A variation of the Azéma martingale and drawdown options

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17th September 2018

Abstract

In this paper, we derive a variation of the Azéma martingale using two approaches - a direct probabilistic method and another by projecting the Kennedy martingale onto the filtration generated by the drawdown duration. This martingale links the time elapsed since the last maximum of the Brownian motion with the maximum process itself. We derive explicit formulas for the joint densities of (τ, W_τ, M_τ), which are the first time the drawdown period hits a pre-specified duration, the value of the Brownian motion, and the maximum up to this time. We use the results to price a new type of drawdown option, which takes into account both dimensions of drawdown risk - the magnitude and the duration.

Keywords: Drawdown duration, Brownian excursions, Local time, Azéma martingale, Drawdown options

1 Introduction

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathcal{P})$ be a filtered probability space and let W be a standard Brownian motion adapted to $(\mathcal{F}_t)_{t\geq 0}$ with $W_0 = 0$. Set

$$g_t := \sup\{s \leq t \mid W_s = 0\},$$

and we define the slow filtration $\mathcal{G} = (\sigma(\text{sgn}(W_s); s \leq t) \vee \mathcal{F}_{g_t})_{t\geq 0}$. Also, we let L_t be the local time at 0 of the Brownian motion. We can obtain $\mathcal{G}_t$-martingales by projecting $\mathcal{F}_t$-martingales onto the filtration

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$\mathcal{G}_t$. In particular, the most well known are Azéma's first and second martingales,

$$\xi_t^1 := \sqrt{\frac{\pi}{2}}\,\mathrm{sgn}(W_t)\sqrt{t-g_t} \tag{1.1}$$

$$\xi_t^2 := \sqrt{t-g_t} - \sqrt{\frac{2}{\pi}}L_t, \tag{1.2}$$

which are the projections of the martingales W_t and $\sqrt{\frac{2}{\pi}}\,(|W_t| - L_t)$ respectively. They are closely related to the excursions of Brownian motion away from 0. They first appeared in Azéma [1] and possesses the chaos representation property (see Yor [26]). One of the first papers to introduce the use of Azéma martingales in finance is Dritschel and Protter [11]. Further, the Azéma martingale is used to construct solutions to the Skorokhod embedding problem for the length and height of excursions (see Obloj and Yor [22]), and to model the default probability of a firm with only information on whether the firm's cash balances are negative or not (see Cetin et al [4]). Filtered Azéma martingales, which are obtained by projecting the Brownian motion onto the signs of another observation process, was studied in Cetin [3]. If we project the exponential martingale $e^{\lambda W_t - \frac{\lambda^2}{2}t}$ onto $\mathcal{G}_t$, we obtain the $\mathcal{G}_t$-martingale,

$$\xi_t^3 := E(e^{\lambda W_t - \frac{\lambda^2}{2}} \mid \mathcal{G}_t) = \Psi(\lambda\,\mathrm{sgn}(W_t)\sqrt{t-g_t})e^{-\frac{\lambda^2}{2}t}, \tag{1.3}$$

where we have used the notation

$$\Psi(x) := 1 + x\sqrt{2\pi}e^{\frac{x^2}{2}}\mathcal{N}(x).$$

This martingale was used in Chesney et al [5] to price Parisian barrier options.

We are interested in studying the maximum and drawdown processes, so we define the following notations for the maximum process,

$$M_t := \max_{s\le t} W_s,$$

and the drawdown process

$$Y_t := M_t - W_t.$$

We denote by U_t the time elapsed since the last time the maximum was achieved, or the drawdown period,

$$U_t := t - \sup\{s \le t \mid M_t = W_s\} = t - \sup\{s \le t \mid Y_t = 0\}.$$

In this paper, we prove that the process ν_t defined by

$$\nu_t := e^{-\gamma M_t} e^{-\beta t} \left(1 + \gamma \sqrt{\frac{\pi U_t}{2}} e^{\beta U_t} + \sqrt{U_t} \beta e^{\beta U_t} \int_0^{U_t} e^{-\beta w} \frac{1}{\sqrt{w}} dw \right),$$

is an $\mathcal{F}_{\mathcal{U}}$-martingale, where $\mathcal{F}_{\mathcal{U}}$ is the filtration generated by $(\mathcal{U}_t)_{t\geq 0}$. The filtration shrinkage introduced here is along the same lines as Sezer [24] who considers it for more general diffusions and multiple crossing points. The martingale ν_t is a function of (M_t, U_t), the running maximum and the time elapsed since the last maximum. Furthermore, by the reflection principle of a standard Brownian motion, $M \overset{D}{=} Y \overset{D}{=} |W|$. Hence, the excursions away from 0 are also the excursions of the Brownian motion away from its maximum. Lévy's identity also states that if L_t is the local time of the Brownian motion W_t at 0, then

$$(|W_t|, L_t, t \geq 0) \overset{law}{=} (M_t - W_t, M_t, t \geq 0).$$

Hence this martingale can also be written in terms of the pair $(L_t, t - g_t)$. This martingale shares a common construction with the Azéma martingales, as it is also a projection of the Kennedy martingale (see Kennedy [17]) onto the filtration generated by $(U_t)_{t\geq 0}$.

We will focus on the representation of the martingale ν_t as a process involving M_t and U_t. In particular, we are interested in looking at the drawdown period U_t and the first time the drawdown period hits a certain length D. Hence, we define the Parisian drawdown time τ_D as

$$\tau_D := \inf\{t \geq 0 \mid U_t = D\}.$$

This is the first time the duration of drawdowns exceeds a certain threshold $D > 0$. Without loss of generality, we let $D = 1$ and simplify the notation by setting $\tau := \tau_1$. In Landriault et al [19], the Laplace transform of the Parisian drawdown time for spectrally negative Lévy processes was obtained using a perturbation approach. Furthermore, we note that for a standard Brownian motion, due to Lévy's identity and the symmetry of Brownian motion, its Parisian drawdown time is equal in distribution to its two-sided Parisian stopping time, which is the first time the length of the excursions around 0 for a standard Brownian motion exceed a certain threshold D. The Parisian stopping time of Brownian motion has been well studied in the literature. For example, the joint Laplace transform of the Parisian stopping time and the value of the Brownian motion at the time is derived in Dassios and Wu [7], and the density of the Parisian stopping time is obtained in Dassios and Lim [8, 9]. Czarna and

Palmowski [6] also studied the Parisian stopping time for a spectrally negative Lévy process. Here, we derive a recursive formula for the joint density of the triplet (τ, M_τ, W_τ) for a standard Brownian motion, and then use Girsanov's theorem to derive a similar result for a drifted Brownian motion.

Drawdowns are a measure of an investment's financial risk. Both the Calmar and Sterling ratios use drawdown as a metric (see Eling [13] for a description of these drawdown measures). Large market drawdowns lead to portfolio losses and liquidity shocks, and several studies have examined the maximum drawdown $MDD_t = \max_{s \le t} Y_s$. This is equivalent to the study of the stopping time

$$T_a := \inf\{t \ge 0 \mid Y_t > a\},$$

which is the first time the magnitude of drawdowns exceeds a certain threshold $a > 0$. Taylor [25] first derived the Laplace transform of T_a and M_{T_a} for Brownian motion processes. The distributional properties of T_a and M_{T_a} are also studied in Douady et al [10] and Magdon-Ismail et al [20]. Furthermore, studies that involve both the drawdown and drawup processes have been done by Popisil et al [23], Zhang and Hadjiliadis [27] and Gapeev and Rodosthenous [14]. This stopping time is commonly referred to as drawdown time in the literature but we note the distinction between this and τ_D. The magnitude of drawdowns is, however, insufficient as a measure of risk, as a prolonged drawdown period can also lead to high sustained losses. In portfolio management, investors usually quit an investment fund after either a single large drawdown or a small but prolonged drawdown. Thus it is useful to also consider the duration of each drawdown. Landriault et al [18] recently studied the frequency of drawdowns for a Brownian motion, and in [19], the duration of drawdowns for Lévy models.

In this paper, we apply our results to the pricing of drawdown options. These are financial derivatives based on the drawdown process, which have drawn much attention in recent years. Drawdown insurance was first introduced in Carr et al [2]. They proposed in their paper two kinds of digital drawdown insurance, the first one has payoff conditional on the maximum drawdown amount hitting a strike K, and the second conditional on the drawdown amount hitting level K before the drawup amount does. Zhang and Hadjiliadis [28] introduced a new variable called the speed of market crash, which is U_{T_a} in our notation. It is the time elapsed between the last maximum of the process and the first time the drawdown hits a certain level, and is a measure of how fast the crash has occurred.

They priced claims based on that payout only when the drawdown time T_a is reached, and the speed of market crash is smaller than a certain strike. Zhang et al. [29] introduced a drawdown insurance contract whereby the protection buyer pays a constant premium over time to insure against a drawdown of a pre-specified amount, with features such as early cancellation and drawup contingencies. Zhang [30] studied the law of the occupation times of the drawdown and drawup processes and used the results to price cumulative Parisian options and α-quantile options on the drawdown process. American options on maximum drawdown were also considered in the recent paper by Gapeev and Rodosthenous [15].

Here, we propose three new variations of drawdown options. The first is a digital drawdown call option which pays off a unit amount when the drawdown duration hits a prespecified qualifying period of length $D = 1$, given that this happens before maturity time T, and that the drawdown amount exceeds a prespecified strike K. The second is a drawdown barrier bond with qualifying period, which pays off the value of the drawdown amount, when the drawdown duration hits a prespecified length $D = 1$, if it occurs before maturity time T and given that the drawdown amount exceeds a prespecified strike K. Finally, we describe the Parisian drawdown call option, which is a Parisian option with the drawdown process as the underlying. In all of these options, the payoff is made at time τ_D, the time when the Parisian drawdown time is first hit. These options provide insurance against both the magnitude and the duration of the drawdown.

This paper will be organised as follows. Section 2 presents the proof of the extended Azéma martingale. We also demonstrate that it is the projection of the Kennedy martingale on the filtration $\mathcal{F}_U$. Section 3 derives the joint density of the triplet (τ, W_τ, M_τ) for the standard and drifted Brownian motion. In Section 4, we introduce new types of drawdown options, and show how the results can be applied to price these options. Section 5 concludes the paper.

2 A variation of the Azéma martingale

In this section, we introduce the following martingale and demonstrate that it is also the projection of the Kennedy martingale onto the filtration $\mathcal{F}_U$.

Theorem 2.1 *Let ν_t be the process defined by*

$$\nu_t := e^{-\gamma M_t} e^{-\beta t} \left(1 + \gamma \sqrt{\frac{\pi U_t}{2}} e^{\beta U_t} + \sqrt{U_t} \beta e^{\beta U_t} \int_0^{U_t} e^{-\beta w} \frac{1}{\sqrt{w}} dw \right).$$

Then ν_t is an $\mathcal{F}_U$-martingale, where $\mathcal{F}_U$ is the filtration generated by $(U_t)_{t \geq 0}$.

Proof. We start by considering a martingale of the form

$$e^{-\gamma M_t} e^{-\beta t} f(U_t).$$

We need to find an integrable function f with $f(0) = 1$ (without loss of generality), such that

$$E\left(e^{-\gamma M_{t+h}} e^{-\beta(t+h)} f(U_{t+h}) \mid U_t, M_t\right) = e^{-\gamma M_t - \beta t} f(U_t), \tag{2.1}$$

for all $t \geq 0$ and $h > 0$. In particular, we need

$$E\left(e^{-\gamma M_t - \beta t} f(U_t) \mid U_0 = 0, M_0 = 0\right) = 1. \tag{2.2}$$

Using the joint density of M_t and U_t (Karatzas and Shreve [16]), this is equivalent to finding f such that

$$\int_0^t \int_0^\infty e^{-\gamma y} f(v) \frac{y}{\pi \sqrt{v} (t-v)^{3/2}} e^{-\frac{y^2}{2(t-v)}} dy dv = e^{\beta t}.$$

Now, set $g(u) = \frac{f(u)}{\sqrt{u}}$, so that we need to find g for which

$$\int_0^t \int_0^\infty e^{-\gamma y} g(v) \frac{y}{\pi (t-v)^{3/2}} e^{-\frac{y^2}{2(t-v)}} dy dv = e^{\beta t}.$$

Taking Laplace transforms over t, we have

$$\hat{g}(\epsilon) \int_0^\infty \sqrt{\frac{2}{\pi}} e^{-\gamma y} e^{-y\sqrt{2\epsilon}} dy = \frac{1}{\epsilon - \beta},$$

for $\epsilon > \beta$. Hence, we have

$$\hat{g}(\epsilon) \frac{1}{\gamma + \sqrt{2\epsilon}} = \frac{1}{\epsilon - \beta} \sqrt{\frac{\pi}{2}},$$

and thus

$$\hat{g}(\epsilon) = \sqrt{\frac{\pi}{2}} \frac{\gamma + \sqrt{2\epsilon}}{\epsilon - \beta} = \sqrt{\frac{\pi}{2}} \frac{\gamma}{\epsilon - \beta} + \epsilon \frac{\sqrt{\pi}}{\sqrt{\epsilon}(\epsilon - \beta)}.$$

Inverting the Laplace transforms, we have

$$\begin{aligned} g(u) &= \gamma\sqrt{\frac{\pi}{2}}e^{\beta u} + \frac{d}{du}\left(\int_0^u \frac{1}{\sqrt{w}}e^{\beta(u-w)}dw\right) \\ &= \gamma\sqrt{\frac{\pi}{2}}e^{\beta u} + \frac{1}{\sqrt{u}} + \beta e^{\beta u}\int_0^u e^{-\beta w}\frac{1}{\sqrt{w}}dw, \end{aligned}$$

and thus

$$f(u) = 1 + \gamma\sqrt{\frac{\pi u}{2}}e^{\beta u} + \sqrt{u}\beta e^{\beta u}\int_0^u e^{-\beta w}\frac{1}{\sqrt{w}}dw \tag{2.3}$$

would satisfy equation (2.2). We now have a candidate for f and it remains now to verify (2.1). We need to prove

$$E\left(e^{-\gamma M_{t+h}}e^{-\beta(t+h)}f(U_{t+h}) \mid U_t = u, M_t = m\right) = e^{-\gamma m}e^{-\beta t}f(u),$$

which is equivalent to

$$e^{-\beta h}f(u+h)\frac{\sqrt{u}}{\sqrt{u+h}} + \int_0^h e^{-\beta s}\frac{\sqrt{u}}{2(u+s)^{3/2}}E\left(e^{-\gamma M_{h-s}}e^{-\beta(h-s)}f(U_{h-s}) \mid U_0 = 0, M_0 = 0\right)ds = f(u).$$

Rearranging the left hand side, we have

$$\begin{aligned} f(u) &= e^{-\beta h}f(u+h)\frac{\sqrt{u}}{\sqrt{u+h}} + \int_0^h e^{-\beta s}\frac{\sqrt{u}}{2(u+s)^{3/2}}ds \\ &= e^{-\beta h}f(u+h)\frac{\sqrt{u}}{\sqrt{u+h}} + 1 - e^{-\beta h}\frac{\sqrt{u}}{\sqrt{u+h}} - \beta\int_0^h e^{-\beta s}\frac{\sqrt{u}}{\sqrt{u+s}}ds \\ &= e^{-\beta h}f(u+h)\frac{\sqrt{u}}{\sqrt{u+h}} + 1 - e^{-\beta h}\frac{\sqrt{u}}{\sqrt{u+h}} - \beta e^{\beta u}\int_u^{u+h} e^{-\beta w}\frac{\sqrt{u}}{\sqrt{w}}dw \\ &= e^{-\beta h}f(u+h)\frac{\sqrt{u}}{\sqrt{u+h}} + 1 - e^{-\beta h}\frac{\sqrt{u}}{\sqrt{u+h}} + \beta e^{\beta u}\int_0^u e^{-\beta w}\frac{\sqrt{u}}{\sqrt{w}}dw - \beta e^{\beta u}\int_0^{u+h} e^{-\beta w}\frac{\sqrt{u}}{\sqrt{w}}dw. \end{aligned}$$

Finally, this simplifies to

$$e^{-\beta(u+h)}\frac{f(u+h)-1}{\sqrt{u+h}} - \beta\int_0^{u+h}\frac{e^{-\beta w}}{\sqrt{w}}dw = e^{-\beta u}\frac{f(u)-1}{\sqrt{u}} - \beta\int_0^u \frac{e^{-\beta w}}{\sqrt{w}}dw,$$

which is true by (2.3). Finally, as $\mathcal{F}_{M_t} \subseteq \mathcal{F}_{U_t}$, ν_t is an $\mathcal{F}_U$-martingale. ■

Corollary 2.2 *For $\beta = 0$, we have*

$$\nu_t^1 := e^{-\gamma M_t}\left(\gamma\sqrt{\frac{\pi U_t}{2}} + 1\right)$$

is an $\mathcal{F}_U$-martingale.

Corollary 2.3 *For $\gamma = 0$, we obtain a martingale similar to ξ_t^3 in (1.1),*

$$\nu_t^2 := e^{-\beta t}\left(1 + \sqrt{U_t}\beta e^{\beta U_t}\int_0^{U_t} e^{-\beta w}\frac{1}{\sqrt{w}}dw\right).$$

Using the notation in (1.1) with $\lambda = \sqrt{2\beta}$, this martingale can be rewritten as

$$\nu_t^2 = \frac{\Psi\left(\sqrt{2\beta U_t}\right) - \Psi\left(-\sqrt{2\beta U_t}\right)}{2}e^{-\beta t}.$$

It is thus the two-sided version of ξ_t^3.

Remark 2.4 (A projection of the Kennedy martingale onto the filtration $\mathcal{F}_U$) *The Kennedy martingale was introduced by Kennedy [17] and more recently studied in Nguyen-Ngoc and Yor [21]. It is given by*

$$\zeta_t := e^{-\gamma M_t - \beta t}\left(\sqrt{2\beta}\cosh\left(\sqrt{2\beta}|Y_t|\right) + \gamma\sinh\left(\sqrt{2\beta}|Y_t|\right)\right). \tag{2.4}$$

Projecting this martingale onto the filtration $\mathcal{F}_{U_t}$, and using that $M_t \in \mathcal{F}_{U_t}$, and that $|Y_t|$ given $\mathcal{F}_{U_t}$, has the law of a Brownian meander, we have

$$E\left(e^{-\gamma M_t - \beta t}\left(\sqrt{2\beta}\cosh\left(\sqrt{2\beta}|Y_t|\right) + \gamma\sinh\left(\sqrt{2\beta}|Y_t|\right)\right) \mid \mathcal{F}_{U_t}\right) \tag{2.5}$$

$$= e^{-\gamma M_t - \beta t}\int_0^\infty \frac{x}{U_t}e^{-\frac{x^2}{2U_t}}\left(\sqrt{2\beta}\cosh\left(\sqrt{2\beta}x\right) + \gamma\sinh\left(\sqrt{2\beta}x\right)\right)dx \tag{2.6}$$

$$= \sqrt{2\beta}e^{-\gamma M_t - \beta t}\left(1 + \sqrt{2\beta}\int_0^\infty e^{-\frac{x^2}{2U_t}}\left(\frac{e^{\sqrt{2\beta}x} - e^{-\sqrt{2\beta}x}}{2}\right)dx + \gamma\int_0^\infty e^{-\frac{x^2}{2U_t}}\left(\frac{e^{\sqrt{2\beta}x} + e^{-\sqrt{2\beta}x}}{2}\right)dx\right) \tag{2.7}$$

$$= \sqrt{2\beta}e^{-\gamma M_t - \beta t}\left(1 + \beta\sqrt{U_t}e^{\beta U_t}\int_0^{U_t} e^{-\beta w}\frac{1}{\sqrt{w}}dw + \gamma\sqrt{\frac{\pi U_t}{2}}e^{\beta U_t}\right), \tag{2.8}$$

where integration by parts was used from (2.6) *to* (2.7), *and from* (2.7) *to* (2.8) *is a result of*

$$\int_0^\infty e^{-\frac{x^2}{2U_t}}e^{\sqrt{2\beta}x}\mathrm{d}x = e^{\beta U_t}\int_0^{\sqrt{2\beta}U_t} e^{-\frac{(x-\sqrt{2\beta}U_t)^2}{2U_t}}\mathrm{d}x + e^{\beta U_t}\int_{\sqrt{2\beta}U_t}^\infty e^{-\frac{(x-\sqrt{2\beta}U_t)^2}{2U_t}}\mathrm{d}x \tag{2.9}$$

$$= \sqrt{\frac{\beta U_t}{2}}e^{\beta U_t}\int_0^{U_t} e^{-\beta w}\frac{1}{\sqrt{w}}\mathrm{d}w + \sqrt{\frac{\beta U_t}{2}}e^{\beta U_t}\int_0^\infty e^{-\beta w}\frac{1}{\sqrt{w}}\mathrm{d}w, \tag{2.10}$$

and

$$\int_0^\infty e^{-\frac{x^2}{2U_t}}e^{-\sqrt{2\beta}x}\mathrm{d}x = e^{\beta U_t}\int_0^\infty e^{-\frac{(x+\sqrt{2\beta}U_t)^2}{2U_t}}\mathrm{d}x \tag{2.11}$$

$$= \sqrt{\frac{\beta U_t}{2}} e^{\beta U_t} \int_{U_t}^{\infty} e^{-\beta w} \frac{1}{\sqrt{w}} \mathrm{d}w, \tag{2.12}$$

so that

$$\int_0^{\infty} e^{-\frac{x^2}{2U_t}} \left(e^{\sqrt{2\beta} x} - e^{-\sqrt{2\beta} x} \right) dx = \sqrt{2\beta U_t} e^{\beta U_t} \int_0^{U_t} e^{-\beta w} \frac{1}{\sqrt{w}} \mathrm{d}w, \tag{2.13}$$

$$\int_0^{\infty} e^{-\frac{x^2}{2U_t}} \left(e^{\sqrt{2\beta} x} + e^{-\sqrt{2\beta} x} \right) dx = \sqrt{2\beta U_t} e^{\beta U_t} \int_0^{\infty} e^{-\beta w} \frac{1}{\sqrt{w}} \mathrm{d}w = \sqrt{2\pi U_t} e^{\beta U_t}. \tag{2.14}$$

Hence, the projection of the Kennedy martingale onto the filtration $\mathcal{F}_U$ gives us exactly the martingale ν_t which we defined in Theorem 2.1, up to a constant.

3 Joint density of τ, W_τ and M_τ

Now, applying the optional stopping theorem on this martingale, we obtain the joint Laplace transform of the stopping time τ and the maximum of the Brownian up to this time, M_τ. Inverting the Laplace transform gives us the following joint density.

Theorem 3.1 *Let $f(t,m)$ be the joint density of the stopping time τ, and the maximum up to this time, M_τ. Then we have the following formula for the joint density*

$$f(t,m) = \sqrt{\frac{2}{\pi}} \sum_{k=0}^{n-1} \frac{1}{k!} \left(-\sqrt{\frac{2}{\pi}} m \right)^k L_k(m, t-1), \qquad \text{for } n < t \leq n+1, n = 1, 2, ...,$$

for $m \geq 0$, where the functions $L_k(m,t)$ are defined recursively below

$$L_0(m,t) = \frac{m}{\sqrt{2\pi t^3}} e^{-\frac{m^2}{2t}}, \qquad \text{for } t > 0,$$

$$L_k(m,t) = \int_1^{t-k} L_k(m, t-s) \frac{1}{2s^{3/2}} ds \qquad \text{for } t > k+1.$$

Furthermore, the joint density of the stopping time τ, maximum M_τ, and Brownian motion W_τ is

$$\mathcal{P}\left(\tau \in dt, W_\tau \in dx, M_\tau \in dm\right) = (m-x) e^{-\frac{(m-x)^2}{2}} f(t,m) dt dm,$$

for $t \geq 1$, $x \leq m$.

Proof. As $U_{t\wedge\tau} \le 1$, $|\nu_{t\wedge\tau}| \le K$ for some constant K a.s., for all t. Hence, the optional stopping theorem applies. As $U_\tau = 1$ (we have let $D = 1$), we have

$$E\left(e^{-\beta\tau}e^{-\gamma M_\tau}\right) = \frac{1}{1+\gamma\sqrt{\frac{\pi}{2}}e^{\beta}+\beta e^{\beta}\int_0^1 e^{-\beta w}\frac{1}{\sqrt{w}}dw}.$$

We use the same method of Laplace inversion which was used in Dassios and Lim [8, 9] to obtain the densities of the one and two-sided Parisian stopping times. Rearranging the expression, we have

$$\begin{aligned}
E\left(e^{-\beta\tau}e^{-\gamma M_\tau}\right) &= \frac{e^{-\beta}}{\gamma\sqrt{\frac{\pi}{2}}+2\sqrt{\pi\beta}\int_0^{\sqrt{2\beta}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx+e^{-\beta}} \\
&= \frac{e^{-\beta}}{\gamma\sqrt{\frac{\pi}{2}}+\sqrt{\pi\beta}+\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds} \\
&= \frac{e^{-\beta}}{\left(\sqrt{\pi\beta}+\gamma\sqrt{\frac{\pi}{2}}\right)\left(1+\frac{1}{\sqrt{\pi\beta}+\gamma\sqrt{\frac{\pi}{2}}}\int_1^\infty\frac{e^{\beta s}}{2s^{3/2}}ds\right)} \\
&= e^{-\beta}\frac{1}{\sqrt{\pi\beta}+\gamma\sqrt{\frac{\pi}{2}}}\sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{\sqrt{\pi\beta}+\gamma\sqrt{\frac{\pi}{2}}}\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds\right)^k \\
&= e^{-\beta}\sum_{k=0}^{\infty}(-1)^k\left(\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds\right)^k\left(\sqrt{\frac{2}{\pi}}\right)^{k+1}\left(\frac{1}{\gamma+\sqrt{2\beta}}\right)^{k+1},
\end{aligned}$$

where the infinite series expansion is valid for a sufficiently large β. Inverting the joint Laplace transform with respect to γ, we have

$$\begin{aligned}
E\left(e^{-\beta\tau};M_\tau\in dm\right) &= e^{-\beta}\sum_{k=0}^{\infty}(-1)^k\left(\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}\right)^k\left(\sqrt{\frac{2}{\pi}}\right)^{k+1}e^{-\sqrt{2\beta}m}\frac{m^k}{\Gamma(k+1)}dm &(3.1)\\
&= e^{-\beta}\sqrt{\frac{2}{\pi}}e^{-\sqrt{2\beta}m}\sum_{k=0}^{\infty}\frac{1}{k!}\left(-\sqrt{\frac{2}{\pi}}m\right)^k\left(\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds\right)^k dm. &(3.2)
\end{aligned}$$

Now, it remains to invert the Laplace transform,

$$\hat{L}_k(m,\beta) = e^{-\sqrt{2\beta}m}\left(\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds\right)^k. \tag{3.3}$$

We have the following Laplace inversions with respect to β,

$$\mathcal{L}^{-1}\left(e^{-\sqrt{2\beta}m}\right) = \frac{m}{\sqrt{2\pi t^3}}e^{-\frac{m^2}{2t}}, \tag{3.4}$$

$$\mathcal{L}^{-1}\left(\int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}}ds\right) = \frac{1}{2t^{3/2}}\mathbf{1}_{\{t>1\}}. \tag{3.5}$$

Inverting (3.2) with respect to β, we have that the k^{th} term in the summation, $L_k(m,t)$, is the k^{th} convolution of (3.4), and $L_0(m,t)$ is the RHS of (3.2). We also note that for $n < t < n+1$, $L_k(m,t)$ is zero for $k > n$ due to the indicator $\mathbf{1}_{\{t>1\}}$ in (3.4), so we only need a finite sum up to n. This gives us a recursive formula for the joint density of τ and M_τ.

Finally, to obtain the trivariate density of τ, M_τ and W_τ, we observe that given $M_\tau = m$ is the maximum and is attained at time $\tau - 1$, $(W_{(\tau-1)+t)})_{0\le t\le 1}$ is a Brownian motion starting at m, conditioned to stay below m. Hence, given τ and M_τ, $(M_\tau - W_{(\tau-1)+t})_{0\le\tau\le 1}$ is a Brownian motion starting at 0, conditioned to stay positive and independent of τ. It has the same law as the Brownian meander (see Durrett et al [12]),

$$\begin{aligned}\mathcal{P}(M_\tau - W_\tau \in dy \mid M_\tau = m, \tau = t) &= \mathcal{P}(W_1 \in dy \mid W_s \ge 0 \ \forall \ s \le 1) \\ &= ye^{-\frac{y^2}{2}}dy, \ \ y \ge 0.\end{aligned}$$

Combining this with the joint density of τ and M_τ, we obtain the result. ■

Using Girsanov's Theorem, we can also obtain a similar result for Brownian motion with drift, which we denote by W^μ.

Theorem 3.2 *We denote by τ^μ, $M^\mu_{\tau^\mu}$ the Parisian drawdown time and the running maximum for a Brownian motion with drift. Then we have*

$$\mathcal{P}\left(\tau^\mu \in dt, M^\mu_{\tau^\mu} \in dm\right) = e^{\mu m}e^{-\frac{1}{2}\mu^2 t}f(t,m)\left(1 - \sqrt{2\pi}\mu e^{\frac{\mu^2}{2}}\mathcal{N}(-\mu)\right)dtdm.$$

Furthermore, the joint density of τ^μ, $M^\mu_{\tau^\mu}$ and $W^\mu_{\tau^\mu}$ is

$$\mathcal{P}\left(\tau^\mu \in dt, W^\mu_{\tau^\mu} \in dx, M^\mu_{\tau^\mu} \in dm\right) = e^{\mu m}e^{-\frac{1}{2}\mu^2(t-1)}f(t,m)(m-x)e^{-\frac{(m-x+\mu)^2}{2}}dtdm.$$

Proof. From Theorem 2.4, we have for dt and dm in some infinitesimal interval,

$$E\left(\mathbf{1}_{\{M_t\in dm;\tau\in dt\}}\right) = f(t,m)dtdm.$$

Applying Girsanov with a change of measure

$$\left.\frac{d\mathcal{P}^{\mu}}{d\mathcal{P}}\right|_{\mathcal{F}_t} = e^{\mu W_t - \frac{1}{2}\mu^2 t}$$

such that $W_t^{\mu} = W_t - \mu t$ is a $\mathcal{P}^{\mu}$-Brownian motion. We have

$$\mathcal{P}^{\mu}\left(M_t \in dm; \tau \in dt\right) = \frac{e^{-\frac{1}{2}\mu^2 t} f(t,m) dt dm}{E^{\mu}\left(e^{-\mu W_t} \mid M_t \in dm; \tau \in dt\right)},$$

where $E^{\mu}(\cdot)$ denotes the expectation under the measure $\mathcal{P}^{\mu}$. Now, under $\mathcal{P}^{\mu}$, given τ and M_{τ}, W_t depends on a Brownian meander with drift μ, so that

$$\mathcal{P}^{\mu}\left(W_t \in dx \mid W_s \geq 0 \;\; \forall \;\; s \leq 1\right) = \frac{\frac{x}{\sqrt{2\pi}} e^{-\frac{(x+\mu)^2}{2}} dx}{\frac{1}{\sqrt{2\pi}} e^{-\frac{\mu^2}{2}} - \mu \mathcal{N}(-\mu)}. \tag{3.6}$$

Hence, we have

$$E^{\mu}\left(e^{-\mu W_t} \mid \tau = t; M_t = m\right) = \frac{\frac{1}{\sqrt{2\pi}} e^{-\mu m} e^{-\frac{\mu^2}{2}}}{\frac{1}{\sqrt{2\pi}} e^{-\frac{\mu^2}{2}} - \mu \mathcal{N}(-\mu)}, \tag{3.7}$$

and

$$\mathcal{P}^{\mu}\left(\tau \in dt; M_t \in dm\right) = e^{\mu m} e^{-\frac{1}{2}\mu^2 t} f(t,m)\left(1 - \sqrt{2\pi} e^{\frac{\mu^2}{2}} \mu \mathcal{N}(-\mu)\right) dt dm.$$

Multiplying this by the density of the Brownian meander with drift gives us the trivariate density of τ^{μ}, W_{τ}^{μ}, and M_{τ}^{μ}, for a Brownian motion with drift μ. ■

4 Applications: Pricing of drawdown options

In this section, we give some examples of how the above results can be used to price new types of drawdown options designed to hedge against large drawdowns over a prolonged period. Expressions of the option prices in terms of a single integral are obtained. We assume the underlying asset S follows a geometric brownian motion. Let $\mathcal{Q}$ denote the risk neutral probability measure. The dynamics of S under Q is

$$dS_t = S_t(r dt + \sigma dW_t), S_0 = x,$$

so that

$$S_t = S_0 e^{\sigma W_t^{\mu}},$$

where

$$
\begin{aligned}
W_t^\mu &:= \mu t + W_t, \\
\mu &:= \frac{1}{\sigma}(r - \frac{\sigma^2}{2}).
\end{aligned}
$$

We also define

$$
\begin{aligned}
M_t^\mu &:= \max_{0 \le s \le t} W_s^\mu, \\
\overline{S}_t &:= \max_{0 \le s \le t} S_s = S_0 e^{\sigma M_t^\mu}, \\
U_t^S &:= t - \sup\{0 \le s \le t \mid \overline{S}_t = S_s\} = U_t, \\
\tau^S &:= \inf\{t \ge 0 \mid U_t^S = 1\}.
\end{aligned}
$$

We note that $U_t^S = U_t$ a.s., and hence the first time the drawdown period of S reaches length 1 is also equal to τ^μ.

4.1 Digital drawdown call option with qualifying period

The digital drawdown call option with qualifying period is an option which pays off a unit amount when the drawdown period reaches the qualifying period 1, if this happens before fixed maturity T, but only if the size of drawdown at this stopping time is larger than a prespecified K. The payout of the option is made at τ^μ, immediately after the qualifying period is reached. This provides an insurance against a prolonged drawdown, if the drawdown amount is large. Specifically, under the risk neutral measure $\mathcal{Q}$, and denoting by $C_1(K,T)$ the no-arbitrage price of this option, we have

$$
\begin{aligned}
C_1(K,T) &= E_{\mathcal{Q}}\left(e^{-r\tau^\mu}\mathbf{1}_{\{\tau^\mu \le T\}}\mathbf{1}_{\{\overline{S}_{\tau^\mu} - S_{\tau^\mu} \ge K\}}\right) \\
&= E_{\mathcal{Q}}\left(e^{-r\tau^\mu}\mathbf{1}_{\{\tau^\mu \le T\}}\mathbf{1}_{\{S_0(e^{\sigma M_\tau^\mu} - e^{\sigma W_\tau^\mu}) \ge K\}}\right) \\
&= E_{\mathcal{Q}}\left(e^{-r\tau^\mu}\mathbf{1}_{\{\tau^\mu \le T\}}\mathbf{1}_{\{W_\tau^\mu \le \frac{1}{\sigma}\ln(e^{\sigma M_\tau^\mu} - \frac{K}{S_0})\}}\right),
\end{aligned}
$$

where $E_{\mathcal{Q}}(\cdot)$ denotes the expectation under the risk-neutral measure $\mathcal{Q}$. Using the explicit expression for the joint density from Theorem 2.5, we obtain

$$
C_1(K,T) = \int_0^T \int_{m=0}^\infty \int_{x=-\infty}^{\frac{1}{\sigma}\ln(e^{\sigma m} - \frac{K}{S_0})} e^{-rt} e^{\mu m} e^{-\frac{1}{2}\mu^2(t-1)} f(t,m)(m-x) e^{-\frac{(m-x+\mu)^2}{2}} dx dm dt
$$

$$= \int_0^T \int_{m=0}^{\infty} e^{-rt} e^{\mu m} e^{-\frac{1}{2}\mu^2(t-1)} f(t,m) \left(e^{-\frac{k_m^2}{2}} - \mu\sqrt{2\pi}\mathcal{N}(-k_m) \right) dmdt,$$

where to simplify notations, we have

$$k_m := m + \mu - \frac{1}{\sigma} \ln \left(e^{\sigma m} - \frac{K}{S_0} \right).$$

Once the density is computed using the recursion in Theorem 3.1, simple numerical integration will then give us the price of the option. In the table below, we have computed the prices of the digital options for parameters $r = 0.05$, $\sigma = 0.2$, $S_0 = 100$, window length $D = 1$ year and $1/2$ year, and a range of values of strike K and maturity T.

Table 1: Price of Digital drawdown option, $P_{digital}(K, T)$

	$D = 1$ year			$D = 1/2$ year		
K	$T = 3$	$T = 5$	$T = 10$	$T = 3$	$T = 5$	$T = 10$
50	0.0296580	0.0636077	0.0833472	0.00756906	0.0113523	0.0117774
40	0.0920706	0.147000	0.171030	0.0294364	0.0355767	0.0360833
30	0.232098	0.312667	0.340787	0.118276	0.127419	0.128002
20	0.439482	0.545175	0.576661	0.368242	0.380481	0.381129
10	0.627703	0.751485	0.785158	0.725585	0.740162	0.740852

For small values of K, the options with the same maturity increase in value when the window length is shortened, as it is easier to achieve the short drawdown period D. However, for large values of K, there is a trade-off between having a shorter drawdown period or window length, and achieving a drawdown of K within this short time frame. Hence the option value decreases with the window length for larger K.

4.2 Drawdown barrier bond with qualifying period

The second derivative we discuss here is a drawdown barrier bond with qualifying period. The bondholder receives an amount proportional to the drawdown amount, when the drawdown period reaches the qualifying period length 1, if this happens before maturity T, again only if the size of the drawdown at this stopping time is larger than a prespecified K, and this payout is made at τ^μ, immediately after the qualifying period is reached. The digital drawdown call option in the previous section pays off a unit amount but the payoff for this bond is proportional to the drawdown amount.

The payoff of this bond at time τ^μ is

$$\text{Payoff} = \left(\overline{S}_{\tau^\mu} - S_{\tau^\mu}\right) \mathbf{1}_{\{\tau^\mu \le T\}} \mathbf{1}_{\{\overline{S}_{\tau^\mu} - S_{\tau^\mu} \ge K\}}.$$

We denote the price of this option by $C_2(K, T)$ and using the same method as before, we have

$$\begin{aligned} C_2(K,T) &= E_Q\left(e^{-r\tau^\mu}(\overline{S}_{\tau^\mu} - S_{\tau^\mu})\mathbf{1}_{\{\tau^\mu \le T\}}\mathbf{1}_{\{\overline{S}_{\tau^\mu} - S_{\tau^\mu} \ge K\}}\right) \\ &= \int_0^T \int_{m=0}^\infty \int_{x=-\infty}^{\frac{1}{\sigma}\ln(e^{\sigma m} - \frac{K}{S_0})} e^{-rt} e^{\mu m} e^{-\frac{1}{2}\mu^2(t-1)} f(t,m)(m-x)^2 e^{-\frac{(m-x+\mu)^2}{2}} dx dm dt \\ &= \int_0^T \int_{m=0}^\infty e^{-rt} e^{\mu m} e^{-\frac{1}{2}\mu^2(t-1)} f(t,m)\psi(k_m,\mu) dm dt, \end{aligned}$$

where we denote

$$\psi(k_m, \mu) := \left(k_m e^{-\frac{k_m^2}{2}} + \sqrt{2\pi}\mathcal{N}(-k_m)(1+\mu^2) - 2\mu e^{-\frac{k_m^2}{2}}\right).$$

We compute the prices of the digital options for parameters $r = 0.05$, $\sigma = 0.2$, $S_0 = 100$, window length $D = 1$ year and $1/2$ year, and a range of values of strike K and maturity T.

Table 2: Price of drawdown option with qualifying period, $P(K, T)$

	$D = 1$ year			$D = 1/2$ year		
K	$T = 3$	$T = 5$	$T = 10$	$T = 3$	$T = 5$	$T = 10$
50	0.0728402	0.137864	0.168582	0.0197508	0.0271523	0.0278135
40	0.203153	0.298327	0.333323	0.0729139	0.0838260	0.0845650
30	0.435782	0.560098	0.598316	0.262627	0.277132	0.277930
20	0.680480	0.825283	0.865409	0.663211	0.680418	0.681252
10	0.813264	0.967000	1.007882	1.013994	1.032454	1.033304

This option has a higher payoff than the digital option of the same maturity, window length and strike, hence it is more expensive. As in the case of Parisian stopping times, the recursion in Theorem 3.1 works particularly well for small values of t. Hence, the computation is fast and particularly efficient for small values of maturity T relative to the window length D.

4.3 Parisian drawdown call option

Finally, we discuss the drawdown call option with qualifying period, which is equivalent to a Parisian call option with the drawdown process as the underlying. The payoff occurs at time τ^μ, and is

$$\text{Payoff} = \left(\overline{S}_{\tau^\mu} - S_{\tau^\mu} - K\right)^+ \mathbf{1}_{\{\tau^\mu \le T\}}.$$

This is a combination of the digital drawdown call and the drawdown barrier bond, because

$$\left(\overline{S}_{\tau^\mu} - S_{\tau^\mu} - K\right)^+ \mathbf{1}_{\{\tau^\mu \le T\}} = \left(\overline{S}_{\tau^\mu} - S_{\tau^\mu}\right) \mathbf{1}_{\{\tau^\mu \le T\}} \mathbf{1}_{\{\overline{S}_{\tau^\mu} - S_{\tau^\mu} \ge K\}} - K \mathbf{1}_{\{\tau^\mu \le T\}} \mathbf{1}_{\{\overline{S}_{\tau^\mu} - S_{\tau^\mu} \ge K\}}. \tag{4.1}$$

The price of this option, which we denote by $C_3(K, T)$ is

$$C_3(K, T) = C_2(K, T) - KC_1(K, T). \tag{4.2}$$

5 Conclusions

In this paper, we prove a variation of the Azéma martingale. This turns out to be a projection of the Kennedy martingale onto the filtration $\mathcal{F}_U$. We use this to study the Parisian drawdown time τ, and derive an analytical formula for the joint density of the triplet (τ, M_τ, W_τ). We introduce two new kinds of drawdown type options which take into account both the duration and magnitude of drawdowns.

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