.31) on p. 30. Let $(\beta^1_p)_q$ and $(\beta^2_p)_q$ be the $(r+1)$ vectors given in (3.36):

$$\begin{aligned} &(U, V)_E = a(u_1, v_1) + (u_2, v_2)_H, \forall U, V \in Z^h \times Z^h, \\ &(\beta^1_p)_q = (Y^{-1})_{pq}\, x^1_q = (Y^{-1})_{pq}\sqrt{(q + 1/2)}, \text{ and} \\ &(\beta^2_p)_q = (Y^{-1})_{pq}\, x^2_q = (Y^{-1})_{pq}\sqrt{(q + 1/2)}(-1)^q, \text{ respectively.} \end{aligned} \tag{6.20}$$

For

$$\mathfrak{f}^1(\mathfrak{v}_{2,p}) := -k^2 \sum_{q=0}^{r} \Big\{ (\beta_p^1)_q \int_{-1}^{1} L_q(\hat{t}) a(f_1 \circ Q, \mathfrak{v}_{2,p}) d\hat{t} \Big\}$$
$$\mathfrak{f}^2(\mathfrak{v}_{2,p}) := 2k\lambda_p \sum_{q=0}^{r} \Big\{ (\beta_p^1)_q \int_{-1}^{1} L_q(\hat{t}) (f_2 \circ Q, \mathfrak{v}_{2,p})_H d\hat{t} \Big\}, \tag{6.21}$$

and with a form and a linear functional $\mathfrak{f} \in (Z_0 \times H)' \equiv Z_0' \times H$ *(since* $H \equiv H'$ *is a pivot space):*

$$b(\mathfrak{u}_{2,p}, \mathfrak{v}_{2,p}) := 4\lambda_p^2 (\mathfrak{u}_{2,p}, \mathfrak{v}_{2,p})_H + k^2 a(\mathfrak{u}_{2,p}, \mathfrak{v}_{2,p})$$
$$\mathfrak{f}(\mathfrak{v}_{2,p}) := \mathfrak{f}^1(\mathfrak{v}_{2,p}) + \mathfrak{f}^2(\mathfrak{v}_{2,p}), \tag{6.22}$$

now reads
Find $\{\mathfrak{u}_{2,p}\}_{p=0}^{r} \subset Z_0$:

$$b(\mathfrak{u}_{2,p}, \mathfrak{v}_{2,p}) = \mathfrak{f}(\mathfrak{v}_{2,p}) + \sum_{q=0}^{r} \Big\{ (\beta_p^2)_q \Big\} \Big(4\lambda_p (u^0_{2,DG,-}, \mathfrak{v}_{2,p})_H - 2ka(u^0_{1,DG,-}, \mathfrak{v}_{2,p}) \Big),$$

for each $p = 0, \cdots, r,$ *for all* $\{\mathfrak{v}_p\}_{p=0}^{r_n} \in Z_0$.

(6.23)

After solving for the unknowns $\{\mathfrak{u}_{2,p}\}_{p=0}^{r} \subset Z_0$ *with given data* $f_1 \in L^2(I; Z_0)$, $f_2 \in L^2(I; H)$, *and* $\bar{u}_{1,0}$, $\bar{u}_{2,0} \in Z_0 \subset Z$, *then comes to update the values of* $\{\mathfrak{u}_{1,p}\}_{p=0}^{r} \subset Z_0$*:*

$$\mathfrak{u}_{1,p} = \frac{k}{2\lambda_p} \mathfrak{u}_{2,p} + \frac{k}{2\lambda_p} \sum_{i=0}^{r} \Big\{ (\beta_p^1)_i \int_{-1}^{1} L_i(\hat{t}) f_1 \circ Q d\hat{t} \Big\} + \frac{1}{\lambda_p} \sum_{i=0}^{r} \Big\{ (\beta_p^2)_i \Big\} \bar{u}_{1,0}, \tag{6.24}$$

which need to be done in every given time step I.

Lemma 6.2 (Existence, uniqueness of continuous reduced spatial problem's for given $r.h.s$**)** *Let* Z *be the Hilbert space given in Definition 5.1 on p. 101. Let* Z_0 *be the separable Hilbert subspace of* Z *given in Definition 5.2 with Lemma 5.1 on p. 103. There exist unique functions* $\{\mathfrak{u}_{2,p}\}_{p=0}^{r} \subset Z_0$ *which solve the* $(r+1)$ *spatial problems (6.23), as well as* $\{\mathfrak{u}_{1,p}\}_{p=0}^{r} \subset Z_0$ *in (6.24), for given* $f_1 \in L^2(I; Z_0)$, $f_2 \in L^2(I; H)$, *and initial data* $\bar{u}_{1,0}, \bar{u}_{2,0} \in Z_0 \subset Z$.

Proof:
Firstly, the form $b(\mathfrak{u}_{2,p}, \mathfrak{v}_{2,p})$ in (6.23) for $\mathfrak{u}_{2,p}$, and $\mathfrak{v}_{2,p}$ in the subspace $(Z_0, \| \cdot \|_Z)$ is continuous with using Lemma 3.5 on p. 34 after letting $c_1 = 4\lambda_p$ and $c_2 = k^2$ such that for $p = 0, \cdots, r$

$$b(\mathfrak{u}_{2,p}, \mathfrak{v}_{2,p}) = 4\lambda_p^2 (\mathfrak{u}_{2,p}, \mathfrak{v}_{2,p})_H + k^2 a(\mathfrak{u}_{2,p}, \mathfrak{v}_{2,p}) \leq (4|\lambda_p| c^2 + k^2 M) \parallel \mathfrak{u}_{2,p} \parallel_Z \parallel \mathfrak{v}_{2,p} \parallel_Z .$$

Secondly, with using Definition 5.3 on p. 104 which says that the form $a(\,,\,)$ is elliptic in Z_0 $\| \cdot \|_Z$, then the form $b(\mathfrak{u}_{2,p}, \mathfrak{v}_{2,p})$ in (6.23) for $\mathfrak{u}_{2,p}$, and $\mathfrak{v}_{2,p}$ in the subspace $(Z_0, \| \cdot \|_Z)$ satisfies the Inf-sup condition in Lemma 3.4 on p. 33 after letting $c_1 = 4\lambda_p$ and $c_2 = k^2$ where the Inf-sup constant would be $k^2\alpha$.
Also in the r.h.s of (6.23) we have

$$|r.h.s| = |\mathfrak{f}^1(\mathfrak{v}_{2,p}) + \mathfrak{f}^2(\mathfrak{v}_{2,p}) + \sum_{q=0}^{r} \Big\{ (\beta_p^2)_q \Big\} \Big(4\lambda_p (u^0_{2,DG,-}, \mathfrak{v}_{2,p})_H - 2ka(u^0_{1,DG,-}, \mathfrak{v}_{2,p}) \Big)|. \tag{6.25}$$

With repeating the same steps done in the proof of Theorem 4.2.1 on p. 84 to get the estimate in (4.36) then

$$|r.h.s| \leq \Bigg(c\sqrt{2^3 k}|\lambda_p| \sum_{q=0}^{r} |(\beta_p^1)_q| \hat{x}_q \parallel f_2 \parallel_{L^2(I;H)} + |\sum_{q=0}^{r} (\beta_p^2)_q| \Big(4c|\lambda_p| \parallel u_{2,DG,-}^0 \parallel_H + 2kM \parallel u_{1,DG,-}^0 \parallel_Z \Big) + \sqrt{2k^3} M \sum_{q=0}^{r_n} |(\beta_p^1)_q| \hat{x}_q \parallel f_1 \parallel_{L^2(I;Z)} \Bigg) \parallel \mathfrak{v}_{2,p} \parallel_Z . \tag{6.26}$$

Now, going back to (Babuška) Theorem A.4 on p. 179, and with having Lemmas 3.4 and 3.5 and (6.26), then there exist a function $\{\mathfrak{u}_{2,p}\}_{p=0}^{r} \subset Z_0$ which solves (6.23). For the identity (6.24) on p. 127:

$$\mathfrak{u}_{1,p} = \frac{k}{2\lambda_p}\mathfrak{u}_{2,p} + \frac{1}{\lambda_p}\sum_{q=0}^{r}\Big\{(\beta_p^2)_q\Big\}u_{1,DG,-}^0 + \frac{k}{2\lambda_p}\sum_{q=0}^{r}\Big\{(\beta_p^1)_q\int_{-1}^{1} L_q(\hat{t}) f_1 \circ Q d\hat{t}\Big\},$$

where

$$\parallel \frac{k}{2\lambda_p}\sum_{q=0}^{r}\Big\{(\beta_p^1)_q\int_{-1}^{1} L_q(\hat{t}) f_1 \circ Q d\hat{t}\Big\} \parallel_Z \leq \frac{k}{2|\lambda_p|}\sum_{q=0}^{r}\Big\{|(\beta_p^1)_q|\int_{-1}^{1} L_q(\hat{t}) \parallel f_1 \circ Q \parallel_Z d\hat{t}\Big\} < \infty,$$

for each $p = 0, \cdots, r$, then from the given existence and uniqueness of $\{\mathfrak{u}_{2,p}\}_{p=0}^{r} \subset Z_0$ and the given forcing data $f_1 \in L^2(J; Z_0)$, $f_2 \in L^2(J; H)$ and initial data $u_{1,DG,-}^0, u_{2,DG,-}^0 \in Z_0$ which are well defined, that also concludes that $\{\mathfrak{u}_{1,p}\}_{p=0}^{r} \subset Z_0$ at a given time step I also exists and is unique. □

Theorem 6.2 (Existence, uniqueness of semi-discrete formulation with linear constraints) *Let Z_g and $Z_{g'}$ be the subsets given in Definition 5.2 on p. 102. Let $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z_{g_{CG}} \times Z_{g'_{CG}})$ be the semi-discrete space given in Definition 6.2 on p. 123. There exists a unique semi-discrete vector-valued function $U_{DG} \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z_{g_{CG}} \times Z_{g'_{CG}})$ which solves the formulation in (6.12) on p. 124.*

Proof:
The semi-discrete vector-valued function U_{DG} is given in the identity (6.14) on p. 125 that is

$$U_{DG} = \bar{U}_{DG} + \hat{G}_{CG}, \tag{6.27}$$

for the existing unique vector-valued function $\hat{G}_{CG} \in W_{G_{CG}}^{\underline{r}} \subset \mathcal{V}^{\underline{r}}(\mathcal{N}; Z_{g_{CG}} \times Z_{g'_{CG}}) \subset L^2(J; Z \times Z)$ which solves (6.2) on p. 121 and for $\bar{U}_{DG} \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z_0 \times Z_0)$ which solves the formulation (6.33) on p. 130:

$$\bar{U}_{DG} = \sum_{n=0}^{N-1} \bar{U}_{DG}|_{I_n},$$

where

$$\bar{U}_{DG}|_{I_n} = \sum_{j=0}^{r_n} \varphi_j(t)\bar{U}_j, \text{ for } Y^{r_n} \in \mathbb{C}^{(r_n+1)\times(r_n+1)} : \bar{U}_j = \sum_{i=0}^{r_n} Y_{ji}^{r_n}\mathfrak{U}_i \in Z_0 \times Z_0,$$

and with using the shown existence and uniqueness of $\{\mathfrak{U}_i\}_{i=0}^{r_n} \subset Z_0 \times Z_0$ in Lemma 6.2 on p. 127 which implies that $\bar{U}_{DG} \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z_0 \times Z_0)$ exists and is unique and from the identity (6.27) thus $U_{DG} \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z_{g_{CG}} \times Z_{g'_{CG}})$ exists and is unique. □

6.2.2 Stability estimates of semi-discrete formulation with linear constraints

In this Section, local and global stability estimates of the vector-valued function $U_{DG} \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z_{g_{CG}} \times Z_{g'_{CG}})$ which solves the semi-discrete formulation in (6.12) on p. 124 will be shown in Lemma 6.3 on p. 129 and Theorem 6.4 on p. 132, respectively. The latter uses the resulting estimates in Lemmas 6.3 and 6.4 on p. 129 - 130. Here since the transformed formulation in (6.33) is considered as a particular example of the general formulation in Problem 4.2.1 on p. 82, then the estimates in Section 4.3.2 are going to be recalled here mainly to proof Lemmas 6.3 and 6.4.

Lemma 6.3 (Local estimate for semi-discrete transformed formulation) *Let Z and H be the Hilbert spaces given in Definition 5.1 on p. 101. Let Z_0 be the separable Hilbert subspace of Z given in Definition 5.2 with Lemma 5.1 on p. 103. At a generic time step $I = (t_0, t_1)$ and given forcing data $F \in L^2(I; Z_0 \times H)$ with initial data $\bar{U}_0 \in Z_0 \times Z_0$, then the vector-valued function $\bar{U}_{DG} \in P^r(I; Z_0 \times Z_0)$ which solves Problem 6.2.4 on p. 125 satisfies*

$$\| \bar{U}_{DG} \|_{L^2(I;E)} \leq \frac{k \| Y \|_2^2 (2r+1)}{2 \min\limits_p \; Re\, \lambda_p} \Big(\| F \|_{L^2(I;E)} + \| \bar{U}_0 \|_E \Big), \tag{6.28}$$

for normed-product space $(L^2(J; Z_0 \times Z_0), \| \cdot \|_{L^2(J;E)})$:

$$\| U \|^2_{L^2(J;E)} = \int_J \overbrace{a(u_1, u_1) + \| u_2 \|_H^2}^{\|U\|_E^2} dt, \forall\, U \in L^2(J; Z_0 \times Z_0), \tag{6.29}$$

where k is the time step size at I. λ_p are the eigenvalues of the matrix $\mathcal{A}^L$ given in (3.31) on p. 30 for $p = 0, \cdots, r$, $\| Y^r \|_2^2$ is the square of the spectral norm (see (A.6) on p. 173) of the transformation matrix $Y \in \mathbb{C}^{(r+1)\times(r+1)}$ and r is the approximation order and the Hermitian form $a(\,,)$ given in Definition 5.3 on p. 104:

$$\begin{aligned} |a(u,v)| &\leq M \| u \|_Z \| v \|_Z, && \forall \quad u, v \in Z, \\ a(u,u) &\geq \alpha \| u \|_Z^2, && \forall \quad u \in Z_0. \end{aligned}$$

Proof:
With knowing that Z_0 is separable Hilbert space as shown in Lemma 5.1 on p. 103 and the form $a(\,,)$ in Z_0 is *elliptic*, then at a generic time step I the formulation (6.19) on p. 126 with

$$F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} -\partial_t \hat{g}_{1,CG} + \hat{g}_{2,CG} \\ f - \partial_t \hat{g}_{2,CG} - A\hat{g}_{1,CG} \end{pmatrix} \in L^2(J; Z_0 \times H),$$

is considered as a particular example of the general formulation in (4.28). This concludes that the solution of (6.19) satisfies shown local stability estimate in Lemma 4.1 on p. 87 i.e.

$$\begin{aligned} \| \bar{U}_{DG} \|_{L^2(I;E)} &\leq \frac{k \| Y \|_2^2}{2(\min\limits_p \; \mathrm{Re}\, \lambda_p)} \sup_{V \in P^r(I; Z_0 \times Z_0)} \frac{\Big| b_{DG}(U_{DG}, V) \Big|}{\| V \|_{L^2(I;E)}} \\ &= \frac{k \| Y \|_2^2}{2(\min\limits_p \; \mathrm{Re}\, \lambda_p)} \sup_{V \in P^r(I; Z_0 \times Z_0)} \frac{\Big| \overbrace{\int_I (F, V)_E dt + (\bar{U}^0_{DG,-}, V_+)_E}^{F_{DG}(V)=} \Big|}{\| V \|_{L^2(I;E)}}, \end{aligned} \tag{6.30}$$

and that completes the proof. □

Lemma 6.4 (Left-sided limits estimate of semi-discrete transformed formulation) *Let D^A be the Hilbert space given in Definition 5.1 with Assumption 5.1 on p. 101. Let $D^{A,tr}$ be the normed-space given in Definition 5.2 on p. 102.*
Let $W^{tr} = \bar{\gamma}[W] = H^1(J; D^{A,tr} \times D^{A,tr})$ be the normed-product space with the norm $\| \cdot \|_{W^{tr}}$ given in Corollary 5.5 on p. 107, where $\bar{\gamma} : H^1(J; D^A \times D^A) \to H^1(J; D^{A,tr} \times D^{A,tr})$ is time independent surjective matrix operator. Let Z and H be the Hilbert spaces given in Definition 5.1 on p. 101. Let Z_0 be separable Hilbert subspace of Z given in Definition 5.2 with Lemma 5.1 on p. 103. For given data $f \in L^2(J; H)$, $G_{CG} \in W^{\underline{r},tr} \subset W^{tr}$ and initial data $\bar{U} \in Z_0 \times Z_0$, the semi-discrete vector-valued function $\bar{U}_{DG} \in P^{r_m}(I_m; Z_0 \times Z_0)$ which solves the time-stepping formulation in (6.19) on p. 126 for $I_m = (t_m, t_{m+1})$, $m = 0, \cdots, n-1$ and $n \leq N$, satisfies

$$\| \bar{U}^n_{DG,-} \|_E \leq (1 + \frac{\mathcal{C}}{N})^n \big(4\sqrt{\max(M, c^2, 1)} \| G_{CG} \|_{W^{tr}} + \sqrt{2} \| f \|_{L^2(J;H)} + \| \bar{U}_0 \|_E \big), \tag{6.31}$$

for normed-product space $(Z_0 \times Z_0, \| \cdot \|_E)$, with Lemma 5.1:

$$\| U \|^2_E = a(u_1, u_1) + \| u_2 \|^2_H, \forall\, U \in Z_0 \times Z_0, \tag{6.32}$$

and $\mathcal{C} = \max_m \big(\dfrac{T \| Y^{r_m} \|^2_2 (2r_m + 1)}{2 \min_p \; Re\, \lambda_p} \big)$, is the same constant appearing in Lemma 4.3 on p. 89, and the form $a(\,,\,)$ given in Definition 5.3 on p. 104:

$$\begin{aligned} |a(u, v)| &\leq M \| u \|_Z \| v \|_Z, && \forall \quad u, v \in Z, \\ a(u, u) &\geq \alpha \| u \|^2_Z, && \forall \quad u \in Z_0. \end{aligned}$$

Proof:
With knowing that Z_0 is separable Hilbert space as shown in Lemma 5.1 on p. 103 and the form $a(\,,\,)$ in Z_0 is *elliptic*, then the formulation in (6.19) on p. 126 with it's linear functional $\mathcal{F}^n_{DG} \in (L^2(I_m; Z_0' \times H))'$, is a particular example of the general formulation in (4.47) on p. 89 where here

$$F = \begin{pmatrix} -\partial_t \hat{g}_{1,CG} + \hat{g}_{2,CG} \\ f - \partial_t \hat{g}_{2,CG} - A\hat{g}_{1,CG} \end{pmatrix} \in L^2(J; Z_0 \times H),$$

with given initial data $\bar{U}_0 \in Z_0 \times Z_0 \subset Z \times Z$. Thus the proof of Lemma 4.3 on p. 89 holds for $\bar{U}_{DG} \in P^{r_m}(I_m; Z_0 \times Z_0)$ and with using the same steps done in (5.40) on p. 117 and the stability estimate of $\hat{G}$ shown in Lemma 6.1 on p. 122 to get the bound of F in terms of the given data G and f then that completes the proof. □

Remark 6.1 *Before going to prove the Last left-sided limit of the semi-discrete solution, we will state the estimate if we allow jumps in the data at the trace i.e. with considering non zero jump terms for the function $\hat{G}$. Then the formulation (6.33) on p. 130 would now be*

$$B_{DG}(\bar{U}_{DG}, V) = \mathcal{F}_{DG}(V) + (\bar{U}^0_{DG,-}, V^0_+)_E - ([\hat{G}_{CG}]^n, V^0_+)_E, \tag{6.33}$$

and when we try to derive the stability estimate of the last left sided limit we get the factor 2^N of the norm of the jumps of $\hat{G}_{CG}$ and that is why we considered only smooth enough data at the trace.

Theorem 6.3 (Last left-sided limit of the semi-discrete formulation)
Let D^A be the Hilbert space given in Definition 5.1 with Assumption 5.1 on p. 101. Let $D^{A,tr}$ be the normed-space given in Definition 5.2 on p. 102.
Let $W^{tr} = \bar{\gamma}[W] = H^1(J; D^{A,tr} \times D^{A,tr})$ be the normed-product space with the norm $\| \cdot \|_{W^{tr}}$ given in Corollary 5.5 on p. 107, where $\bar{\gamma} : H^1(J; D^A \times D^A) \to H^1(J; D^{A,tr} \times D^{A,tr})$ is time independent surjective matrix operator. Let Z and H be the Hilbert spaces given in Definition 5.1 on p. 101. Let Z_g and $Z_{g'}$, be the subsets given in Definition 5.2 on p. 102. Let $P^{r_n}(I_n; Z_{g_{CG}} \times Z_{g'_{CG}})$ be the semi-discrete finite dimensional subset given in Definition 6.2 on p. 123. With given initial data $U_0 \in Z_g \times Z_{g'}$ and $G_0 \in D^{A,tr} \times D^{A,tr}$ and the last left sided limit $G_-^N \in D^{A,tr} \times D^{A,tr}$, the semi-discrete vector-valued function $U_{DG} \in P^{r_n}(I_n; Z_{g_{CG}} \times Z_{g'_{CG}})$ which solves the time-stepping formulation in (6.13) on p. 124 satisfies

$$\| U_{DG,-}^N \|_X \leq \frac{(1+\frac{\mathcal{C}}{N})^n}{\sqrt{\min(\alpha,1)}} \big(4\sqrt{\max(M,c^2,1)} \| G_{CG} \|_{W^{tr}} + \sqrt{2} \| f \|_{L^2(J;H)} + \| U_0 \|_E + \sqrt{\max(M,c^2)} \| G_0 \|_{D^{A,tr} \times D^{A,tr}} \big) + \sqrt{\max(1,c^2)} \| G_{CG,-}^N \|_{D^{A,tr} \times D^{A,tr}}, \tag{6.34}$$

where $\mathcal{C} = \max_n(\frac{T \| Y^{r_n} \|_2^2 (2r_n+1)}{2 \min_p \ Re\ \lambda_p})$, is the same constant appearing in Lemma 4.3 on p. 89, with the norm

$$\| U \|_E^2 = a(u_1,u_1) + \| u_2 \|_H^2, \forall\, U \in Z \times Z, \tag{6.35}$$

for the form $a(\,,\,)$ given in Definition 5.3 on p. 104:

$$|a(u,v)| \leq M \| u \|_Z \| v \|_Z, \qquad \forall \quad u,v \in Z,$$
$$a(u,u) \geq \alpha \| u \|_Z^2, \qquad \forall \quad u \in Z_0.$$

Proof:
With using the identity given in (6.14) on p. 125 and the triangle inequality then

$$\| U_{DG,-}^N \|_X \leq \| \bar{U}_{DG,-}^N \|_X + \| \hat{G}_{CG,-}^N \|_X,$$

where

$$\| \bar{U}_{DG,-}^N \|_X \leq \frac{1}{\sqrt{\min(\alpha,1)}} \| \bar{U}_{DG,-}^N \|_E,$$
$$\| \hat{G}_{CG,-}^N \|_X \leq \sqrt{\max(1,c^2)} \| \hat{G}_{CG,-}^N \|_{D^A \times D^A},$$

and with using the resulting estimate in Lemma 6.4 on p. 130 then

$$\| U_{DG,-}^N \|_X \leq \frac{(1+\frac{\mathcal{C}}{N})^n}{\sqrt{\min(\alpha,1)}} \big(4\sqrt{\max(M,c^2,1)} \| G_{CG} \|_{W^{tr}} + \sqrt{2} \| f \|_{L^2(J;H)} + \| \bar{U}_0 \|_E \big) + \sqrt{\max(1,c^2)} \| \hat{G}_{CG,-}^N \|_{D^A \times D^A}.$$

Moreover, since

$$\| \bar{U}_0 \|_E = \| U_0 - \hat{G}_0 \|_E \leq \| U_0 \|_E + \| \hat{G}_0 \|_E \leq \| U_0 \|_E + \sqrt{\max(M,c^2)} \| \hat{G}_0 \|_{D^A \times D^A}, \tag{6.36}$$

and with recalling the estimate done in (5.41) on p. 117 and (5.44) on p. 119 that is

$$\| \bar{U}_0 \|_E \leq \| U_0 \|_E + \sqrt{\max(M, c^2)} \| G_0 \|_{D^{A,tr} \times D^{A,tr}},$$

and

$$\inf_{\substack{\hat{G}^N_{CG,-} \in D^A \times D^A \\ \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix} \hat{G}^N_{CG,-} = G^N_{CG,-}}} \| \hat{G}^N_{CG-} \|_{D^A \times D^A} = \| G^N_{CG,-} \|_{D^{A,tr} \times D^{A,tr}},$$

that completes the proof. $\square$

Theorem 6.4 (Global stability of semi-discrete formulation with linear constraints) *Let D^A be the Hilbert space given in Definition 5.1 with Assumption 5.1 on p. 101. Let $D^{A,tr}$ be the normed-space given in Definition 5.2 on p. 102.*
Let W be the Hilbert space given in Definition 5.4 on p. 104 and $W^{tr} = \bar{\gamma}[W] = H^1(J; D^{A,tr} \times D^{A,tr})$ be the normed-product space with the norm $\| \cdot \|_{W^{tr}}$ given in Corollary 5.5 on p. 107, where $\bar{\gamma} : H^1(J; D^A \times D^A) \to H^1(J; D^{A,tr} \times D^{A,tr})$ is time independent surjective matrix operator. Let Z and H be the Hilbert spaces given in Definition 5.1 on p. 101. Let Z_g and $Z_{g'}$, be the subsets given in Definition 5.2 on p. 102. Let $P^{r_n}(I_n; Z_{g_{CG}} \times Z_{g'_{CG}})$ be the semi-discrete finite dimensional subset given in Definition 6.2 on p. 123. With given forcing data $f \in H^1(J; H)$, compatible initial data $U_0 \in Z_{g_{CG}} \times Z_{g'_{CG}}$ and semi-discrete linear constraints $G_{CG} \in W^{\underline{r},tr} \subset W^{tr}$, the solution $U_{DG} \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z_{g_{CG}} \times Z_{g'_{CG}})$ of the formulation (6.12) on p. 124 satisfies

$$\begin{aligned} \| U_{DG} \|_{L^2(J;X)} \leq & \left(2\sqrt{\max(1, c^2)} + 4\sqrt{\max(M, c^2, 1)} \frac{(e^{\mathcal{C}} - 1)}{\sqrt{\min(\alpha, 1)}}\right) \| G_{CG} \|_{W^{tr}} \\ & + \frac{(e^{\mathcal{C}} - 1)}{\sqrt{\min(\alpha, 1)}} \left(\sqrt{2} \| f \|_{L^2(J;H)} + \| U_0 \|_E + \sqrt{\max(M, c^2)} \| G_0 \|_{D^{A,tr} \times D^{A,tr}} \right), \end{aligned} \tag{6.37}$$

for normed-product space $(L^2(J; Z \times Z), \| \cdot \|_{L^2(J;X)})$:

$$\| U \|^2_{L^2(J;X)} = \int_J \overbrace{\| u_1 \|^2_Z + \| u_2 \|^2_H}^{\| U \|^2_X} dt, \forall\, U \in L^2(J; Z \times Z), \tag{6.38}$$

where $\mathcal{C} = \max_n \left(\frac{T \| Y^{r_n} \|^2_2 (2r_n + 1)}{2 \min_p \ Re\ \lambda_p}\right)$, is the same constant which appear in Lemma 4.5 on p. 91. $M < \infty$ the lower bounds of the form $a(\,,)$ which satisfies the properties given in Definition 5.3 on p. 104:

$$|a(u, v)| \leq M \| u \|_Z \| v \|_Z, \qquad \forall \quad u, v \in Z.$$

c the constant in the inclusion inequality $\| u \|_H \leq c \| u \|_Z, \forall u \in Z$ (see [22, (6.)]), and the norm $\| \cdot \|_{W^{tr}}$ given in Corollary 5.5 on p. 107.

Proof:
Here, from the identity (6.14) on p. 125

$$U_{DG} = \hat{G}_{CG} + \bar{U}_{DG},$$

with using the *triangle inequality* and from the inequality in Lemma 5.6 on p. 108, then

$$\begin{aligned}
\| U_{DG} \|_{L^2(J;X)} &= \| \hat{G}_{CG} + \bar{U}_{DG} \|_{L^2(J;X)} \\
&\leq \| \hat{G}_{CG} \|_{L^2(J;X)} + \| \bar{U}_{DG} \|_{L^2(J;X)} \\
&\leq \sqrt{\max(1,c^2)} \underbrace{\| \hat{G}_{CG} \|_{L^2(J;D^A\times D^A)}}_{\leq \|\hat{G}_{CG}\|_{H^1(J;D^A\times D^A)}} + \frac{1}{\sqrt{\min(\alpha,1)}} \| \bar{U}_{DG} \|_{L^2(J;E)} \\
&= \sqrt{\max(1,c^2)} \| \hat{G}_{CG} \|_W + \frac{1}{\sqrt{\min(\alpha,1)}} \sum_{n=0}^{N-1} \| \bar{U}_{DG} \|_{L^2(I_n;E)} .
\end{aligned}$$

Now, with using the resulting estimate in Lemma 6.3 on p. 129 after letting $I = I_n$ and the correspondence initial data accordingly with the use of Lemma 6.4, and the same steps used in the proof of Theorem 4.5 on p. 91 yields

$$\| U_{DG} \|_{L^2(J;E)} \leq \sqrt{\max(1,c^2)} \| \hat{G}_{CG} \|_W + \frac{(e^{\mathcal{C}}-1)}{\sqrt{\min(\alpha,1)}} (\| F \|_{L^2(J;E)} + \| \bar{U}_0 \|_E), \tag{6.39}$$

and repeating the same steps done in the proof of Theorem 5.3 on p. 116 to get the upper bound of the $\| F \|_{L^2(J;E)}$ and $\| \bar{U}_0 \|_E$ in terms of the forcing data f, linear constraints G_{CG}, and the given initial data and with returning to the estimate in (6.39) and with using the stability estimate in Lemma 6.1 on p. 122 yield

$$\begin{aligned}
\| U_{DG} \|_{L^2(J;X)} \leq &\big(2\sqrt{\max(1,c^2)} + 4\sqrt{\max(M,c^2,1)} \frac{(e^{\mathcal{C}}-1)}{\sqrt{\min(\alpha,1)}}\big) \| G_{CG} \|_{W^{tr}} \\
&+ \frac{(e^{\mathcal{C}}-1)}{\sqrt{\min(\alpha,1)}} \big(\sqrt{2} \| f \|_{L^2(J;H)} + \| U_0 \|_E + \sqrt{\max(M,c^2)} \| G_0 \|_{D^{A,tr}\times D^{A,tr}} \big),
\end{aligned}$$

and that completes the proof. □

6.3 The fully-discrete auxiliary (elliptic) problem

In Section 6.3.1, existence and uniqueness of the solution of the fully-discrete auxiliary (elliptic) problem (6.43) on p.135 will be shown in Lemma 6.5 on p. 135. The stability estimate of the solution of (6.43) is shown in Lemma 6.6 on p. 136. Lemma 6.5 and Lemma 6.6 will be used in order to show existence, uniqueness and stability estimate of the solution of the fully-discrete variational formulation with given linear constraints.

Definition 6.3 *Let Z and D^A be the Hilbert spaces given in Definition 5.1 with Assumption 5.1 on p. 101. Also, let Z^{tr} and $D^{A,tr} = \gamma[D^A]$ be the separable normed spaces with the sets Z_g and D_g given in Definition 5.2 on p. 102 with Corollary 5.2 on p. 102:*

- *$(Z^h, \| \cdot \|_Z)$ is a conformal finite dimensional subspace of the Hilbert space Z. $(D^{h,A}, \| \cdot \|_{D^A})$ is a conformal finite dimensional subspace of the Hilbert space $(D^A, \| \cdot \|_{D^A})$.*

- *$Z^{h,tr} := \gamma[Z^h]$ is a conformal finite dimensional subspace of Z^{tr} and $D^{h,A,tr} := \gamma[D^{h,A}]$ is a conformal finite dimensional subspace of $D^{A,tr}$.*

- *For any $g^h_{CG} \in Z^{h,tr}$ and $g^h_{CG} \in D^{h,A,tr}$, $Z^h_{g^h_{CG}}$ is a conformal finite dimensional subset of Z_g and $D^h_{g^h_{CG}}$ is a conformal finite dimensional subset of D_g.*

Corollary 6.1 *$(D^{h,A}, \| \cdot \|_{D^A})$ is a subspace of $(Z^h, \| \cdot \|_Z)$ and $D^{h,A,tr}$ is a subspace of $Z^{h,tr}$, where all are separable.*

Proof:
Firstly, since D^A is a subspace of Z, and the finite dimensional spaces Z^h and $D^{h,A}$ are also subspaces of Z and D^A, that implies that $D^{h,A}$ is also a subspace of Z^h where all are Hilbert spaces and separable (see [51, Proof of Theorem 2.C.]).
Secondly, with using Corollary A.5 on p. 175 i.e.from Definition 6.3 on p. 133:

$$Z^{h,tr} = \gamma[Z^h], \text{ and } D^{h,A,tr} = \gamma[D^{h,A}],$$

and since Z^h and $D^{h,A}$ are separable Hilbert spaces and γ is linear and bounded operator, then that completes the proof. □

Definition 6.4 *Let $D^{h,A}$ and Z^h be the finite dimensional subspaces of the Hilbert spaces $D^{h,A}$ and Z^h, respectively, given in Definition 6.3. Let $Z^{h,tr}$ and $D^{h,A,tr}$ be the conformal finite dimensional subspaces of the normed vector space Z^{tr} and $D^{A,tr}$, respectively and let $Z^h_{g^h_{CG}}$ and $D^h_{g^h_{CG}}$ be the conformal finite dimensional subsets of Z_g and D^A_g given in Definition 6.3. Let W_G, and $W_{\mathbf{0}}$ be the subset and the subspace given in Definition 5.4 on p. 104. Let $W^{\underline{r}}$, $W^{\underline{r}}_{G_{CG}}$, $W^{\underline{r}}_{\mathbf{0}}$ be the semi-discrete finite dimensional spaces given in Definition 6.1 on p. 120. Let $(W^{\underline{r},tr}, \| \cdot \|_{W^{tr}})$ be the semi-discrete finite normed-product space given in Definition 6.1 on p. 120. With following the same time partitioning in the previous sections, the following fully-discrete finite dimensional spaces in time and space are introduced*

$$W^{h,\underline{r}} := \Big\{ U | U \in H^1(J; D^{h,A} \times D^{h,A}) : U_{I_n} \in P^{r_n}(I_n; D^{h,A} \times D^{h,A}), 0 \leq n \leq N-1 \Big\}, \tag{6.40}$$

which is a fully-discrete conformal finite-dimensional subspace of $W = H^1(J; D^A \times D^A)$.

$$W^{h,\underline{r},tr} := \Big\{ U | U \in H^1(J; D^{h,A,tr} \times D^{h,A,tr}) : U|_{I_n} \in P^{r_n}(I_n; D^{h,A,tr} \times D^{h,A,tr}), 0 \leq n \leq N-1 \Big\},$$

with the defined norm $\| \cdot \|_{W^{tr}}$ for $\bar{\gamma} : W^{h,\underline{r}} \to W^{h,\underline{r},tr}$:

$$\| G^h_{CG} \|_{W^{tr}} = \inf_{\substack{U \in W^{h,\underline{r}} \\ \bar{\gamma} U = G^h_{CG}}} \| U \|_W, \quad \forall G^h_{CG} \in W^{h,\underline{r},tr},$$

which is the fully-discrete conformal finite dimensional space of the normed-space $(W^{tr}, \| \cdot \|_{W^{tr}})$. Moreover, for fully-discrete vector-valued function $G^h_{CG} = \binom{g^h_{1,CG}}{g^h_{2,CG}} = \binom{g^h_{CG}}{g^{h,\prime}_{CG}} \in W^{h,\underline{r},tr}$ with

$$\partial_t g^h_{CG} = g^{h,\prime}_{CG}. \tag{6.41}$$

$$W^{h,\underline{r}}_{G^h_{CG}} := \Big\{ U | U \in W^{h,\underline{r}} \text{ with } \bar{\gamma} U = G^h_{CG} = \begin{pmatrix} g^h_{CG} \\ g^{h,\prime}_{CG} \end{pmatrix} \text{ and } \partial_t u_1 \in L^2(J; D^h_{g^{h,\prime}_{CG}}), \partial_t u_2 \in L^2(J; D^{h,A}) \Big\}$$
$$\text{and } W^{h,\underline{r}}_{\mathbf{0}} := \Big\{ U | U \in W^{h,\underline{r}} \text{ with } \bar{\gamma} U = \mathbf{0} \Big\}, \tag{6.42}$$

are fully-discrete conformal finite-dimensional subset of W_G and Hilbert space $W_{\mathbf{0}}$, respectively.

The fully-discrete (continuous in time and space) auxiliary problem of (6.2) on p. 121 reads

Problem 6.3.1 (Fully-discrete auxiliary (elliptic) problem)
Let $W^{h,\underline{r}}_{G^h_{CG}}$ *be the fully-discrete finite dimensional subset given in Definition 6.4. Let* $(\,,\,)_W$ *be the inner product given in Lemma 5.2 on p. 104:*

$$(U,V)_W := \int_J \{(\partial_t U, \partial_t V)_{D^A\times D^A} + (U,V)_{D^A\times D^A}\}dt = (U,V)_{H^1(J;D^A\times D^A)},\ \forall U, V \in W.$$

Find $\hat{G}^h_{CG} \in W^{h,\underline{r}}_{G^h_{CG}}$ *:*

$$(\hat{G}^h_{CG}, V^h)_W = 0, \tag{6.43}$$

for all $V^h \in W^{h,\underline{r}}_{\mathbf{0}}$ *where* $\hat{G}^h_{CG} \in W^{h,\underline{r}} \subset C([0,T]; D^{h,A}\times D^{h,A})$ *and*

$$\partial_t \hat{g}^h_1 \in L^2(J; D_{g^{h,\prime}_{CG}}),\ \text{and}\ \hat{g}_2 \in L^2(J; D_{g^{h,\prime}_{CG}}). \tag{6.44}$$

6.3.1 Existence, uniqueness, and stability estimate of the fully-discrete auxiliary (elliptic) problem

Lemma 6.5 (Existence, uniqueness of fully-discrete auxiliary(elliptic) problem) *Let* W *be the Hilbert space given in Definition 5.4 on p. 104. Let* $W^{h,\underline{r}}_{G^h_{CG}}$ *be the fully-discrete finite dimensional subset of* W *given in Definition 6.4. There exists a unique solution* $\hat{G}^h_{CG} \in W^{h,\underline{r}}_{G^h_{CG}} \subset W$ *which solves (6.43).*

Proof:
Firstly, to show existence, let

$$\hat{G}^h_{CG} := \bar{G}^h_{CG} + \tilde{G}^h_{CG}, \tag{6.45}$$

where $\bar{G}^h_{CG} \in W^{h,\underline{r}}$ is an existing vector-valued function which satisfies the fully-discrete data at the trace space and

$$\tilde{G}^h_{CG} \in W^{h,\underline{r}}_{\mathbf{0}}, \tag{6.46}$$

then the formulation in (6.43) written in terms of $\bar{G}^h_{CG}$ and $\tilde{G}^h_{CG}$ would be
Find $\tilde{G}^h_{CG} \in W^{h,\underline{r}}_{\mathbf{0}}$:

$$(\tilde{G}^h_{CG}, V^h)_W = -(\bar{G}^h_{CG}, V^h)_W, \qquad \forall V^h \in W^{h,\underline{r}}_{\mathbf{0}}. \tag{6.47}$$

Based on the choice of the conformal finite dimensional space $D^{h,A}$ which is a subspace of D^A, thus the form in (6.47) is the same one in (6.4) on p. 122 and with using the steps of the proof of Theorem 5.1 on p. 109 i.e. since $W^{h,\underline{r}}_{\mathbf{0}}$ is a subspace of W which is a Hilbert space (see [51, Proof of Theorem 2.C.]), then by Lax-Milligram theorem A.3 on p. 178 there exists a unique solution $\tilde{G}^h_{CG} \in W^{h,\underline{r}}_{\mathbf{0}}$ which solves the formulation in (6.47).
Secondly, with the given identity in (6.45) that is

$$\hat{G}^h_{CG} = \bar{G}^h_{CG} + \tilde{G}^h_{CG},$$

and since $\tilde{G}^h_{CG}$ exists, the identity concludes that $\hat{G}^h_{CG} \in W^{h,\underline{r}}_{G_{CG}}$ exists.
Finally, for the uniqueness, let $\hat{G}^{h,1}_{CG} = \begin{pmatrix}\hat{g}^{h,1}_1\\ \hat{g}^{h,1}_2\end{pmatrix} \in W^{h,\underline{r}}_{G_{CG}}$ and $\hat{G}^{h,2} = \begin{pmatrix}\hat{g}^{h,2}_1\\ \hat{g}^{h,2}_2\end{pmatrix} \in W^{h,\underline{r}}_{G^h_{CG}}$ be two solutions

which solve (6.43) on p. 135, and with taking the difference of the two solutions i.e. let $\bar{W} := \hat{G}_{CG}^{h,1} - \hat{G}_{CG}^{h,2} \in W_{\mathbf{0}}^{h,\underline{r}}$ which solves

$$(\bar{W}, V)_W = 0,$$

and with choosing $V = \bar{W} \in W_{\mathbf{0}}^{h,\underline{r}}$, then

$$(\bar{W}, \bar{W})_W = \| \bar{W} \|_W^2 \geq 0 \Longrightarrow \bar{W} = \hat{G}_{CG}^{h,1} - \hat{G}_{CG}^{h,2} = 0,$$

which means $\hat{G}_{CG}^{h,1} = \hat{G}_{CG}^{h,2}$, and that completes the proof. □

Lemma 6.6 (Stability estimate of the fully-discrete auxiliary (elliptic) problem) *Let W be the Hilbert space given in Definition 5.4 on p. 104. Let $W^{h,\underline{r}}$ be the fully-discrete conformal finite-dimensional subspace of W. Let $W^{h,\underline{r},tr}$ be the fully-discrete conformal finite dimensional subspace of the normed-space $(W^{tr}, \| \cdot \|_{W^{tr}})$ given in Definition 6.4 with the defined norm $\| \cdot \|_{W^{tr}}$ for $\bar{\gamma} : W^{h,\underline{r}} \to W^{h,\underline{r},tr}$:*

$$\| G_{CG}^h \|_{W^{tr}} = \inf_{\substack{U \in W^{h,\underline{r}} \\ \bar{\gamma} U = G_{CG}^h}} \| U \|_W, \quad \forall G_{CG}^h \in W^{h,\underline{r},tr}.$$

Let $W_{G_{CG}^h}^{h,\underline{r}}$ be the fully-discrete conformal finite-dimensional subset of W given in Definition 6.4 on p. 134. For given data $G_{CG}^h \in W^{h,\underline{r},tr}$ the solution $\hat{G}_{CG}^h \in W_{G_{CG}}^{h,\underline{r}}$ which solves (6.43) satisfies

$$\| \hat{G}_{CG}^h \|_W \leq 2 \| G_{CG}^h \|_{W^{tr}} . \tag{6.48}$$

Proof:
Starting with the identity given in (6.45) on p. 135 and the use of the *triangle inequality* yields

$$\| \hat{G}_{CG}^h \|_W = \| \bar{G}_{CG}^h + \tilde{G}_{CG}^h \|_W \leq \| \bar{G}_{CG}^h \|_W + \| \tilde{G}_{CG}^h \|_W . \tag{6.49}$$

Now, to bound $\| \tilde{G}_{CG}^h \|_W$ is by finding the stability estimate of the solution of the formulation (6.43) and this starts with choosing $V = \tilde{G}_{DG}^h$, in (6.47) and using the same steps in the proof of Lemma 6.1 on p. 122 implies

$$\| \hat{G}_{CG}^h \|_W \leq 2 \inf_{\substack{\bar{G}_{CG}^h \in W^{h,\underline{r}} \\ \bar{\gamma} \bar{G}_{CG}^h = G_{CG}^h}} \| \bar{G}_{CG} \|_W = 2 \| G_{CG} \|_{W^{tr}}, \tag{6.50}$$

and that completes the proof. □

6.4 The high-order in time DGFEM with conformal spatial discretisation, fully-discrete formulation

Definition 6.5 *Let Z be the Hilbert space given in Definition 5.1 on p. 101. Let Z_g be the subset of the Hilbert space Z and Z_0 be the Hilbert subspace of Z given in Definition 5.2 on p. 102 with Lemma 5.1 on p. 103. Let $Z_{g_{CG}^h}^h$ be the conformal finite dimensional subset of Z and subspace of Z_g. Then with recalling the semi-discrete spaces given in Definition 6.1 the following fully-discrete spaces are defined, accordingly*

$$\mathcal{V}^{\underline{r}}(\mathcal{N}; Z_{g_{CG}^h}^h \times Z_{g_{CG}^{h,\prime}}^h) := \Big\{ U : J \to Z_{g_{CG}^h}^h \times Z_{g_{CG}^{h,\prime}}^h \, : U|_{I_n} \in P^{r_n}(I_n; Z_{g_{CG}^h}^h \times Z_{g_{CG}^{h,\prime}}^h), 0 \leq n \leq N-1 \Big\},$$

and

$$\mathcal{V}^{\underline{r}}(\mathcal{N}; Z_0^h \times Z_0^h) := \Big\{ U : J \to Z_0^h \times Z_0^h \,:\, U|_{I_n} \in P^{r_n}(I_n; Z_0 \times Z_0),\ 0 \leq n \leq N-1 \Big\},$$

where $Z^h_{g^h_{CG}} \times Z^h_{g^{h,\prime}_{CG}}$ *is the conformal finite dimensional subset and* $Z_0^h \times Z_0^h$ *a finite dimensional subspace of* $Z \times Z$*, respectively.*

Let $\hat{a}(\,,)$ be the skew-symmetric form given in Definition 2.7 on p. 14 and $(\,,)_E$ be the inner product given in Corollary 2.2 on p. 10:

$$\hat{a}(U,V) = -a(u_2, v_1) + a(u_1, v_2), \text{ and } (U,V)_E = a(u_1, v_1) + (u_2, v_2)_E.$$

With using Definition 6.5 on p. 136, and for the given form

$$B_{DG}(U^h_{DG}, V^h) = \sum_{n=0}^{N-1} \int_{I_n} \Big\{ (\partial_t U^h_{DG}, V)_E + \hat{a}(U^h_{DG}, V^h) \Big\}\, dt + \sum_{n=1}^{N-1} ([U_{DG}]^{h,n}, V_+^{h,n})_E + (U^{h,0}_{DG,+}, V_+^{h,0})_E, \tag{6.51}$$

and for given $f \in L^2(J; H)$ the linear functional $F_{DG} \in (L^2(J; Z \times H))'$:

$$F_{DG}(V^h) = \sum_{n=0}^{N-1} \int_{I_n} (f, v_2^h)_H dt = \int_J (f, v_2^h)_H dt, \forall v_2^h, \in L^2(J; H), \tag{6.52}$$

where all vector-valued functions in semi-discrete space $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z_0^h \times Z_0^h)$ are also included in this linear map since $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z_0^h \times Z_0^h) \subset L^2(J; Z \times H)$.
Then the fully-discrete formulation of (6.12) on p. 124 reads:

Problem 6.4.1 (Fully-discrete formulation with linear constraints)
Let Z *and* H *be the Hilbert spaces given in Definition 5.1 on p. 101. Let* $Z^h_{g_{CG}}$*,* $Z^h_{g'_{CG}}$*, and* Z_0^h *be the conformal finite dimensional subsets and subspace given in Definition 6.5 on p. 136. Let* $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h_{g^h_{CG}} \times Z^h_{g^{h,\prime}_{CG}})$ *be the fully-discrete finite dimensional subset given in Definition 6.5. Let* $B_{DG}(\,,)$ *be the form given in* (6.51) *and* F_{DG} *be the linear functional given in* (6.52)*.*
Find $U^h_{DG} \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h_{g^h_{CG}} \times Z^h_{g^{h,\prime}_{CG}})$ *such that*

$$B_{DG}(U^h_{DG}, V^h) = F_{DG}(V^h) + (U^{h,0}_{DG,-}, V_+^{h,0})_E, \quad \forall\ V^h \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z_0^h \times Z_0^h), \tag{6.53}$$

for given compatible initial data $U^{h,0}_{DG,-} \in Z^h_{g^h_{CG}} \times Z^h_{g^{h,\prime}_{CG}}$ *and forcing data* $f \in H^1(J; H)$ *with the fully-discrete data* $G^h_{CG} \in W^{h,\underline{r},tr}$*.*

The time-stepping scheme for (6.53) would read:

Problem 6.4.2 (Fully-discrete time-stepping formulation with linear constraints)
Let Z *and* H *be the Hilbert spaces given in Definition 5.1 on p. 101. Let* $Z^h_{g_{CG}}$*,* $Z^h_{g'_{CG}}$*, and* Z_0^h *be the conformal finite dimensional subsets and subspace given in Definition 6.5 on p. 136. Let* $P^{r_n}(I_n; Z^h_{g^h_{CG}} \times Z^h_{g^{h,\prime}_{CG}})$ *be the fully-discrete finite dimensional subset given in Definition 6.5. Let* $\hat{a}(\,,)$ *be the skew-symmetric form given in Definition 2.7 on p. 14 and* $(\,,)_E$ *be the inner product given in Corollary 2.2 on p. 10:*

$$\hat{a}(U,V) = -a(u_2, v_1) + a(u_1, v_2), \text{ and } (U,V)_E = a(u_1, v_1) + (u_2, v_2)_E.$$

Find $U^h_{DG} \in P^{r_n}(I_n; Z^h_{g^h_{CG}} \times Z^h_{g^{h,\prime}_{CG}})$ *such that*

$$\overbrace{\int_{I_n} \Big\{(\partial_t U^h_{DG}, V^h)_E + \hat{a}(U^h_{DG}, V^h)\Big\} dt + (U^{h,n}_{DG,+}, V^{h,n}_+)_E}^{b^n_{DG}(U^h_{DG}, V^h)=} = \overbrace{\int_{I_n} (f, v^h_2)_H dt}^{F^n_{DG}(v^h_2)=} + (U^{h,n}_{DG,-}, V^{h,n}_+)_E,$$

$\forall V^h \in P^{r_n}(I_n; Z^h_0 \times Z^h_0)$, *with* $U^{h,n}_{DG,-} \in Z^h_{g^h_{CG}} \times Z^h_{g^{h,\prime}_{CG}}$ *to be the initial data at a given time step* I_n (6.54)

and forcing data $f \in H^1(J; H)$ *with the fully-discrete data* $G^h_{CG} \in W^{h,\underline{r},tr}$.

6.4.1 Existence and uniqueness

The splitting technique will be used here again in order to show existence and uniqueness of the fully-discrete solution which solves (6.12) on p. 137. Theorem 6.5 on p. 141 shows existence and uniqueness of the solution of (6.12). The latter theorem uses Lemma 6.7 on p. 140.

6.4.1.1 Time decoupling and the discrete reduced spatial problems of the fully-discrete transformed formulation with linear constraints

Let $\hat{G}^h_{CG} \in W^{h,\underline{r}}_{G^h_{CG}} \subset \mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h_{g^h_{CG}} \times Z^h_{g^{h,\prime}_{CG}})$ be the existing fully-discrete real vector-valued function which solves the formulation in (6.43) on p. 135 such that

$$U^h_{DG} := \hat{G}^h_{CG} + \bar{U}^h_{DG}. \tag{6.55}$$

With the defined fully-discrete real vector-valued function $\bar{U}^h_{DG}$, such that $\bar{U}^h_{DG} \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h_0 \times Z^h_0)$, and for

$$F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} -\partial_t \hat{g}^h_{1,CG} + \hat{g}^h_{2,CG} \\ f - \partial_t \hat{g}^h_{2,CG} - A\hat{g}^h_{1,CG} \end{pmatrix} \in L^2(J; D_0 \times H) \in L^2(J; Z_0 \times H), \tag{6.56}$$

the linear functional $\mathcal{F}_{DG} \in (L^2(J; Z_0 \times H))'$:

$$\mathcal{F}_{DG}(V^h) := \sum_{n=0}^{N-1} \int_{I_n} \{\overbrace{a(f_1, v^h_1) + (f_2, v^h_2)_H}^{(F, V^h)_E}\} dt = \int_J (F, V^h)_E dt, \forall V^h \in L^2(J; Z^h_0 \times H), \tag{6.57}$$

where all vector-valued functions in fully-discrete space $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h_0 \times Z^h_0)$ are also included in this linear map since $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h_0 \times Z^h_0) \subset L^2(J; Z_0 \times H)$.
then the formulation in (6.53) on p. 137 written in terms of $\hat{G}^h_{CG}$ and $\bar{U}^h_{DG}$ would be

Problem 6.4.3 (Fully-discrete transformed formulation)
Let Z *and* H *be the Hilbert spaces given in Definition 5.1 on p. 101. Let* Z^h_0 *be the conformal finite dimensional subspace given in Definition 6.5 on p. 136. Let* $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h_0 \times Z^h_0)$ *be the fully-discrete finite dimensional subspace given in Definition 6.5. Let* $B_{DG}(\,,)$ *be the form given in (6.51) on p. 137 and* $\mathcal{F}_{DG}$ *the linear functional in (6.57).*
Find $\bar{U}^h_{DG} \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h_0 \times Z^h_0)$ *such that*

$$B_{DG}(\bar{U}^h_{DG}, V^h) = \mathcal{F}_{DG}(V^h) + (\bar{U}^{h,0}_{DG,-}, V^{h,0}_+)_E, \forall\ V^h \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h_0 \times Z^h_0), \tag{6.58}$$

for given initial data $\bar{U}^h_0 \in Z^h_0 \times Z^h_0$.

Problem 6.4.4 (Fully-discrete time-stepping transformed formulation)
Let Z and H be the Hilbert spaces given in Definition 5.1 on p. 101. Let Z_0^h be the conformal finite dimensional subspace given in Definition 6.5 on p. 136. Let $P^{r_n}(I_n; Z_0^h \times Z_0^h)$ be the fully-discrete finite dimensional subspace given in Definition 6.5. Let $(\,,)_E$ be the inner product given in Corollary 2.2 on p. 10. Let $\hat{a}(\,,)$ be the skew-symmetric form given in Definition 2.7 on p. 14 and $(\,,)_E$ be the inner product given in Corollary 2.2 on p. 10:

$$\hat{a}(U,V) = -a(u_2,v_1) + a(u_1,v_2), \text{ and } (U,V)_E = a(u_1,v_1) + (u_2,v_2)_E.$$

Find $\bar{U}_{DG}^h \in P^{r_n}(I_n; Z_0^h \times Z_0^h)$ such that

$$\overbrace{\int_{I_n} \Big\{(\partial_t \bar{U}_{DG}, V)_E + \hat{a}(\bar{U}_{DG}^h, V^h)\Big\}dt + (\bar{U}_{DG,+}^{h,n}, V_+^{h,n})_E}^{b_{DG}^n(\bar{U}_{DG}^h, V^h)=} = \overbrace{\int_{I_n} (F, V^h)_E dt + (\bar{U}_{DG,-}^{h,n}, V_+^{h,n})_E}^{\mathcal{F}_{DG}^n(V^h)=}, \quad (6.59)$$

for all $V^h \in P^{r_n}(I_n; Z_0^h \times Z_0^h)$, with $\bar{U}_{DG,-}^{h,n}$ to be the initial data at a given time step I_n, and $F \in L^2(J; Z^h \times H) \subset L^2(J; Z \times H)$.

Here, (6.59) is the fully-discrete formulation of the semi-discrete one in (6.19). Now, since the latter is a particular example of the general semi-discrete formulation given in (4.28) on p. 83 and with knowing that Z_0^h and D_0^h are the conformal finite dimensional subspaces of Z_0 and D_0 which also fit the properties of the discrete spatial spaces in Chapter 4, then (6.59) is a particular example of the general fully-discrete formulation given in (4.56) on p. 94, where now the terms in $F = \binom{f_1}{f_2}$ are

$$f_1 = -\partial_t \hat{g}_{1,CG}^h + \hat{g}_{2,CG}^h \in L^2(J; D_0) \subset L^2(J; Z_0), \text{ and } f_2 = f - \partial_t \hat{g}_{2,CG}^h - A\hat{g}_{1,CG}^h \in L^2(J; H).$$

With repeating the same steps in Section 4.3 and considering the formulation (4.56) on p. 94 with its equivalent $(r+1)$ discrete reduced spatial problems in (4.60) in Theorem 4.7 on p. 96, the formulation in (6.59) reads the same as (4.56), and accordingly, the $(r+1)$ discrete reduced spatial problems of (6.59) with referring to (4.59) on p. 95:

Problem 6.4.5 (Discrete reduced spatial problems for the transformed formulation)
Let Z and H be the Hilbert spaces given in Definition 5.1 on p. 101. Let Z_0 be the subspace of Z given in Definition 5.2 with Lemma 5.1 on p. 102. Let Z^h be the conformal finite dimensional subspace of Z. Let Z_0^h be the conformal finite dimensional subspace of Z_0. Let λ_p be the eigenvalues of the matrix $\mathcal{A}^L$ given in (3.31) on p. 30. Let $(\beta_p^1)_q$ and $(\beta_p^2)_q$ be the $(r+1)$ vectors given in (3.36):

$$\begin{aligned}(\beta_p^1)_q &= (Y^{-1})_{pq}\, x_q^1 = (Y^{-1})_{pq}\sqrt{(q+1/2)}, \text{ and}\\ (\beta_p^2)_q &= (Y^{-1})_{pq}\, x_q^2 = (Y^{-1})_{pq}\sqrt{(q+1/2)}(-1)^q, \text{ respectively.}\end{aligned} \quad (6.60)$$

For

$$\begin{aligned}\mathfrak{f}^1(\mathfrak{v}_{2,p}^h) &= 2\lambda k \sum_{i=0}^{r} \Big\{(\beta_p^1)_i \int_{-1}^{1} L_i(\hat{t})(f_2 \circ Q, \mathfrak{v}_{2,p}^h)_H d\hat{t}\Big\}, \text{ and}\\ \mathfrak{f}^2(\mathfrak{v}_{2,p}^h) &= -k^2 \sum_{i=0}^{r} \Big\{(\beta_p^1)_i \int_{-1}^{1} L_i(\hat{t}) a(f_1 \circ Q, \mathfrak{v}^{\mathfrak{h}}{}_{2,p}) d\hat{t}\Big\},\end{aligned}$$

and with a form and a linear functional $\mathfrak{f} \in (Z_0 \times H)'$:

$$\begin{aligned}b(\mathfrak{u}_{2,p}^h, \mathfrak{v}_{2,p}^h) &= 4\lambda_p^2(\mathfrak{u}_{2,p}^h, \mathfrak{v}_{2,p})_H + k^2 a(\mathfrak{u}_{2,p}^h, \mathfrak{v}_{2,p}^h),\\ \mathfrak{f}(\mathfrak{v}_{2,p}^h) &= \mathfrak{f}^1(\mathfrak{v}_{2,p}^h) + \mathfrak{f}^2(\mathfrak{v}_{2,p}^h),\end{aligned}$$

where all test functions in the conformal finite dimensional subspace Z_0^h are also included in this linear map since $Z_0^h \subset Z_0$, now reads:
Find $\{\mathfrak{u}_{2,p}^h\}_{p=0}^{r_n} \subset Z_0^h$:

$$b(\mathfrak{u}_{2,p}^h, \mathfrak{v}_{2,p}^h) = \mathfrak{f}(\mathfrak{v}_{2,p}^h) + 4\lambda \sum_{i=0}^{r_n} \Big\{(\beta_p^2)_i\Big\}(\bar{u}_{2,0}^h, \mathfrak{v}_{2,p}^h)_H - 2k \sum_{i=0}^{r} \Big\{(\beta_p^2)_i\Big\} a(\bar{u}_{1,0}^h, \mathfrak{v}_{2,p}^h),$$

for each $p = 0, \cdots, r$, for all $\{\mathfrak{v}_p^h\}_{p=0}^{r_n} \subset Z_0^h$.

(6.61)

After solving for the unknowns $\{\mathfrak{u}_{2,p}^h\}_{p=0}^{r} \subset Z_0^h$ with given data $f_1 \in L^2(I; Z_0^h)$, $f_2 \in L^2(I; H)$, and $\bar{u}_{1,0}^h$, $\bar{u}_{2,0}^h \in Z_0^h \subset Z^h \subset Z$, then comes to update the values of $\{\mathfrak{u}_{1,p}^h\}_{p=0}^{r} \subset Z_0^h$:

$$\mathfrak{u}_{1,p}^h = \frac{k}{2\lambda_p}\mathfrak{u}_{2,p}^h + \frac{k}{2\lambda_p} \sum_{i=0}^{r} \Big\{(\beta_p^1)_i \int_{-1}^{1} L_i(\hat{t}) f_1 \circ Q d\hat{t}\Big\} + \frac{1}{\lambda_p} \sum_{i=0}^{r} \Big\{(\beta_p^2)_i\Big\}\bar{u}_{1,0}^h, \tag{6.62}$$

which need to be done at the given generic time step I.

Lemma 6.7 (Existence, uniqueness of discrete reduced spatial problem's for given $r.h.s$**)**

Let Z be the Hilbert space given in Definition 5.1 on p. 101. Let Z_0 be the subspace of Z given in Definition 5.2 with Remark 5.1 on p. 102. Let Z^h be the conformal finite dimensional subspace of Z. Let Z_0^h be the conformal finite subspace of Z_0. There exist unique functions $\{\mathfrak{u}_{2,p}^h\}_{p=0}^{r} \subset Z_0^h$ which solve the $(r_n + 1)$ discrete spatial problems (6.61), as well as $\{\mathfrak{u}_{1,p}^h\}_{p=0}^{r} \subset Z_0^h$, for given $f_1 \in L^2(I; Z)$, $f_2 \in L^2(I; H)$, and initial data $\bar{u}_{1,0}^h, \bar{u}_{2,0}^h \in Z_0^h \subset Z_0 \subset Z$.

Proof:
Firstly, the form $b(\mathfrak{u}_{2,p}^h, \mathfrak{v}_{2,p}^h)$ in (6.61) for $\mathfrak{u}_{2,p}^h$, and $\mathfrak{v}_{2,p}^h$ in the subspace $(Z_0^h, \|\cdot\|_Z)$ is continuous with using Lemma 3.5 on p. 34 after letting $c_1 = 4\lambda_p$ and $c_2 = k^2$, thus, for $p = 0, \cdots, r_n$:

$$b(\mathfrak{u}_{2,p}^h, \mathfrak{v}_{2,p}^h) = 4\lambda_p^2(\mathfrak{u}_{2,p}^h, \mathfrak{v}_{2,p}^h)_H + k^2 a(\mathfrak{u}_{2,p}^h, \mathfrak{v}_{2,p}^h) \leq (4|\lambda_p|c^2 + k^2 M) \parallel \mathfrak{u}_{2,p}^h \|_Z \| \mathfrak{v}_{2,p}^h \|_Z .$$

Secondly, with using Definition 5.3 on p. 104 the form $b(\mathfrak{u}_{2,p}^h, \mathfrak{v}_{2,p}^h)$ in (6.61) for $\mathfrak{u}_{2,p}^h$, and $\mathfrak{v}_{2,p}^h$ in the subspace $(Z_0^h, \|\cdot\|_Z)$ which is again a subspace of $(Z_0, \|\cdot\|_Z)$ satisfies the discrete Inf-sup condition in Lemma 3.11 on p. 47 after letting $c_1 = 4\lambda_p$ and $c_2 = k^2$. The Inf-sup constant would be $k^2\alpha$. Then for the $r.h.s$ of (6.61) and with recalling the proof of Theorem 4.6 on p. 96, then

$$|r.h.s| \leq \Bigg(c\sqrt{2^3k}|\lambda_p| \sum_{q=0}^{r} |(\beta_p^1)_q|\hat{x}_q \parallel f_2 \|_{L^2(I;H)} + |\sum_{q=0}^{r}(\beta_p^2)_q|\Big(4c|\lambda_p| \parallel u_{2,DG,-}^{h,0} \|_H + 2kM \parallel u_{1,DG,-}^{h,0} \|_Z \Big) + \sqrt{2k^3}M \sum_{q=0}^{r_n} |(\beta_p^1)_q|\hat{x}_q \parallel f_1 \|_{L^2(I;Z)} \Bigg) \parallel \mathfrak{v}_{2,p}^h \|_Z . \tag{6.63}$$

Then, going back to (Babuška) Theorem A.4 on p. 179, and with having Lemmas 3.11 and 3.5 and (6.63), then there exist a function $\{\mathfrak{u}_{2,p}^h\}_{p=0}^{r} \subset Z_0^h$ which solves (6.61).
For the identity (6.62) on p. 140:

$$\mathfrak{u}_{1,p}^h = \frac{k}{2\lambda_p}\mathfrak{u}_{2,p}^h + \frac{1}{\lambda_p} \sum_{q=0}^{r} \Big\{(\beta_p^2)_q\Big\} u_{1,DG,-}^{h,0} + \frac{k}{2\lambda_p} \sum_{q=0}^{r} \Big\{(\beta_p^1)_q \int_{-1}^{1} L_q(\hat{t}) f_1 \circ Q d\hat{t}\Big\},$$

where

$$\| \frac{k}{2\lambda_p} \sum_{q=0}^{r} \Big\{ (\beta_p^1)_q \int_{-1}^{1} L_q(\hat{t}) f_1 \circ Q d\hat{t} \Big\} \|_Z \leq \frac{k}{2|\lambda_p|} \sum_{q=0}^{r} \Big\{ |(\beta_p^1)_q| \int_{-1}^{1} L_q(\hat{t}) \| f_1 \circ Q \|_Z \, d\hat{t} \Big\} < \infty,$$

for each $p = 0, \cdots, r$, then from the given existence and uniqueness of $\{\mathfrak{u}_{2,p}^h\}_{p=0}^r \subset Z_0^h$ and the given forcing data $f_1 \in L^2(J; Z_0^h)$, $f_2 \in L^2(J; H)$ and initial data $u_{1,DG,-}^{h,0}, u_{2,DG,-}^{h,0} \in Z_0^h$ which are well defined, that also concludes that $\{\mathfrak{u}_{1,p}^h\}_{p=0}^r \subset Z_0^h$ at a given time step I also exists and is unique. □

Theorem 6.5 (Existence, uniqueness of fully-discrete formulation with linear constraints) *Let Z and H be the Hilbert spaces given in Definition 5.1 on p. 101. Let Z^h be the conformal finite dimensional subspace of Z. Let $Z_{g_{CG}^h}^h$ and $Z_{g_{CG}^{h,\prime}}^h$ be the conformal finite dimensional subsets given in Definition 6.5 on p. 136. Let $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z_{g_{CG}^h}^h \times Z_{g_{CG}^{h,\prime}}^h)$ be the fully-discrete space given in Definition 6.5 on p. 136. There exists a unique fully-discrete vector-valued function $U_{DG}^h \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z_{g_{CG}^h}^h \times Z_{g_{CG}^{h,\prime}}^h)$ which solves the formulation in (6.53) on p. 137.*

Proof:
The fully-discrete vector-valued function U_{DG}^h is given in the identity (6.55) on p. 138 that is

$$U_{DG}^h = \bar{U}_{DG}^h + \hat{G}_{CG}^h,$$

for the existing unique vector-valued function $\hat{G}_{CG}^h \in W_{G_{CG}^h}^{h,\underline{r}} \subset \mathcal{V}^{\underline{r}}(\mathcal{N}; Z_{g_{CG}^h}^h \times Z_{g_{CG}^{h,\prime}}^h)$ which solves (6.43) on p. 135 and for $\bar{U}_{DG}^h \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z_0^h \times Z_0^h)$ which solves the formulation (6.58) on p. 138:

$$\bar{U}_{DG}^h = \sum_{n=0}^{N-1} \bar{U}_{DG}^h|_{I_n},$$

where

$$\bar{U}_{DG}^h|_{I_n} = \sum_{j=0}^{r_n} \varphi_j(t) \bar{U}_j^h, \text{ for } Y^{r_n} \in \mathbb{C}^{(r_n+1)\times(r_n+1)} : \bar{U}_j^h = \sum_{i=0}^{r_n} Y_{ji}^{r_n} \mathfrak{U}_i^h \in Z_0^h \times Z_0^h,$$

and with using the shown existence and uniqueness of $\{\mathfrak{U}_i^h\}_{i=0}^{r_n} \in Z_0^h \times Z_0^h$ in Lemma 6.7 which implies that $\bar{U}_{DG}^h \in P^{r_n}(I_n; Z_0^h \times Z_0^h)$ exists and is unique and from the identity (6.55) thus $U_{DG}^h \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z_{g_{CG}^h}^h \times Z_{g_{CG}^{h,\prime}}^h)$ exists and is unique. □

6.4.2 Stability estimates of fully-discrete formulation with linear constraints

In this Section, global stability estimate of the vector-valued function $U_{DG}^h \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z_{g_{CG}^h}^h \times Z_{g_{CG}^{h,\prime}}^h)$ which solves the fully-discrete formulation in (6.53) on p. 137 will be shown in Theorem 6.7 on p. 144. The latter uses the resulting estimates in Lemmas 6.8 and 6.9 on p. 141. The stability estimates shown in Section 4.3.2 are used here, since the transformed formulation in this section is a particular example of the given general formulation in Section 4.3.2.

Lemma 6.8 (Local estimate for fully-discrete transformed formulation) *Let Z and H be the Hilbert spaces given in Definition 5.1 on p. 101. Let Z_0 be the subspace of Z given in Definition 5.2 with Lemma 5.1 on p. 102. Let Z^h be the conformal finite dimensional subspace of Z. Let Z_0^h be the* *conformal finite dimensional subspace of* Z_0. *With given vector-valued forcing data* $F \in L^2(J, Z_0 \times H)$, *initial data* $\bar{U}_0^h \in Z_0^h \times Z_0^h$, *then the vector-valued function* $\bar{U}_{DG} \in P^r(I; Z_0^h \times Z_0^h)$ *at a generic time step* $I = (t_0, t_1)$ *satisfies*

$$\| \bar{U}_{DG}^h \|_{L^2(I;E)}^2 \leq \frac{k \| Y \|_2^2 \, (2r+1)}{2 \min\limits_p \ Re \, \lambda_p} \Big(\| F \|_{L^2(I;E)}^2 + \| \bar{U}_0^h \|_E^2 \Big), \tag{6.64}$$

for normed-product space $(L^2(J; Z_0^h \times Z_0^h), \| \cdot \|_{L^2(J;E)})$:

$$\| U \|_{L^2(J;E)}^2 = \int_J \overbrace{a(u_1, u_1) + \| u_2 \|_H^2}^{\|U\|_E^2} dt, \forall \, U \in L^2(J; Z_0^h \times Z_0^h) \subset L^2(J; Z_0 \times Z_0), \tag{6.65}$$

where k *is the time step size at* I. λ_p *are the eigenvalues of the matrix* $\mathcal{A}^L$ *given in* (3.31) *on p. 30 for* $p = 0, \cdots, r$, $\| Y^r \|_2^2$ *is the square of the spectral norm (see* (A.6) *on p. 173) of the transformation matrix* $Y \in \mathbb{C}^{(r+1)\times(r+1)}$ *and* r *is the approximation order and the form* $a(\, ,)$ *satisfies the properties given in Definition 5.3 on p. 104.*

Proof:
At generic time step I, with the *ellipticity* property given in Definition 5.3 on p. 104 for $a(\, ,)$, the form $b_{DG}(\bar{U}_{DG}^h, V^h)$ of Problem 6.4.4 on p. 139 for $\bar{U}_{DG}^h$, and V^h in $P^r(I; Z_0^h \times Z_0^h) \subset P^r(I; Z_0 \times Z_0)$ satisfies Lemma 3.10 on p. 45. Thus

$$\begin{aligned} \| \bar{U}_{DG}^h \|_{L^2(I;E)} &\leq \frac{k \| Y \|_2^2}{2(\min\limits_p \ \mathrm{Re} \ \lambda_p)} \sup_{V^h \in P^r(I; Z_0^h \times Z_0^h)} \frac{\left| b_{DG}(U_{DG}^h, V^h) \right|}{\| V^h \|_{L^2(I;E)}} \\ &= \frac{k \| Y \|_2^2}{2(\min\limits_p \ \mathrm{Re} \ \lambda_p)} \sup_{V^h \in P^r(I; Z_0^h \times Z_0^h)} \frac{\left| \overbrace{\int_I (F, V^h)_E dt}^{F_{DG}(V^h)=} + (\bar{U}_{DG,-}^{h,0}, V_+^h)_E \right|}{\| V^h \|_{L^2(I;E)}}, \end{aligned} \tag{6.66}$$

and then with using the same steps in the proof Lemma 4.4 on p. 97 with

$$F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} -\partial_t \hat{g}_{1,CG}^h + \hat{g}_{2,CG}^h \\ f - \partial_t \hat{g}_{2,CG}^h - A \hat{g}_{1,CG}^h \end{pmatrix} \in L^2(J; Z_0 \times H),$$

that completes the proof. □

Lemma 6.9 (Left-sided limits estimate of fully-discrete transformed formulation)
Let Z *and* H *be the Hilbert spaces given in Definition 5.1 on p. 101. Let* Z_0 *be the subspace of* Z *given in Definition 5.2 with Remark 5.1 on p. 102. Let* Z^h *be the conformal finite dimensional subspace of* Z. *Let* Z_0^h *be the conformal finite subspace of* Z_0. *Let* $D^{h,A,tr}$ *be the conformal finite dimensional space of* $D^{A,tr}$ *given in Definition 6.5. Let* $W^{h,\underline{r},tr}$ *be the conformal finite dimensional space given in Definition 6.4 on p. 134. With given forcing data* $f \in L^2(J, H)$, *initial data* $\bar{U}_0^h \in Z_0^h \times Z_0^h$, *and the data* $G_{CG}^h \in W^{h,\underline{r},tr}$, *then the fully-discrete vector-valued function* $\bar{U}_{DG}^h \in P^{r_m}(I_m; Z_0^h \times Z_0^h)$ *which solves the time-stepping formulation in* (6.59) *on p. 139 satisfies*

$$\| \bar{U}_{DG,-}^{h,n} \|_E \leq (1 + \frac{\mathcal{C}}{N})^n \big(4\sqrt{\max(M, c^2, 1)} \, \| G_{CG}^h \|_{W^{tr}} + \sqrt{2} \, \| f \|_{L^2(J;H)} + \| \bar{U}_0^h \|_E \big), \tag{6.67}$$

for normed-product space $(Z_0^h \times Z_0^h, \|\cdot\|_E)$ *which is a finite dimensional subspace of* $(Z_0 \times Z_0, \|\cdot\|_E)$*, with Remark 5.1:*

$$\| U \|_E^2 = a(u_1, u_1) + \| u_2 \|_H^2, \forall\, U \in Z_0^h \times Z_0^h \subset Z_0 \times Z_0, \tag{6.68}$$

and $\mathcal{C} = \max_m \left(\frac{T \| Y^{r_m} \|_2^2 (2r_m + 1)}{2 \min_p \; Re\, \lambda_p} \right)$*, is the same constant appearing in Lemma 4.5 on p. 98, and the form* $a(\,,\,)$ *satisfies the properties given in Definition 5.3 on p. 104.*

Proof:
With knowing that Z_0 is separable Hilbert space as shown in Lemma 5.1 on p. 103 and the form $a(\,,\,)$ in Z_0 is *elliptic*, then at a generic time step I the formulation in (6.59) on p. 139 is a particular example of the general formulation in (4.66) on p. 98 where here

$$F = \begin{pmatrix} -\partial_t \hat{g}^h_{1,CG} + \hat{g}^h_{2,CG} \\ f - \partial_t \hat{g}^h_{2,CG} - A\hat{g}^h_{1,CG} \end{pmatrix} \in L^2(J; Z_0 \times H),$$

with given initial data $\bar{U}_0^h \in Z_0^h \times Z_0^h \subset Z_0 \times Z_0 \subset Z \times Z$. Thus the proof of Lemma 4.5 on p. 98 holds for $\bar{U}_{DG}^h \in P^{r_m}(I_m; Z_0^h \times Z_0^h)$ and also repeating the same steps done in Lemma 6.4 on p. 130 which don't depend on the spatial spaces that completes the proof. □

Theorem 6.6 (Last left-sided limit of the fully-discrete formulation)
Let Z *and* H *be the Hilbert spaces given in Definition 5.1 on p. 101. Let* Z^h *be the conformal finite dimensional subspace of* Z*. Let* $Z^h_{g^h_{CG}}$ *and* $Z^h_{g^{h,\prime}_{CG}}$ *be the conformal finite dimensional subsets given in Definition 6.5 on p. 136. Let* $D^{h,A,tr}$ *be the conformal finite dimensional space of* $D^{A,tr}$ *given in Definition 6.5. Let* $W^{h,\underline{r},tr}$ *be the conformal finite dimensional space given in Definition 6.4 on p. 134. With given forcing data* $f \in L^2(J, H)$*, initial data* $U_0^h \in Z^h_{g^h_{CG}} \times Z^h_{g^{h,\prime}_{CG}} \subset Z_g \times Z_{g'}$*,* $G_0^h \in D^{h,A,tr} \times D^{h,A,tr} \subset D^{A,tr} \times D^{A,tr}$*, and the last left sided limit* $G_-^{h,N} \in D^{h,A,tr} \times D^{h,A,tr} \subset D^{A,tr} \times D^{A,tr}$*, then the full-discrete vector-valued function* $U_{DG} \in P^{r_n}(I_n; Z^h_{g^h_{CG}} \times Z_{g^{h,\prime}_{CG}})$ *which solves the time-stepping formulation in (6.54) on p. 138 satisfies*

$$\begin{aligned} \| U^{h,N}_{DG,-} \|_X \le (1 + \frac{\mathcal{C}}{N})^n \big(4\sqrt{\max(M, c^2, 1)} \, \| G^h_{CG} \|_{W^{tr}} + \sqrt{2} \, \| f \|_{L^2(J;H)} \\ + \| U_0^h \|_E + \sqrt{\max(M, c^2)} \, \| G_0^h \|_{D^{A,tr} \times D^{A,tr}} \big) + \sqrt{\max(1, c^2)} \, \| G^{h,N}_{CG,-} \|_{D^{A,tr} \times D^{A,tr}}, \end{aligned} \tag{6.69}$$

where $\mathcal{C} = \max_n \frac{T \| Y^{r_n} \|_2^2 (2r_n + 1)}{2 \min_p \; Re\, \lambda_p}$*, is the same constant appearing in Lemma 4.3 on p. 89, with the norm*

$$\| U \|_X^2 = \| u_1 \|_Z^2 + \| u_2 \|_H^2, \forall\, U \in Z^h \times Z^h \subset Z \times Z, \tag{6.70}$$

for the form $a(\,,\,)$ *which satisfies the properties given in Definition 5.3 on p. 104.*

Proof:
With the identity given in (6.14) on p. 125, the triangle inequality, then

$$\| U^{h,N}_{DG,-} \|_X \le \| \bar{U}^{h,N}_{DG,-} \|_X^2 + \| \hat{G}^{h,N}_{DG,-} \|_X \le \frac{1}{\sqrt{\min(\alpha, 1)}} \| \bar{U}^{h,N}_{DG,-} \|_E^2 + \sqrt{\max(1, c^2)} \, \| \hat{G}^{h,N}_{DG,-} \|_{D^A \times D^A}$$

Since the spatial discretisation is done using conformal finite subspaces of the continuous ones, then the steps done in the proof of Theorem 6.3 on p. 131 still holds with considering now the discrete initial data i.e.

$$\| \bar{U}_0^h \|_E \leq \| U_0^h \|_E + \sqrt{\max(M, c^2)} \| G_0^h \|_{D^{A,tr} \times D^{A,tr}},$$

and that completes the proof. □

Theorem 6.7 (Global stability of fully-discrete formulation with linear constraints) *Let Z, H, and D^A be the Hilbert spaces given in Definition 5.1 on p. 101. Let Z^h be the conformal finite dimensional subspace of Z. Let $Z^h_{g^h_{CG}}$ and $Z^h_{g^{h,\prime}_{CG}}$ be the conformal finite dimensional subsets given in Definition 6.5 on p. 136. Let $D^{h,A,tr}$ be the conformal finite dimensional space of $D^{A,tr}$ given in Definition 6.5. Let $W^{h,\underline{r},tr}$ be the conformal finite dimensional space given in Definition 6.4 on p. 134. With given forcing data $f \in L^2(J;H)$, initial data $U_0^h \in Z^h_{g^h_{CG}} \times Z^h_{g^{h,\prime}_{CG}}$, and given data $G^h_{CG} \in W^{h,\underline{r},tr}$ with $G_0^h \in D^{h,A,tr} \times D^{h,A,tr}$, then the solution $U^h_{DG} \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h_{g^h_{CG}} \times Z^h_{g^{h,\prime}_{CG}})$ of the formulation (6.53) on p. 137 satisfies*

$$\begin{aligned} \| U^h_{DG} \|_{L^2(J;X)} \leq & \big(2\sqrt{\max(1,c^2)} + 4\sqrt{\max(M,c^2,1)}\frac{(e^{\mathcal{C}}-1)}{\sqrt{\min(\alpha,1)}}\big) \| G^h_{CG} \|_{W^{tr}} \\ & + \frac{(e^{\mathcal{C}}-1)}{\sqrt{\min(\alpha,1)}}\big(\sqrt{2} \| f \|_{L^2(J;H)} + \| U_0^h \|_E + \sqrt{\max(M,c^2)} \| G_0^h \|_{D^{A,tr}\times D^{A,tr}} \big), \end{aligned} \tag{6.71}$$

for normed-product space $(Z^h \times Z^h, \| \cdot \|_X)$:

$$\| U \|_X^2 = \| u_1 \|_Z^2 + \| u_2 \|_H^2, \forall\, U \in Z^h \times Z^h, \tag{6.72}$$

where $\mathcal{C} = \max_n \big(\frac{T \| Y^{r_n} \|_2^2 (2r_n+1)}{2 \min_p \ Re\ \lambda_p}\big)$, is the same constant which appear in Lemma 4.8 on p. 98. $M < \infty$ the lower bounds of the form $a(\,,)$ which satisfies the properties given in Definition 5.3 on p. 104. c the constant in the inclusion inequality $\| u \|_H \leq c \| u \|_Z, \forall u \in Z$, and the norm $\| \cdot \|_{W^{tr}}$ given in Corollary 5.5 on p. 107.

Proof:
Here

$$U^h_{DG} = \hat{G}^h_{CG} + \bar{U}^h_{DG},$$

with using the *triangle inequality* and from the inequality in Lemma 5.6 on p. 108 but this time with considering the correspondent norm of the D^A in Definition 5.1 on p. 101, then

$$\begin{aligned} \| U^h_{DG} \|_{L^2(J;X)} &= \| \hat{G}^h_{CG} + \bar{U}^h_{DG} \|_{L^2(J;X)} \\ &\leq \| \hat{G}^h_{CG} \|_{L^2(J;X)} + \| \bar{U}^h_{DG} \|_{L^2(J;X)} \\ &\leq \sqrt{\max(1,c^2)} \underbrace{\| \hat{G}^h_{CG} \|_{L^2(J;D^A\times D^A)}}_{\leq \| \hat{G}^h_{CG} \|_{H^1(J;D^A\times D^A)}} + \frac{1}{\sqrt{\min(\alpha,1)}} \| \bar{U}^h_{DG} \|_{L^2(J;E)} \\ &= \sqrt{\max(1,c^2)} \| \hat{G}^h_{CG} \|_W + \frac{1}{\sqrt{\min(\alpha,1)}} \sum_{n=0}^{N-1} \| \bar{U}^h_{DG} \|_{L^2(I_n;E)} . \end{aligned}$$

Now, with using the resulting estimate in Lemma 6.8 on p. 141 after letting $I = I_n$ and the correspondence initial data accordingly with the use of Lemma 6.9 on p. 142, and the same steps used in the proof of Theorem 4.8 on p. 98 and repeating the same steps done in the proof of Theorem 6.4 on p. 132 in order to get the upper bound of the $\| F \|^2_{L^2(J;E)}$ in terms of the forcing data f and G^h_{CG}; the smooth data at the trace space where

$$\bar{U}^h_0 = U^h_0 - \hat{G}^h_0,$$

implies

$$\| U^h_{DG} \|_{L^2(J;X)} \leq \big(2\sqrt{\max(1,c^2)} + 4\sqrt{\max(M,c^2,1)}\frac{(e^{\mathcal{C}}-1)}{\sqrt{\min(\alpha,1)}}\big) \| G^h_{CG} \|_{W^{tr}} + \frac{(e^{\mathcal{C}}-1)}{\sqrt{\min(\alpha,1)}}\big(\sqrt{2} \| f \|_{L^2(J;H)} + \| U^h_0 \|_E + \sqrt{\max(M,c^2)} \| G^h_0 \|_{D^{A,tr}\times D^{A,tr}} \big),$$

and that completes the proof. ☐

6.5 The a priori error estimate

6.5.1 An auxiliary problem and its stability estimate

With recalling the subsets in Definitions 5.1, 5.2 on p. 101, and 6.5 on p. 136, the following auxiliary problem is considered

Problem 6.5.1 (The fully-discrete auxiliary problem)
Let Z and H be the Hilbert spaces given in Definition 5.1 on p. 101. Let Z^h_0 be the conformal finite subspace given in Definition 6.5 on p. 136. Let $\mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h_0 \times Z^h_0)$ be the fully-discrete finite dimensional subspace given in Definition 6.5. Let $\hat{a}(\,,\,)$ be the skew-symmetric form given in Definition 2.7 on p. 14 and $(\,,\,)_E$ be the inner product given in Corollary 2.2 on p. 10:

$$\hat{a}(U,V) = -a(u_2,v_1) + a(u_1,v_2), \text{ and } (U,V)_E = a(u_1,v_1) + (u_2,v_2)_E.$$

Let $B_{DG}(\,,\,)$ be the form given in (6.51) on p. 137:

$$B_{DG}(U_{DG},V) = \sum_{n=0}^{N-1}\int_{I_n}\{(\partial_t U_{DG},V)_E + \hat{a}(U_{DG},V)\}\,dt + \sum_{n=1}^{N-1}([U_{DG}]^n, V^n_+)_E + (U^0_{DG,+},V^0_+)_E.$$

Find $\Phi^h \in \mathcal{V}^r(\mathcal{N}; Z^h_0 \times Z^h_0)$, which is a real vector-valued function, satisfies

$$B_{DG}(V^h,\Phi^h) = (V^{h,N}_-, \Theta^{h,N}_-)_E, \forall V^h \in \mathcal{V}^r(\mathcal{N}; Z^h_0 \times Z^h_0), \tag{6.73}$$

for given data $\Theta^{h,N}_- \in Z^h_0 \times Z^h_0$.

Lemma 6.10 (Equivalence between Problem 6.5.1 and Problem 3.5.2) *Let Z and H be the Hilbert spaces introduced in Definition 5.1 on p. 101. Let Z^h and Z^h_0 be the conformal finite dimensional subspaces of the Hilbert space Z and Z_0, respectively given in Definition 6.5 on p. 136. Problem 6.5.1 and Problem 3.5.2 on p. 55 are equivalent.*

Proof:
Firstly, Lemma 5.1 on p. 103 states that Z_0 is a separable Hilbert space which satisfies all the properties given in Part I of the theses, and the same holds also for the product space $Z_0 \times Z_0$ (see [7, 1.3 Hilbert Spaces from Hilbert spaces] i.e. $Z_0 \times Z_0$ is a separable Hilbert space. Secondly, with knowing that the product space $Z_0^h \times Z_0^h$ is a conformal finite dimensional subspace of $Z_0 \times Z_0$ and that renders that product space $Z_0^h \times Z_0^h$ satisfies the properties of the product space $Z_0 \times Z_0$, then Problem 6.5.1 is equivalent to Problem 3.5.2 on p. 55 and that completes the proof. □

Lemma 6.11 (Solvability of Problem 6.5.1)
Let Z and H be the Hilbert spaces introduced in Definition 5.1 on p. 101. Let Z^h and Z_0^h be the conformal finite dimensional subspaces of the Hilbert space Z and Z_0, respectively given in Definition 6.5 on p. 136. Problem 6.5.1 on p. 145 is solvable and its solution satisfies

$$\| \Phi^h \|_{L^2(J;E)} \leq \mathcal{C} \| \Theta_-^{h,N} \|_E, \tag{6.74}$$

$$\| \Phi_-^{h,n} \|_E^2 \leq \| \Theta_-^{h,N} \|_E^2, \tag{6.75}$$

and

$$\sum_{m=0}^{n-1} \Big(\| [\Phi^h]^m \|_E^2 \Big) \leq \| \Theta_-^{h,N} \|_E^2, \tag{6.76}$$

for $n \leq N$ and given data $\Theta_-^{h,N} \in Z^h \times Z^h$, and $\mathcal{C} = \max_n (T \dfrac{\| Y^{r_n} \|_2^2 \, (2r_n + 1)}{2 \min_p \; Re \; \lambda_p}) < \infty$ is the same real constant appearing in Lemma 3.18 on p. 56.

Proof:
From the shown equivalence in Lemma 6.10 on p. 145 then this implies that Problem 6.5.1 on p. 145 is solvable and its solution satisfies the same stability of the solution of Problem 3.5.2 on p. 55 and that completes the proof. □

6.5.2 The full a priori error estimate

Definition 6.6 (The spatial matrix projection operator with linear constraints) *Let Z and H be the Hilbert spaces introduced in Definition 5.1 on p. 101. Let Z^h and Z_0^h be the conformal finite dimensional Hilbert subspaces of the Hilbert space Z and Z_0, respectively given in Definition 6.5 on p. 136.*
For $u_1 \in Z$, the spatial projection operator $\hat{\Pi}_{g^h} : Z \to Z_{g^h}^h$ is defined:

$$a(\hat{\Pi}_{g^h} u_1 - u_1, v) = 0, \;\; \forall \, v \in Z_0^h, \tag{6.77}$$

and for $u_2 \in Z$, the spatial projection operator $\hat{\Pi}_{g',h} : Z \to Z_{g',h}^h$ is defined:

$$a(\hat{\Pi}_{g',h} u_2 - u_2, v) = 0, \;\; \forall \, v \in Z_0^h, \tag{6.78}$$

and with coupling them would be $\hat{\Pi}_{G^h} := \begin{pmatrix} \hat{\Pi}_{g^h} & 0 \\ 0 & \hat{\Pi}_{g',h} \end{pmatrix} : Z \times Z \to Z_{g^h}^h \times Z_{g^{h,\prime}}^h$.

Lemma 6.12 *The spatial projection operator $\hat{\Pi}$ given in Definition 6.6 is well defined.*

Proof:
Firstly, for $\hat{\Pi}_{g^h} : Z \to Z^h_{g^h}$, assume that $u^1 \in Z^h_{g^h}$ and $u^2 \in Z^h_{g^h}$ to be functions which satisfy (6.77) for $u \in Z$ i.e.

$$a(u^1 - u, v) = 0, \text{ and } a(u^2 - u, v) = 0, \ \forall v \in Z^h_0.$$

with letting $\bar{w} := u^1 - u^2 \in Z^h_0$ then with using the *continuity* of the form $a(\,,)$ as given in Definition 5.3 on p. 104,

$$a(\bar{w}, v) \leq M \parallel \bar{w} \parallel_Z \parallel v \parallel_Z,$$

and with choosing $v = \bar{w} \in Z^h_0$, and using *ellipticity* of the form $a(\,,)$, then

$$\alpha \parallel \bar{w} \parallel_Z \leq a(\bar{w}, \bar{w}) = 0, \Longrightarrow 0 = \bar{w} = u^1 - u^2,$$

that concludes the uniqueness, and with the use of Lax-Milgram theorem also concludes the existence. Secondly, for $\hat{\Pi}_{g',h} : Z \to Z^h_{g',h}$ and with repeating the same steps again that completes the proof. □

Definition 6.7 (Spatio-time projection operator) *Let $\hat{\Pi}_{G^h} := \begin{pmatrix} \hat{\Pi}_{g^h} & 0 \\ 0 & \hat{\Pi}_{g',h} \end{pmatrix} : Z \times Z \to Z^h_{g^h} \times Z^h_{g',h}$ be the spatial projection given in Definition 6.6. Let $I = (-1, 1)$ be the given time step, Z and H be the Hilbert spaces given in Definition 2.1 on p. 6. Let $P^r(I; Z^h_{g^h_{CG}} \times Z^h_{g'^{,h}_{CG}})$ be the fully-discrete space of rth order polynomials in the time step $I = (-1, 1)$ with coefficients in $Z^h_{g^h_{CG}} \times Z^h_{g'^{,h}_{CG}}$. For $U \in L^2(I; \times Z_g \times Z_{g'})$ which is continuous at $t = 1$. With using the defined spatial projection in Definition 6.6, the following two projection operators are defined, via the $r + 1$, $r \geq 1$ conditions:*

$$\tilde{\Pi}^r_{g^h_{CG}} : L^2(J; Z) \to P^r(I; Z^h_{g^h_{CG}}), \text{ and } \tilde{\Pi}^r_{g'^{,h}_{CG}} : L^2(J; Z) \to P^r(I; Z^h_{g'^{,h}_{CG}}) \tag{6.79}$$

such that

$$\int_I a(\tilde{\Pi}^r_{g^h_{CG}} u_1 - u_1, v_1) dt = 0, \forall v_1 \in P^{r-1}(I; Z^h_0), \text{ with } \tilde{\Pi}^r_{g^h_{CG}(+1)} u_1(+1) = \hat{\Pi}_{g^h(+1)} u_1(+1) \in Z^h_{g^h(+1)}, \tag{6.80}$$

and

$$\int_I a(\tilde{\Pi}^r_{g'^{,h}_{CG}} u_2 - u_2, v_2) dt = 0, \forall v_2 \in P^{r-1}(I; Z^h_0), \text{ with } \tilde{\Pi}^r_{g'^{,h}_{CG}(+1)} u_2(+1) = \hat{\Pi}_{g'^{,h}_{CG}(+1)} u_2(+1) \in Z^h_{g'^{,h}_{CG}(+1)}, \tag{6.81}$$

Lemma 6.13 *The spatio-time projection $\tilde{\Pi}^r_{G^h_{CG}} = \begin{pmatrix} \tilde{\Pi}^r_{g^h_{CG}} & 0 \\ 0 & \tilde{\Pi}^r_{g'^{,h}_{CG}} \end{pmatrix}$ given in Definition 6.7 is well-defined.*

Proof:
To show uniqueness:

Firstly, for given $u_1 \in L^2(I;Z)$, let $u_1^1 \in P^r(I; Z^h_{g^h_{CG}})$ and $u_1^2 \in P^r(I; Z^h_{g^h_{CG}})$ where both satisfy (6.80) such that

$$\int_I a(u_1^1 - u_1, v_1)dt = 0, \text{ and } \int_I a(u_1^2 - u_1, v_1)dt = 0, \forall\, v_1 \in P^{r-1}(I; Z_0^h), \tag{6.82}$$

with

$$u_1^1(+1) = \hat{\Pi}_{g^h(+1)} u_1(+1) \text{ and } u_1^2(+1) = \hat{\Pi}_{g^h(+1)} u_1(+1). \tag{6.83}$$

Then from taking the difference of $u_1^1 - u_1^2$ which can be written in the following finite series

$$u_1^1 - u_1^2 = \sum_{i=0}^{r} \bar{v}_i L_i, \text{ for } \bar{v}_i = \int_I (u_1^1 - u_1^2) L_i\, dt \in Z_0^h, (\text{ see [44, Lemma 1.10]}), \tag{6.84}$$

and for any $j = \{0, \cdots, r-1\}$ and choosing $v_1 = L_j v \in P^{r-1}(I, Z_0^h)$ and using the orthogonal property of Legendre polynomials, yield

$$\begin{gathered} 0 = \int_I a(u_1^1 - u_1^2, L_j v)dt = \sum_{i=0}^{r} a(\bar{v}_i, v) \int_I L_i L_j dt = \sum_{i=0}^{r} a(\bar{v}_i, v)\delta_{ij} \frac{2}{2j+1} = a(\bar{v}_j, v)\frac{2}{2j+1}, \\ \Longrightarrow 0 = a(\bar{v}_j, v), \text{ for } j = \{0, \cdots, r-1\}. \end{gathered} \tag{6.85}$$

Now, with choosing $v = \bar{v}_j \in Z_0^h$ and using the *ellipticity* of the form $a(\,,)$, imply

$$M \parallel \bar{v}_j \parallel_Z^2 \leq a(\bar{v}_j, \bar{v}_j) = 0, \tag{6.86}$$

and, from (6.86), implies that $\bar{v}_j = 0$, for $j = \{0, \cdots, r-1\}$, and with going back to (6.84), yields that

$$u_1^1 - u_1^2 = \overbrace{\sum_{j=0}^{r-1} \bar{v}_j L_j}^{=0} + \bar{v}_r L_r \Longrightarrow u_1^1 - u_1^2 = \bar{v}_r L_r,$$

and finally from (6.80) i.e.

$$0 = u_1^1(+1) - u_1^2(+1) = \bar{v}_r L_r(+1) = \bar{v}_r \Longrightarrow \bar{v}_r = 0,$$

which concludes that $u_1^1 = u_1^2$.
Secondly, for $u_2 \in L^2(I;Z)$, let $u_2^1 \in P^r(I; Z^h_{g'^{,h}_{CG}})$ and $u_2^2 \in P^r(I; Z^h_{g'^{,h}_{CG}})$ where both satisfy (6.81) such that

$$\int_I a(u_2^1 - u_2, v_2)dt = 0, \text{ and } \int_I a(u_2^2 - u_2, v_2)dt = 0, \forall\, v_2 \in P^{r-1}(I; Z_0^h), \tag{6.87}$$

with

$$u_2^1(+1) = \hat{\Pi}_{g'^{,h}(+1)} u_2(+1) \text{ and } u_2^2(+1) = \hat{\Pi}_{g'^{,h}(+1)} u_2(+1). \tag{6.88}$$

Then from taking the difference of $u_2^1 - u_2^2$ which can be written in the following finite series

$$u_1^1 - u_1^2 = \sum_{i=0}^{r} \bar{v}_i L_i, \text{ for } \bar{v}_i = \int_I (u_1^1 - u_1^2) L_i\, dt \in Z_0^h, (\text{ see [44, Lemma 1.10]}), \tag{6.89}$$

and with repeating the same steps done at the first part of this proof, the uniqueness is also shown. To show existence and again with following the same argument again in [44, Lemma 1.10] such that, from Lemma 2.2 on p. 11, for $U \in L^2(J; Z_g \times Z'_g)$ it can be written as

$$U = \bar{U} + \hat{G}, \text{ where } \bar{U} \in L^2(J; Z_0 \times Z_0), \text{ and } \hat{G} \in H^1(J; Z \times Z),$$
$$\text{which satisfies the data at the trace s.t. } \bar{\gamma}\hat{G} = G,$$

and since both $L^2(J; Z_0 \times Z_0)$ and $H^1(J; Z \times Z)$ are Hilbert spaces thus the following exists

$$U = \bar{U} + \hat{G} = \sum_i^\infty L_i U_i + \sum_i^\infty L_i \hat{G}_i,$$

and setting

$$\Pi^r_{G^h_{CG}} U = \sum_{i=0}^{r-1} L_i U_i + \big(\hat{\Pi}_{G^h} U(1) - \sum_{i=0}^{r-1} U_i\big) L_r,$$

then that also concludes that $\Pi^r_{G^h_{CG}} U \in P^r(I; Z^h_{g^h_{CG}} \times Z^h_{g'^{,h}_{CG}})$, which satisfies (3.104) and (3.105) on p. 59, does exist and that completes the proof. □

Assumption 6.1 *Let Z and H be the Hilbert spaces introduced in Definition 5.1 on p. 101. Let $A \in L(Z, Z')$ be the linear bounded operator. Let $\tilde{\Pi}^r_1 : L^2(J; Z) \to P^r(I; Z^h_{g^h_{CG}})$ be the spatio-time projection operator given in Definition 6.7 on p. 147. The following is assumed*

$$A(\tilde{\Pi} u_1 - u_1) \in H. \tag{6.90}$$

Theorem 6.8 (Last left-sided limit of the a priori error estimate with linear constraints) *Let Z and H be the Hilbert spaces given in Definition 5.1 on p. 101. Let Z_g be the subset and Z_0 the subspace of the Hilbert space Z given in Definition 5.2 on p. 102. Let $U \in L^2(J; Z_g \times Z_{g'})$ be the continuous solution of (5.25) on p.112 with initial data $U_0 \in Z_g \times Z_{g'}$. Let $U^h_{DG} \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h_{g^h_{CG}} \times Z^h_{g^{h,\prime}_{CG}})$ be the fully-discrete solution of (6.53) on p. 137 with initial data $U^h_0 \in Z^h_{g^h} \times Z^h_{g^{h,\prime}}$ for $X = Z \times H$, then there holds*

$$\begin{aligned} \| (U - U^h_{DG})^N_- \|^2_X \leq \| (U - \hat{\Pi}_{G^h} U)^N_- \|_X + \frac{1}{\sqrt{\min(1,\alpha)}} \Bigg(& \frac{\sqrt{2\mathcal{P}_{r_n}}}{k} \mathcal{C} \| \tilde{\Pi}^r_{g'^{,h}_{CG}} u_2 - u_2 \|_{L^2(J;H)} \\ & + \sum_{n=1}^{N-1} \| (\hat{\Pi}^{r_n}_{g',h} u_2 - u_2)^n_- \|_H + \sqrt{2} \| (\hat{\Pi}^{r_N}_{g',h} u_2 - u_2)^N_- \|_H \\ + \frac{\max(1,M)\mathcal{C}}{\sqrt{\min(\alpha,1)}} \big(\| A(\tilde{\Pi}^r_{g^h_{CG}} u_1 - u_1) \|_{L^2(J;H)} & + \| \tilde{\Pi}^r_{g'^{,h}_{CG}} u_2 - u_2 \|_{L^2(J;Z)} \big) \\ & + \sqrt{\frac{2}{k}} (2r_n + 1) \mathcal{C} \| U_0 - U^h_0 \|_E \Bigg), \end{aligned} \tag{6.91}$$

where $\hat{\Pi}_{G^h} := \begin{pmatrix} \hat{\Pi}_{g^h} & 0 \\ 0 & \hat{\Pi}_{g',h} \end{pmatrix} : Z \times Z \to Z^h_{g^h} \times Z^h_{g',h}$ *is the spatial projection given in Definition 6.6 and* $\tilde{\Pi}^r_{G^h_{CG}} = \begin{pmatrix} \tilde{\Pi}^r_{g^h_{CG}} & 0 \\ 0 & \tilde{\Pi}^r_{g'^{,h}_{CG}} \end{pmatrix}$ *is the spatio-time projection given in Definition 6.7. $M < \infty$ and $\alpha > 0$, the* *upper and lower bounds of the form $a(\,,\,)$, respectively given in Definition 5.3 on p. 104 and $\mathcal{P}_r$ is a real constant which also depends on the maximum approximation order r.*

Proof:
Firstly, with the *triangle inequality*

$$\| (U - U^h_{DG})^N_- \|_X \leq \| (U - \tilde{\Pi}^r_{G^h_{CG}} U)^N_- \|_X + \| (\tilde{\Pi}^r_{G^h_{CG}} U - U^h_{DG})^N_- \|_X \,.$$

Since $\tilde{\Pi}^r_{G^h_{CG}} U \in \mathcal{V}^r_-(\mathcal{N}; Z^h_{g^h_{CG}} \times Z^h_{g'^{,h}_{CG}})$ then $\tilde{\Pi}^r_{G^h_{CG}} U - U^h_{DG} \in \mathcal{V}^r_-(\mathcal{N}; Z^h_0 \times Z^h_0)$ which concludes

$$\| (\tilde{\Pi}^r_{G^h_{CG}} U - U^h_{DG})^N_- \|_X \leq \frac{1}{\sqrt{\min(1,\alpha)}} \| (\tilde{\Pi}^r_{G^h_{CG}} U - U^h_{DG})^N_- \|_E,$$

and from (6.80) on p. 147 where $\tilde{\Pi}^r_{g^h_{CG}(+1)} u_1(+1) = \hat{\Pi}_{g^h(+1)} u_1(+1) \in Z^h_{g^h(+1)}$ then

$$\| (U - U^h_{DG})^N_- \|_X \leq \| (U - \hat{\Pi}_{G^h} U)^N_- \|_X + \frac{1}{\sqrt{\min(1,\alpha)}} \| (\tilde{\Pi}^r_{G^h_{CG}} U - U^h_{DG})^N_- \|_E \,. \tag{6.92}$$

Now, for finding the upper bound of $\| (\tilde{\Pi}^r_{G^h_{CG}} U - U^h_{DG})^N_- \|_E$ and with $\Theta^h = \tilde{\Pi}^r_{G^h_{CG}} U - U^h_{DG} \in Z^h_0 \times Z^h_0$, by going back to Problem 6.5.1 on p. 55 with recalling the proof of Theorem 3.6 on p. 61.
Such that $\Theta^{h,N}_- = (\tilde{\Pi}^r_{G^h_{CG}} U - U^h_{DG})^N_- \in Z^h_0 \times Z^h_0$, for $U \in L^2(J; Z_g \times Z_{g'})$ the solution of continuous formulation (5.25) on p. 112, and $U^h_{DG} \in \mathcal{V}^r_-(\mathcal{N}; Z^h_{g^h_{CG}} \times Z^h_{g^{h,\prime}_{CG}})$ be the fully-discrete solution of (6.53) on p. 137.
Problem 6.5.1 in (6.73) on p. 145 that is
Find $\Phi^h \in \mathcal{V}^r_-(\mathcal{N}; Z^h_0 \times Z^h_0)$, which is a real vector-valued function, satisfies

$$B_{DG}(V^h, \Phi^h) = (V^{h,N}_-, \Theta^{h,N}_-)_E, \forall V^h \in \mathcal{V}^r_-(\mathcal{N}; Z^h_0 \times Z^h_0), \tag{6.93}$$

for given data $\Theta^{h,N}_- \in Z^h_0 \times Z^h_0$. With choosing $V^h = \Theta^h \in \mathcal{V}^r_-(\mathcal{N}; Z^h_0 \times Z^h_0)$ in (6.93):

$$\begin{aligned}
\| \Theta^{h,N}_- \|^2_E = B_{DG}(\Theta^h, \Phi^h) &= B_{DG}(\tilde{\Pi}^r_{G^h_{CG}} U - U^h_{DG}, \Phi^h) \\
&= B_{DG}(\tilde{\Pi}^r_{G^h_{CG}} U - U + U - U^h_{DG}, \Phi^h) \\
&= B_{DG}(\tilde{\Pi}_{G^h_{CG}} U - U, \Phi^h) + \underbrace{B_{DG}(U - U^h_{DG}, \Phi^h)}_{=(U_0 - U^h_0, \Phi^{h,0}_+)_E}.
\end{aligned}$$

Firstly, in $B_{DG}(\tilde{\Pi}^r_{G^h_{CG}} U - U, \Phi^h)$ with using Lemma 3.3 on p. 26, yields

$$\begin{aligned}
B_{DG}(\tilde{\Pi}^r_{G^h_{CG}} U - U, \Phi^h) = &- \sum_{n=0}^{N-1} \int_{I_n} a(\tilde{\Pi}^{r_n}_{g^h_{CG}} u_1 - u_1, \partial_t \Phi^h_1) dt - \sum_{n=0}^{N-1} \int_{I_n} (\tilde{\Pi}^{r_n}_{g'^{,h}_{CG}} u_2 - u_2, \partial_t \Phi^h_2)_H dt \\
&- \sum_{n=1}^{N-1} ((\tilde{\Pi}^{r_n}_{G^h_{CG}} U - U)^n_-, [\Phi^h]^n)_E + ((\tilde{\Pi}^{r_N}_{G^h_{CG}} U - U)^N_-, \Phi^{h,N}_-)_E \\
&+ \sum_{n=0}^{N-1} \int_{I_n} a(\tilde{\Pi}^{r_n}_{g^h_{CG}} u_1 - u_1, \Phi^h_2) dt - \sum_{n=0}^{N-1} \int_{I_n} a(\tilde{\Pi}^{r_n}_{g'^{,h}_{CG}} u_2 - u_2, \Phi^h_1) dt \\
&= \sum_{j=1}^{6} \mathcal{E}_j.
\end{aligned}$$

Now, with using (6.80) on p. 147 where $u_1 \in L^2(J; Z_g)$:

$$\mathcal{E}_1 = 0,$$

$$\mathcal{E}_2 = -\int_J (\tilde{\Pi}^r_{g',h_{CG}} u_2 - u_2, \partial_t \Phi_2^h)_H dt,$$

and

$$\mathcal{E}_3 = \sum_{n=1}^{N-1} ((\tilde{\Pi}^{r_n}_{G^h_{CG}} U - U)^n_-, [\Phi^h]^n)_E = \sum_{n=1}^{N-1} a((\tilde{\Pi}^{r_n}_{g^h_{CG}} u_1 - u_1)^n_-, [\Phi_1^h]^n) + \sum_{n=1}^{N-1} ((\tilde{\Pi}^{r_n}_{g^{\prime,h}_{CG}} u_2 - u_2)^n_-, [\Phi_2^h]^n)_H,$$

and from (6.80) on p. 147 where $\tilde{\Pi}^r_{g^h_{CG}(+1)} u_1(+1) = \hat{\Pi}_{g^h(+1)} u_1(+1) \in Z^h_{g^h(+1)}$, and then using (6.78) on p. 146, imply

$$\sum_{n=1}^{N-1} a((\tilde{\Pi}^{r_n}_{g^h_{CG}} u_1 - u_1)^n_-, [\Phi_1^h]^n) = \sum_{n=1}^{N-1} a((\hat{\Pi}^{r_n}_{g^h} u_1 - u_1)^n_-, [\Phi_1^h]^n) = 0,$$

thus

$$\mathcal{E}_3 = \sum_{n=1}^{N-1} ((\hat{\Pi}^{r_n}_{g',h} u_2 - u_2)^n_-, [\Phi_2^h]^n)_H.$$

The same thing hold for $\mathcal{E}_4$ s.t.

$$\mathcal{E}_4 = ((\hat{\Pi}^{r_N}_{g',h} u_2 - u_2)^N_-, \Phi^{h,N}_{2,-})_H.$$

Similar steps to get the upper bounds of $\mathcal{E}_2$, $\mathcal{E}_3$, $\mathcal{E}_4$, $\mathcal{E}_5$ and $\mathcal{E}_6$ given in the proof of Theorem 3.6 on p. 61 are going to be used with considering the change in the projection used this time:

$$\mathcal{E}_2 = -\int_J (\tilde{\Pi}^r_{g^{\prime,h}_{CG}} u_2 - u_2, \partial_t \Phi_2^h)_H dt \leq \| \tilde{\Pi}^r_{g^{\prime,h}_{CG}} u_2 - u_2 \|_{L^2(J;H)} \| \partial_t \Phi_2^h \|_{L^2(J;H)},$$

and with recalling the estimate of $\| \partial_t \Phi_2^h \|_{L^2(J;H)}$ given in (**??**) o p. **??**:

$$\begin{aligned}
\| \partial_t \Phi_2^h \|^2_{L^2(J;H)} &\leq \frac{1}{k} \mathcal{P}_{r_n} \sum_{n=0}^{N-1} \underbrace{\sum_{j=0}^{r_n} \| \Phi^h_{2,j} \|^2_H}_{= \frac{2}{k} \| \Phi_2^h \|^2_{L^2(I_n;H)}} = \frac{2}{k^2} \mathcal{P}_{r_n} \| \Phi_2^h \|^2_{L^2(J;H)} \\
&\leq \frac{2}{k^2} \mathcal{P}_{r_n} \Big(\int_J a(\Phi_1^h, \Phi_1^h) dt + \| \Phi_2^h \|^2_{L^2(J;H)} \Big) \\
&= \frac{2}{k^2} \mathcal{P}_{r_n} \| \Phi^h \|^2_{L^2(J;E)} \leq \frac{2}{k^2} \mathcal{P}_{r_n} \mathcal{C}^2 \| \Theta^{h,N}_- \|^2_E,
\end{aligned}$$

then

$$\mathcal{E}_2 \leq \frac{\sqrt{2\mathcal{P}_{r_n}}}{k} \mathcal{C} \| \tilde{\Pi}_{g^{\prime,h}_{CG}} u_2 - u_2 \|_{L^2(J;H)} \| \Theta^{h,N}_- \|_E . \tag{6.94}$$

For $\mathcal{E}_3$ and $\mathcal{E}_4$, with using *Cauchy inequality*, and then the estimates in (6.76) and (6.75), then

$$\begin{aligned}
\mathcal{E}_3 + \mathcal{E}_4 &= \sum_{n=1}^{N-1} ((\hat{\Pi}^{r_n}_{g',h} u_2 - u_2)^n_-, [\Phi_2^h]^n)_H + ((\hat{\Pi}^{r_N}_{g',h} u_2 - u_2)^N_-, \Phi^{h,N}_{2,-})_H \\
&\leq \sqrt{\sum_{n=1}^{N-1} \| (\hat{\Pi}^{r_n}_{g',h} u_2 - u_2)^n_- \|^2_H} \sqrt{\sum_{n=1}^{N-1} \| \Phi^{h,N}_{2,-} \|^2_H} + \| (\hat{\Pi}^{r_N}_{g',h} u_2 - u_2)^N_- \|_H \| \Phi^{h,N}_{2,-} \|_H \qquad (6.95) \\
&\leq \Big(\sum_{n=1}^{N-1} \| (\hat{\Pi}^{r_n}_{g',h} u_2 - u_2)^n_- \|_H + \sqrt{2} \| (\hat{\Pi}^{r_N}_{g',h} u_2 - u_2)^N_- \|_H \Big) \| \Theta^{h,N}_- \|_E
\end{aligned}$$

For $\mathcal{E}_5$ and $\mathcal{E}_6$ and with using Assumption 6.1 on p. 149, *Cauchy inequality*, and then the estimate in (6.74) on p. 146, then

$$\begin{aligned}\mathcal{E}_5+\mathcal{E}_6 &= \int_J a(\tilde{\Pi}^r_{g^h_{CG}} u_1 - u_1, \Phi^h_2)dt - \int_J a(\tilde{\Pi}^r_{g^{\prime,h}_{CG}} u_2 - u_2, \Phi^h_1)dt \\ &\leq \frac{\max(1,M)\mathcal{C}}{\sqrt{\min(\alpha,1)}}\big(\| A(\tilde{\Pi}^r_{g^h_{CG}} u_1 - u_1) \|_{L^2(J;H)} + \| \tilde{\Pi}^r_{g^{\prime,h}_{CG}} u_2 - u_2 \|_{L^2(J;Z)} \big) \| \Theta^{h,N}_- \|_E \,.\end{aligned} \tag{6.96}$$

Secondly, with using *Cauchy inequality*

$$(U_0 - U^h_0, \Phi^{h,0}_+)_E \leq \| U_0 - U^h_0 \|_E \| \Phi^{h,0}_+ \|_E,$$

and with also recalling the steps done (3.59) with (3.60) on p. 42 and the estimate in (6.74) on p. 146, thus

$$\begin{aligned}\| \Phi^{h,0}_+ \|_E \leq (2r_n+1)\sqrt{\sum_{p=0}^{r_n} \| \Phi^h_p \|^2_E} = \sqrt{\frac{2}{k}}(2r_n+1) \| \Phi^h \|_{L^2(I_n;E)} \leq \sqrt{\frac{2}{k}}(2r_n+1) \| \Phi^h \|_{L^2(J;E)} \\ \leq \sqrt{\frac{2}{k}}(2r_n+1)\mathcal{C} \| \Theta^{h,N}_- \|_E,\end{aligned}$$

then

$$(U_0 - U^h_0, \Phi^{h,0}_+)_E \leq \sqrt{\frac{2}{k}}(2r_n+1)\mathcal{C} \| U_0 - U^h_0 \|_E \| \Theta^{h,N}_- \|_E \,. \tag{6.97}$$

Finally, with adding (6.97), (6.96), (6.95), and (6.94) and going back to (6.92) on p. 150, then

$$\begin{aligned}\| \Theta^{h,N}_- \|^2_E &= \| (\tilde{\Pi}^r_{G^h_{CG}} U - U^h_{DG})^N_- \|^2_E \\ &\leq \Big(\frac{\sqrt{2\mathcal{P}_{r_n}}}{k}\mathcal{C} \| \tilde{\Pi}^r_{g^{\prime,h}_{CG}} u_2 - u_2 \|_{L^2(J;H)} + \sum_{n=1}^{N-1} \| (\hat{\Pi}^{r_n}_{g^{\prime,h}} u_2 - u_2)^n_- \|_H \\ &+ \sqrt{2} \| (\hat{\Pi}^{r_N}_{g^{\prime,h}} u_2 - u_2)^N_- \|_H + \frac{\max(1,M)\mathcal{C}}{\sqrt{\min(\alpha,1)}}\Big(\| A(\tilde{\Pi}^r_{g^h_{CG}} u_1 - u_1) \|_{L^2(J;H)} \\ &+ \| \tilde{\Pi}^r_{g^{\prime,h}_{CG}} u_2 - u_2 \|_{L^2(J;Z)} \Big) + \sqrt{\frac{2}{k}}(2r_n+1)\mathcal{C} \| U_0 - U^h_0 \|_E \Big) \| \Theta^{h,N}_- \|_E,\end{aligned}$$

and that completes the proof. □

Theorem 6.9 (Local a priori error estimate in the $L^2(I;X)$-norm)
Let Z and H be the Hilbert spaces given in Definition 5.1 on p. 101. Let Z_g be the subset and Z_0 the subspace of the Hilbert space Z given in Definition 5.2 on p. 102. Let $U \in L^2(J; Z_g \times Z_{g'})$ be the continuous solution of (5.25) on p.112 with initial data $U_0 \in Z_g \times Z_{g'}$. Let $U^h_{DG} \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h_{g^h_{CG}} \times Z^h_{g^{h,\prime}_{CG}})$ be the fully-discrete solution of (6.53) on p. 137 with initial data $U^h_0 \in Z^h_{g^h} \times Z^h_{g^{h,\prime}}$, then, for $X = Z \times H$,

at a generic time step $I = (t_0, t_1)$ there holds

$$
\begin{aligned}
\| U - U^h_{DG} \|_{L^2(I;X)} \leq & \| U - \tilde{\Pi}^r_{G^h_{CG}} U \|_{L^2(I;X)} \\
& + \left(\frac{k \| Y \|^2_2}{2 \min_p Re\ \lambda_p \sqrt{\min(\alpha, 1)}}\right)\Bigg(\frac{\sqrt{2\mathcal{P}_r}}{k} \| \tilde{\Pi}^r_{g'^{,h}_{CG}} u_2 - u_2 \|_{L^2(I;H)} \\
& \quad + \sqrt{\frac{2}{k}}(2r+1) \| (U - U^h_{DG})^0_- \|_E + \sqrt{\frac{2}{k}}(2r+1) \| (\tilde{\Pi}^r_{g'^{,h}_{CG}} u_2 - u_2)_- \|_H \\
& + \frac{\max(1, M)}{\sqrt{\min(\alpha, 1)}} (\| A(\tilde{\Pi}^r_{g^h_{CG}} u_1 - u_1) \|_{L^2(I;H)} + \| \tilde{\Pi}^r_{g'^{,h}_{CG}} u_2 - u_2 \|_{L^2(I;Z)}) \Bigg),
\end{aligned}
\tag{6.98}
$$

where $\hat{\Pi}_{G^h} := \begin{pmatrix} \hat{\Pi}_{g^h} & 0 \\ 0 & \hat{\Pi}_{g',h} \end{pmatrix} : Z \times Z \to Z^h_{g^h} \times Z^h_{g',h}$ is the spatial projection given in Definition 6.6 and $\tilde{\Pi}^r_{G^h_{CG}} = \begin{pmatrix} \tilde{\Pi}^r_{g^h_{CG}} & 0 \\ 0 & \tilde{\Pi}^r_{g'^{,h}_{CG}} \end{pmatrix}$ is the spatio-time projection given in Definition 6.7. $M < \infty$ and $\alpha > 0$, the upper and lower bounds of the form $a(,)$, respectively given in Definition 5.3 on p. 104 and $\mathcal{P}_r$ is a real constant which also depends on the maximum approximation order r.

Proof:
Starting with the triangle inequality

$$
\| U - U^h_{DG} \|_{L^2(I;X)} \leq \| U - \tilde{\Pi}^r_{G^h_{CG}} U \|_{L^2(I;X)} + \| \tilde{\Pi}^r_{G^h_{CG}} U - U^h_{DG} \|_{L^2(I;X)} .
$$

Now, since $\tilde{\Pi}^r_{G^h_{CG}} U - U^h_{DG} \in P^r(I; Z^h_0 \times Z^h_0)$, then

$$
\| \tilde{\Pi}^r_{G^h_{CG}} U - U^h_{DG} \|_{L^2(I;X)} \leq \frac{1}{\sqrt{\min(\alpha, 1)}} \| \tilde{\Pi}^r_{G^h_{CG}} U - U^h_{DG} \|_{L^2(I;E)},
$$

and as shown earlier in the proof of Lemma 6.10 on p. 145 $Z_0 \times Z_0$ is a product separable Hilbert space as well as $Z^h_0 \times Z^h_0$ which is the conformal finite dimensional subspace of $Z_0 \times Z_0$, then with using the discrete inf-sup condition in Lemma 3.10 on p. 45 since all properties their holds, then

$$
\begin{aligned}
\| U - U^h_{DG} \|_{L^2(I;X)} \leq & \| U - \tilde{\Pi}^r_{G^h_{CG}} U \|_{L^2(I;X)} \\
& + \frac{1}{\sqrt{\min(\alpha, 1)}} \Bigg(\left(\frac{k \| Y \|^2_2}{2 \min_p \mathrm{Re}\ \lambda_p}\right) \sup_{V^h \in P^r(I; Z^h_0 \times Z^h_0)} \frac{b_{DG}(\tilde{\Pi}^r_{G^h_{CG}} U - U^h_{DG}, V^h)}{\| V^h \|_{L^2(I;E)}} \Bigg).
\end{aligned}
$$

For finding the upper bounds of the supremum and with recalling the same steps done in (3.138) on p. 69, then

$$
\begin{aligned}
\sup_{V^h \in P^r(I; Z^h_0 \times Z^h_0)} \frac{b_{DG}(\tilde{\Pi}^r_{G^h_{CG}} U - U^h_{DG}, V^h)}{\| V^h \|_{L^2(I;E)}} \leq & \sup_{V^h \in P^r(I; Z^h_0 \times Z^h_0)} \frac{b_{DG}(\tilde{\Pi}^r_{G^h_{CG}} U - U, V^h)}{\| V^h \|_{L^2(I;E)}} \\
& + \sup_{V^h \in P^r(I; Z^h_0 \times Z^h_0)} \frac{((U - U^h_{DG})^0_-, V^{h,0}_+)_E}{\| V^h \|_{L^2(I;E)}}.
\end{aligned}
$$

Here and with recalling the steps done in (3.139) on p. 69, then

$$\sup_{V^h \in P^r(I;Z_0^h \times Z_0^h)} \frac{((U-U_{DG}^h)_-^0, V_+^{h,0})_E}{\| V^h \|_{L^2(I;E)}} \leq \sup_{V^h \in P^r(I;Z_0^h \times Z_0^h)} \frac{\| (U-U_{DG}^h)_-^0 \|_E \| V_+^{h,0} \|_E}{\| V^h \|_{L^2(I;E)}}$$
$$\leq \sqrt{\frac{2}{k}}(2r+1) \| (U-U_{DG}^h)_-^0 \|_E . \tag{6.99}$$

Recalling Corollary 3.2 on p. 28, then

$$b_{DG}(\tilde{\Pi}_{G_{CG}^h} U - U, V^h) = \int_I -a(\tilde{\Pi}^r_{g_{CG}^h} u_1 - u_1, \partial_t v_1^h) dt - \int_I (\tilde{\Pi}^r_{g_{CG}^{\prime,h}} u_2 - u_2, \partial_t v_2^h)_H dt$$
$$+ ((\tilde{\Pi}^r_{G_{CG}^h} U - U)_-, V_-^h)_E + \int_I a(\hat{\Pi}^r_{g_{CG}^h} u_1 - u_1, v_2^h) dt$$
$$- \int_I a(\hat{\Pi}^r_{g_{CG}^{\prime,h}} u_2 - u_2, v_1^h) dt = \sum_{j=0}^{5} \mathcal{E}_j. \tag{6.100}$$

Now, with using the resulting estimates for $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_5, \mathcal{E}_6$, and $\mathcal{E}_4$ given in the proof of Theorem 6.8 on p. 149 but this time being at a generic time step I, then

$$\mathcal{E}_1 = \int_I -a(\tilde{\Pi}^r_{g_{CG}^h} u_1 - u_1, \partial_t v_1^h) dt = 0,$$

$$\mathcal{E}_2 = -\int_I (\tilde{\Pi}^r_{g_{CG}^{\prime,h}} u_2 - u_2, \partial_t v_2^h)_H dt \leq \frac{\sqrt{2\mathcal{P}_r}}{k} \| \tilde{\Pi}^r_{g_{CG}^{\prime,h}} u_2 - u_2 \|_{L^2(I;H)} \| V \|_{L^2(I;E)},$$
$$\mathcal{E}_3 = ((\tilde{\Pi}^r_{G_{CG}^h} U - U)_-, V_-^h)_E = a(\tilde{\Pi}^r_{g_{CG}^h} u_1 - u_1)_-, v_{1,-}^h) + (\tilde{\Pi}^r_{g_{CG}^{\prime,h}} u_2 - u_2)_-, v_{2,-}^h)_H,$$

but since $a(\tilde{\Pi}^r_{g_{CG}^h} u_1 - u_1)_-, v_{1,-}^h) = a(\hat{\Pi}_{g^h} u_1 - u_1)_-, v_{1,-}^h) = 0$, with recalling the estimates for $v_{2,-}^h$ done in the proof of Theorem 3.7 on p. 68, then

$$\mathcal{E}_3 \leq \| (\tilde{\Pi}^r_{g_{CG}^{\prime,h}} u_2 - u_2)_- \|_H \| v_{2,-}^h \|_H \leq \sqrt{\frac{2}{k}}(2r+1) \| (\tilde{\Pi}^r_{g_{CG}^{\prime,h}} u_2 - u_2)_- \|_H \| V^h \|_{L^2(I;E)},$$

with Assumption 6.1 on p. 149, *Cauchy inequality*, and then the estimate in (6.74) on p. 146, then

$$\mathcal{E}_5 + \mathcal{E}_6 = \int_I a(\tilde{\Pi}^r_{g_{CG}^h} u_1 - u_1, v_2^h) dt - \int_I a(\tilde{\Pi}^r_{g_{CG}^{\prime,h}} u_2 - u_2, v_1^h) dt$$
$$\leq \frac{\max(1,M)}{\sqrt{\min(\alpha,1)}} \left(\| A(\tilde{\Pi}^r_{g_{CG}^h} u_1 - u_1) \|_{L^2(I;H)} + \| \tilde{\Pi}^r_{g_{CG}^{\prime,h}} u_2 - u_2 \|_{L^2(I;Z)} \right) \| V^h \|_{L^2(I;E)} .$$

Then we have

$$\| U - U_{DG}^h \|_{L^2(I;X)} \leq \| U - \tilde{\Pi}^r_{G_{CG}^h} U \|_{L^2(I;X)}$$
$$+ \left(\frac{k \| Y \|_2^2}{2 \min_p \mathrm{Re}\ \lambda_p \sqrt{\min(\alpha,1)}} \right) \Bigg(\frac{\sqrt{2\mathcal{P}_r}}{k} \| \tilde{\Pi}^r_{g_{CG}^{\prime,h}} u_2 - u_2 \|_{L^2(I;H)}$$
$$+ \sqrt{\frac{2}{k}}(2r+1) \| (U - U_{DG}^h)_-^0 \|_E + \sqrt{\frac{2}{k}}(2r+1) \| (\tilde{\Pi}^r_{g_{CG}^{\prime,h}} u_2 - u_2)_- \|_H$$
$$+ \frac{\max(1,M)}{\sqrt{\min(\alpha,1)}} \left(\| A(\tilde{\Pi}^r_{g_{CG}^h} u_1 - u_1) \|_{L^2(I;H)} + \| \tilde{\Pi}^r_{g_{CG}^{\prime,h}} u_2 - u_2 \|_{L^2(I;Z)} \right) \Bigg).$$

and that completes the proof. □

Lemma 6.14 (The a priori error estimate in the max $L^2(I_m;X)$-norm)
Let Z and H be the Hilbert spaces given in Definition 5.1 on p. 101. Let Z_g be the subset and Z_0 the subspace of the Hilbert space Z given in Definition 5.2 on p. 102. Let $U \in L^2(J; Z_g \times Z_{g'})$ be the continuous solution of (5.25) on p.112 with initial data $U_0 \in Z_g \times Z_{g'}$. Let $U^h_{DG} \in \mathcal{V}^{\underline{r}}(\mathcal{N}; Z^h_{g^h_{CG}} \times Z^h_{g^{h,\prime}_{CG}})$ be the fully-discrete solution of (6.53) on p. 137 with initial data $U^h_0 \in Z^h_{g^h} \times Z^h_{g^{h,\prime}}$ there holds for $X = Z \times H$

$$
\begin{aligned}
\max_m \| U - U^h_{DG} \|_{L^2(I_m;X)} \leq\ & \max_m \| U - \tilde{\Pi}^{r_m}_{G^h_{CG}} U \|_{L^2(I_m;X)} \\
& + \frac{\sqrt{2}\mathcal{C} \max_m(\sqrt{\mathcal{P}_{r_m}})}{T\sqrt{\min(\alpha,1)}} \max_m \| \tilde{\Pi}^{r_m}_{g^{\prime,h}_{CG}} u_2 - u_2 \|_{L^2(I_m;H)} \\
& + \frac{\mathfrak{C}_2}{\sqrt{TN\min(\alpha,1)}} \max_m \| (U - \hat{\Pi}_{G^h} U)^m_- \|_X \\
& + \frac{\sqrt{N\max(1,M)}\mathfrak{C}_1}{\sqrt{T^3}\min(\alpha,1)} \| \tilde{\Pi}^{r}_{g^{\prime,h}_{CG}} u_2 - u_2 \|_{L^2(J;H)} \\
& \frac{\mathfrak{C}_2}{\sqrt{TN\min(\alpha,1)}} \big(\max_m \sum_{m=1}^{N-1} \| (\hat{\Pi}^{r_m}_{g^{\prime,h}} u_2 - u_2)^m_- \|_H + \sqrt{2} \| (\hat{\Pi}^{r_N}_{g^{\prime,h}} u_2 - u_2)^N_- \|_H \big) \\
& + \frac{\mathfrak{C}_2 \mathcal{C}\max(M,1)}{\sqrt{TN\min(\alpha,1)^3}} \big(\| A(\tilde{\Pi}^{r}_{g^h_{CG}} u_1 - u_1) \|_{L^2(J;H)} + \| \tilde{\Pi}^{r}_{g^{\prime,h}_{CG}} u_2 - u_2 \|_{L^2(J;Z)} \big) \\
& \frac{\mathcal{C}\max(1,M)}{N\min(\alpha,1)} \max_m \big(\| A(\tilde{\Pi}^{r_m}_{g^h_{CG}} u_1 - u_1) \|_{L^2(I_m;H)} + \| \tilde{\Pi}^{r_m}_{g^{\prime,h}_{CG}} u_2 - u_2 \|_{L^2(I_m;Z)} \big) \\
& + \frac{\sqrt{2}\mathfrak{C}_2\mathcal{C}\max_m(2r_m+1)}{T\min(\alpha,1)} \| U_0 - U^h_0 \|_E \\
& + \frac{\mathcal{C}\max_m(2r_m+1)}{\sqrt{TN\min(\alpha,1)}} \max_m \| (\tilde{\Pi}^{r_m}_{g^{\prime,h}_{CG}} u_2 - u_2)^{m+1}_- \|_H,
\end{aligned}
\tag{6.101}
$$

where $\hat{\Pi}_{G^h} := \begin{pmatrix} \hat{\Pi}_{g^h} & 0 \\ 0 & \hat{\Pi}_{g^{\prime,h}} \end{pmatrix} : Z \times Z \to Z^h_{g^h} \times Z^h_{g^{\prime},h}$ is the spatial projection given in Definition 6.6 and $\tilde{\Pi}^{r_m}_{G^h_{CG}} = \begin{pmatrix} \tilde{\Pi}^{r_m}_{g^h_{CG}} & 0 \\ 0 & \tilde{\Pi}^{r_m}_{g^{\prime,h}_{CG}} \end{pmatrix}$ is the spatio-time projection given in Definition 6.7. $M < \infty$ and $\alpha > 0$, the upper and lower bounds of the form $a(,)$, respectively given in Definition 5.3 on p. 104 and $\mathcal{P}_{r_m}$ is a real constant which also depends on the maximum approximation order r and

$$
\mathfrak{C}_1 = \mathcal{C}^2 \max_m(\sqrt{\mathcal{P}_{r_m} r_m}) \max_m(2r_m+1),\ \mathfrak{C}_2 = \mathcal{C}\sqrt{\max(1,M)} \max_m(2r_m+1),
$$

Proof:
Starting with the resulting estimate in Theorem 6.9 on p. 152 with $k = \frac{T}{N}$, $\mathcal{C} = \max_m(\frac{T \| Y^{r_m} \|^2_2}{2\min_p \operatorname{Re} \lambda_p})$,

and taking the maximum at the given $m = 0, \cdots, N-1$:

$$\begin{aligned}
\max_m \| U - U_{DG}^h \|_{L^2(I_m;X)} \leq \max_m \Bigg(& \| U - \tilde{\Pi}_{G_{CG}^h}^{r_m} U \|_{L^2(I_m;X)} \\
& + (\frac{T \| Y^{r_m} \|_2^2}{2N \min_p \mathrm{Re}\, \lambda_p \sqrt{\min(\alpha,1)}}) \Big(\frac{\sqrt{2\mathcal{P}_{r_m}} N}{T} \| \tilde{\Pi}_{g_{CG}^{\prime,h}}^{r_m} u_2 - u_2 \|_{L^2(I_m;H)} \\
& + \sqrt{\frac{2N}{T}} (2r_m + 1) \| (U - U_{DG}^h)_-^m \|_E + \sqrt{\frac{2N}{T}} (2r_m+1) \| (\tilde{\Pi}_{g_{CG}^{\prime,h}}^{r_m} u_2 - u_2)_-^{m+1} \|_H \\
& + \frac{\max(1,M)}{\sqrt{\min(\alpha,1)}} (\| A(\tilde{\Pi}_{g_{CG}^h}^{r_m} u_1 - u_1) \|_{L^2(I_m;H)} + \| \tilde{\Pi}_{g_{CG}^{\prime,h}}^{r_m} u_2 - u_2 \|_{L^2(I_m;Z)}) \Big) \Bigg) \\
& \leq \max_m \| U - \tilde{\Pi}_{G_{CG}^h}^{r_m} U \|_{L^2(I_m;X)} + \frac{\mathcal{C} \max_m (\sqrt{2\mathcal{P}_{r_m}})}{T\sqrt{\min(\alpha,1)}} \max_m \| \tilde{\Pi}_{g_{CG}^{\prime,h}}^{r_m} u_2 - u_2 \|_{L^2(I_m;H)} \\
& + \frac{\mathcal{C} \max_m (2r_m+1)}{\sqrt{TN\min(\alpha,1)}} \max_m \| (U - U_{DG}^h)_-^m \|_E + \frac{\mathcal{C} \max_m (2r_m+1)}{\sqrt{TN\min(\alpha,1)}} \max_m \| (\tilde{\Pi}_{g_{CG}^{\prime,h}}^{r_m} u_2 - u_2)_-^{m+1} \|_H \\
& \frac{\mathcal{C} \max(1,M)}{N \min(\alpha,1)} \max_m (\| A(\tilde{\Pi}_{g_{CG}^h}^{r_m} u_1 - u_1) \|_{L^2(I_m;H)} + \| \tilde{\Pi}_{g_{CG}^{\prime,h}}^{r_m} u_2 - u_2 \|_{L^2(I_m;Z)}),
\end{aligned} \tag{6.102}$$

and also using the resulting estimate in Theorem 6.8 on p. 149 s.t. with

$$\mathfrak{C}_1 = \mathcal{C}^2 \max_m (\sqrt{\mathcal{P}_{r_m} r_m}) \max_m (2r_m+1), \;\; \mathfrak{C}_2 = \mathcal{C} \sqrt{\max(1,M)} \max_m (2r_m+1),$$

and rearranging all terms after taking the max over m time steps then

$$\begin{aligned}
\max_m \| U - U_{DG}^h \|_{L^2(I_m;X)} \leq & \max_m \| U - \tilde{\Pi}_{G_{CG}^h}^{r_m} U \|_{L^2(I_m;X)} \\
& + \frac{\sqrt{2}\mathcal{C} \max_m (\sqrt{\mathcal{P}_{r_m}})}{T\sqrt{\min(\alpha,1)}} \max_m \| \tilde{\Pi}_{g_{CG}^{\prime,h}}^{r_m} u_2 - u_2 \|_{L^2(I_m;H)} \\
& + \frac{\mathfrak{C}_2}{\sqrt{TN\min(\alpha,1)}} \max_m \| (U - \hat{\Pi}_{G^h} U)_-^m \|_X \\
& + \frac{\sqrt{N\max(1,M)}\mathfrak{C}_1}{\sqrt{T^3}\min(\alpha,1)} \| \tilde{\Pi}_{g_{CG}^{\prime,h}}^{r} u_2 - u_2 \|_{L^2(J;H)} \\
& \frac{\mathfrak{C}_2}{\sqrt{TN}\min(\alpha,1)} (\max_m \sum_{m=1}^{N-1} \| (\hat{\Pi}_{g^{\prime,h}}^{r_m} u_2 - u_2)_-^m \|_H + \sqrt{2} \| (\hat{\Pi}_{g^{\prime,h}}^{r_N} u_2 - u_2)_-^N \|_H) \\
& + \frac{\mathfrak{C}_2 \mathcal{C} \max(M,1)}{\sqrt{TN\min(\alpha,1)^3}} (\| A(\tilde{\Pi}_{g_{CG}^h}^{r} u_1 - u_1) \|_{L^2(J;H)} + \| \tilde{\Pi}_{g_{CG}^{\prime,h}}^{r} u_2 - u_2 \|_{L^2(J;Z)}) \\
& \frac{\mathcal{C}\max(1,M)}{N\min(\alpha,1)} \max_m (\| A(\tilde{\Pi}_{g_{CG}^h}^{r_m} u_1 - u_1) \|_{L^2(I_m;H)} + \| \tilde{\Pi}_{g_{CG}^{\prime,h}}^{r_m} u_2 - u_2 \|_{L^2(I_m;Z)}) \\
& + \frac{\sqrt{2}\mathfrak{C}_2 \mathcal{C} \max_m (2r_m+1)}{T\min(\alpha,1)} \| U_0 - U_0^h \|_E \\
& + \frac{\mathcal{C} \max_m (2r_m+1)}{\sqrt{TN\min(\alpha,1)}} \max_m \| (\tilde{\Pi}_{g_{CG}^{\prime,h}}^{r_m} u_2 - u_2)_-^{m+1} \|_H,
\end{aligned}$$

and that completes the proof. □

Part III: Application and conclusion

Chapter 7

Practical examples and Conclusion

In this Chapter, it will be shown that Linear Acoustic wave equation in Section 7.1 and the elastic wave equation in Section 7.2 starting on p. 163 do fit our abstract approximation theory given in Part II with I. Here the dimension of discretised spatial space with the choices of the bases are kept open.

7.1 Linear Acoustic wave equation

In Part II of this thesis, the first-order system in time variational formulation with linear constraints is discussed in abstract spatial spaces. It is approximated using high-order in time DGFEM to discretise in time and then conformal spatial discretisation to discretise in space. All that is done in abstract subsets, subspaces and spatial operators. This section shows that the acoustic wave equation does fit our abstract theory shown in Parts II with I:

- with $J = (0, T)$ for $T < 0$, and Ω to be a smooth simple connected Lipschitz bounded domain in $\mathbb{R}^n$ for $n \geq 2$:

$$\Gamma_D = \Gamma \backslash \Gamma_N, \text{ for } \partial\Omega := \Gamma = \Gamma_D \cup \Gamma_N, \tag{7.1}$$

  and the Neumann-Dirichlet transition points are Dirichlet nodes.

Then the acoustic linear wave equation would read

$$\partial_t^2 u - \Delta u = f \qquad \text{in } J \times \Omega,$$

for given compatible initial data and boundary conditions

$$\begin{aligned} u &= g && \text{on } J \times \Gamma_D, \\ \frac{\partial u}{\partial n} &= 0 && \text{on } J \times \Gamma_N \\ u|_{t=0} &= u_0 && \text{in } \Omega, \\ \partial_t u|_{t=0} &= \bar{u}_0 && \text{in } \Omega, \end{aligned}$$

such that:

- The following Sobolev spaces (see [8, The Sobolev Space $W^{1,p}$]) and subsets are chosen for the abstract ones:

$$
\begin{aligned}
&Z = H^1(\Omega), \text{ and } H = L^2(\Omega).\\
&Z_0 = H_0^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega\}, (\text{ which is a subspace of } H^1(\Omega)),\\
&H = L^2(\Omega), \text{ and}\\
&Z' = H^{-1}(\Omega)(\text{ the dual space of } H_0^1(\Omega)).
\end{aligned}
$$

(For simplification they will be noted as H^1, L^2, H_0^1, and the same would be applied for the rest of the spaces).
These spaces fit the properties given in Definitions 5.1 and 5.2 on p. 101, respectively, which are:

  - H^1 and L^2 are separable Hilbert spaces over the field $\mathbb{C}$ (see [8, Proposition 8.1]), H^1 is a subspace of L^2, dense in L^2, and is separable with the *Gelfand triple* ($H^1 \subset L^2 \subset (H^1)'$) for $(H^1)' =: \tilde{H}^{-1}$ (see [28, 2.2 References from Functional Analysis]).
  - Also in Lemma 5.1 on p. 103; H_0^1 by definition is a subspace of H^1as well as L^2, dense in L^2, and is separable and also satisfies the *Gelfand triple* $H_0^1 \subset L^2 \subset H^{-1}$ where

$$H_0^1 \subset H^1 \subset L^2 \subset \tilde{H}^{-1} \subset H^{-1}.$$

- The abstract spatial operator $A \in L(Z, Z')$ to be the Laplace operator $-\Delta \in L(H^1, \tilde{H}^{-1})$, where all functions in H_0^1 are involved in this linear map since $H_0^1 \subset H^1$.

- With choosing

$$D^A = H^1(\Delta) = \{u \in H^1 : \Delta u \in L^2\}\big(\text{ see } [4, (1-11)]\big), \tag{7.2}$$

which is a subspace of H^1, endowed with the norm

$$\| u \|_{H^1(\Delta)}^2 = \| u \|_{H^1}^2 + \| \Delta u \|_{L^2}^2,$$

which is as assumed in Assumption 5.1 and shown in Corollary 5.1 on p. 101 is a separable Hilbert space, see [4, Section 2.1-10].

- With considering Dirichlet boundary data in $\partial\Omega$ such that

$$
\begin{aligned}
Z^{tr} &= H^{1-1/2}(\Gamma_D) = H^{1/2}(\Gamma_D \text{ (see [28, 2.2.4 Trace of a Function]) and}\\
D^{A,tr} &= H^{1/2}(\Gamma_D) \text{ see [4, (1-40) of Theorem 1-2]},
\end{aligned}
$$

and for any $g \in H^{1/2}(\Gamma_D)$

$$Z_g = H_g^1 = \{u \in H^1 : u = g \text{ on } \Gamma_D\},$$

which is the subset of H^1.

- Also, from Definition 5.2 on p. 102 with (7.2) such that for any $g \in H^{1/2}$

$$
\begin{aligned}
D_g &= H_g^1(\Delta) = \{u \in H^1(\Delta) : u = g \text{ on } \Gamma_D\},\\
D_0 &= H_0^1(\Delta) = \{u \in H^1(\Delta) : u = 0 \text{ on } \Gamma_D\}.
\end{aligned}
$$

From the definition of such subset and subspace and referring to Corollary 5.3 on p. 102 which hold true, i.e. $H_g^1(\Delta)$ is a subset of $H^1(\Delta)$ and $H_0^1(\Delta)$ is a subspace of $H^1(\Delta)$.

Here, for the higher regularity of the first component of the solution of Problem 5.4.1 on p. 112, since the forcing data $f \in H^1(J;L^2)$ and $\partial_t u_2 = \partial_t^2 u_1 \in L^2(J;L^2)$ thus

$$-\Delta u_1 \in L^2(J;L^2),$$

i.e. $u_1 \in H^1(\Delta) \subset H^1$.
Moreover, the form in (5.24) on p. 112 which now would be

$$\int_J \hat{a}(U,V)dt = \int_J \{-a(u_2,v_1) + a(u_1,v_2)\}dt, \quad \forall V \in L^2(J;H_0^1 \times H_0^1). \tag{7.3}$$

Here for $u, v \in L^2(J;H^1)$

$$\int_J a(u,v)dt := -\int_J \langle \Delta u, v\rangle_{\tilde{H}^{-1}\times H^1} dt = (\nabla u, \nabla v)_{L^2(J;L^2)} \forall v \in L^2(J;H_0^1), \tag{7.4}$$

where

$$\| u \|^2_{L^2(J;H^1)} := \| u \|^2_{L^2(J;L^2)} + \| \nabla u \|^2_{L^2(J;L^2)} \tag{7.5}$$

then with also referring to Corollary A.9 on p. 178

$$|\int_J a(u,v)dt| = |(\nabla u, \nabla v)_{L^2(J;L^2)}| \leq \| \nabla u \|_{L^2(J;L^2)} \| \nabla v \|_{L^2(J;L^2)} \leq M \| u \|_{H^1} \| v \|_{H^1}, \quad \forall u, v \in H^1,$$

and with also using Poincare-Friedrichs inequality (see [24, 2.4 Abstract Variational Problem]), then

$$a(u,u) \geq \alpha \| u \|^2_{H^1}, \qquad \forall u \in H_0^1,$$

and that fit all properties given in Definition 5.3 on p. 104.

From the choices of subspaces and subsets of Hilbert spaces:

- With having Dirichlet boundary data in $J \times \Gamma_D$ that is $G = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} := \begin{pmatrix} g \\ g' \end{pmatrix} \in W^{tr} = H^1(J;H^{1/2} \times H^{1/2})$, where

  $$g_1 = g \text{ and } g_2 = \partial_t g_1 = \partial_t g \text{ on } J \times \Gamma_D,$$

  and $\bar{\gamma} : W = H^1(J;H^1(\Delta) \times H^1(\Delta)) \to W^{tr} = H^1(J;H^{1/2} \times H^{1/2})$.
- Forcing data $F = \binom{0}{f} \in L^2(J;H^1 \times L^2)$ with $f \in H^1(J;L^2)$.
- Compatible initial data $U_0 = \binom{u_{1,0}}{u_{2,0}} \in H_g^1 \times H_{g'}^1 \subset H^1 \times H^1$ with $u_{1,0} \in H_g^1(\Delta)$.

The vector-valued function $U = \binom{u_1}{u_2} \in L^2(J;H_g^1 \times H_{g'}^1)$ with $\partial_t U \in L^2(J;H_{g'}^1 \times L^2)$ in Definition 5.8 which solves Problem 5.4.1 on p. 112, are all well defined.

Also, with the separable Hilbert space $W = H^1(J;H^1(\Delta) \times H^1(\Delta))$, the solution of the auxiliary (elliptic) problem (5.13) on p. 109 which is

$$\hat{G} \in W_G = \left\{ U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} | \, U \in W \text{ with } \bar{\gamma} U = \begin{pmatrix} g \\ g' \end{pmatrix}, \, \partial_t u_1 \in L^2(J;H_{g'}^1(\Delta)) \text{ and } \partial_t u_2 \in L^2(J;H^1(\Delta)) \right\},$$

is well defined since Theorem 5.1 on p. 109 holds true, because via the definition of the normed space in $H^1(J; H^{1/2} \times H^{1/2})$ there exists a vector-valued function $\bar{G} \in H^1(J; H^1(\Delta) \times H^1(\Delta))$ such that $\bar{G}|_{J\times\partial\Omega} = G$ and a vector-valued function $\tilde{G} \in W_{\mathbf{0}} = \left\{ U = \binom{u_1}{u_2} | \, U \in W \text{ with } \bar{\gamma} U = \mathbf{0} \right\}$:

$$\hat{G} = \tilde{G} + \bar{G}. \tag{7.6}$$

So, firstly, by Lax-Milgram theorem and since $W_{\mathbf{0}} \subset W$ is a Hilbert space, there exists a unique vector-valued function $\tilde{G} \in W_{\mathbf{0}}$ and secondly, from (7.6) also $\hat{G}$ does exist.
Also, Lemma 5.7 on p. 110 holds true for the normed-vector space $W^{tr} = H^1(J; H^{1/2} \times H^{1/2})$:

$$\| \hat{G} \|_{H^1(J;H^1(\Delta)\times H^1(\Delta))} \leq 2 \| G \|_{H^1(J;H^{1/2}\times H^{1/2})} . \tag{7.7}$$

Thus, Theorem 5.2 on p. 115 holds true, i.e. firstly, for $\bar{U} \in L^2(J; H_0^1 \times H_0^1)$ which solves Problem 5.4.2 on p. 115 and since such formulation is a particular example of the general continuous one in Problem 2.2.2 on p. 15 with given

$$F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} -\partial_t \hat{g}_1 + \hat{g}_2 \\ f - \partial_t \hat{g}_2 + \Delta \hat{g}_1 \end{pmatrix} \in L^2(J; H_0^1 \times L^2), \text{ initial data } \bar{U}_0 = U_0 - \hat{G}_0 \in H_0^1 \times H_0^1 \subset H^1 \times H^1, \tag{7.8}$$

where here we have $-\partial_t \hat{g}_1 + \hat{g}_2 \in H_0^1(\Delta) \subset H_0^1$, and $\bar{u}_{1,0} \in H_0^1(\Delta) \subset H_0^1$ which all are well defined since all such Hilbert spaces fit all assumptions given in Part II as well as I. Secondly, with the splitting technique i.e.

$$U = \hat{G} + \bar{U} \tag{7.9}$$

Thus $U \in L^2(H_g^1 \times H_{g'}^1)$ does exist.
Theorem 5.3 on p. 116 holds also true, for given the following norm

$$\| U \|_{L^2(J;E)}^2 \leq \int_J \| U \|_E^2 \, dt = \int_J \{a(u_1, u_1) + \| u_2 \|_{L^2}^2\} dt, \forall U \in L^2(J; H^1 \times H^1), \tag{7.10}$$

such that

$$\begin{aligned} \| U \|_{L^2(J;E)} \leq 2(4\sqrt{TC \max(M, c^2, 1)} + \sqrt{\max(M, c^2)}) \| G \|_{H^1(J;H^{1/2}\times H^{1/2})} + \sqrt{2TC}\big(\| f \|_{L^2(J;L^2)} \\ + \sqrt{TC} \| U_0 \|_E + \sqrt{\max(c^2, M)} \| G_0 \|_{H^{1/2}\times H^{1/2}} \big), \end{aligned} \tag{7.11}$$

and Theorem 5.4 on p. 119 that is

$$\begin{aligned} \| U_-^N \|_E \leq C\big(4\sqrt{\max(M, c^2, 1)} \| G \|_{H^1(J;H^{1/2}\times H^{1/2})} + \sqrt{2} \| f \|_{L^2(J;L^2)} + \| U_0 \|_E^2 \\ + \sqrt{\max(c^2, M)} \| G_0 \|_{H^{1/2}\times H^{1/2}} \big) + \sqrt{\max(M, c^2)} \| G_-^N \|_{H^{1/2}\times H^{1/2}} . \end{aligned}$$

Finally, with letting

- $V_M \times V_M$ to be the conformal finite dimensional subspace of $H^1 \times H^1$ with $M^2 = \dim(V_M \times V_M)$.
- $\bar{V}_M \times \bar{V}_M$ to be the conformal finite dimensional subset of $H_g^1 \times H_{g'}^1$ with $\bar{M}^2 = \dim(\bar{V}_M \times \bar{V}_M)$.
- $\hat{V}_M \times \hat{V}_M$ to be the conformal finite dimensional subspace of $H_0^1 \times H_0^1$ with $\hat{M}^2 = \dim(\hat{V}_M \times \hat{V}_M)$, where $\hat{V}_M \times \hat{V}_M \subset V_M \times V_M$ with $\hat{M}^2 < M^2$.

- $\tilde{V}_M$ to be the conformal finite dimensional subset of $H^1_g(\Delta)$ with $\tilde{M} = \dim \tilde{V}_M$.

- $\mathfrak{V}_M \times \mathfrak{V}_M$ is a conformal finite dimensional subspace of $H^{1/2} \times H^{1/2}$ with $\mathfrak{M} = \dim(\mathfrak{V}_M \times \mathfrak{V}_M)$.

Then the a priori error from the continuous to the fully-discrete solution in the maximum global L^2- norm over the whole time interval steps given in Lemma 6.14 on p. 155 with their initial data $U_0 \in H^1_g \times H^1_{g'}$ and $U_0^h \in \bar{V}_M \times \bar{V}_M$, respectively would be

$$\begin{aligned}
\max_m \| U - U^h_{DG} \|_{L^2(I_m;E)} \leq & \| U - \Pi U \|_{L^2(J;E)} + \sqrt{2} \| (\tilde{\Pi} U - U)^0_- \|_E + \max_m \| (\tilde{\Pi} U - U)^m_- \|_E \\
& + \big(4\sqrt{2} C \mathfrak{C}_2 \frac{N}{T^2} + \frac{2CN}{T}\big(\max(\sqrt{r_m}) + \max(\frac{\| Y^{r_m} \|_2^2 \sqrt{\mathcal{P}_{r_m} r_m}}{2 \min_p \mathrm{Re}\, \lambda_p}) \big) \big) \| U - \tilde{\Pi} U \|_{L^2(J;E)} \\
& + \left(\frac{\sqrt{2^3}}{2N} \mathfrak{C}_2 + 2C\mathcal{C}\mathfrak{C}_3 + C \max(\frac{1}{\sqrt{r_n}}) \right) \max_m \| (U - \tilde{\Pi} U)^{m+1}_- \|_E \\
& + (\sqrt{2^3} \mathcal{C}^2 C \mathfrak{C}_3 + 2\frac{T}{N} \mathfrak{C}_2) \sum_{n=1}^{N-1} \| (\tilde{\Pi} U - U)^n_- \|_E + \sqrt{2^3} \mathfrak{C}_2 (C\mathcal{C}\mathfrak{C}_3 + \frac{T}{N}) \| (\tilde{\Pi} U - U)^N_- \|_E \\
& + \frac{\max(1, M)}{\sqrt{\min(\alpha, 1)}} \mathcal{C} (\frac{\mathcal{C}}{N} + 1) \big(\| \Delta(\Pi_1 u_1 - u_1) \|_{L^2(J;L^2)} + \| \Pi_2 u_2 - u_2 \|_{L^2(J;H^1)} \big) \\
& + \sqrt{2^3} \mathfrak{C}_2 \max(2r_m + 1) \| U_0 - U_0^h \|_E,
\end{aligned} \tag{7.12}$$

where Π is the interpolant in time related to the projection operator given in Definition 3.12 on p. 59. $\tilde{\Pi} = \binom{\hat{\Pi}_1 \; 0}{0 \; \hat{\Pi}_2}$ is the spatial matrix projection operator given in Definition 6.6 on p. 146 which is now chosen very carefully so that also the data at the boundary of the given domain are interpolated in order to be as close as possible to the given continuous data. $M < \infty$ and $\alpha \in \mathbb{R}$, the upper and lower bounds of the form $a(\, ,)$, respectively given in (7.4) on p. 160 and

$$\mathcal{C} = \max_n (\frac{T \| Y^{r_n} \|_2^2}{2 \min_p \mathrm{Re}\, \lambda_p}), \mathfrak{C}_1 = \mathcal{C} \max_n (\sqrt{\mathcal{P}_{r_m} r_m}), \;\; \mathfrak{C}_2 = \mathcal{C} \max_n ((2r_m + 1)), \;\; \mathfrak{C}_3 = \max_n (\frac{\sqrt{\mathcal{P}_{r_m}}}{\sqrt{\min(r_m)}}),$$

are depending on the maximum value of the approximation order r_n for $n = 0, \cdots, N-1$, the square of the spectral norm (see (A.6) on p. 173) of the matrix $Y^{r_n} \in \mathbb{C}^{(r_n+1)\times(r_n+1)}$, and the eigenvalues λ_p of the matrix $\mathcal{A}^L$ given in (3.31) on p. 30 for $p = 0, \cdots, r_n$. $\mathcal{P}_{r_n}$ is a real constant which also depends on the maximum approximation order r_n and C is a generic constant (see [44, Lemma 1.16]).
Here we use the approximation results of the time Projection Π^r from ([44, Corollary 1.20]) and assuming that we have analytic functions in time i.e. with

$$\begin{aligned}
&\| U - \Pi U \|_{L^2(J;E)} \leq \sqrt{\max(M, c^2)} \| U - \Pi U \|_{L^2(J;H^1 \times H^1)}, \text{ from Lemma 2.1 on p. 9, and} \\
&\| \Pi_2 u_2 - u_2 \|_{L^2(J;H^1)} \leq \sqrt{2} \| \Pi U - U \|_{L^2(J;H^1 \times H^1)},
\end{aligned} \tag{7.13}$$

where now the Sobolev spaces are used and c is the constant in the inclusion inequality

$$\| u \|_{L^2} \leq c \| u \|_{H^1}, \forall u \in H^1, \tag{7.14}$$

letting $r \geq 1$, and $U \in H^1(J; Z \times Z)$ with $k = \frac{T}{N}$, then

$$\begin{aligned} \| U - \Pi U \|_{L^2(J;E)} &\leq \sqrt{\max(M, c^2)} \| U - \Pi U \|_{L^2(J;H^1 \times H^1)} \\ &\leq C\sqrt{\max(M, c^2)} \sum_{n=0}^{N-1} \frac{T}{2\sqrt{r_n}N} \| U \|_{H^1(I_n;H^1 \times H^1)} \\ &\leq C\sqrt{\max(M, c^2)} \frac{T}{2N} \max(\frac{1}{\sqrt{r_n}}) \| U \|_{H^1(J;H^1 \times H^1)} \\ &= C\sqrt{\max(M, c^2)} \frac{k}{2} \max(\frac{1}{\sqrt{r_n}}) \| U \|_{H^1(J;H^1 \times H^1)}, \end{aligned}$$

and

$$\| \Pi_2 u_2 - u_2 \|_{L^2(J;H^1)} \leq C\sqrt{2} \frac{T}{2N} \max(\frac{1}{\sqrt{r_n}}) \| U \|_{H^1(J;H^1 \times H^1)} = C\frac{k}{\sqrt{2}} \max(\frac{1}{\sqrt{r_n}}) \| U \|_{H^1(J;H^1 \times H^1)}, \tag{7.15}$$

also with letting $\Delta u_1 \in H^1(J; L^2)$, then

$$\begin{aligned} \| \Delta(\Pi_1 u_1 - u_1) \|_{L^2(J;L^2)} = \| (\Pi_1 \Delta u_1 - \Delta u_1) \|_{L^2(J;L^2)} &\leq C \sum_{n=0}^{N-1} \frac{T}{2\sqrt{r_n}N} \| \Delta u_1 \|_{H^1(I_n;L^2)} \\ &\leq \frac{T}{2N} \max(\frac{1}{\sqrt{r_n}}) \| \Delta u_1 \|_{H^1(J;L^2)} . \end{aligned} \tag{7.16}$$

Now, for approximating the spatial projection in (7.12) on p. 162, comes the question of how to choose the finite dimensional spatial spaces appropriately with the choice of bases function in space and the mesh size in space to get the most optimal a priori estimate possible. This is not going to be investigated any further here, but there are many varieties of conformal finite element methods and techniques which leads to optimal or nearly optimal a priori error estimates.

7.2 The elastic wave equation

Here, the same domain $\Omega \in \mathbb{R}^d$, which is an elastic body, and time interval $J = (0, T)$ considered in Section 7.1 are going to be used. The displacement vector $\mathbf{u}(t, \mathbf{x})$ where $t \in [0, T]$ and $\mathbf{x} \in \overline{\Omega}$, $\rho = \rho(x) = \bar{c} > 0$ is the density which is constant in this case, and the stress is determined by the generalised Hook's law:

$$\sigma(\nabla \mathbf{u}) = \mathfrak{c} \cdot \nabla \mathbf{u}, \tag{7.17}$$

or, in components,

$$\sigma_{ij} = \mathfrak{c}_{ijmn} u_{m,n}, \tag{7.18}$$

for $1 \leq i, j,\ m, n \leq d$, $u_{m,n} = \frac{\partial u_m}{\partial x_n}$, where summation over repeated indices is implied. The elastic coefficients $\mathfrak{c}_{ijmn} = \mathfrak{c}_{ijmn}(\mathbf{x})$ are assumed to satisfy the following conditions:

$$\begin{aligned} &\mathfrak{c}_{ijmn} = \mathfrak{c}_{mnij} && \text{(major symmetry)}, \\ &\mathfrak{c}_{ijmn}\phi_{ij}\phi_{mn} > 0 \quad \forall \phi_{ij} = \phi_{ji} \neq 0 && \text{(positive-definiteness)}, \end{aligned} \tag{7.19}$$

see [25, 2. Classical linear elastodynamics]. This section shows that the elastic wave equation e.g.

$$\rho \partial_t^2 \mathbf{u} - \nabla \cdot \sigma(\nabla \mathbf{u}) = f \qquad \text{in } J \times \Omega,$$

with compatible initial data and boundary conditions

$$\begin{aligned} \mathbf{u} &= \mathbf{g} && \text{on } J \times \Gamma_D, \\ \mathbf{n} \cdot \sigma(\nabla \mathbf{u}) &= 0 && \text{on } J \times \Gamma_N \\ \mathbf{u}|_{t=0} &= \mathbf{u}_0 && \text{in } \Omega, \\ \partial_t \mathbf{u}|_{t=0} &= \bar{\mathbf{u}}_0 && \text{in } \Omega, \end{aligned} \tag{7.20}$$

does fit our abstract theory shown in Parts II with I. The same Soblolev spaces, subsets, and subspaces chosen in Section 7.1 are also chosen here, except for the space which depend on the spatial operator. Firstly, with (7.18) let the abstract spatial operator A given in Part II:

$$A = -\Lambda = -\nabla \cdot \sigma \nabla, \text{ such that } a(\mathbf{u}, \mathbf{v}) := -\langle \Lambda \mathbf{u}, \mathbf{v} \rangle_{\tilde{H}^{-1} \times H^1} = \int_\Omega \nabla \mathbf{u} \cdot \mathfrak{c} \cdot \nabla \mathbf{v} \, d\Omega, \forall \mathbf{v} \in H_0^1. \tag{7.21}$$

Secondly, to show that $\Lambda \in L(H^1, \tilde{H}^{-1})$ is continuous and elliptic for all $\mathbf{u} \in H_0^1 \subset H^1$, $\mathbf{v} \in H_0^1$, the following is used:

Definition 7.1 *[35, 1.12 Korn's Inequalities] For defined strain tensor* $\mathbf{e}$*:*

$$\mathbf{e}_{ab} = \frac{1}{2}(u_{a|b} + u_{b|a}), \text{ i.e. } \sigma = \mathfrak{c} \cdot \mathbf{e}, \text{ see [35, 3.4 Definition]}. \tag{7.22}$$

There exists a constant $\tilde{c} > 0$ *such that*

$$\| \mathbf{e} \|_{L^2}^2 + \| \mathbf{u} \|_{L^2}^2 \geq \tilde{c} \| \mathbf{u} \|_{H^1}^2, \tag{7.23}$$

for $\mathbf{u} \in H^1$*.*

Form Definition 7.1, the major symmetry and positive-definiteness of the elastic coefficient $\mathfrak{c}$ in (7.19) then

$$a(\mathbf{u}, \mathbf{u}) = -\langle \Lambda \mathbf{u}, \mathbf{u} \rangle_{\tilde{H}^{-1} \times H^1} = \int_\Omega \mathbf{e} \cdot \mathfrak{c} \cdot \mathbf{e} \, d\Omega \geq \eta \| \mathbf{e} \|_{L^2}^2 \geq \tilde{c} \| \mathbf{u} \|_{H^1}^2, \tag{7.24}$$

and the form $a : H^1 \times H^1 \to \mathbb{R}$, with real displacement vectors is a continuous symmetric form, see [35, 1.4 Definition]. Thus, the operator $\Lambda \in L(H^1, \tilde{H}^{-1})$ is linear and continuous, then

$$D^A = H^1(\Lambda)\{u \in H^1 : \Lambda u \in L^2\}\big(\text{ see [4, (2-4)]}\big), \tag{7.25}$$

which is a subspace of H^1, endowed with the norm

$$\| u \|_{H^1(\Lambda)}^2 = \| u \|_{H^1}^2 + \| \Lambda u \|_{L^2}^2,$$

is a Hilbert space as assumed in Assumption 5.1 on p. 101, see [4, 2-2. Formal Operator Λ Associated with $a(u, v)$].
From here, the rest follows as given in Section 7.1 such that the a priori error from the continuous to

the fully-discrete solution in the last time step $T := t_N$ given in Lemma 6.14 on p. 155 would be

$$\max_m \| U - U^h_{DG} \|_{L^2(I_m;E)} \leq \| U - \Pi U \|_{L^2(J;E)} + \sqrt{2} \| (\tilde{\Pi} U - U)^0_- \|_E + \max_m \| (\tilde{\Pi} U - U)^m_- \|_E$$
$$+ \big(4\sqrt{2} C \mathfrak{C}_2 \frac{N}{T^2} + \frac{2CN}{T} \big(\max(\sqrt{r_m}) + \max(\frac{\| Y^{r_m} \|^2_2 \sqrt{\mathcal{P}_{r_m} r_m}}{2 \min_p \mathrm{Re}\, \lambda_p})\big)\big) \| U - \tilde{\Pi} U \|_{L^2(J;E)}$$
$$+ \left(\frac{\sqrt{2^3}}{2N} \mathfrak{C}_2 + 2C \mathcal{C} \mathfrak{C}_3 + C \max(\frac{1}{\sqrt{r_n}}) \right) \max_m \| (U - \tilde{\Pi} U)^{m+1}_- \|_E$$
$$+ (\sqrt{2^3} \mathcal{C}^2 C \mathfrak{C}_3 + 2 \frac{T}{N} \mathfrak{C}_2) \sum_{n=1}^{N-1} \| (\tilde{\Pi} U - U)^n_- \|_E + \sqrt{2^3} \mathfrak{C}_2 (C \mathcal{C} \mathfrak{C}_3 + \frac{T}{N}) \| (\tilde{\Pi} U - U)^N_- \|_E$$
$$+ \frac{\max(1, M)}{\sqrt{\min(\alpha, 1)}} \mathcal{C} (\frac{\mathcal{C}}{N} + 1) \big(\| \Lambda (\Pi_1 u_1 - u_1) \|_{L^2(J;L^2)} + \| \Pi_2 u_2 - u_2 \|_{L^2(J;H^1)} \big)$$
$$+ \sqrt{2^3} \mathfrak{C}_2 \max(2 r_m + 1) \| U_0 - U^h_0 \|_E .$$

7.3 Conclusion

As a conclusion, the abstract linear wave equation with smooth enough data which give its solution the high-regularity we need is rewritten as a first-order in time system that is

$$\partial_t \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 & -A \\ A & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 \\ f \end{pmatrix}, \tag{7.26}$$

and its equivalent variational formulation is approximated with using the High-order in time DGFEM introduced by D Schötzau (PhD Thesis, 1999) to discrete in time and a conformal spatial discretisation to discretise in space. Our approximation analysis is a generalisation of Claes Johnson (CMAME, 1993), since:

- All the stability and a priori error estimate theorems are shown in abstract Hilbert spaces with continuous functions in time.

- It covers the scalar acoustic and elastic wave equation governed by Lamé-Navier equations of linear elasticity theory.

- It also covers the approximation analysis of a variational formulation of the first-order system in time with time-dependent linear constraints and showing its solvability, stability, and a priori error estimates. The linear constraints model is in-homogeneous boundary condition.

- Using the two step discretisation, with firstly, discretising in time by applying the high-order in time DGFEM approach introduced by D Schötzau [44] to get the semi-discrete scheme. Secondly, using a conformal spatial discretisation to get the fully-discrete scheme. This approach is not common since in literature usually the approximation is done first by discretising in space and the resulting schemes are discretised in time with using implicit or explicit conformal or non-conformal methods or having space-time approximation methods where it is done simultaneously in time and space.

Our approximation analysis approach can be implemented to any variational formulation of first-order system in time of linear hyperbolic (second-order in time) evolution problems with smooth enough data which satisfy our local Inf-sup condition and not necessary having symmetric form.

The initial data in our semi and fully-discrete formulations are weakly imposed in our both formulations in part I and II. In part I, the discrete initial data were defined by a certain projection so that we have Galerkin orthogonality and they don't appear in the bounds of the a priori error estimate. In part II and since they are considered to be compatible with the linear constraints e.g. compatible with the given boundary data then the difference between the continuous and the discretised initial data do contributes in the bounds of the a priori error estimate.

In the case of considering boundary data which are not continuous in time and allowing discontinuity over the time nodes as mentioned in Remark 6.1 on p. 130, no stability estimate can be achieved since the resulted left-sided limit of such data will have an exploding parameter 2^N as $N \to \infty$.

In future, and as a recommendation it would be interesting to investigate the generalisation of our analysis with the case of spatial operators which depend on time and also when considering spatial discretisation with allowing jumps in the spatial space. Also, finding out weather our analysis also covers problems with mixed boundary conditions, where it would probably have to have enough regularity as what we have here to fulfil at a minimum our stability and a priori error estimates.

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Appendix

Appendix A

Foundations

This part includes common and fundamental definition, theorems, and inequalities in linear functional analysis.

A.1 Metric and Normed spaces

A metric is simply a way of measuring distances between points of the space. In general, a *metric space* (M, d) is defined to be a set M together with a function $d : M \times M \to \mathbb{R}$ called a *metric* satisfying four conditions:

(i) $d(x, y) \geq 0$ for all $x, y \in M$.

(ii) $d(x, y) = 0$ if and only if $x = y$.

(iii) $d(x, y) = d(y, x)$, for all $x, y \in M$.

(iv) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in M$.

See [43, 1.1 Basic Definitions and Theorems].

Definition A.1 (Norm) *c.f. [41, Definition 2.1]*

(a) Let X be a vector space over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). A norm on X is a function $\| \cdot \|: X \to \mathbb{R}$ such that for all x, $y \in X$ and $\alpha \in \mathbb{F}$,

(i) $\| x \| \geq 0$.

(ii) $\| x \| = 0$ if and only if $x = 0$.

(iii) $\| \alpha x \| = |\alpha| \cdot \| x \|$.

(iv) $\| x + y \| \leq \| x \| + \| y \|$.

(b) A vector space X on which there is a norm is called a normed vector space or just a normed space, where every normed space is a metric space, see [43, 1.1 Basic Definitions and Theorems].

(c) A subset vector M is a subspace if:

1. The zero vector is in M.

2. For $\alpha \in \mathbb{C}$ and $x \in M$, then $\alpha x \in M$.

3. For $x, y \in M$, then $x + y \in M$.

Definition A.2 *[12, c.f. Definition 2.1.5 (Convergence in normed spaces)] A sequence $\{v_n\}$ in a normed vector space X converges to $v \in X$ if*

$$\lim_{n \to 0} \| v - v_n \|_X = 0. \tag{A.1}$$

Definition A.3 *[10, c.f. Definition 1.2] Let X be a normed vector-space. $\{u_n\} \subset X$ is called Cauchy sequence, if there holds*

$$\forall \varepsilon > 0, \ \exists N \in \mathbb{N}, \ \forall \ n, \ m \geq N : \ \| u_n - u_m \|_X \leq \varepsilon. \tag{A.2}$$

Every convergent sequence in a normed vector space is a Cauchy sequence, but the converse is not true, see [48, B.2 Banach and Hilbert Spaces].

Definition A.4 *[41, c.f. Definition 1.4] A normed vector space $(X, \| \cdot \|_X)$ is a Banach space if it is complete, i.e. if every Cauchy sequence $\{u_n\} \subset X$ converges in X.*

Definition A.5 *[21, c.f. 4.1.2 Basic facts about Banach Spaces]*

- *Let $(X, \| \cdot \|)$ be a Banach space. A subset $X_0 \subset X$ is dense, if for any $x \in X$ there is a sequence $x_i \in X_0$ with $\| x - x_i \| \to 0$.*
- *A Banach space is separable, if there is a countable dense subset.*

Corollary A.1 *A vector subspace M of a normed vector space X is itself a normed space under the norm of X and so normed is called a subspace of X, see [1, Normed Spaces].*

Definition A.6 (Product space) *Let $N \in \mathbb{N}$. Let $X_1, \cdots, X_N$ be vector spaces. Then a product space is defined as*

$$X := X_1 \times \cdots \times X_N,$$

and $u \in X$ means that

$$u = (u_1, \cdots, u_N),$$

for $u_1 \in X_1, \cdots, u_N \in X_N$.
For all $u, v \in X$, $\alpha \in \mathbb{R}$ the following operation are defined

$$\begin{aligned} u + v &:= (u_1 + v_1, \cdots, u_N + v_N), \ \text{and} \\ \alpha \cdot u &:= (\alpha\, u_1, \cdots, \alpha\, u_N). \end{aligned}$$

If $X_1, \cdots, X_N$ are normed vector spaces, the following is defined for $u \in X = X_1 \times \cdots \times X_N$

$$\| u \|_X^2 := \| u_1 \|_{X_1}^2 + \cdots + \| u_N \|_{X_N}^2 .$$

If $X_1, \cdots, X_N$ have inner products, the following is defined for $u, v \in X = X_1 \times \cdots \times X_N$

$$(u, v)_X := (u_1, v_1)_{X_1} + \cdots + (u_N, v_N)_{X_N}.$$

Corollary A.2 *Let $X = X_1 \times \cdots \times X_N$ be a product space with $X_1, \cdots, X_N$ being normed spaces. Then $(X, \| \cdot \|_X)$ is a normed space.*

Proof:
For $u \in X$, $\alpha \in \mathbb{R}$

$$\| \alpha \cdot u \|_X^2 := \sum_{i=1}^N \| \alpha \cdot u_i \|_{X_i}^2 = \sum_{i=1}^N |\alpha|^2 \| u_i \|_{X_i}^2 = |\alpha|^2 \sum_{i=1}^N \| u_i \|_{X_i}^2 = |\alpha|^2 \| u \|_X^2 . \tag{A.3}$$

For $u, v \in X$,

$$\begin{aligned} \| u+v \|_X^2 := \sum_{i=1}^N \| u_i + v_i \|_{X_i}^2 &\leq \sum_{i=1}^N (\| u_i \|_{X_i} + \| v_i \|_{X_i})^2 \\ &= \sum_{i=1}^N \| u_i \|_{X_i}^2 + \sum_{i=1}^N \| v_i \|_{X_i}^2 + 2 \sum_{i=1}^N \| u_i \|_{X_i} \| v_i \|_{X_i} \\ &\leq \sum_{i=1}^N \| u_i \|_{X_i}^2 + \sum_{i=1}^N \| v_i \|_{X_i}^2 + 2 \sqrt{\sum_{i=1}^N \| u_i \|_{X_i}^2} \sqrt{\sum_{i=1}^N \| v_i \|_{X_i}^2} \\ &= \left(\sqrt{\sum_{i=1}^N \| v_i \|_{X_i}^2} + \sqrt{\sum_{i=1}^N \| u_i \|_{X_i}^2} \right)^2 = (\| u \|_X + \| v \|_X)^2. \end{aligned} \tag{A.4}$$

For $\| u \|_X = 0$ then

$$0 = \| u \|_X^2 = \sum_{i=1}^N \| u_i \|_{X_i}^2 . \tag{A.5}$$

Consequently, for $\| u_i \|_{X_i}^2 = 0$, $i = 1, \cdots, N$ and therefore $u_i = 0$ for $i = 1, \cdots, N$, i.e. $u = 0$. Therefore $\| \cdot \|_X$ is a norm. □

Definition A.7 (Distance) *[49, Def. 757] Let X be a normed vector-space. The distance of two sets S_1 $S_2 \subset X$ is defined by*

$$\mathrm{dist}(S_1, S_2) := \inf\{\| u_1 - u_2 \|_X : u_1 \in S_1, u_2 \in S_2\},$$

with the special case $\mathrm{dist}(S, \phi) = \infty$ for $S \subset X$, where ϕ is the empty set.
For $u \in X$, $S \subset X$ the following is defined

$$\mathrm{dist}(u, S) := \mathrm{dist}(\{u\}, S).$$

For $\epsilon > 0$ the ϵ-neighbourhood of S is defined by

$$B_\epsilon(S) := \{v \in X : \mathrm{dist}(v, S) < \epsilon\},$$

$B_\epsilon(u) = B_\epsilon(\{u\}) = \{v \in X : \mathrm{dist}(v, u) < \epsilon\}$ is called ϵ-ball around u.

Definition A.8 *[29, c.f. Frobenius Norm, The Usual Norm] Let A be a square matrix. The Frobenius norm is defined as*

$$\| A \|_F = \left(\sum_{i,j} |a_{ij}|^2 \right)^{1/2}.$$

The spectral norm of the matrix A is

$$\| A \|_2 = \sigma(A) := \sqrt{\varrho((A)^*(A))}, \text{ and } \varrho(A) := \max\{|\lambda| : \lambda \text{ is the eigenvalue of } A\}, \tag{A.6}$$

and

$$||A||_2 \leq ||A||_F \leq \sqrt{r}||A||_2, \text{ where } r = \mathrm{rank(A)}.$$

A.2 Operators

Definition A.9 *[41, Notation in 4. Linear Operator]*
Let X and Y be normed linear spaces then $L(X,Y)$ denotes the set of all linear transformations from X to Y.

Definition A.10 *Let X and Y be two vector spaces. An operator A is a mapping $A : X \to Y$, which assigns to every $u \in X$ an element $v = Au \in Y$.*
X is called domain of A denoted as $\mathcal{D}(A)$ and Y is called the co-domain of A. The image or range of A is defined as

$$A[X] = \{v \in Y | v = Au \text{ for some } u \in X\},$$

and the kernel of A

$$\ker(A) = \{u \in X | Au = 0\},$$

see [38, Linear Operators and Linear Functionals].

Definition A.11 *[27, c.f. 3.2.6 Definition] Suppose A is an operator from X into Y. A is said to be injective"one-to-one" iff for each $g \in A[X]$ (range of A), there is exactly one $f \in \mathcal{D}(A)$ such that $Af = g$. A is called surjective"onto" iff $A[X] = Y$, when saying that A maps $\mathcal{D}(A)$ onto Y. A is known as bijective"one-to-one and onto" iff it is both injective and surjective.*

Definition A.12 (Linear operator) *Let X and Y be two vector spaces, then the operator $A : X \to Y$ is called linear if there holds for all $u, v \in X$ and $\alpha \in \mathbb{R}$*

$$A(u + v) = Au + Av,$$
$$A(\alpha u) = \alpha Au,$$

see [38, Linear Operators and Linear Functionals].

Corollary A.3 *Let X, Y be two normed vector spaces. Let $A \in L(X,Y)$. Then $\ker(A)$ is closed.*

Proof:
Let $u \in \text{clos}(\ker(A))$, then there exists a sequence $\{u_n\} \subset \ker(A)$ such that $u_n \to u$. Consequently, $0 = Au_n \to Au$, i.e. $Au = 0$ and $u \in \ker(A)$. □

Definition A.13 (Continuity) *Let X and Y be two normed vector-spaces. An operator $A : X \to Y$ is called continuous in $u \in X$ if there holds*

$$\forall \epsilon > 0\, \exists \delta > 0, \forall\, v \in B_\delta(u) :\| Av - Au \|_Y \leq \epsilon,$$

see [38, Linear Operators and Linear Functionals].

Definition A.14 *Let X and Y be two normed vector-spaces. Let A be an operator $A : X \to Y$ is a continuous linear operator from X into Y. The norm of the operator A is defined by*

$$\| A \|_{X,Y} := \sup_{u \in X} \frac{\| Au \|_Y}{\| u \|_X},$$

see [38, Linear Operators and Linear Functionals].

Theorem A.1 *Let X, Y be normed vector spaces. Let $\gamma \in L(X,Y)$ be surjective. A norm on Y is defined as*

$$\| u \|_\gamma := \inf_{\substack{U \in X \\ \gamma U = u}} \| U \|_X, \qquad \forall u \in Y, \tag{A.7}$$

Proof:

N_1 : Absolute scalability
Let $u \in Y$ and $\alpha \in \mathbb{R}$. With the linearity of γ it follows

$$\| \alpha u \|_\gamma = \inf_{\substack{U \in X \\ \gamma U = \alpha u}} \| U \|_X = \inf_{\substack{U \in X \\ Z = \frac{1}{\alpha} U, \gamma Z = u}} \| \alpha Z \|_X = \inf_{\substack{Z \in X \\ \gamma Z = u}} |\alpha| \| Z \|_X = |\alpha| \| u \|_\gamma \tag{A.8}$$

N_2 : Triangle inequality
Let $u_1, u_2 \in Y$. Again with the linearity of γ it follows that

$$\begin{aligned} \| u_1 + u_2 \|_\gamma = \inf_{\substack{U \in X \\ \gamma U = u_1 + u_2}} \| U \|_X = \inf_{\substack{U_1, U_2 \in X, U = U_1 + U_2 \\ \gamma U_1 = u_1, \gamma U_2 = u_2}} \| U \|_X &= \inf_{\substack{U_1, U_2 \in X \\ \gamma U_1 = u_1, \gamma U_2 = u_2}} \| U_1 + U_2 \|_X \\ &\leq \inf_{\substack{U_1, U_2 \in X \\ \gamma U_1 = u_1, \gamma U_2 = u_2}} \| U_1 \|_X + \| U_2 \|_X \\ &= \| u_1 \|_\gamma + \| u_2 \|_\gamma \end{aligned} \tag{A.9}$$

N_3 : If $\| u \|_\gamma = 0$, then $u = 0$
By contradiction, let $u \neq 0$ and $\| u \|_\gamma := \inf_{\substack{U \in X \\ \gamma U = u}} \| U \|_X = 0$. Therefore there exists a sequence $\{U_n\}$, $n \in \mathbb{N}$ with $\gamma U_n = u$ such that $\| U_n \|_X \to 0$. Therefore with $\gamma \in L(X,Y)$ it follows $\| u \|_Y = \| \gamma U_n \|_Y \leq C \| U_n \|_X \to 0$, i.e. $\| u \|_Y = 0$ and consequently $u \equiv 0$, which is a contradiction,

and that complete the proof. □

Corollary A.4 *Let X and Y be two normed vector spaces and $A \in L(X,Y)$. Let W be dense in X. Then $A[W]$ is dense in $A[X]$.*

Proof:
Since W is dense in X i.e. for every $u \in X$ there exists $u_n \subset W$ such that

$$\| u - u_n \|_X \to 0.$$

Now, for every $v \in A[X]$ there exists $u \in X$ such that $v = Au$. Then $Au_n \in A[W]$ and

$$\| v - Au_n \|_Y = \| Au - Au_n \|_Y \leq \| A \|_{X,Y} \| u - u_2 \|_X \to 0.$$

Therefore $A[W]$ is dense in $A[X]$. □

Corollary A.5 *Let X and Y be two normed vector space and $A \in L(X,Y)$. Let X be separable, then $A[X]$ is separable.*

Proof:
X is separable, i.e. there exists a dense and countable subset $W \subset X$. Due to Corollary A.4 $A[W]$ is dense in $A[X]$. W is countable, therefore $A[W]$ is also countable. □

A.3 Inner product, Hilbert, and dual spaces

Definition A.15 *[32, Definition 1.3.1] Let X be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. An inner product $(\cdot,\cdot)$ is a function from $X \times X$ to $\mathbb{F}$ with the following properties:*

1. *For any $v \in X$, $(v, v) \geq 0$ and $(v, v) = 0$ if and only if $v = 0$.*
2. *For any $u, v \in X$:*

$$\begin{aligned} (u, v) &= \overline{(v, u)}, \text{ conjugate symmetry if } u, v \text{ are complex,} \\ (u, v) &= (v, u), \text{ symmetry if } u, v \text{ are real.} \end{aligned} \tag{A.10}$$

3. *For any $u, v, w \in X$, any α, $\beta \in \mathbb{F}$, $(\alpha u + \beta v, w) = \alpha(u, w) + \beta(v, w)$.*

The space X together with the inner product $(\cdot,\cdot)$ is called an inner product space.

Lemma A.1 *[41, Lemma 3.13] Let X be an inner product space and let $x, y \in X$. Then:*

(a) $|(x, y)|^2 \leq (x, x)(y, y)$, $x, y \in X$.

(b) The function $\| \cdot \| : X \to \mathbb{R}$ defined by $\| x \| = (x, x)^{1/2}$, is a norm on X.

Also, the inequality in part (a) of Lemma A.1 can be rewritten as

$$|(x, y)| \leq \| x \| \| y \|, \tag{A.11}$$

and it is also called the *Cauchy-Schwarz inequality.*

Definition A.16 *[41, Definition 3.23] An inner product space which is complete with respect to the metric associated with the norm induced by the inner product is called a Hilbert space.*

Definition A.17 *[41, Definition 4.28] Let X be a normed space over $\mathbb{F}$. The space $L(X, \mathbb{F})$ is called the dual space of X and is denoted by X'.*

Definition A.18 *[45, c.f. 3.92 Definition linear functional] Let X be a normed vector space. A linear functional on X is a linear map from X to $F = \mathbb{R}$ or $\mathbb{C}$. In other words, a linear functional is an element of $L(X, \mathbb{F})$.*

Corollary A.6 *[41, Corollary 4.29] If X is a normed vector-space, then X' is a Banach space.*

Definition A.19 *Let X and Y be separable Hilbert spaces over the field $\mathbb{C}$. Let X be dense in Y and a subspace of Y with the Gelfand triple such that $X \subset Y \subset X'$. For linear and bounded operator $A \in L(X, X')$ we define the set*

$$D_A := \{u \in X : Au \in Y\}. \tag{A.12}$$

Corollary A.7 *The set D_A given in Definition A.19, is a non-empty set and is a subspace of X.*

Proof:
Firstly, from the *Gelfand triple* and the linearity of the operator $A \in L(X, X')$ then the zero element is in D_A.
Secondly, for any u_1, $u_2 \in D_A$, and $\forall\, \alpha_1, \alpha_2 \in \mathbb{C}$, with knowing that X and Y are Hilbert spaces and A is linear,

$$u_1 + u_2 \in X : Au_1 + Au_2 = A(u_1 + u_2) \in Y \Longrightarrow u_1 + u_2 \in D_A, \text{ and },$$
$$\alpha_1 u_1 + \alpha_2 u_2 \in X : \alpha_1 Au_1 + \alpha_2 Au_2 = A(\alpha_1 u_1 + \alpha_2 u_2) \in Y \Longrightarrow \alpha_1 u_1 + \alpha_2 u_2 \in D_A. \qquad \square$$

Corollary A.8 *For D_A a subspace of X as shown in Corollary A.7. For all $u, v \in D_A$, the following inner product is defined*

$$(u, v)_{D_A} := (u, v)_X + (Au, Av)_Y \text{ and } \| u \|_{D_A} = (u, u)_{D_A}^{1/2},$$

then $(D_A, \| \cdot \|_{D_A})$ as defined above is a pre-Hilbert space. There also holds

$$\| u \|_X^2 \leq \| u \|_{D_A}^2, \quad \forall u \in D_A \subset X,$$

Proof:
First, is to show that $(\cdot, \cdot)_{D_A}$ is a inner product:

1. Bilinearity, for any u^1, $u^2, v \in D_A$ and $\alpha_1\, \alpha_2 \in \mathbb{C}$:

$$\begin{aligned}(\alpha_1 u^1 + \alpha_2 u^2, v)_{D_A} &= ((\alpha_1 u^1 + \alpha_2 u^2), v)_X + (A(\alpha_1 U^1 + \alpha_2 U^2), Av)_Y \\ &= \alpha_1 (u^1, v)_X + \alpha_2 (u^2, v)_X + \alpha_1 (Au^1, Av)_Y + \alpha_2 (Au^2, Av)_Y, \text{(linearity of } A) \\ &= \alpha_1 (u^1, v)_X + \alpha_1 (Au^1, Av)_Y + \alpha_2 (u^2, v)_X + \alpha_2 (Au^2, Av)_Y, \\ &= \alpha_1 (u^1, v)_{D_A} + \alpha_2 (u^2, v)_{D_A} = (u^1, \alpha_1 v)_{D_A} + (u^2, \alpha_2 v)_{D_A}.\end{aligned}$$

2. Symmetry:
By definition and the symmetries of its individual terms

$$(u, v)_{D_A} = (u, v)_X + (Au, Av)_Y = (v, u)_X + (Av, Au)_Y = (v, u)_{D_A}.$$

3. Positive definiteness:
For $u \neq 0$ and by definition

$$(u, u)_{D_A} = (u, u)_X + (Au, Au)_Y = \| u \|_X^2 + \| Au \|_Y^2 > 0,$$

and this concludes that $(u, u)_{D_A} = 0$ if and only if $u \equiv 0$.

Second, from above they imply that $\| u \|_{D_A} = (u, u)_{D_A}^{1/2}$ is a norm in D_A for all $u \in D_A$ and

$$(u, v)_{D_A} \leq \| u \|_{D_A} \| v \|_{D_A}.$$

Also, since X and Y are Hilbert spaces such that

$$\begin{aligned}&\| u \|_X^2 < \infty, \ \forall u \in X \text{ and } \| Au \|_Y^2 < \infty, \ Au \in Y \text{ then} \\ &\| u \|_{D_A}^2 = \| u \|_X^2 + \| Au \|_Y^2 < \infty \Longrightarrow \| u \|_{D_A} \text{ is defined on the whole } D_A.\end{aligned}$$

Finally, since $\| Au \|_Y^2 \geq 0$ for all $u \in D_A$, then

$$\| u \|_X^2 \leq \| u \|_X^2 + \| Au \|_Y^2 = \| u \|_{D_A}^2 \Longrightarrow \| u \|_X \leq \| u \|_{D_A}. \qquad \square$$

Theorem A.2 *[19, Theorem 17.2] A closed subspace of a separable Hilbert space is separable.*

Theorem A.3 *(Lax-Milgram)[42, c.f. Theorem 4.20.] Let Z be a Hilbert space. Let $a : Z \times Z \to \mathbb{C}$ be a Hermitian form which satisfies the following:*

$$|a(u, v)| \leq M \parallel u \parallel_Z \parallel v \parallel_Z, \qquad \forall \quad u, v \in Z, \text{ continuity}, \tag{A.13}$$

and

$$a(u, u) \geq \alpha \parallel u \parallel_Z^2, \qquad \forall \quad u \in Z, \text{ ellipticity}, \tag{A.14}$$

for $0 < M, \alpha \in \mathbb{R}$. Then with $L \in Z'$ there is a unique $u \in Z$:

$$a(u, v) = L(v), \ \ \forall v \in Z. \tag{A.15}$$

Corollary A.9 *Let $A : Z \to Z'$; $A \in L(Z, Z')$. Let $a(\,,)$ be the form which satisfies the properties given in Theorem A.3:*

$$\langle A\,u, v\rangle_{Z' \times Z} := a(u, v), \quad \forall u,\, v \in Z, \tag{A.16}$$

then

$$\parallel A \parallel_{Z,Z'} \leq M \text{ and } \parallel A^{-1} \parallel_{Z',Z} \leq \frac{1}{\alpha}. \tag{A.17}$$

Proof of (A.17):
Firstly, with defining

$$\parallel A \parallel_{Z,Z'} := \sup_{u \in Z} \frac{\parallel Au \parallel_{Z'}}{\parallel u \parallel_Z},$$

and using the properties in (A.13) and (A.14) given in Theorem A.3:

$$\begin{aligned} \langle A\,u, v\rangle_{Z' \times Z} &:= a(u, v) \leq M \parallel u \parallel_Z \parallel v \parallel_Z \ \text{ and} \\ \langle A\,u, v\rangle_{Z' \times Z} &:= a(u, u) \geq \alpha \parallel u \parallel_Z^2, \end{aligned} \tag{A.18}$$

then

$$\begin{aligned} \parallel A \parallel_{Z,Z'} := \sup_{u \in Z} \frac{\parallel Au \parallel_{Z'}}{\parallel u \parallel_Z} = \sup_{u \in Z} \frac{\sup_{v \in Z} \frac{\langle Au, v\rangle_{Z' \times Z}}{\parallel v \parallel_Z}}{\parallel u \parallel_Z} \leq \sup_{u \in Z} \frac{\sup_{v \in Z} \frac{M \parallel u \parallel_Z \cdot \parallel v \parallel_Z}{\parallel v \parallel_Z}}{\parallel u \parallel_Z} &\leq \sup_{u \in Z} \frac{M \parallel u \parallel_Z}{\parallel u \parallel_Z} \\ &= M. \end{aligned}$$

Secondly, from (A.18) and Theorem A.3, there exists $A^{-1} \neq 0$: such that with $u = A^{-1}(Au) := A^{-1}(l) \in Z$, and with letting $l \in Z'$, then

$$\begin{aligned} \parallel A^{-1} \parallel_{Z',Z} &:= \sup_{l \in Z'} \frac{\parallel A^{-1} l \parallel_Z}{\parallel l \parallel_{Z'}} = \sup_{Au \in Z'} \frac{\parallel A^{-1} Au \parallel_Z}{\parallel Au \parallel_{Z'}} = \sup_{u \in Z} \frac{\parallel u \parallel_Z}{\parallel Au \parallel_{Z'}}, \\ \Longleftrightarrow \frac{1}{\parallel A^{-1} \parallel_{Z',Z}} &= \inf_{u \in Z} \frac{\parallel Au \parallel_{Z'}}{\parallel u \parallel_Z} = \inf_{u \in Z} \sup_{v \in Z} \frac{\langle Au, v\rangle_{Z' \times Z}}{\parallel u \parallel_Z \parallel v \parallel_Z} \geq \inf_{u \in Z} \frac{\langle Au, u\rangle_{Z' \times Z}}{\parallel u \parallel_Z \parallel u \parallel_Z} \geq \inf_{u \in Z} \frac{\alpha \parallel u \parallel_Z^2}{\parallel u \parallel_Z \parallel u \parallel_Z} \\ &= \alpha, \end{aligned}$$

and that implies that

$$\parallel A^{-1} \parallel_{Z',Z} \leq \frac{1}{\alpha}.$$

□

Theorem A.4 *[28, c.f. Theorem 2.15 (Babuška)] Assume that a Hermitian form $b : V_1 \times V_2 \to \mathbb{C}$ on Hilbert spaces V_1, V_2 satisfies*

1. *Continuity:*
$$\exists M > 0 : |b(u,v)| \leq M \parallel u \parallel_{V_1} \parallel v \parallel_{V_2}, \qquad \forall\, u \in V_1, v \in V_2. \tag{A.19}$$

2. *Inf-sup Condition:*
$$\exists \beta > 0 : \beta \leq \sup_{0 \neq v \in V_2} \frac{|b(u,v)|}{\parallel u \parallel_{V_1} \parallel v \parallel_{V_2}}, \qquad \forall\, 0 \neq u \in V_1. \tag{A.20}$$

3. *"Transposed" Inf-sup Condition:*
$$\sup_{0 \neq u \in V_1} |b(u,v)| > 0, \qquad \forall\, 0 \neq v \in V_2,$$

and let $f : V_2 \to \mathbb{C}$ be bounded functional defined on V_2. Then there exists a unique element $u_0 \in V_1$ such that
$$b(u_0, v) = f(v), \qquad \forall v \in V_2.$$

Corollary A.10 *[17, c.f. Corollary 3.1.6] Let X be a normed space and $u \in X$. If $\langle l, u \rangle_{X' \times X} = 0$ for all $l \in X'$, then there holds $u = 0$.*

Theorem A.5 *[41, c.f. Theorem 1.52] A infinite-dimensional Hilbert space $\mathcal{H}$ is separable if and only if it has an orthonormal basis.*

Definition A.20 *[16, c.f. D.2. Hilbert spaces] Let H be a Hilbert space:*

(i) Two elements $u, v \in H$ are orthogonal if $(u,v) = 0$.

(ii) A countable basis $\{w_k\}_{k=1}^{\infty} \subset H$ is called orthonormal if
$$\begin{cases} (w_k, w_l) = 0, & (\text{ at } k \neq l) \\ \parallel w_k \parallel = 1. \end{cases}$$

If $u \in H$ and $\{w_k\}_{k=1}^{\infty} \subset H$ is an orthonormal basis, we can write
$$u = \sum_{k=1}^{\infty} (u, w_k) w_k.$$

In addition
$$\parallel u \parallel^2 = \sum_{k=1}^{\infty} (u, w_k)^2.$$

A.4 Bochner spaces; Spaces involving time

Definition A.21 *(Bochner integral) [36, Definition 1.2.9] Let* $-\infty < a < b < \infty$ *and let* $(X, \|\cdot\|_X)$ *be a real Banach space.*

a) A function $s : [a, b] \to X$ *is called simple if there are* $\varphi_1, \cdots, \varphi_N \in X$ *and Lebesgue-measurable disjoint sets* $\mathcal{M}_1, \cdots, \mathcal{M}_N \subset [a, b]$ *with* $|\mathcal{M}_i| < \infty, i = 1, \cdots, N$*, such that*

$$s(t) = \sum_{i=1}^{N} \mathcal{X}_{\mathcal{M}_i}(t)\varphi_i \qquad t \in [a, b], \tag{A.21}$$

$\mathcal{X}_{\mathcal{M}_i}$ *denotes the indicator function of the set* $\mathcal{M}_i \subset [a, b]$.

b) For a simple function $s(t) = \sum_{i=1}^{N} \mathcal{X}_{\mathcal{M}_i}(t)\varphi_i$ *the following is defined*

$$\int_a^b s(t)\, dt := \sum_{i=1}^{N} |\mathcal{M}_i|\varphi_i \in X. \tag{A.22}$$

c) A function $f : [a, b] \to X$ *is Bochner-measurable if there exists a sequence of simple functions* $\{s_k\}_{k\in\mathbb{N}}$ *such that*

$$\lim_{k\to\infty} s_k(t) = f(t) \qquad \text{for almost all } t \in [a, b]. \tag{A.23}$$

d) Beyond that, a Bochner-measurable function $f : [a, b] \to X$ *is Bochner-summable if*

$$\lim_{k\to\infty} \int_a^b \| s_k(t) - f(t) \|_X \; dt = 0. \tag{A.24}$$

e) For a Bochner-summable function $f : [a, b] \to X$ *the following is defined*

$$\int_a^b f(t)\, dt := \lim_{k\to\infty} \int_a^b s_k(t)\, dt. \tag{A.25}$$

Theorem A.6 *[36, Theorem 1.2.10] A Bochner-measurable function* $f : [a, b] \to X$ *is Bochner-summable if and only if* $t \mapsto \| f(t) \|_X$ *is Lebesgue-summable. In this case*

$$\| \int_a^b f(t)\, dt \|_X \leq \int_a^b \| f(t)\, dt \|_X \; dt. \tag{A.26}$$

Definition A.22 *[36, Definition 1.2.11] For* $-\infty < a < b < \infty$, $p \in [1, \infty]$ *and a Banach space* $(X, \|\cdot\|_X)$ *the following is defined:*

a) The Lebesgue space

$$L^p(a, b; X) := \{f : [a, b] \to X : f \text{ is Bochner-measurable and } \| f \|_{L^p(a,b;X)} < \infty\}, \tag{A.27}$$

where the corresponding norm is defined as

$$\| f \|_{L^p(a,b;X)} := \left[\int_a^b \| f(t) \|_X^p \, dt \right]^{1/p} < \infty, \qquad \text{for } p < \infty. \tag{A.28}$$

The spaces $L^p(a,b;X)$ *equipped with its respective norm* $\| \cdot \|_{L^p(a,b;X)}$ *is a Banach space and Hölder's inequality (to be mentioned in the proceeding sections) holds in the Lebesgue space* $L^p(a,b;X)$.

Definition A.23 *For* $k \geq 1$. *The following Bochner space is defined*

$$W^{k-1,2}(J;Z) := \{u \in L^2(J;Z), \frac{d^{k-1}}{dt^{k-1}}u \in L^2(J;Z)\}, \tag{A.29}$$

see [52, c.f. 23.6. The Sobolev Space $W_p^1(0,T;V,H)$*].*

Definition A.24 *(Weak derivative)[36, Definition 1.2.13] Let* $(X, \| \cdot \|_X)$ *be a real Banach space and suppose* $f, g \in L^2(a,b;X)$. *Then, for* g *as the weak derivative of* f, *provided*

$$\int_a^b f(t)\varphi'(t)\,dt = -\int_a^b g(t)\varphi(t)\,dt, \qquad \text{for all } \varphi \in C_0^\infty(a,b), \tag{A.30}$$

where where $C_0^\infty(a,b)$, *denotes the* C_0^∞*-functions with compact support in* (a,b). *For the weak derivative* $f' := g$

Theorem A.7 *[26, Theorem 6.41] Let* $Z \hookrightarrow H \hookrightarrow Z'$ *be a Hilbert triple. If* $u \in L^2(J;Z)$ *and* $\partial_t u \in L^2(J;Z')$, *then* $u \in C([0,T];H)$. *Moreover:*

(1) For any $v \in Z$, *the real-valued function* $t \mapsto (u(t),v)_H$ *is weakly differentiable in* J *and*

$$\frac{d}{dt}(u(t),v)_H = \langle \partial_t u, v\rangle_{Z'\times Z}.$$

(2) The real-valued function $t \mapsto \| u(t) \|_H^2$ *is weakly differentiable in* J, *and*

$$\frac{d}{dt} \| u \|_H^2 = 2\langle \partial_t u, u\rangle_{Z'\times Z}.$$

A.5 Important inequalities

Lemma A.2 *(Young's inequality), Peter-Paul inequality[15, Lemma 0.5] If* $s,\ q \in (1,+\infty)$, $\frac{1}{s}+\frac{1}{q} = 1$ *and* $a,\ b \geq 0$, *then*

$$ab \leq \frac{a^s}{s} + \frac{b^q}{q}.$$

In particular, if $s = q = 2$, *and* $\lambda = 1$, *then*

$$ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2.$$

Lemma A.3 *For any* $a, b \geq 0$, *the following holds*

$$a + b \geq \sqrt{a^2+b^2}, \text{ and } a + b \leq \sqrt{2}\sqrt{a^2+b^2},$$

(see [1, 2.2 Lemma] with $p = 2$*).*

Theorem A.8 *[16, Gronwall's inequality, differential form]* *Let* $u(\cdot)$, *be a non-negative, absolutely continuous function on* $[0, T]$, *which satisfies*

$$u'(t) \leq q(t)u(t) + h(t),$$

$q(t)$, *and* $h(t)$ *are non-negative, summable functions on* $[0, T]$.
Then

$$u(t) \leq e^{\int_0^t q(s)\,ds}\Big[u(0) + \int_0^t h(s)\,ds\Big], \text{ for all } 0 \leq t \leq T.$$