Optimal Design of Uptime-Guarantee Contracts under IGFR Valuations and Convex Costs

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Abstract

An uptime-guarantee contract commits a service provider to maintain the functionality of a customer's equipment at least for certain fraction of working time during a contracted period. This paper addresses the optimal design of uptime-guarantee contracts for the service provider when the customer's valuation of a contract with a given guaranteed uptime level has an Increasing Generalized Failure Rate (IGFR) distribution. We first consider the case where the service provider proposes only one contract and characterize the optimal contract in terms of price as well as guaranteed uptime level assuming that the service provider's cost function is convex. In the second part, the case where the service provider offers a menu of contracts is considered. Given the guaranteed uptime levels of different contracts in the menu, we calculate the corresponding optimal prices. We also give the necessary and sufficient conditions for the existence of optimal contract menus with positive expected profits.

Keywords: Revenue management, Pricing, Game theory, Maintenance, Contracts, Servitization

1. Introduction

In many industries, the steady execution of value added activities depends on uninterrupted performance of sophisticated equipment whose maintenance must be outsourced. The imperfect reliability of critical equipment threatens

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the revenue stream of their owners. To hedge the risk of break downs, owners would be willing to pay premiums for services which increase the reliability of their equipment. On the other hand, faced by saturated markets and fierce competition, many traditional core manufacturing companies are reinventing their services as key sources of revenue (Sawhney et al., 2003). When managed correctly, maintenance contracts can leverage customers' requirements for more reliable equipment and service providers' revenue diversification strategies.

A standard quality control indicator for determining the effectiveness of a maintenance service contract is the equipment's *uptime*, that is, the fraction of time that the device will be operational and ready to use (Chan, 2003). An *uptime-guarantee contract* commits a service provider to maintain functionality of some equipment at least for certain fraction of working time during a contracted period. In this sense, an uptime-guarantee contract is a special form of performance-based contracts (see Selviaridis and Wynstra (2015) for a review). This paper addresses the optimal design of uptime-guarantee maintenance contracts.

This study is particularly motivated by the contracting practices in the high-tech medical equipment industry such as medical imaging devices. The annual maintenance service cost for a medical imaging device can be as much as 8.5 percent of the initial purchase cost and it has become a key competitive factor among different manufacturers (Sferrella, 2012). Not surprisingly, specific patterns of servitization can be observed among the manufacturers of medical technology (Schröter and Lay, 2014). Van Brunschot (2015) gives a detailed report on a major European manufacturer's attempts to systematically overhaul its uptime-guarantee contracts. Although contractual commitments on uptime are quite common with medical equipment (Mancino and Siachos, 2007), the research into maintenance service contracts in the health care sector is still in its infancy stages (Cruz and Rincon, 2012).

Among all the terms and conditions of an uptime-guarantee contract, we focus on two main features: the guaranteed uptime level and the price. The service provider's goal in designing an uptime-guarantee contract is to maximize his (expected) profit. While charging too high a price deters the customer from purchasing the contract, too low a price leaves the service provider at loss. The key to the optimal design of such contracts is understanding a customer's valuation of a contract which is the basis for her decision to purchase the contract or rely solely on corrective maintenance. The customer's valuation is considered to involve two terms: (a) expected utilization time, i.e., the amount of time that the customer can make use of the equipment given its uptime level, and (b) unit revenue rate, i.e., the revenue obtained by utilizing the equipment for a unit of time. In many cases, including our motivating scenario, the service provider has a very good understanding of the customer's utilization function—in light of standard warranty periods where the performance of the equipment is constantly monitored by the service provider. However, in almost all real life situations the service provider does not have the exact information about the customers' unit revenue rates. We assume that the service provider knows the distribution of the customer's unit revenue rate. As a technical requirement for tractability we assume that the customer's unit revenue has an Increasing Generalized Failure Rate (IGFR) distribution function. Many well-known distribution functions satisfy the IGFR condition, e.g., exponential, normal, logistic, Weibull, gamma, beta, Cauchy, etc. (Banciu and Mirchandani, 2013). The IGFR distribution assumption is common in supply chain revenue management and pricing literature (Lariviere, 2006).

In the first part of this paper, we consider the case where the service provider offers a single contract to the customer. We focus on contracts which are interesting from the perspective of both parties. That is, we search for a contract that has a positive chance of being purchased by the customer and generates a positive profit for the service provider. As we show, the existence of such a contract can be ensured by enforcing a condition on the guaranteed uptime level. We then develop closed-form formulas for the optimal price as well as the optimal guaranteed uptime level. In doing so we assume that the service provider's cost function for providing contracts with different uptime levels is convex, that is, guaranteeing higher uptime levels are increasingly costly.

The second part of the paper extends the analysis to situations where the service provider offers menus of contracts. A contract menu enables the service provider to extract more profit from a customer with high valuation without risking the potential profit that can be obtained from a customer with low valuation. Given the guaranteed uptime levels of the contracts in a menu, we optimize their corresponding prices to maximize the expected profit from the contract menu. We provide the necessary and sufficient conditions on the guaranteed uptime levels that ensure the existence of optimal contract menus with positive expected profits.

The rest of this paper is organized as follows. Section 2 overviews the previous relevant studies in the literature. Section 3 outlines the elements of our mathematical model. The design of singular contracts is discussed in Section 4. The analysis of contract menus is carried out in Section 5. Section 6 concludes the paper. All proofs are presented in the appendix.

2. Literature Review

We overview the literature on maintenance contracts. A number of papers in this domain focus on either customer' or service providers' decision problems. In a multi-criteria decision making framework, de Almeida (2001) investigates a customer's optimal choice from a given set of contracts by incorporating their risks, costs, and other consequences on the performance of equipment. Wang (2010) emphasizes on the interplay between inspections and repair services. Assuming fix contract prices, he analyzes the relationship between an equipment owner's choice of contract and inspection intervals offered by a service provider.

Several papers in the literature study the design of optimal service contracts drawing upon the Stackelberg game while making the critical assumption that all information is commonly known by the parties involved. Murthy and Yeung (1995) discuss the optimal strategies for a customer and a service provider with two types of maintenance services: pre-planned and immediate. The optimal contract prices are calculated based on the customer's unit revenue rate of workable equipment time. Murthy and Asgharizadeh (1999) and Ashgarizadeh and Murthy (2000) study the optimal decisions of one or more equipment owners and a service provider in terms of right choices of contract, contract prices, and service channels. Rinsaka and Sandoh (2006) extend the work of Murthy and Asgharizadeh (1999) to the time after the initial warranty period. Within a multi-stage decision making framework, Hartman and Laksana (2009) examine the optimal strategies for equipment owners with regards to the types of extended warranty contracts as well as pricing policies. They show that by offering multiple contracts a service provider can dramatically increase its profit. Tong et al. (2014) discuss the pricing strategies for a provider of two-dimensional warranty contracts. Esmaeili et al. (2014) study the choices of various attributes in a three-level warranty service contract among a manufacturer, a service provider and a customer. Gallego et al. (2014) analyze the residual value extended warranty contracts where a customer would receive a bonus if no claims are maid during the term of contract. They study the pricing problems associated with single and menu contracts for strategic customers with different risk attitudes.

Although the Stackelberg game is the most common approach taken in the literature, few authors use cooperative and/or bargaining games to study the design of maintenance contracts. Hamidi et al. (2014) analyze a two-player cooperative game with an equipment owner, who announces the time of replacement of a part, and a service provider, who chooses the order time of the part. Jackson and Pascual (2008) consider a negotiation scenario where contract prices, preventive maintenance intervals, and response times are set to equally divide the profits between an equipment owner and a service provider.

Another stream of research underlines the fact that interacting parties may not be fully aware of each other's attributes. Taking into account the indeterministic nature of customers' attitude to risk, Huber and Spinler (2012) formulate a model which allows a service provider to manage his revenue by setting the prices of full-service or on-call maintenance contracts. They give a closed-form formula for the service provider's optimal contract prices under the assumption that the customer's attitude toward risk is uniformly distributed. Huber and Spinler (2014) extend the latter model to account for learning, optimized maintenance, and information asymmetry between customers and service providers. In an alternative setting where customers design the maintenance contracts, Kim et al. (2010) introduce contractual structures that mitigate service providers' moral hazards associated with their capacity investments. Zeng and Dror (2015) extend the analysis of this problem in several directions.

Another approach to managing the relationship between customers and service providers seeks to coordinate the parties' efforts to optimize the system-wide profit—as opposed to focusing on either customers' or service providers' decision making problems. Within a deterministic framework, Tarakci et al. (2006a) and Tarakci et al. (2006b) analyze several mechanisms, including a pricing scheme for maintenance contracts, to ensure that the optimal intervals for preventive maintenance from the perspectives of both service provider and equipment owner coincide. Tseng and Yeh (2013) extend the single processor case discussed in Tarakci et al. (2006a) for risk-averse customers.

Instead of relying on game theory to analyze the interactions between agents involved, some authors build models that use demand elasticity to capture the effects of contracts with different terms and conditions on service providers' profit. Drawing upon a non-linear deterministic demand function for approximating customers' sensitivity to contract price and delivery time, So and Song (1998) propose a model to analyze pricing, delivery time guarantee and capacity expansion decisions of a service provider. In a holistic approach to optimize contract prices and uptime levels, as well as the network of maintenance facilities and personnel, Lieckens et al. (2015) use a multinomial logit model to estimate the probabilities of the customers choosing among full-service, on-call, or no-contract options. They obtain contract choice probabilities indirectly as functions of customer's price elasticity and downtime sensitivity.

3. Model

The basic situation consists of a service provider (he) who offers an uptime-guarantee contract to a customer (she) for a specific piece of equipment over a period of time. A guaranteed uptime-level, $d \in (0, 1]$, represents the minimum fraction of time in the contracted period that the equipment must be operational. An uptime-guarantee contract t charges a price p for providing the services that guarantee an uptime level d and is denoted with the pair $t = (p, d)$. Without purchasing an uptime-guarantee contract and relying on corrective maintenance services of the service provider only, the customer's equipment has an expected uptime level of d_0. Naturally, the guaranteed uptime level of the contract must be higher than the base uptime level, i.e., $d \in (d_0, 1]$.

The service provider incurs different types of costs, such as preventive and corrective maintenance costs as well as default penalties, to uphold a contract. We consolidate such costs and let c_d be the expected cost of the contract with uptime level d. Guaranteeing higher uptime levels generates higher costs. We assume that the cost function c_d is increasing and twice differentiable on d. The expected cost associated with the corrective maintenance services is denoted by c_0.

The customer has a certain *valuation* for having the equipment in workable conditions. We model the customer's valuation as a function of two elements. The first element λ_d is the expected utilization time of the equipment at uptime level d during the period of contract, that is, the expected time that the equipment actively generates revenue. The second element is the unit revenue rate v, i.e., the revenue that the customer obtains if she utilizes the equipment for one unit of time. The customer's valuation of a contract with uptime level d as a function of her unit revenue is denoted by $\nu_d(v) = v\lambda_d$.

As the uptime level of the equipment increases, the customer can utilize it further. However, the equipment may not be utilized whenever it is functional. Thus we assume that λ_d is an increasing and concave function of d. The service provider knows the utilization function of the customer—such information is usually obtained during the preliminary warranty period. The unit revenue rate, however, is the private knowledge of the customer. Although the service provider is unaware of the exact customer's unit revenue rate, he knowns its probability distribution. We assume that the customer's unit revenue rate posses a density function f with its support on $V = [0, v_{\max}]$ and is differentiable at all interior points of V. The probability distribution is denoted with F and its complement with $\overline{F}$. Furthermore, we assume that the customer's unit revenue has an Increasing Generalized Failure Rate (IGFR) distribution.

3.1. IGFR distributions

A probability distribution F satisfies the IGFR condition if $xf(x)/\overline{F}(x)$ is non-decreasing (or weakly increasing) everywhere in its support such that $F(x) < 1$ (Lariviere, 2006). The family of IGFR distributions includes exponential, normal, log-normal, logistic, Weibull, gamma, beta, Cauchy, etc. (Banciu and Mirchandani, 2013). The following lemma presents a technical property of IGFR distribution functions which is used later in this paper.

Lemma 1. *Let F be an IGFR distribution. Define $G(x) = \overline{F}(x) - (x - a)f(x)$ with $a > 0$ and suppose $x^* = G^{-1}(0)$ exists such that $x^* > a$. We have $dG(x)/dx|_{x^*} < 0$.*

Lemma 1 shows that for IGFR distributions, the critical point of the function G occurring after a is a maximum.

4. Singular Contracts

In this section we study the situation where the service provider offers a single contract to the customer. We start by stating the customer's utility as well as the service provider's profit functions. Afterwards, we draw upon a Stackelberg game to combine these two functions in order to tackle the contract design problem.

4.1. The customer's utility

Before purchasing an uptime-guarantee contract, the rational customer compares her profit under the contract with that in situation of operating without a contract and relying on corrective maintenance. We define the utility function of the customer under the choice of contract $t = (p, d)$ as the additional profit obtained from purchasing t, that is,

$$\begin{aligned} u(v,t) &= \nu_d(v) - p - (\nu_0(v) - c_0) \\ &= v(\lambda_d - \lambda_0) - (p - c_0), \end{aligned} \tag{1}$$

where ν_0 and λ_0 represent the respective values at uptime level d_0. The customer would certainly purchase t if it results in a positive utility, i.e., $u(v,t) > 0$. Hence, a customer with unit revenue rate v purchases t whenever $v > v_0 = (p - c_0)/(\lambda_d - \lambda_0)$. In case $v = v_0$, which implies $u(v,t) = 0$, purchasing the contract has no additional benefit so the customer may or may not purchase t.

4.2. The service provider's profit

The service provider can expect two outcomes by offering contract t. Either the customer rejects the contact or she purchases it. If the contract is not purchased then the profit of zero would be obtained by the service provider. Otherwise, the service provider obtains the profit of $p - c_d$.

4.3. Singular contract design problem

To tackle the contract design problem, the service provider's profit and the customer's utility must be considered simultaneously. This situation can be handled by a Stackelberg game where the service provider, acting as the leader, proposes the contract in anticipation of the optimal response of the customer. The expected profit of the risk-neutral service provider upon offering contract t is the associated profit multiplied by the probability of the customer's unit revenue be such that she purchases the contract:

$$\Pi(t) = (p - c_d)Pr\{v|t \text{ is purchased}\}. \tag{2}$$

Since the customer purchases the contract whenever $v > v_0$, the probability of contract being purchased is $\overline{F}(v_0)$. Note that with a continuous probability function the customer's choice at $v = v_0$ is irrelevant. By incorporating the customer's optimal choice into the service provider's problem, his expected profit can be denoted as follows:

$$\Pi(t) = \begin{cases} (p - c_d)\overline{F}\left(\frac{p-c_0}{\lambda_d-\lambda_0}\right) & \text{if } 0 \le \frac{p-c_0}{\lambda_d-\lambda_0} < v_{\max} \\ 0 & \text{if } \frac{p-c_0}{\lambda_d-\lambda_0} \ge v_{\max} \end{cases} . \tag{3}$$

4.4. Desirable contracts

Not all possible contracts are worth considering. For example, if the contract is such that $v_0 \ge v_{\max}$, there is zero chance that the customer purchases it and accordingly the service provider's profit would be zero. The essential condition for a contract, defined below, reflects the requirement that an offered contract must have a positive chance of being purchased, i.e., $Pr\{v|t \text{ is purchased}\} > 0$.

Definition 1. *A contract $t = (p, d)$ is* essential *if $(p - c_0)/(\lambda_d - \lambda_0) < v_{\max}$.*

The service provider must also ensure that the offered contract generates profit. Therefore, a contract is only worthwhile to offer if it is profitable.

Definition 2. *A contract $t = (p, d)$ is* profitable *if $p - c_d > 0$.*

Since $c > c_0$, profitability of a contract also means that $v_0 > 0$. Therefore with an essential and profitable contract the service provider's profit in (3) boils down to:

$$\Pi(t) = (p - c_d)\overline{F}\left(\frac{p - c_0}{\lambda_d - \lambda_0}\right). \tag{4}$$

4.5. Admissible uptime levels

We introduce an important condition for uptime levels.

Definition 3. *An uptime level d is* admissible *if $(c_d - c_0)/(\lambda_d - \lambda_0) < v_{\max}$.*

For the customer who purchases a contract with uptime level d, the value $(c_d - c_0)/(\lambda_d - \lambda_0)$ is the lowest unit revenue which leaves the service provider with no loss. The admissibility condition on uptime levels is required to ensure that an essential and profitable contract can be found.

Lemma 2. *A necessary condition for the existence of an essential and profitable contract with guaranteed uptime level d is that d be an admissible uptime level.*

4.6. Optimal prices

Suppose that the uptime level is fixed to d. In order to obtain the optimal price we solve the following program:

$$\max_{p} \quad (p - c_d)\overline{F}\left(\frac{p - c_0}{\lambda_d - \lambda_0}\right) \tag{5}$$

$$s.t. \quad \frac{p - c_0}{\lambda_d - \lambda_0} < v_{\max} \tag{6}$$

$$p - c_d > 0 \tag{7}$$

In the above program, constraint (6) ensures that the selected price together with d yield an essential contract. Constraint (7) ascertains that $t = (p, d)$ is a profitable contract. When these two constraints are met, the program maximizes the service provider's expected profit considering the customer's optimal response.

The objective function is not necessarily concave. However, the main result of this section obtains a unique solution for this program.

Theorem 1. *Let $d \in (d_0, 1]$ be an admissible guaranteed uptime level. There exists a unique price, p^*, such that essential and profitable contract $t_d^* = (p^*, d)$ maximizes the service provider's expected profit considering the customer's optimal response. The optimal price satisfies:*

$$\overline{F}\left(\frac{p^* - c_0}{\lambda_d - \lambda_0}\right) = \frac{p^* - c_d}{\lambda_d - \lambda_0} f\left(\frac{p^* - c_0}{\lambda_d - \lambda_0}\right). \tag{8}$$

Theorem 1 shows that if d is admissible, one can find a unique optimal price such that the corresponding contract is essential and profitable. Equation (8) is the first order condition for the relaxed program. Hence, the constraints in the above program are never binding.

In conjunction with Lemma 2, which introduces the admissibility of an uptime level as a necessary condition for the existence of an essential and profitable contract, Theorem 1 enables us to present the last result of this section without a formal proof.

Corollary 1. *Let $d \in (d_0, 1]$. A unique essential and profitable contract with the highest expected profit for the service provider exists if and only if d is an admissible uptime level.*

4.7. Optimal contracts

In this section we address the problem of finding the best combination of guaranteed uptime level and price for the service provider. Using Corollary 1, the underlying program for finding the optimal contract can be written as

$$\max_{p,d} \quad (p - c_d)\overline{F}\left(\frac{p - c_0}{\lambda_d - \lambda_0}\right) \tag{9}$$

$$s.t. \quad \frac{c_d - c_0}{\lambda_d - \lambda_0} < v_{\max} \tag{10}$$

$$d \in (d_0, 1] \tag{11}$$

In the above program, constraint (10) ensures the admissibility of the selected uptime level while constraint (11) obtains a feasible guaranteed uptime level for the contract. Once these constraints are met, maximizing the objective function attains an essential and profitable contract.

A challenging aspect of the above program is that the feasible region is not necessarily a convex set. However, in case c_d is convex—which implies that the rate of growth of costs associated with providing contracts which higher uptime levels is increasing—the feasible region is a convex set. The next lemma demonstrates this.

Lemma 3. *Assume that c_d is a convex function. Either every uptime level $d \in (0, 1]$ is admissible, or there exists a limit uptime level $\hat{d} \in (0, 1]$ such that every $d \in (0, \hat{d})$ is admissible and every $d \in [\hat{d}, 1]$ is not admissible.*

To check the existence of limit uptime level $\hat{d}$ it is sufficient to test if $d = 1$ is admissible or not. Admissibility of $d = 1$ implies that the limit uptime level does not exist while the opposite manifests the existence of $\hat{d}$. The main result of this section characterizes the unique optimal contract under convex costs.

Theorem 2. *Assume that c_d is a convex function. If any admissible uptime level exists in $(d_0, 1]$, then a unique essential and profitable contract $t^* = (p^*, d^*)$ exists that maximizes the service provider's expected profit considering the customer's optimal response. The optimal contract is obtained via the following conditions:*

(i) *In case $d = 1$ is not admissible: the unique optimal contract satisfies*

$$\overline{F}\left(\frac{c'_{d^*}}{\lambda'_{d^*}}\right) = \left(\frac{c'_{d^*}}{\lambda'_{d^*}} - \frac{c_{d^*} - c_0}{\lambda_{d^*} - \lambda_0}\right) f\left(\frac{c'_{d^*}}{\lambda'_{d^*}}\right), \tag{12}$$

$$p^* = \frac{c'_{d^*}}{\lambda'_{d^*}}(\lambda_{d^*} - \lambda_0) + c_0 \tag{13}$$

where c'_d and λ'_d are the first derivatives of c_d and λ_d.

(ii) *In case $d = 1$ is admissible: if (12) has a solution then (12) and (13) together yield the unique optimal contact. Otherwise, the unique optimal contract has the guaranteed uptime level $d^* = 1$ with the optimal price corresponding to this uptime level obtained from (8).*

Theorem 2 includes several observations. First, it gives a condition for the existence of optimal contracts in terms of limit uptime level $\hat{d}$. If $\hat{d} \leq d_0$ offering an uptime-guarantee contract does not make sense for the service provider since no profitable and essential contract can be found. When $\hat{d} > d_0$, Theorem 2 determines the unique optimal contract. In case the limit uptime level $\hat{d}$ exists, i.e., when $d = 1$ is not admissible, the optimal uptime level is the solution to condition (12) and the optimal price is obtained from (13). In case the limit uptime level does not exist, i.e., $d = 1$ is admissible, the optimal contract either happens at the critical point, which satisfies conditions (12) and (13), or in case $\Pi(p, d)$ does not have a critical point, it happens at the extreme uptime level of 1.

5. Contract Menus

In this section we extend our analysis to study situations where the service provider offers a collection of contracts, i.e., a *contract menu*. The service provider's incentive for doing so is to take advantage of different possible types of the customer to get the most out of a high-valuation customer who is willing to pay more to purchase contracts with higher guaranteed uptime levels while avoiding the risk of losing a low-valuation customer.

Formally, a contract menu is a non-empty set of singular contracts denoted by $\mathbf{t} = \{t_1, ..., t_m\}$. We refer to m as the menu's size. We assume hereafter that the indices of contracts in the menu are arranged according to increasing order of their uptime levels, i.e., for $k = 1, ..., m - 1$ we have $d_k < d_{k+1}$. The (ordered) vector of uptime levels is denoted with $\mathbf{d} = (d_1, ..., d_m)$. Naturally, we require that $d_1 > d_0$ and $d_m \leq 1$. To compare the choice of different contracts with the choice of no contract, we define the dummy contract $t_0 = (p_0, d_0)$ with $p_0 = c_0$ and let $\mathbf{t}^+ = \mathbf{t} \cup t_0$ be the augmented contract menu.

The optimal design of contract menus is more complicated than that of singular contracts since (a) the customer's optimal choice in this case is less straightforward, and (b) the service provider's optimization problem has more variables that should be simultaneously optimized.

5.1. The customer's utility

In line with equation (1), the utility of contract $t_k \in \mathbf{t}^+$ to the customer with unit revenue v can be formulated as $u(v, t_k) = v(\lambda_{d_k} - \lambda_0) - (p_k - c_0)$. Contract t_k is an optimal choice for the customer if $u(v, t_k) \geq u(v, t_{k'})$ for all $t_{k'} \in \mathbf{t}^+$. In case there are multiple optimal contracts, the customer chooses randomly among them.

The customer's utility functions associated with every pair of distinct contracts intersect at a single point. Hence, we can find a threshold for each pair of contracts such that the customer's preference over the pair changes at the threshold. For $t_k, t_{k'} \in \mathbf{t}^+$ such that $k < k'$ we define the *pivot of k and k'* by

$$v_{k,k'} = \frac{p_{k'} - p_k}{\lambda_{d_{k'}} - \lambda_{d_k}}. \tag{14}$$

The customer with unit revenue rate v prefers $t_{k'}$ to t_k whenever $v > v_{k,k'}$ since $u(v, t_{k'}) > u(v, t_k)$. t_k would be chosen over $t_{k'}$, i.e. $u(v, t_{k'}) < u(v, t_k)$, whenever $v < v_{k,k'}$. In case $v = v_{k,k'}$, we have $u(v, t_{k'}) = u(v, t_k)$ which implies that the customer is indifferent between t_k and $t_{k'}$ and chooses randomly between the two contracts.

5.2. The service provider's profit

The profit to the service provider when the customer purchases contract $t_k = (p_k, d_k) \in \mathbf{t}^+$ is $p_k - c_{d_k}$. We assume hereafter that c_d is convex, that is, the rate of growth of costs associated with providing contracts which higher uptime levels is increasing.

5.3. Contract menu design problem

Similar to the case of singular contracts, we can express the expected profit of the service provider, while taking into account the customer's optimal response, in terms of the probabilities of unit revenue rates:

$$\Pi(\mathbf{t}) = \sum_{k=1}^{m} (p_k - c_{d_k}) Pr\{v | t_k \text{ is purchased}\}. \quad (15)$$

5.4. Desirable contract menus

We extend the notions of essential and profitable contracts to contract menus. For a contract menu to be essential, every contract in the menu must have a positive chance of being purchased by the customer so that no contract is redundant.

Definition 4. *A contract menu* $\mathbf{t}$ *of size* $m \geq 1$ *is essential if for every* $k = 1, ..., m$*, it holds that* $Pr\{v|t_k \textit{ is purchased}\} > 0$.

Note that in order to have $Pr\{v|t_k \text{ is purchased}\} > 0$ there must exists $v \in V$ such that for all $t_{k'} \in \mathbf{t}^+$ we have $u(v, t_k) \geq u(v, t_{k'})$. On the other hand, with a continuous distribution the probability of the customer's unit revenue having a single value is zero. Therefore, in order for a contract menu to be essential, every contract in the menu must be an optimal choice for the customer over a non-empty interval (i.e., an interval with non-identical extreme points) of unit revenue rates. Thus essential contract menus can be defined in an alternative way.

Definition 4$'$. *A contract menu* $\mathbf{t}$ *of size* $m \geq 1$ *is essential if and only if for every* $k = 1, ..., m$ *there exists a non-empty interval* $V_k \subseteq V$ *such that for all* $v \in V_k$ *we have* $u(v, t_k) \geq u(v, t_{k'})$ *for every* $k' = 0, ..., m$.

Due to their importance in the rest of our analysis, we provide a complete characterization of essential contract menus of size two and larger. For singular contracts such a characterization is already given in Definition 1.

Lemma 4. *A contract menu* $\mathbf{t}$ *of size* $m \geq 2$ *is essential, if and only if the following conditions hold: (i)* $v_{0,1} < v_{1,2} < ... < v_{m-1,m}$*, (ii)* $v_{1,2} > 0$*, (iii)* $v_{m-1,m} < v_{\max}$.

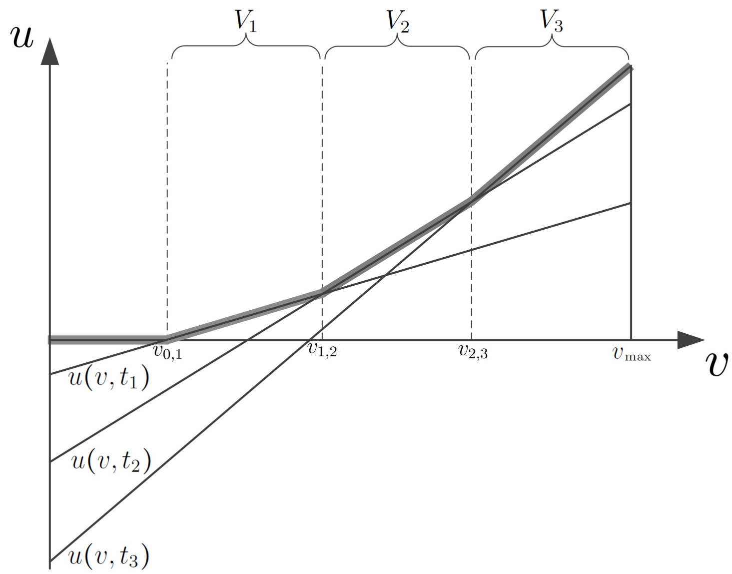

Figure 1: Acceptance regions of an essential contract menu of size 3

The first condition in Lemma 4 requires that the pivots of consecutive contracts be ordered consecutively. The second condition prescribes that the pivot of first two contracts be situated in positive real numbers. Finally, the third condition necessitates that the pivot of last two contracts happens prior to the upper bound of the customer's unit revenue. The proof of Lemma 4 immediately establishes the following observation regarding essential contract menus which we provide without a formal proof.

Corollary 2. *An essential contract menu* $\mathbf{t}$ *of size* $m \geq 2$ *generates* m *non-empty intervals* $V_1 = [\max\{0, v_{0,1}\}, v_{1,2}]$, $V_k = [v_{k-1,k}, v_{k,k+1}]$ *for* $k = 2, ..., m-1$, *and* $V_m = [v_{m-1,m}, v_{\max}]$ *such that for the customer with* $v \in V_k$, $k = 1, ..., m$, *an optimal choice of contract is* t_k. *The choice of* t_k *is uniquely optimal when* v *is an interior point of* V_k.

As the result of Corollary 2, with an essential contract menu the optimal choice of the customer depends on the position of her unit revenue rate with respect to m non-empty acceptance regions of the contract menu. Figure 1 illustrates this for an essential contract menu of size 3. This figure shows the customer's utility function under three different contracts for different unit revenue rates. For a customer with $v < v_{0,1}$, all contracts in the menu have negative utilities so her best choice is not to buy any contract. For a customer with $v \in V_1$, $u(v, t_1)$ is positive and larger than $u(v, t_2)$ and $u(v, t_3)$ so t_1 is the best choice. With similar arguments it can be seen that the customer's best choices of contracts in regions V_2 and V_3 are t_2 and t_3 respectively.

The service provider has sufficient incentives to offer a contract menu if it is profitable. Offering multiple contracts by the service provider is only reasonable if contracts with higher guaranteed uptime levels result in more profit for him. Otherwise, the contracts with higher guaranteed uptime levels cannibalize the profits of those with lower uptime levels.

Definition 5. *A contract menu* $\mathbf{t}$ *of size* $m \geq 2$ *is profitable if (a) every contract* $t_k \in \mathbf{t}$ *is profitable, and (b) for every* $t_k, t_{k'} \in \mathbf{t}^+$ *such that* $k' > k$ *it holds that* $p_{k'} - c_{d_{k'}} > p_k - c_{d_k}$.

Thus, a contract menu is profitable if the price of every singular contract in it is greater than the corresponding cost, and contracts with higher uptime levels generate more profits than the contracts with lower uptime levels.

We formulate the expected profit of the service provider specifically for essential and profitable contract menus. Using Corollary 2 and the fact that for a profitable contract menu it holds that $v_{0,1} > 0$, it can be seen that the probability of a contract being purchased is equal to the probability that the unit revenue rate falls within the corresponding acceptance region. Therefore, we have $Pr\{v|t_k \text{ is purchased}\} = Pr\{v_{k-1,k} < v < v_{k,k+1}\} = F(v_{k,k+1}) - F(v_{k-1,k})$ for $k = 1, ..., m-1$ and $Pr\{v|t_m \text{ is purchased}\} = \overline{F}(v_{m-1,m})$. Subsequently, for the essential and profitable contract menu $\mathbf{t}$ of size $m \geq 2$, (15) can be written as

$$\Pi(\mathbf{t}) = \sum_{k=1}^{m-1} (p_k - c_{d_k}) \left[F(v_{k,k+1}) - F(v_{k-1,k})\right] + (p_m - c_{d_m}) \overline{F}(v_{m-1,m}). \quad (16)$$

5.5. Admissible uptime vectors

We extend the notion of admissible uptime levels to admissible uptime vectors.

Definition 6. *An (ordered) uptime vector* $\mathbf{d} = (d_1, ..., d_m)$ *is admissible if* $(c_{d_m} - c_{d_{m-1}})/(\lambda_{d_m} - \lambda_{d_{m-1}}) < v_{\max}$.

The admissibility condition for an uptime vector is expressed as an inequality for the pair of contracts with the highest uptime levels in the menu. Similar to the case of singular contracts, the possibility of having essential and profitable contract menus depends on the admissibility of guaranteed uptime vectors.

Lemma 5. *A necessary condition for the existence of an essential and profitable contract menu with uptime vector* $\mathbf{d}$ *is that* $\mathbf{d}$ *be an admissible uptime vector.*

5.6. Optimal pricing of contract menus

In this section we assume that the service provider has already fixed the guaranteed uptime vector to $\mathbf{d}$ and focus on the problem of finding the optimal prices of the menu, i.e., $\mathbf{p}$. The following program yields the prices of an essential and profitable contract menu that maximize the service provider's expected profit incorporating the customer's optimal response while keeping the conditions of essential and profitable contract menus in place:

$$
\begin{aligned}
\max_{\mathbf{p}} \quad & \sum_{k=1}^{m-1} (p_k - c_{d_k}) \left[F\left(\frac{p_{k+1} - p_k}{\lambda_{d_{k+1}} - \lambda_{d_k}} \right) - F\left(\frac{p_k - p_{k-1}}{\lambda_{d_k} - \lambda_{d_{k-1}}} \right) \right] \\
& + (p_m - c_m)\, \overline{F}\left(\frac{p_m - p_{m-1}}{\lambda_{d_m} - \lambda_{d_{m-1}}} \right) && (17) \\
s.t. \quad & \frac{p_{k+1} - p_k}{\lambda_{d_{k+1}} - \lambda_{d_k}} > \frac{p_k - p_{k-1}}{\lambda_{d_k} - \lambda_{d_{k-1}}} \qquad \forall k = 1, ..., m-1 && (18) \\
& \frac{p_m - p_{m-1}}{\lambda_{d_m} - \lambda_{d_{m-1}}} < v_{\max} && (19) \\
& p_k - c_{d_k} > p_{k-1} - c_{d_{k-1}} \qquad \forall k = 1, ..., m && (20)
\end{aligned}
$$

The group of constraints in (20) ascertains that the obtained contract menu is profitable. The groups of constraints in (18) and (19) correspond to the necessary and sufficient conditions of essential contract menus in Lemma 4—in particular conditions (i) and (iii) respectively. Condition (ii) in Lemma 4 is implied by constraint (20) for $k = 2$.

As a technical comment, it should be noted that our method is slightly different from that in classic contract design literature which draws upon incentive compatibility constraints to ensure that the service provider takes into account the customer's optimal response (see for example Laffont and Martimort (2002)). As the result of Corollary 2 which characterizes the customer's best choice in essential menus, the objective function in (17) already incorporates the customer's optimal response so additional constraints for incentive compatibility are not needed.

To solve the non-concave constrained optimization program above, we first focus on the relaxed program and investigate its possible critical points using multi-variable calculus.

Lemma 6. *Let* $\mathbf{d}$ *be an admissible uptime vector. The following system yields the unique critical point of (17). Set* $p_0^* = c_0$ *and for* $k = 1, ..., m$*:*

$$\overline{F}\left(\frac{p_k^* - p_{k-1}^*}{\lambda_{d_k} - \lambda_{d_{k-1}}}\right) = \frac{p_k^* - c_{d_k} - p_{k-1}^* + c_{d_{k-1}}}{\lambda_{d_k} - \lambda_{d_{k-1}}} f\left(\frac{p_k^* - p_{k-1}^*}{\lambda_{d_k} - \lambda_{d_{k-1}}}\right). \tag{21}$$

For $k = 1$, equation (21) reduces to (8). For $k = 2, ..., m$, the values of p_k^* can be obtained recursively. The next observation establishes the fact that price vector $\mathbf{p}^*$ corresponding to the critical point of $\Pi(\mathbf{t})$ obtained in (21) satisfies the constraints in the original optimization program.

Lemma 7. *The vector of prices obtained via implicit equations in (21) upon existence satisfies the constraints in (18)–(20).*

So far we have found a unique critical point of $\Pi(\mathbf{t})$ which is also a feasible point for our optimization program. What remains to establish is that the critical point is in fact a maximum. This is proven in the main result of this section regarding the optimal prices of uptime-guarantee contract menus.

Theorem 3. *Given an admissible uptime vector* $\mathbf{d}$*, the system of equations in (21) obtains the unique vector of prices which results in the highest expected profit for the service provider among all essential and profitable contract menus with uptime vector* $\mathbf{d}$*.*

Theorem 3 establishes the fact that for every admissible guaranteed uptime vector, a unique vector of optimal prices exists. On the other hand, Lemma 5 asserts that the admissibility of a guaranteed uptime vector is necessary for having an essential and profitable contract menu. The final conclusion, which we provide without a formal proof, follows immediately.

Corollary 3. *Let* $\mathbf{d}$ *be an uptime vector with* $d_1 > d_0$ *and* $d_m \leq 1$*. A unique essential and profitable contract menu with the highest expected profit for the service provider exists if and only if* $\mathbf{d}$ *is an admissible uptime vector.*

Although we have solved the pricing problem of the contract menus with given uptime vectors, we still need to comment on the problem of finding optimal guaranteed uptime vectors. The latter problem is considerably more

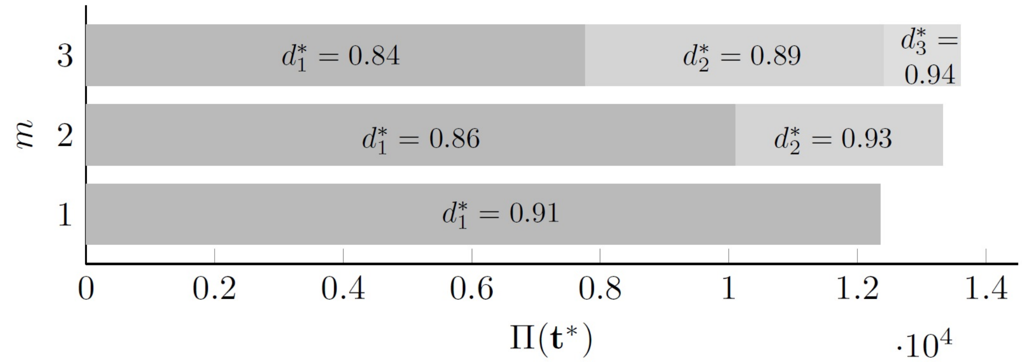

Figure 2: Expected profit of contract menus in Example 1

complicated than that in the case of singular contracts. Nevertheless, if the size of the menu is small and the possible uptime levels are chosen from a finite set—which is a rather reasonable assumption in practical scenarios—one can enumerate all possible combinations of contracts and find the most profitable. We finish the paper with a numerical example which calculates the optimal contract menus with maximum size three.

Example 1. *Assume that the customer's unit revenue v is distributed uniformly between* 0 *and* 1000000*. Without a contract the expected uptime of the equipment is* %80*, i.e.* $d_0 = 0.80$*. We normalize the preventive maintenance costs to zero and let* $c_d = 300000(\lambda_d - \lambda_0)^2$*. The optimal choice of guaranteed uptime level for a singular contract is* $d_1^* = 0.91$*. Accordingly, the optimal singular contract has the expected profit of* 12345*. By offering a menu of two contracts, our numerical calculations indicate that the optimal contract menu consists of* $t_1^* = (50700, 0.86)$ *and* $t_2^* = (90350, 0.93)$*, with the expected profit of* 13322*. With a contract menu of size three, the service provider can capture the expected profit of* 13597 *when offering the optimal menu with contracts* $t_1^* = (22400, 0.84)$*,* $t_2^* = (57150, 0.89)$*, and* $t_3^* = (99400, 0.94)$*.* $\triangle$

Figure 2 depicts the expected profits of optimal contract menus in Example 1. Each rectangle corresponds to the expected profit of a specific contract. As illustrated in this figure, offering additional contracts will increase the expected profit of the service provider (as long as the guaranteed uptime vector remains admissible). The underlying reason is that by including more contracts in a menu, the service provider can better target different customer types in terms of possible unit revenue rates.

6. Concluding Remarks

In this paper we examined the profit maximization problem for a service provider who offers uptime-guarantee maintenance contracts. The novelty of our analysis is to take into consideration the fact that the service provider does not have complete information on the customer's actual valuation for contracts with different guaranteed uptime levels. We determined the unique optimal contract as well as the unique vector of optimal prices for menus of contracts for the service provider to offer. As the service provider increases size of the menu, he can expect higher profits. The reason is that offering more contracts in a menu enables the service provider to better target the customer's possible valuation and charge proportionally higher prices for contracts with higher uptime levels. Finding the optimal guaranteed uptime vectors for contract menus is a complex problem. However, when the number of feasible uptime levels are finite and the size of the menu is not too large, enumeration techniques can be used to find close to optimal uptime vectors. The latter is a practical solution, especially since managing contract menus which include large number of contracts is complicated and expensive. Finding the optimal uptime vectors analytically remains an intriguing question for future research.

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Appendix

Proof of Lemma 1. By IGFR assumption, $xf(x)/\overline{F}(x)$ is non-decreasing so

$$\frac{d\frac{xf(x)}{\overline{F}(x)}}{dx} = \frac{f(x)}{\overline{F}(x)} + \frac{x\frac{df(x)}{dx}}{\overline{F}(x)} + x\frac{f^2(x)}{\overline{F}^2(x)} \geq 0.$$

Equivalently, we have,

$$-\frac{df(x)}{dx} \leq f(x)\frac{\overline{F}(x) + xf(x)}{x\overline{F}(x)}.$$

Multiplying both sides by $x - a$, assuming $x > a$, and subtracting $2f(x)$ yields

$$-2f(x)-(x-a)\frac{df(x)}{dx} = \frac{dG(x)}{dx} \leq f(x)\frac{-2x\overline{F}(x) + (x-a)\overline{F}(x) + x(x-a)f(x)}{x\overline{F}(x)}.$$

If $x^* = G^{-1}(0)$ exists such that $x^* > a$, then it must be that $(x^* - a)f(x^*) = \overline{F}(x^*)$. Thus, last inequality obtains

$$\left.\frac{dG(x)}{dx}\right|_{x=x^*} \leq f(x^*)\frac{-2x^*\overline{F}(x^*) + (x^*-a)\overline{F}(x^*) + x^*\overline{F}(x^*)}{x^*\overline{F}(x^*)} = f(x^*)\frac{-a}{x^*}.$$

Since $f(x) > 0$ for all $x \in V$ we conclude that $dG(x)/dx|_{x^*} < 0$. □

Proof of Lemma 2. Suppose the contrary, that is, $(c_d - c_0)/(\lambda_d - \lambda_0) \geq v_{\max}$ and there exists a profitable and essential contract $t = (p, d)$. The profitability of (p, d) implies that $p > c_d$ which in turn results in having $(p - c_0)/(\lambda_d - \lambda_0) > (c_d - c_0)/(\lambda_d - \lambda_0)$. By the contrary assumption we conclude that $(p - c_0)/(\lambda_d - \lambda_0) > v_{\max}$. However, the latter contradicts t being essential. Therefore, to have an essential and profitable contract d must be admissible. □

Proof of Theorem 1. Consider the relaxation of the program in (5)-(7). The first derivative of the objective function in (5) with respect to p can be written as $\overline{F}(x) - (x - a)f(x)$, where $x = (p - c_0)/(\lambda_d - \lambda_0)$ and $a = (c_d - c_0)/(\lambda_d - \lambda_0)$. The critical point, upon existence, happens wherever two functions $xf(x)/\overline{F}(x)$ and $x/(x - a)$ intersect such that $x > 0$. The IGFR assumption implies that $xf(x)/\overline{F}(x)$ is non-decreasing for every $x \in V$ such that $\overline{F}(x) > 0$. Due to properties of distribution functions we know that $xf(x)$ is bounded and we can find $x \in V$ such that $\overline{F}(x)$ is arbitrary close to zero. Thus $xf(x)/\overline{F}(x)$ has the range $[0, \infty)$. On the other hand, the function $x/(x - a)$ is strictly decreasing for $x > a$ and has the range $(1, \infty)$. Note that with an admissible d it is the case that $a < v_{\max}$. Thus, there must exist a unique $a < x^* < v_{\max}$ such that $x^*f(x^*)/\overline{F}(x^*) = x^*/(x^* - a)$ which implies that $\overline{F}(x^*) - (x^* - a)f(x^*) = 0$. By lemma 1 it follows that x^* is the unique maximum.

To complete the proof, we show that constraints (6) and (7) are satisfied at x^*. Since $x^* < v_{\max}$ we have $(p^* - c_0)/(\lambda_d - \lambda_0) < v_{\max}$ thus (6) holds, and as $x^* > a$ we have $p^* - c_d > 0$ thus (7) holds. □

Proof of Lemma 3. Let d and d' be such that $d_0 < d < d' \leq 1$. By convexity of c_d it follows that $(c_d - c_0)/(d - d_0) < (c_{d'} - c_0)/(d' - d_0)$. By concavity of λ_d it must be that $(\lambda_d - \lambda_0)/(d - d_0) > (\lambda_{d'} - \lambda_0)/(d' - d_0)$ which implies that $(d - d_0)/(\lambda_d - \lambda_0) < (d' - d_0)/(\lambda_{d'} - \lambda_0)$. Therefore we have

$$\frac{c_d - c_0}{\lambda_d - \lambda_0} < \frac{c_{d'} - c_0}{\lambda_{d'} - \lambda_0}. \tag{22}$$

If d is not admissible then $(c_d - c_0)/(\lambda_d - \lambda_0) \geq v_{\max}$. In the case inequality (22) implies that $(c_{d'} - c_0)/(\lambda_{d'} - \lambda_0) > v_{\max}$ thus if d is not admissible d' is not admissible either. On the other hand, if d' is admissible, inequality (22) establishes that d must also be admissible. The claim of the lemma follows. □

Proof of Theorem 2. If an admissible uptime level within $(d_0, 1]$ exists, then the feasible region for uptime levels is non-empty. By Lemma 3, the optimization problem is to maximize $\Pi(p, d)$ in part by selecting d in non-empty intervals: (i) $(d_0, \hat{d})$ when $\hat{d}$ exists (when $d = 1$ is not admissible), or (ii) $(d_0, 1]$ when $\hat{d}$ does not exist (when $d = 1$ is admissible).

Case (i): In this case $\hat{d} \leq 1$ exists. We first show that $\Pi(p, d)$ has at least one critical point within $(d_0, \hat{d})$. Since for an essential and profitable contract we have $c_d < p < v_{\max}(\lambda_d - \lambda_0) + c_0$, as $d \to d_0$ we have $p \to c_0$. Equation (8) obtains that as $p \to c_0$ we have $\Pi(p, d) \to 0$. Thus, $\Pi(p, d)$ decreases to zero as d approaches d_0. With the same argument it can be seen that as $d \to \hat{d}$ we have $p \to c_{\hat{d}}$ and consequently $\Pi(p, d) \to 0$ so $\Pi(p, d)$ also decreases to zero as d approaches $\hat{d}$. Since $\Pi(p, d)$ is a continuous, bounded, and positive function everywhere within $(d_0, \hat{d})$ and it approaches zero close to the boundaries, $\Pi(p, d)$ has a maximum within $(d_0, \hat{d})$ which must happen at a critical point.

The conditions for critical points of $\Pi(p, d)$ are as follows. While the first order condition with regard to p yields (8), the first order condition with regard to d yields

$$c'_{d^*} \overline{F}\left(\frac{p - c_0}{\lambda_{d^*} - \lambda_0}\right) = \lambda'_{d^*} \frac{p - c_0}{\lambda_{d^*} - \lambda_0} \underbrace{\frac{(p - c_{d^*})}{\lambda_{d^*} - \lambda_0} f\left(\frac{p - c_0}{\lambda_{d^*} - \lambda_0}\right)}. \tag{23}$$

At a critical point equations (8) and (23) hold simultaneously. Thus at a critical point $p = p^*$ and $d = d^*$, the last two terms of (23) can be replaced by

$\overline{F}\left((p^{*}-c_0)/(\lambda_{d^*}-\lambda_0)\right)$ from (8) and (23) simplifies to $(p^{*}-c_0)/(\lambda_{d^*}-\lambda_0) = c^{'}_{d^*}/\lambda^{'}_{d^*}$. While the latter can be rewritten as (13), replacing $(p^{*}-c_0)/(\lambda_{d^*}-\lambda_0)$ by $c^{'}_{d^*}/\lambda^{'}_{d^*}$ in (8) yields (12).

In the final step we show that the critical point is unique so (12) and (13) together correspond to a unique maximum. It suffices to show that (12) has a single solution. Similar to the proof of Theorem 1, let $c^{'}_{d}/\lambda^{'}_{d} = x$ and $(c_d - c_0)/(\lambda_d - \lambda_0) = a$. Since c_d is convex and increasing and λ_d is concave and increasing, it is straightforward to see that as d increases, x increases as well. In order for condition (12) to have a solution, we must have $xf(x)/\overline{F}(x) = x/(x-a)$. The function $xf(x)/\overline{F}(x)$ is positive and non-decreasing under IGFR assumption. The function $x/(x-a)$ is also decreasing whenever it is positive. Therefore, the two functions can have at most one intersection. On the other hand, since we have already established that $\Pi(p,d)$ has at least one critical point in $(d_0,\hat{d})$, the last two functions must intersect exactly once. Thus, condition (12) has a single solution.

Case (ii): The optimization problem is to maximize $\Pi(p,d)$ over the half-open interval $(d_0,1]$. Since $\Pi(p,d)$ is a continuous, bounded, and positive function everywhere within $(d_0,1]$, and it approaches zero when d approaches d_0, $\Pi(p,d)$ must have its maximum either at a critical point within $(d_0,1]$, which is unique upon existence, or at the boundary value of $d=1$. □

Proof of Lemma 4. [If part] By Definition $4^{'}$, we must show that when (i), (ii), and (iii) hold, for every $k=1,...,m$ there exists a non-empty interval $V_k \subseteq V$ such that for every $v \in V_k$ we have $u(v,t_k) \geq u(v,t_{k^{'}})$ for all $k^{'}=0,...,m$. Since (i) holds, we can define the non-empty intervals $\dot{V}_k = [v_{k-1,k}, v_{k,k+1}]$ for $k=1,...,m-2$, and $\dot{V}_m = [v_{m-1,m},\infty)$. Imposing the conditions (ii) and (iii) assures that $\mathbf{t}$ divides V into the non-empty intervals $V_0 = [0,\max\{0,v_{0,1}\}]$, $V_1 = [\max\{0,v_{0,1}\}, v_{1,2}]$, $V_k = \dot{V}_k$ for $k=2,...,m-1$, and $V_m = [v_{m-1,m}, v_{\max}]$.

Let $k \in \{1,...,m\}$. By the definition of $v_{k-1,k}$ we know that $u(v,t_k) \geq u(v,t_{k-1})$ whenever $v \geq v_{k-1,k}$. From condition (i) we can conclude that for $k^{'} < k$ it also holds that that $u(v,t_k) \geq u(v,t_{k^{'}})$ whenever $v \geq v_{k-1,k}$. On the other hand, by definition of $v_{k,k+1}$ we know that $u(v,t_k) \geq u(v,t_{k+1})$ whenever $v \leq v_{k,k+1}$. From condition (i) we can conclude that for $k^{'} > k$ it also holds that that $u(v,t_k) \geq u(v,t_{k^{'}})$ whenever $v \leq v_{k,k+1}$. Therefore, for every $v \in [v_{k-1,k}, v_{k,k+1}]$ we have $u(v,t_k) \geq u(v,t_{k^{'}})$ for every $k^{'}=0,...,m$.

[Only-if part] We show if any of the three conditions fails, the contract menu would not be essential.

(i) Consider $l, h, k \in \{0, ..., m\}$ such that $l < h < k$. Suppose the contrary, that is, assume $\mathbf{t}$ is essential and $v_{h,k} \leq v_{l,h}$. We continue in two steps.

(Step I) We show if $v_{h,k} \leq v_{l,h}$, then it must be that $v_{h,k} \leq v_{l,k} \leq v_{l,h}$. Let $A = [0, v_{h,k}]$, $B = [v_{h,k}, v_{l,h}]$, and $C = [v_{l,h}, v_{\max}]$. Considering the property of the pivot points we have:

$v \in A$	$v \in B$	$v \in C$
$u(v, t_h) \geq u(v, t_k)$	$u(v, t_h) \leq u(v, t_k)$	$u(v, t_h) \leq u(v, t_k)$
$u(v, t_l) \geq u(v, t_h)$	$u(v, t_l) \geq u(v, t_h)$	$u(v, t_l) \leq u(v, t_h)$

From the above we can deduce that $u(v, t_l) \geq u(v, t_k)$ for all $v \in A$, and $u(v, t_l) \leq u(v, t_k)$ for all $v \in C$. Thus, it must be that $v_{l,k} \in B$.

(Step II) We show if $v_{h,k} \leq v_{l,k} \leq v_{l,h}$, then there exists no non-empty interval such that t_h is an optimal choice for the customer. Let $B_1 = [v_{h,k}, v_{l,k}]$ and $B_2 = [v_{l,k}, v_{l,h}]$. We have the following cases:

$v \in A$	$v \in B_1$	$v \in B_2$	$v \in C$
$u(v, t_h) \geq u(v, t_k)$	$u(v, t_h) \leq u(v, t_k)$	$u(v, t_h) \leq u(v, t_k)$	$u(v, t_h) \leq u(v, t_k)$
$u(v, t_l) \geq u(v, t_h)$	$u(v, t_l) \geq u(v, t_h)$	$u(v, t_l) \geq u(v, t_h)$	$u(v, t_l) \leq u(v, t_h)$
$u(v, t_l) \geq u(v, t_h)$	$\min\{(v, t_l), (v, t_k)\} \geq u(v, t_h)$	$\min\{(v, t_l), (v, t_k)\} \geq u(v, t_h)$	$u(v, t_h) \leq u(v, t_k)$

The above table shows that everywhere in V the choice of t_h results in a utility for the customer which is at most as high as either t_k or t_l. Considering that any two contracts can only have a single point at which their utilities are equal, we conclude that t_h is among the optimal choices of the customer at most at one point in V thus there is no non-empty interval on V where t_h is the optimal contract. Hence, $\mathbf{t}$ is not an essential contract menu. This is a contradiction so it must be that $v_{h,k} > v_{l,h}$. Extending this logic for all trios of contracts in $\mathbf{t}$ obtains the claim for part (i).

(ii) If $v_{1,2} \leq 0$ then the customer with a unit revenue rates in $(0, v_{\max}]$ prefers t_2 to t_1. Thus t_1 is within the optimal choice set of the customer at most at a single point $v = 0$ and not in a non-empty interval in V which implies that $\mathbf{t}$ is not essential. Hence it must be that $0 \leq v_{1,2}$

(iii) If $v_{m-1,m} \geq v_{\max}$ then the customer with a unit revenue rates in $[0, v_{\max})$ prefers t_{m-1} to t_m. Thus contract t_m is at most optimal at the single point $v_{\max}$ which means that $\mathbf{t}$ is not essential. Hence it must be that $v_{m-1,m} < v_{\max}$. □

Proof of Lemma 5. Suppose the contrary, that is, $\mathbf{t}$ is an essential and profitable contract menu and $\mathbf{d}$ is not admissible, i.e., $(c_{d_m} - c_{d_{m-1}})/(\lambda_{d_m} -$

$\lambda_{d_{m-1}}) \geq v_{\max}$. In order for $\mathbf{t}$ to be profitable it must hold that $p_m - c_{d_m} > p_{m-1} - c_{d_{m-1}}$. This condition can be rewritten as $(p_m - p_{m-1})/(\lambda_{d_m} - \lambda_{d_{m-1}}) > (c_{d_m} - c_{d_{m-1}})/(\lambda_{d_m} - \lambda_{d_{m-1}})$. Thus we get $(p_m - p_{m-1})/(\lambda_{d_m} - \lambda_{d_{m-1}}) > v_{\max}$. The latter violates condition (iii) in Lemma 4 which must hold for every essential contract menu. This is a contradiction. Therefore, the claim of the lemma holds. □

Proof of Lemma 6. For $k = 1, ..., m-1$ the first derivatives of (17) are of the form:

$$\begin{aligned} \tfrac{\delta \Pi}{\delta p_k} = \;& F\left(\frac{p_{k+1}-p_k}{\lambda_{d_{k+1}}-\lambda_{d_k}}\right) + \frac{p_{k+1}-c_{d_{k+1}}-p_k+c_{d_k}}{\lambda_{d_{k+1}}-\lambda_{d_k}} f\left(\frac{p_{k+1}-p_k}{\lambda_{d_{k+1}}-\lambda_{d_k}}\right) \\ & -F\left(\frac{p_k-p_{k-1}}{\lambda_{d_k}-\lambda_{d_{k-1}}}\right) - \frac{p_k-c_{d_k}-p_{k-1}+c_{d_{k-1}}}{\lambda_{d_k}-\lambda_{d_{k-1}}} f\left(\frac{p_k-p_{k-1}}{\lambda_{d_k}-\lambda_{d_{k-1}}}\right) \end{aligned}$$

and for $k = m$ we have,

$$\tfrac{\delta \Pi}{\delta p_m} = \overline{F}\left(\frac{p_m-p_{m-1}}{\lambda_{d_m}-\lambda_{d_{m-1}}}\right) - \frac{p_m-c_m-p_{m-1}+c_{m-1}}{\lambda_{d_m}-\lambda_{d_{m-1}}} f\left(\frac{p_m-p_{m-1}}{\lambda_{d_m}-\lambda_{d_{m-1}}}\right).$$

A critical point $\mathbf{p}^*$, upon existence, is a solution to the system of equations $\{\delta\Pi/\delta p_k = 0, k = 1, ..., m\}$. Starting from the last equation and substituting the corresponding terms backward, the solution to this system of equations is obtained via the implicit functions expressed in (21).

To complete the proof it suffices to show that $\mathbf{p}^*$ exists and is unique. For $k = 1, ..., m$, let $x_k = (p_{k+1}-p_k)/(\lambda_{d_{k+1}} - \lambda_{d_k})$ and $a_k = (c_{d_{k+1}} - c_{d_k})/(\lambda_{d_{k+1}} - \lambda_{d_k})$. With an admissible $\mathbf{d}$ and under the assumption on contract costs, it is the case that $a_k < v_{\max}$. In this case, similar logic as the proof of Theorem 1 obtains that there must exist a unique $a_k < x_k^* < v_{\max}$ such that $\overline{F}(x_k^*) - (x_k^* - a_k) f(x_k^*) = 0$. □

Proof of Lemma 7. To show (18), fix $k \in \{0, ..., m-1\}$ and note that by optimality conditions in (21) we have

$$f\left(\frac{p^*_{k+1}-p^*_k}{\lambda_{d_{k+1}}-\lambda_{d_k}}\right) \Big/ \overline{F}\left(\frac{p^*_{k+1}-p^*_k}{\lambda_{d_{k+1}}-\lambda_{d_k}}\right) = 1 \Big/ \left(\frac{p^*_{k+1}-p^*_k}{\lambda_{d_{k+1}}-\lambda_{d_k}} - \frac{c_{d_{k+1}}-c_{d_k}}{\lambda_{d_{k+1}}-\lambda_{d_k}}\right),$$

and

$$f\left(\frac{p_k^*-p_{k-1}^*}{\lambda_{d_k}-\lambda_{d_{k-1}}}\right)\Big/\overline{F}\left(\frac{p_k^*-p_{k-1}^*}{\lambda_{d_k}-\lambda_{d_{k-1}}}\right)=1\Big/\left(\frac{p_k^*-p_{k-1}^*}{\lambda_{d_k}-\lambda_{d_{k-1}}}-\frac{c_{d_k}-c_{d_{k-1}}}{\lambda_{d_k}-\lambda_{d_{k-1}}}\right).$$

Suppose the contrary, that is, $(p_{k+1}^*-p_k^*)/(\lambda_{d_{k+1}}-\lambda_{d_k})\le(p_k^*-p_{k-1}^*)/(\lambda_{d_k}-\lambda_{d_{k-1}})$. Then, by IGFR assumption the last two equalities yield

$$\frac{p_{k+1}^*-p_k^*}{\lambda_{d_{k+1}}-\lambda_{d_k}}-\frac{p_k^*-p_{k-1}^*}{\lambda_{d_k}-\lambda_{d_{k-1}}}\ge\frac{c_{d_{k+1}}-c_{d_k}}{\lambda_{d_{k+1}}-\lambda_{d_k}}-\frac{c_{d_k}-c_{d_{k-1}}}{\lambda_{d_k}-\lambda_{d_{k-1}}}$$

Since c_d is increasing and convex, it holds that $(c_{d_{k+1}}-c_{d_k})/(d_{k+1}-d_k)>(c_{d_k}-c_{d_{k-1}})/(d_k-d_{k-1})$. Also, by concavity of λ_d we have $(\lambda_{d_{k+1}}-\lambda_{d_k})/(d_{k+1}-d_k)<(\lambda_{d_k}-\lambda_{d_{k-1}})/(d_k-d_{k-1})$. Thus we have $(c_{d_{k+1}}-c_{d_k})/(\lambda_{d_{k+1}}-\lambda_{d_k})>(c_{d_k}-c_{d_{k-1}})/(\lambda_{d_k}-\lambda_{d_{k-1}})$ which implies that $(p_{k+1}^*-p_k^*)/(\lambda_{d_{k+1}}-\lambda_{d_k})>(p_k^*-p_{k-1}^*)/(\lambda_{d_k}-\lambda_{d_{k-1}})$. This is clearly a contradiction. Therefore, it must be that $v_{k-1,k}<v_{k,k+1}$.

To show that (19) holds, note that since $\mathbf{p}^*$ exists, clearly we have $(p_m^*-p_{m-1}^*)/(\lambda_{d_m}-\lambda_{d_{m-1}})\le v_{\max}$, otherwise the cdf would be undefined. It remains to show that equality does not hold. Considering the optimality condition in (21), the case $(p_m^*-p_{m-1}^*)/(\lambda_{d_m}-\lambda_{d_{m-1}})=v_{\max}$ implies that $f((p_m^*-p_{m-1}^*)/(\lambda_{d_m}-\lambda_{d_{m-1}}))=0$. However, this cannot be as f is positive everywhere on its support.

Finally, we show that the solution satisfies constraint (20), i.e. for $k=1,...,m-1$ we have $p_{k+1}^*-c_{d_{k+1}}>p_k^*-c_{d_k}$. Fix k and consider the optimality condition in (21). In this case, we would have

$$F\left(\frac{p_{k+1}^*-p_k^*}{\lambda_{d_{k+1}}-\lambda_{d_k}}\right)=1-\frac{p_{k+1}^*-c_{d_{k+1}}-p_k^*+c_{d_k}}{\lambda_{d_{k+1}}-\lambda_{d_k}}f\left(\frac{p_{k+1}^*-p_k^*}{\lambda_{d_{k+1}}-\lambda_{d_k}}\right).$$

Assume the contrary, i.e. $p_{k+1}^*-c_{d_{k+1}}-p_k^*+c_{d_k}\le 0$. Since $f(\cdot)>0$ for all $v\in V$, the last inequality yields $F\left((p_{k+1}^*-p_k^*)/(\lambda_{d_{k+1}}-\lambda_{d_k})\right)\ge 1$ which is a contradiction since $(p_{k+1}^*-p_k^*)/(\lambda_{d_{k+1}}-\lambda_{d_k})<v_{\max}$ as shown in the last step. Thus, for $k=1,...,m-1$ we have $p_{k+1}^*-c_{d_{k+1}}>p_k^*-c_{d_k}$. □

The following Lemmas are needed to prove Theorem 3.

Lemma 8. *Consider an essential contract menu* $\mathbf{t}$ *with* $m\ge 2$. *For* $k=$

$1, ..., m$ *define*

$$b_k = \frac{2}{\lambda_{d_{k+1}} - \lambda_{d_k}} f\left(\frac{p_{k+1} - p_k}{\lambda_{d_{k+1}} - \lambda_{d_k}}\right) + \frac{p_{k+1} - c_{d_{k+1}} - p_k + c_{d_k}}{\lambda_{d_{k+1}} - \lambda_{d_k}} \frac{\delta f\left(\frac{p_{k+1}-p_k}{\lambda_{d_{k+1}}-\lambda_{d_k}}\right)}{\delta p_{k+1}}.$$

The Hessian matrix associated with $\Pi(\mathbf{t})$ *is*

$$H = \begin{bmatrix} -b_1 - b_2 & b_2 & 0 & 0 & \dots & 0 & 0 \\ b_2 & -b_2 - b_3 & b_3 & 0 & \dots & 0 & 0 \\ 0 & b_3 & -b_3 - b_4 & b_4 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & b_m & -b_m \end{bmatrix}. \qquad (24)$$

Proof of Lemma 8. For $k = 1, ..., m-1$:

$$\begin{aligned} \tfrac{\delta^2 \Pi}{\delta p_k^2} = \; & -\frac{2}{\lambda_{d_{k+1}} - \lambda_{d_k}} f\left(\frac{p_{k+1} - p_k}{\lambda_{d_{k+1}} - \lambda_{d_k}}\right) - \frac{p_{k+1} - c_{d_{k+1}} - p_k + c_{d_k}}{\lambda_{d_{k+1}} - \lambda_{d_k}} \frac{\delta f\left(\frac{p_{k+1}-p_k}{\lambda_{d_{k+1}}-\lambda_{d_k}}\right)}{\delta p_{k+1}} \\ & -\frac{2}{\lambda_{d_k} - \lambda_{d_{k-1}}} f\left(\frac{p_k - p_{k-1}}{\lambda_{d_k} - \lambda_{d_{k-1}}}\right) - \frac{p_k - c_{d_k} - p_{k-1} + c_{d_{k-1}}}{\lambda_{d_k} - \lambda_{d_{k-1}}} \frac{\delta f\left(\frac{p_k-p_{k-1}}{\lambda_{d_k}-\lambda_{d_{k-1}}}\right)}{\delta p_k}, \end{aligned}$$

and for $k = 2, ..., m-1$:

$$\tfrac{\delta^2 \Pi}{\delta p_k \delta p_{k-1}} = \frac{2}{\lambda_{d_k} - \lambda_{d_{k-1}}} f\left(\frac{p_k - p_{k-1}}{\lambda_{d_k} - \lambda_{d_{k-1}}}\right) + \frac{p_k - c_{d_k} - p_{k-1} + c_{d_{k-1}}}{\lambda_{d_k} - \lambda_{d_{k-1}}} \frac{\delta f\left(\frac{p_k-p_{k-1}}{\lambda_{d_k}-\lambda_{d_{k-1}}}\right)}{\delta p_k}.$$

For $k = 3, ..., m-1$ and $l = 2, ..., k-1$ we have $\delta^2\Pi/\delta p_{k-l}^2 = 0$. Finally,

$$\begin{aligned} \tfrac{\delta^2 \Pi}{\delta p_m^2} = \; & -\frac{2}{\lambda_{d_m} - \lambda_{d_{m-1}}} f\left(\frac{p_m - p_{m-1}}{\lambda_{d_m} - \lambda_{d_{m-1}}}\right) - \frac{p_m - c_m - p_{m-1} + c_{m-1}}{\lambda_{d_m} - \lambda_{d_{m-1}}} \frac{\delta f\left(\frac{p_m-p_{m-1}}{\lambda_{d_m}-\lambda_{d_{m-1}}}\right)}{\delta p_m} \\ \tfrac{\delta^2 \Pi}{\delta p_m \delta p_{m-1}} = \; & \frac{2}{\lambda_{d_m} - \lambda_{d_{m-1}}} f\left(\frac{p_m - p_{m-1}}{\lambda_{d_m} - \lambda_{d_{m-1}}}\right) + \frac{p_m - c_m - p_{m-1} + c_{m-1}}{\lambda_{d_m} - \lambda_{d_{m-1}}} \frac{\delta f\left(\frac{p_m-p_{m-1}}{\lambda_{d_m}-\lambda_{d_{m-1}}}\right)}{\delta p_m}, \end{aligned}$$

and for $l = 2, ..., m-1$ we have $\delta^2\Pi/\delta p_{m-l}^2 = 0$. In the above derivations we use the fact that $\delta f((p_2 - p_1)/(d_2 - d_1))/\delta p_1 = -\delta f((p_2 - p_1)/(d_2 - d_1))/\delta p_2$. Replacing the terms obtains the Hessian matrix as indicated in (24). □

To show that $\mathbf{p}^*$ is a maximum for $\Pi(\mathbf{t})$, H should be negative definite at $\mathbf{p}^*$ (Güler, 2010). A matrix is negative definite if and only if all its leading principal minors of odd-numbered order are negative and those of even-numbered order are positive. The k-th order leading principal minor, D_k, is the determinant of the k-th order principal sub-matrix formed by deleting the last $n-k$ rows and columns (Güler, 2010).

Lemma 9. *The leading principal minors of H in (24) are as follows. For $k = 2, ..., m-1$, k-th leading principal minor of H is $D_k = (-1)^k \sum_{l=1}^{k+1} \prod_{j=1}^{k+1} b_j/b_l$. Also, m-th leading principal minor of H is $D_m = (-1)^m \prod_{j=1}^{m} b_j$.*

Proof of Lemma 9. For $k = 2$ we have $D_2 = (-b_1 - b_2)(-b_2 - b_3) - b_2^2 = b_2b_3 - b_1b_3 + b_1b_2$. Thus, the formula works for 2. We use transfinite induction and suppose that the formula works for all $l \leq k$ with $k = 2, ..., m-2$. We show that the formula works for $k+1$. In order to do so, first note that we have

$$D_{k+1} = \begin{vmatrix} -b_1 - b_2 & b_2 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & b_{k-1} & -b_{k-1} - b_k & -b_k & 0 \\ 0 & 0 & \dots & 0 & b_k & -b_k - b_{k+1} & b_{k+1} \\ 0 & 0 & \dots & 0 & 0 & b_{k+1} & -b_{k+1} - b_{k+2} \end{vmatrix}.$$

According to Laplace expansion, the determinant of a matrix can be obtained from its minors. Thus

$$D_{k+1} = (-1)^{2k+1} b_{k+1} D + (-1)^{2k+2}(-b_{k+1} - b_{k+2}) D_k = -b_{k+1} D - (b_{k+1} + b_{k+2}) D_k$$

where

$$D = \begin{vmatrix} -b_1 - b_2 & b_2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & b_{k-1} & -b_{k-1} - b_k & 0 \\ 0 & 0 & \dots & 0 & b_k & b_{k+1} \end{vmatrix}.$$

The matrix whose determinant is D has a column with only one non-zero entry. Thus we have $D = (-1)^{2k} b_{k+1} D_{k-1} = b_{k+1} D_{k-1}$. Therefore, we have $D_{k+1} = -b_{k+1}^2 D_{k-1} - (b_{k+1} + b_{k+2}) D_k$. Based on the assumption of induction, it holds that $D_{k-1} = (-1)^{k-1} \sum_{l=1}^{k} \prod_{j=1}^{k} b_j/b_l$, and $D_k = (-1)^k \sum_{l=1}^{k+1} \prod_{j=1}^{k+1} b_j/b_l$.

By substituting the terms we get:

$$\begin{aligned} D_{k+1} &= -b_{k+1}^2(-1)^{k-1}\sum_{l=1}^{k}\frac{\prod_{j=1}^{k} b_j}{b_l} - (b_{k+1}+b_{k+2})(-1)^k\sum_{l=1}^{k+1}\frac{\prod_{j=1}^{k+1} b_j}{b_l} \\ &= (-1)^{k-1}\left[-b_{k+1}^2\sum_{l=1}^{k}\frac{\prod_{j=1}^{k} b_j}{b_l} + b_{k+1}\sum_{l=1}^{k+1}\frac{\prod_{j=1}^{k+1} b_j}{b_l} + \sum_{l=1}^{k+1}\frac{\prod_{j=1}^{k+2} b_j}{b_l}\right] \\ &= (-1)^{k-1}\left[\frac{\prod_{j=1}^{k+2} b_j}{b_{k+2}} + \sum_{l=1}^{k+1}\frac{\prod_{j=1}^{k+2} b_j}{b_l}\right] = (-1)^{k+1}\sum_{l=1}^{k+2}\frac{\prod_{j=1}^{k+2} b_j}{b_l}. \end{aligned}$$

This proves the induction premise in case $k \leq m-1$. To obtain the formula for case $k = m$, first note that we have

$$D_m = \begin{vmatrix} -b_1-b_2 & b_2 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & b_{m-2} & -b_{m-2}-b_{m-1} & -b_{m-1} & 0 \\ 0 & 0 & \dots & 0 & b_{m-1} & -b_{m-1}-b_m & b_m \\ 0 & 0 & \dots & 0 & 0 & b_m & -b_m \end{vmatrix}.$$

Using Laplace expansion, the determinant can be written as

$$D_m = (-1)^{2m-1}b_m D + (-1)^{2m}(-b_m)D_{m-1} = -b_m D - b_m D_{m-1}$$

Similar to the previous case it can be observed that $D = b_m D_{m-2}$ and consequently $D_m = -b_m^2 D_{m-2} - b_m D_{m-1}$. Using the assumption of induction we have

$$\begin{aligned} D_{k+1} &= -b_m^2(-1)^{m-2}\sum_{l=1}^{m-1}\frac{\prod_{j=1}^{m-1} b_j}{b_l} - b_m(-1)^{m-1}\sum_{l=1}^{m}\frac{\prod_{j=1}^{m} b_j}{b_l} \\ &= (-1)^{m-2}\left[-b_m^2\sum_{l=1}^{m-1}\frac{\prod_{j=1}^{m-1} b_j}{b_l} + b_m^2\sum_{l=1}^{m-1}\frac{\prod_{j=1}^{m-1} b_j}{b_l} + b_m\frac{\prod_{j=1}^{m} b_j}{b_m}\right] = (-1)^m\prod_{j=1}^{m} b_j. \end{aligned}$$

This proves the induction premise in case $k = m$. □

Lemma 10. *Let* $\mathbf{p}^*$ *be the vector of prices obtained via the implicit equations in (21). The Hessian matrix* H *associated with* $\Pi(\mathbf{t})$ *is negative definite at* $\mathbf{p}^*$.

Proof. Considering that all equations in (21) hold, from Lemma 1 it can be inferred that $b_k > 0$ for $k = 1, ..., m$. Given the formulas of leading principal minors of H observe that at $\mathbf{p}^*$, for odd k we have $D_k < 0$ and for even k we have $D_k > 0$. Therefore, $\Pi(\mathbf{t})$ is negative definite at $\mathbf{p}^*$. □

Proof of Theorem 3. Lemmas 8, 9, and 10 establish that $\mathbf{p}^*$, upon existence, is the maximum for $\Pi(\mathbf{t})$. On the other hand, Lemma 6 indicates that given an admissible uptime vector, $\mathbf{p}^*$ always exists. This completes the proof. □

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