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SOLVING ROBBIN'S PROBLEM

by

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ABSTRACT

In this report the numerical integration of elliptic partial differential equations under Robbin's boundary conditions is attempted by means of the Extrapolated form of the Alternating Direction Implicit methods.
A set of varying extrapolation parameters is determined along with Douglas' cycle of acceleration parameters and a comparison between the above two sets of iteration parameters is performed.

1. INTRODUCTION

In [7] the author employed an Extrapolated Alternating Direction Implicit (E.A.D.I.) procedure for solving numerically Robbin's problem on a unit square by determining a fixed acceleration parameter as well as a varying one of Samarskii and Andreyev (S-A) type,

A comparison between the two alternative parameters was carried out and the superiority of the S-A set was clearly established. Thus, the results obtained in [10], wherein only the fixed acceleration parameter was considered, were supplemented.

In the present report the problem is studied on a rectangle and two other alternative sets of accelerating parameters are defined. These are the Douglas parameter cycle [1] and an accelerating set based on Richardson's method [12] as finalised by Young [13]. The last set can be easily shown to be equivalent to using a varying extrapolation parameter whereas all the other sets, mentioned above, assume it to be fixed. Finally, a comparison between the two alternative iterative parameters considered in this report is carried out,

2. STATEMENT AMD DISCRETISATION OF THE PROBLEM

The Robbin's boundary value problem here consists of Laplace's equation

$$\frac{\partial}{\partial} + \frac{\partial}{\partial} = \qquad (1)$$

considered on the region $R \equiv \{(X_1, X_2) \mid 0 \leqq X_1 \leqq S_1, 0 \leqq X_2 \leqq S_2\}$

and subject to the boundary conditions

$$\left.\begin{aligned} \frac{\partial u}{\partial X_1} - p_1 u &= H(0, X_2)\ , \\ \frac{\partial u}{\partial X_1} + q_1 u &= H(1, X_2)\ , \\ \frac{\partial u}{\partial X_1} - p_2 u &= H(X_1, 0)\ , \\ \frac{\partial u}{\partial X_1} + q_2 u &= H(X_1, 1)\ , \end{aligned}\right\} - (X_1, X_2) \in \partial R \qquad (2)$$

along the boundary ∂R of R. The parameters p_i, q_i, $i = 1,2$ are real quantities and the function $H(X_1,X_2)$ is continuous on ∂R .

To discretise the problem, a uniform grid of lines parallel to the sides of the rectangle and of spacings h_1 and h_2 respectively is superimposed on the region R. The number of grid points at which the unknown function u has to be determined is $(M_1+ 1)$-$(M_2+ 1)$ with M_1 , and M_2 given by

$$M_i = \frac{s_i}{h_i},\ i = 1,2,$$

and the co-ordinates of a typical grid point (X_1 ,X_2) are

$$X_1 = (i\text{-}1)h_1,\ i \in \{1,2, ...,M_1 + 1\} \equiv I$$

$$X_1 = (i\text{-}1)h_2,\ j \in \{1,2, ...,M_2 + 1\} \equiv J$$

If we let $u_{1,m}$ represent the difference approximation to

the function u at the grid point (lh_1, mh_2), then the Laplacian operator can be replaced by the following five-point discretisations

$$\left(\delta_{X_1}^2 + \sigma\delta_{X_2}^2\right) u_{i,j} = 0 , \qquad (3)$$

with if $i \in I$, $j \in J$, $\sigma = h_1^2 / h_2^2$ and $\delta_{x_1}, \delta_{x_2}$ the usual centred difference operators in the designated direction.

In the same way the derivative boundary conditions (2) are replaced by the following difference equations :

$$\begin{aligned}
u_{2,j} - u_{-2,j} - 2h_1 p_1 u_{1,j} &= 2h_1 H(0, jh_2 - h_2), \\
u_{i,2} - u_{i,-2} - 2h_2 p_2 u_{i,1} &= 2h_2 H(ih_1 - h_1, 0), \\
u_{-M_1,j} - u_{M_1,j} + 2h_1 q_1 u_{M_1+1,j} &= 2h_1 H(1, jh_2 - h_2), \qquad (4) \\
u_{i,-M_2} - u_{i,M_2} + 2h_2 q_2 u_{i,M_2+1} &= 2h_2 H(ih_1 - h_1, 1),
\end{aligned}$$

with $i \in I$, and $j \in J$. It is readily seen that the values of u which correspond to grid points lying outside the region and of number $\{(M_1+1) + (M_2+1)\}$ are easily traoed, because of the negative subscript that they bear, and they can be eliminated by use of (3) and (4).

In this manner the boundary conditions are incorporated into the difference analogue of the problem (l), (2) which yields $(M_1+1) \cdot (M_2+1)$ linear equations, one for each grid point,

which is written in matrix form as

$$(H + V)u = K. \tag{5}$$

In (5),

$$H = I_{(M_2+1)} \otimes U^{(p_1,q_1)}_{(M_1+1)}, \quad v = \sigma U^{(p_2,q_2)}_{(M_2+1)} \otimes I_{(M_1+1)} \tag{6}$$

with I_K the unit matrix of order K, and $U_K^{(p_i,q_i)}$ a tridiagonal matrix of order K given by

$$U_K^{(p_i,q_i)} = \begin{bmatrix} 2(1+p_ih_i) & -2 & & & & \\ -1 & 2 & -1 & & & \\ & \bullet & \bullet & \bullet & & \\ & & -1 & 2 & -1 & \\ & & & -2 & 2(1+q_ih_i) & \end{bmatrix}, i=1,2, \tag{7}$$

and the vectors u and K are

$$\{u_{1,1}, \ldots, u_{1,M_1+1}; u_{2,1}, \ldots, u_{2,M_1+1}; \ldots u_{M_2+1,1}, \ldots u_{M_2+1,M_2+1}\}'$$

and

$$\{K_{1.1}, \ldots K_{1,M_1+1}; K_{2.1}, \ldots, K_{2,M_1+1}; \ldots; K_{M_2+1,1}, \ldots, K_{M_2+1,M_1+1}\}'$$

respectively. The vector u represents the unknown values of $u_{i,j}$ that we seek, while the vector K depends on the mesh sizes and on the values of $H(X_1,X_2)$ at the grid points on the perimeter of the region [see 7,8].

From the definition of the matrix $U_K^{(p,q)}$ in (7), it is apparent that neither U nor H is symmetric; similarly from (6) neither is V. However, as was shown in [7], it is always possible that a similar equation to (5) can be obtained by means of a similarity transformation, wherein the matrices involved are symmetric. To avoid repetition of this work we refer the reader to [7] and [11] and assume that in (5) the two matrices, H and V are symmetric.

The above assumption is painless because of the similarity transformation through which it is carried out and thereby the eigenvalue spectra remain unchanged.

3. THE E.A.D.I.SCHEME

For the numerical solution of (5) we make use of the following Extrapolated A.D.I, procedure

$$(I+rH)u^* = [(I+rH) - \omega r(H+V)]u^{(n)} + \omega rK$$

$$(I+rV)u^{(n+1)} = u^* + rVu^{(n)}, \qquad (8)$$

where

$u^{(n)}$ is the approximating solution vector at the nth iteration ($u^{(0)}$ is arbitrary); u^* is an intermediate vector iterate; I is the unit matrix of order $(M_1+1)(M_2+1)$; ω is the extrapolation constant and r is the accelerating parameter.

In the following we shall focus our attention on the cases where ω is kept fixed while the acceleration parameter r is allowed to vary in a cyclic way thus getting the known

extrapolated form of the A.D.I, procedures [5]; and the case where r is kept constant while ω varies in a cyclic fashion which case, in fact, corresponds to a Peaceman-Rachford scheme with a built-in Richardson's accelerating process [4].

Before proceeding with the evaluation of the iterative parameters involved in both cases, we point out the results obtained in [7] whereby the evaluation of the upper and lower eigenvalue bounds of the typical matrix (7) is always attainable with any desired accuracy. Accordingly we could well assume that the eigenvalue spectra of H and V, are known and given by (L1,U1), (L2,U2) respectively.

We now examine the two cases.

Case I. Fixed ω and varying r

A "good" parameter sequence $\{r_n\}$ of Douglas type may be defined on the basis of either spectra. For the moment let us proceed on the spectrum of H.

So, we start off by setting

$$\xi^{(1)} = L1$$

and define recursively the following two finite sequences

$$r_n \; \mu/\xi^{(n)} = \underline{\mu}\, L1^{\left\{\frac{\mu}{\nu}\right\}^{n-1}}, \qquad n = 1,2,...,n_0 \tag{9}$$

$$\xi^{(n+1)} = \frac{\nu}{\mu}\xi^{(n)} = L1(\frac{\nu}{\mu})^n \quad \text{with n as above.}$$

The common length of the sequences n_o is determined by

$$\xi^{(n_0)} < U1 \leq \xi^{(n_0+1)} \tag{10}$$

and the positive parameters μ, and ν are such that

$$\mu \leqq 1 \leqq \nu \tag{11}$$

Clearly, because of (10) we have

$$\left[\xi^{(1)}, \xi^{(n_0+1)}\right] \supseteq [L1, U1] ;$$

in addition, it can be proved that the sequence $\{r_n\}$ is such that for any $\xi \in [\xi^{(1)}, \xi^{(n_0+1)}]$ there is at least one index $i \in \{1, 2, \ldots, n_0\} = N$, say i*, such that

$$\mu \leqq r_i * \xi \leqq \nu ,$$

whereas for any other $i \neq i^*$ the relationship

$$\mu \frac{L1}{U1} < r_i \xi < \mu \frac{U1}{L1}$$

holds.

Furthermore, if we consider any pair of eigenvalues $(\lambda^{(1)}, \lambda^{(2)})$ where $\lambda^{(1)}$ and $\lambda^{(2)}$ belong to the spectra of H, and V respectively, then it can readily be shown that if we iterate n_0 times with the parameter sequence r_n, $n = 1, 2, \ldots, n_0$ there exists an index $n = n^* N$ for which the inequalities

$$\begin{aligned} \mu &\leqq r_{n^*}\lambda^{(1)} \leqq \nu \\ \mu \frac{L2}{U1} &\leqq r_{n^*}\lambda^{(2)} \leqq \mu \frac{U2}{L1} \end{aligned} \tag{11}$$

hold; while for any other index n we have

$$\left.\begin{array}{lll} \mu\dfrac{L1}{U1} & \leqq\ r_n\,\lambda^{(1)} & \leqq\ \mu\dfrac{U1}{L1} \\ \mu\dfrac{L2}{U1} & \leqq\ r_n\,\lambda^{(2)} & \leqq\ \mu\dfrac{U2}{L1} \end{array}\right\}\quad n\in N-\{n^*\} \qquad (11^{**})$$

Finally, from the inequalities (10) we can get the following explicit formula for evaluating the cycle length n_0,

$$\ln\left(\frac{L1}{U1}\right)\ n^{-1}\ \left(\frac{\mu}{\nu}\right)\ \leq\ n_0\ <\ln\ \left(\frac{L1}{U1}\right)\ \ln^{-1}\left(\frac{\mu}{\nu}\right)\ +\ 1, \qquad (12)$$

from which an additional restriction on μ and ν can be imposed because of the requirement for a cycle length greater than 1 as was effectively observed by Hadjidimos [5].

Now, the reduction matrix for the procedure (8) at the n+1 iteration is given by

$$^{T}n_{+1}\ =I-\omega r_{n+1}(I+r_{n+1}H)^{-1}(I+r_{n+1}V)^{-1}\ (H+V)$$

and therefore its amplification factor is

$$\rho_{i,j}\ =1-\omega f\,,$$

with

$$f(\lambda_i^{(1)}\ ,\ \lambda_j^{(2)})\ \ \{r_{n+1}(\lambda_i^{(1)}\ +\ \lambda_j^{(2)})\}\Big/\{(1+r_{n+1}(\lambda_i^{(1)})(1+r_{n+1}(\lambda_i^{(2)})\} \qquad (13)$$

Further, the error vectors in two consecutive iterations, say the n^{th} and the $n+1^{th}$, are associated with the equation

$$e^{(n+1)}\ =\ T_{n+1}\ e^{(n)}$$

.which easily can be put in the form

$$e^{(n+1)} = \prod_{i=1}^{n+1} T_i \, e^{(0)}$$

and therefore the error reduction within a whole cycle is successively given by

$$\frac{||e^{(n_0)}||}{||e^{(0)}||} \leqq ||\prod_{i=1}^{n_0} T_i|| = \max_{i,j} |\prod_{n=1}^{n_0} \rho_{i,j}(\omega, r_n)| =$$

$$\max_{i,j} |\rho_{i,j}(r_{n^*}, \omega)| \max_{i,j} \prod_{\substack{n \in N \\ -\{n^*\}}} |\rho_{i,j}(r_n, \omega)| \leqq$$

$$\max\{|1 = \omega f_M^*|, |1 - \omega f_m^*|\} \prod_{n \in N - \{n^*\}} \{|1 = \omega f_M|, |1 - \omega f_m|\}$$

where f_M^*, f_m^* stand for the maximum and minimum value of f under (11*) and f_M f_m denote the extremes of f under (11**).

Moreover, if we set

$$\rho_1^*(\mu, \nu) = \max\{|1 - \omega f_M^*|, 1 - \omega f_m^*|\}$$

and require that

$$\{|1 - \omega f_M|, 1 - \omega f_m|\} \leqq 1$$

then the error reduction within a cycle becomes

$$\frac{||e^{(n_0)}||}{||e^{(0)}||} \leq \rho_1^*(\mu,\nu) \bullet$$

Furthermore, because of (12) the number of iterations needed for the above reduction is

$$n_0 \sim \ln\left(\frac{L1}{U1}\right) \Big/ \ln\left(\frac{\mu}{\nu}\right)$$

Obviously, the number of cycles required for an error reduction below an assigned amount $\in > 0$ is determined by

$$\rho_1^*(\mu,\nu)^K \approx \in$$

and by virtue of (14) we get as the total number of iterations for the completion of the task above the quantity

$$I_1 \sim \frac{\ln\left(\frac{L1}{L1}\right)\ln(\in)}{\ln(\frac{\mu}{\nu})\ln\rho_1^*(\mu,\nu)} \bullet \qquad (15)$$

From the expression (15) it is evident that the quantity I_1, is minimised when the function

$$Z(\mu,\nu) = \ln\left(\frac{\mu}{\nu}\right)\ln\rho_1^*(\mu,\nu)$$

has a maximum.

In fact, the maximisation of $Z(\mu,\nu)$ minimises the calculation needed to reach the preassigned accuracy in the

solution of the problem at hand, and determine the optimum μ and ν.

On the other hand, the optimum m will be given by

$$\omega = 2\min\left\{\frac{1}{f_M^* + f_m^*} \ , \ \frac{1}{f_M}\right\}$$

and consequently for determining the optimum iteration parameters we need the extremes of f for the regions given in (11*) and (11**), After some algebra, we get

$$f_M^* = \max\left\{f(\mu,\ \mu\frac{U_2}{L_1},\ f(\nu,\ \mu\frac{L_2}{U_1})\right\}$$

$$f_m^* = \min\left\{f(\mu,\ \mu\frac{L_2}{U_1},\ f(\nu,\ \mu\frac{U_2}{L_1})\right\}$$

$$f_M = \max\left\{f(\mu\frac{L_1}{U_1},\ \mu\frac{U_2}{L_1}),\ f(\nu\frac{U_1}{L_1},\ \mu\frac{L_2}{U_1})\right\}$$

The above forms for the extremes of f clearly indicate that a new restriction on the parameters μ, and ν can be imposed through the equalisation of the two terms involved in each one of them. However, the extensive analysis made in [9] comes out rather emphatically for the equalisation of the terms in f_m^* which results in faster convergence.

If we comply with this finding we get as the new restriction the equality

$$\upsilon = \frac{\mu(\mu+1)U2(U1+\mu L2) - \mu\,(U1+L2)\ (L1+\mu U2)}{\mu\,(U1+L2)\ (L1+\mu U2) - (\mu+1)L2(U1+\mu L2)} \ . \qquad (16)$$

Obviously the same procedure can be carried out on the basis of the second spectrum, which ends up with the following relationships

$$I_2 \sim \frac{\ln\left(\frac{L2}{U2}\right) \ln(\epsilon)}{\ln\left(\frac{u}{v}\right) \ln \rho^*_2(\mu,\nu)} \tag{17}$$

And

$$\nu = \frac{\mu(\mu+1)U1(U1+\mu L1) - \mu(U2+L1)(L2+\mu U1)}{\mu(U2+L1)(L2+\mu U1) - (\mu+1)L2(U2+\mu L1)}, \tag{18}$$

giving approximately the number of iterations for attaining the same accuracy ϵ, and the extra restriction on the parameters μ, and ν.

CASE II. Varying ω and constant r.

Let us assume that the extrapolation parameter ω varies and that it takes on values from the finite sequence $\{\omega_K\}$ with length K_0 while the r remains fixed during the iterations.

The iteration matrix for scheme (8) during the $K+1^{th}$ iteration will be given this time by

$$T_{K+1} = I - \omega_{K+1} r (I+rH)^{-1} (I+rV)^{-1} (H+V) \tag{19}$$

while the error in the end of the first cycle is associated

with the initial error by

$$e^{(K_0)} = \left(\prod_{K=1}^{K_0} T_K \right) e^{(0)} .$$

Accordingly we obtain the following error reduction within a cycle

$$\frac{|| e^{(K_0)} ||}{|| e^{(0)} ||} \leq || \prod_{K=1}^{K_0} T_K || = || f_K(A) ||$$

(20)

with

$$f_K(x) = \prod_{K=1}^{K_0} (1-\omega_K x) \text{ a polynomial of degree } K_0,$$

and

$$A = r(I+rH)^{-1}(I+rV)^{-1}(H+V) .$$

Since the matrix A is symmetric we have

$$|| f_K(A) || = \max_{a \leq x \leq b} | f_K(x) | \quad , \qquad (21)$$

Where

$$a = \min_{i,j} \left\{ \frac{r(\lambda_1^{(1)} + \lambda_j^{(2)})}{(1+r\,\lambda_i^{(1)})(1+r\lambda_j^{(2)})} \right\}$$

$$b = \min_{i,j} \left\{ \frac{r(\lambda_1^{(1)} + \lambda_j^{(2)})}{(1+r\,\lambda_i^{(1)})(1+r\lambda_j^{(2)})} \right\} \bullet$$

By definition we get.

$$f_K(0) = 1, \text{ and } f_K\left(\frac{1}{\omega_i}\right) = 0, \quad i = 1,2,\ldots,K_0 \bullet \qquad (22)$$

Furthermore, from (20), (21) and (22) we easily see that our problem has now been transformed to the minimization problem

$$\min_{f_K \in P_{K_0}} \left\{ \max_{a<x<b} \left| f_K(x) \right| \right\} \bullet \qquad (23)$$

where P_{K_0} is the set of all real polynomials $f_K(x)$ of degree K_0 satisfying $f_K(0) = 1$•

The solution of problem (23)is unique and given

by [3]

$$f_K(x) \frac{T_{K_0}\left(\frac{b+a-2x}{b-a}\right)}{T_{K_0}\left(\frac{b+a}{b-a}\right)} \qquad (24)$$

with $T_{K_0}(x)$ denoting the ordinary normalised Chebyshev polynomial of degree K_0•

The extrapolation sequence is given by

$$\omega_n = 2\{b + a - (b - a)\, t_n\}^{-1}, \quad n = 1,2,..., K_0$$

where t_n signifies the zeroes of the K_o^{th} Chebyshev polynomials

Finally, the optimum r turns out to be the same as that used in the Peaceman-Raohford scheme with one acceleration parameter [2] and which as we shewed in [6] is as in table 1. The error reduction achieved in the end of the first cycle of K_Q iterations is given by

$$T_{K_0}\left(\frac{b+a}{b-a}\right)^{-1}, \qquad (25)$$

while after m successive cycles with the same sequence

the factor of convergence becomes

$$T_{K_0}\left(\frac{b+a}{b-a}\right)^{-m} \bullet \qquad (26)$$

If we now want to obtain an accuracy $\in$, then the number of iterations I_T required can easily be found to be approximately equal to

$$I_T \sim -K_0 \frac{\ln \in}{\ln T_{K_0}\left(\frac{b+a}{b-a}\right)} \quad \bullet \qquad (27)$$

The properties of the Chebyshev polynomials and the expression (25) clearly suggest that the larger the length of the extrapolation sequence, the more rapid the convergence; consequently one is tempted to think of using sequences with a large length, e.g. a sequence of $K_0.m$ extrapolation parameters instead of using m-times the same sequence with length K_0. Although the above argument is backed up by the inherent property of the process to be independent of the length of the parameter sequence to be used, this may not be so from the computational point of view, because an appreciable building up of round-off errors makes the whole procedure quite susceptible to instabilities. Sometimes this round-off error growth may severely decrease the length of the sequence. For example, in [2] the author claims that in some cases he was getting poor round-off results with even a pair of extrapolation parameters. In [13] and elsewhere, ways of controlling this round-off error accumulation are suggested for alleviating its effect on the actual convergence of the scheme, yet this growth

seems to be a real Achilles' heel for Richardson's acceleration device.

NUMERICAL RESULTS

To compare the efficiency of the two different E.A. D. I. schemes discussed in this report, we set up several computer programs for the evaluation of the optimum parameters and produced effective comparison tables for the two problems considered in [7]. Thus, the comparison tables 2 and 3. one for each problem, were compiled by incorporating the measures of calculation CVD and CTK_0 (for the Douglas parameter cycle and the Chebyshev set respectively) derived from formulae (15), (17) and (27)•

In the second set (the Chebyshev one), cases of various sequence lengths are displayed to the effect of obtaining better convergence for these sequences than the Douglas set, up to a number of subdivisions. For example, if we assume a Chebyshev set of length two, then from tables 2 and 3 we conclude that it is best up to 16 and 7 subdivisions respectively for problems I and II. Further, if we increase the cycle length by one,then the Chebyshev set becomes best up to 25 and 12 subdivisions respectively for problems I and II; and it is a sequence of six extrapolation parameters which for the case of 25 subdivisions can restore for the second problem, the supremacy of the Chebyshev set against the Douglas one. Finally, if we suppose that our region is covered by a grid of 50 subdivisions in each direction, then

varying extrapolation parameter sequences of lengths 6 and 15 are required so that the supremacy of the Chebyshev set may be retained for this case of mesh.

Before closing this report we would like to stress the point that for a fixed vazying extrapolation parameter cycle length the efficiency of the set falls off when compared with Douglas' cycle as we move on to finer nets.

TABLE 1. The Optimum Acceleration Parameter

<table>
<tr><th colspan="3">Cases</th><th>Optimum acceleration parameter</th></tr>
<tr><td colspan="3">L1<L2<U1<U2</td><td>$r^* = \sqrt{\dfrac{U1 + U2 - L1 - L2}{U1U2\,(L1 + L2) - L1L2\,(U1 + U2)}}$</td></tr>
<tr><td colspan="3">L1<L2<U1<U2</td><td>As above.</td></tr>
<tr><td rowspan="4">L1<Ul<L2<U2</td><td colspan="2">L1U2 = L2U1</td><td>The r* is given by the positive root of q(x) below.</td></tr>
<tr><td rowspan="2">L1U2<L2U1</td><td>$q\left(\frac{1}{L2}\right) > 0$</td><td>r* as above</td></tr>
<tr><td>$q\left(\frac{1}{L2}\right) \leq 0$</td><td>$r^* = \frac{1}{L2}$</td></tr>
<tr><td rowspan="2">L1U2>L2U1</td><td>$q\left(\frac{1}{L2}\right) \geq 0$</td><td>$r^* = \frac{1}{U1}$</td></tr>
<tr><td>$q\left(\frac{1}{L2}\right) < 0$</td><td>The r* is given by the positive root of q(x)</td></tr>
<tr><td colspan="4">$q(x) = \{(U2 - L2)U1L1 - (U1 - L1)U2L2\}x^2 + 2(L1U2 - L2U1)x + U2 - U1 + L1 - L2$</td></tr>
</table>

Note: The cases not included in this table can easily be reduced to those given above.

TABLE 2. COMPARISON OF THE METHODS

Problem I. $p_1 = 1 \cdot 0$, $q_i = 1 \cdot 0$, $i = 1, 2$

Sub-divisions	CD	CT2	CT3	CT6
3	1.65173	0.86049	0.78339	0.71840
4	1.97500	1.08397	0.96612	0.86924
5	2.22607	1.29880	1.13631	1.00488
6	2.43069	1.50799	1.29798	1.12977
7	2.60331	1.71347	1.45367	1.35765
8	2.75266	1.91647	1.60504	1.35765
9	2.88396	2.11726	1.75280	1.46349
10	3.00126	2.31648	1.89782	1.56525
11	3.10737	2.51481	2.04088	1.66377
12	3.20422	2.71242	2.18230	1.75951
13	3.29282	2.90832	2.32157	1.85233
14	3.37547	3.10526	2.46077	1.94378
15	3.45176	3.30012	2.59781	2.03262
16	3.52335	3.49527	2.73446	2.12012
17	3.59034	3.68940	2.86987	2.20585
18	3.65379	3.88415	3.00525	2.29066
19	3.71361	4.07810	3.13967	2.37404
20	3.77022	4.27135	3.27324	2.45614
21	3.82462	4.46647	3.40777	2.53813
22	3.87547	4.65759	3.53926	2.61762
23	3.92458	4.85057	3.67176	2.69713
24	3.97284	5.04867	3.80752	2.77802
25	4.01755	5.23996	3.93841	2.85549
26	4.05959	5.42695	4.06617	2.93064
27	4.10272	5.62630	4.20218	3.01017
28	4.14185	5.81388	4.33001	3.08450
29	4.18123	6.80948	4.46314	3.16152
30	4.21779	6.19732	4.59085	3.23503
31	4.25382	6.38858	4.72077	3.30947
32	4.28905	6.58171	4.85183	3.38423
33	4.32315	6.77454	4.98258	3.45849
34	4.35575	6.96449	5.11128	3.53129
35	4.39039	7.17254	5.25213	3.61065
36	4.41891	7.34879	5.37137	3.67759
37	4.44890	7.53961	5.50039	3.74978
38	4.48081	7.74719	5.64065	3.82799
39	4.50961	7.94034	5.77109	3.90048
40	4.53482	8.11346	5.88795	3.96524
41	4.56667	8.33800	6.03943	4.04893
42	4.58897	8.49909	6.14805	4.10877
43	4.61818	8.71501	6.29359	4.18875
44	4.66271	8.90081	6.41877	4.25735
45	4.66838	9.09970	6.55272	4.33059
46	4.68846	9.25848	6.65961	4.38890
47	4.71640	9.48435	6.81163	4.47164
48	4.73833	9.65561	6.93357	4.53786
49	4.76116	9.85821	7.06311	4.60807
50	4.78499	10.06348	7.20113	4.68271

TABLE 3. COMPARISON OF THE METHODS

Problem II. $p_i = -0.25$, $q_i = 1.0$, $i = 1,2$.

Sub-divisions	CD	CT2	CT3	CT6	CT15
3	2.48349	1.56779	1.34358	1.16435	1.07740
4	2.79943	1.98545	1.65358	1.16435	1.27153
5	3.04546	2.39690	1.95597	1.60551	1.44478
6	3.24683	2.80475	2.24804	1.80350	1.60324
7	3.41712	3.21003	2.53453	1.99173	1.75043
8	3.56451	3.61319	2.81677	2.17234	1.88864
9	3.69420	4.01404	3.09532	2.34662	2.01935
10	3.81013	4.41357	3.37132	2.51599	2.14400
11	3.91524	4.81323	3.64613	2.68180	2.26387
12	4.01220	5.21668	3.92249	2.84609	2.38064
13	4.10096	5.61797	4.19650	3.00686	2.49307
14	4.18174	6.01203	4.46487	3.16252	2.60029
15	4.25737	6.40778	4.73380	3.31692	2.70512
16	4.33061	6.81754	5.01172	3.47500	2.81096
17	4.39471	7.19897	5.27002	3.62071	2.90724
18	4.45820	7.59909	5.54060	3.77223	3.00614
19	4.51983	8.01006	5.81816	3.92659	3.10566
20	4.57796	8.41912	6.09413	4.07908	3.20284
21	4.63053	8.80806	6.35628	4.22313	3.29357
22	4.68189	9.20620	6.62442	4.36971	3.38496
23	4.73112	9.60556	6.89318	4.51594	3.47520
24	4.77710	9.99503	7.15511	4.65784	3.56193
25	4.81860	10.36064	7.40085	4.79046	3.64226
26	4.86347	10.77166	7.67697	4.93893	3.73139
27	4.90219	11.14004	7.92432	5.07148	3.81029
28	4.94384	11.55091	8.20009	5.21879	3.89385
29	4.98889	12.01313	8.51018	5.38389	3.99385
30	5.02527	12.40041	8.76990	5.52176	4.07384
31	5.06421	12.82932	9.05743	5.67400	4.10149
32	5.09179	13.14235	9.26723	5.78482	4.22485
33	5.13594	13.65998	9.61404	5.96759	4.32860
34	5.16746	14.04244	9.87021	6.10227	4.40445
35	5.20089	14.46008	10.14988	6.24902	4.48656
36	5.21840	14.68400	10.29980	6.32756	4.53028
37	5.25518	15.16596	10.62243	6.49631	4.62370
38	5.27453	15.42602	10.79648	6.58721	4.67373
39	5.31540	15.99053	11.17423	6.78416	4.78149
40	5.33703	16.29779	11.37980	6.89117	4.83967
41	5.35954	16.62418	11.598.3	7.00469	4.90113
42	5.38301	16.97149	11.83044	7.12533	4.96614
43	5.43318	17.73934	12.34393	7.39152	5.10855
44	5.43317	17.73919	12.34383	7.39147	5.10852
45	5.46008	18.16559	12.62893	7.53900	5.18635
46	5.48836	18.62508	12.93610	7.69776	5.27070
47	5.51818	19.12248	13.26858	7.86937	5.36033
48	5.54970	19.66297	13.62982	8.05559	5.45806
49	5.54970	19.66297	13.62982	8.05559	5.45806
50	5.58312	20.25336	14.02435	8.25871	5.56315

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