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THE DETERMINATION OF CONSISTENT ORDERINGS FOR THE SOR ITERATIVE METHOD

by

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1. Introduction

The concept of consistent ordering is central to the theory of the successive over-relaxation (SOR) iterative method for the solution of the set of equations

$$A\underline{x} = \underline{b} \quad . \tag{1}$$

If the matrix A is consistently ordered and satisfies certain convergence properties then the theoretical determination of the accelerating parameter, which maximizes the asymptotic rate of convergence of the SOR method, is possible.

The earliest definition of consistent ordering was due to Young (1954), who defined a consistently ordered matrix in terms of an ordering vector related to the disposition of zero and non-zero elements in the matrix. Young considered only point iterative methods and his original definition and the related theory applied only to a certain class of matrices, matrices with property (A). More general definitions of consistent ordering have been given by Arms, Gates and Zondek (1956), Forsythe and Wasow (1960), Kjellberg (1961), Varga (1962), Broyden (1964, 1968), and Verner and Bernal (1968). In particular Arms, Gates and Zondek (1956), generalised Young's definitions to block partitioned matrices and extended the theory to block SOR, whilst Varga (1962), with his definition of consistent ordering, extended the theory still further to the much wider class of p-cyclic matrices. Matrices with property (A) are in fact 2-cyclic and Varga's definition includes the definitions of Young and of Arms, Gates and Zondek as special cases.

A well-known property of a p-cyclic matrix is that it can always be transformed, by a permutation similarity transformation, into a consistently ordered form known as a "normal form". In fact this property is often used to define a p-cyclic matrix. Although graph theory may be used to establish the cyclicity of a matrix (Varga (1962), p. 100) and to check, in certain cases, whether a given p-cyclic matrix A is consistently ordered, the graph theoretic approach does not conveniently yield the permutation matrix P which transforms A into a normal and therefore consistently ordered form PAP^T • The purpose of the present paper is to develop techniques for determining P.

In section 2 we consider in detail a definition of a p-cyclic matrix suggested by Varga((1962) , p.103, ex.1) . This definition is a generalization of Young's property (A) and introduces the concept of an ordering vector to the p-cyclic ($p > 2$) case. By generalizing the techniques indicated by young (1954), we show that the use of ordering vectors leads to simple and systematic methods for establishing the cyclicity of a matrix and for determining P.

The more recent definitions of consistent ordering, due to Broyden (1964, 1968) and Verner and Bernal (1968), include Varga's definition a3 a special case but exclude direct reference to p-cyclicity. Many of the matrices encountered in these generalizations are however p-cyclic, and the significance of ordering vectors in these cases is discussed in section 3. In particular we establish formulae to generate ordering vectors for matrices which have the consistently ordered block form considered by Broyden (1964, p.280) and Verner and Bernal (1968, p.219).

We note that most of the results in the present paper may also be deduced from Young's recent generalizations of property (A). (Young (1971), Chap.13). However, our approach highlights the usefulness of ordering vectors in determining consistently ordered forms of a p-cyclic matrix in a relatively simple and direct manner.

2. Ordering vectors and p-cyclic matrices.

We assume throughout that $A = (a_{ij})$ is an nxn matrix and that π partitions A into submatrices A_{ij}, i,j = 1,2 . ...N, so that the diagonal blocks A_{ij}. are square and non-singular.

Definition 1, (Varga (1962), p.103, Ex.1)

The matrix A is p-cyclic, relative to the partitioning π if there exist p disjoint nonempty subsets S_m, m = 0,1, ...,p-1 of W, the set of the first N positive integers, such that

$$\bigcup_{m=0}^{p-1} s_m = W$$

and if $A_{ij} \neq \underset{\sim}{0}$ then either i = j, or if $i \in S_m$ then $j \in S_{m=1}$, subscripts taken modulo p. (i.e., $S_{-1} = S_{p-1}$).

This definition generalises the definition given by Young (1954), p.93t for property (A) matrices and provides a simple method for establishing the cyclicity of a matrix. Thus, given a matrix A we set $1 \in T_m$ and attempt to construct sets T_m, T_{m-1}, T_{m-2}, T_{m-3}, ... having the properties of the sets in Definition 1. If A is p-cyclic this process generates m-p+1 disjoint sets, T_m, T_{m-1} ...., $T_{m\ p+1}$ which may be identified by the relation

$$S_{k-(m-p+1)} = T_k, k = (m-p+1)(1)m,$$

as the sets S_0, S_1, ...,S_{p-1} of Definition 1. If A is not cyclic at some stage the sets T_i cease to be disjoint and the process is terminated.

Example 1. For the matrix

$$A_1 = \begin{bmatrix} a_{11} & 0 & 0 & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & a_{42} & 0 & a_{44} & 0 \\ 0 & a_{52} & 0 & 0 & a_{55} \end{bmatrix},$$

partitioned so that its diagonal sub ma trice 3 are 1×1, we obtain the following distribution of elements in the sets.

	$T_{m-2} = S_0$	$T_{m-1} = S_1$	$T_m = S_2$
Row 1		$j = 4$ 5	$i = 1$
Row 4	$j = 2$	$i = 4$	
Row 2	$i = 2$		$j = 1$ 3
Row 3		$j = 4$ 5	$i = 3$
Row 5	$j = 2$	$i = 5$	

As the sets S_0, S_1 and S_2 are disjoint and

$$S_0 \cup S_1 \cup S_2 = (1,2,3,4,5) = W,$$

the matrix A is 3-cyclic.

The theorem which follows is a generalisation of theorem 2.1 of Young (1954), P.97, and introduces the concept of the ordering vector in the study of p - cyclic matrices.

<u>Theorem 1</u>, (See Varga (1962) ,p. 103,Ex. 2).
The matrix A is p-cyclic, relative to the partitioning π, if and only if there exists a row vector $\underline{y} = \{y_1, y_2, .., y_N\}$ with integral components such that if $A_{ij} \not\equiv \underset{\sim}{0}$ and $i \neq j$ then $y_j - y_i$ is equal to -1 or p - 1 and for each integer r, $0 \leq r \leq p-1$ there exists at least one $y_i \equiv r(\text{mod} p)$.

<u>Proof</u>: Assume that A is p - cyclic and, referring to Definition 1, let $y_i = k$ if $i \in S_k$. If $A_{ij} \not\equiv \underset{\sim}{0}$ and $i \neq j$ then either,

(i) $i \in S_k$ and $j \in S_{k-1}$ $(i \neq 0)$, in which case $y_i = k, y_j = k-1$ and hence $y_j - y_i = -1$, or

(ii) $i \in S_0$ and $j \in S_{p-1}$, in which case $y_i = 0$, $y_j = p-1$ and hence $y_j - y_i = p-1$.

Conversely assume that $\underline{y}$ exists and let S_k, $k=0,1,2,..,p-1$, denote respectively sets of integers i such that $i \in S_k$. if $y_i \equiv k(\text{mod} p)$. If $y_j - y_i = -1$ it follows that $y_i \equiv k(\text{mod} p)$, $y_i \equiv (k-1)(\text{mod} p)$, i.e., $i \in S_k$, $j \in S_{k-1}$, $k = 1, 2, .., p-1$, and if $y_j - y_i = p-1$ it follows that $y_i = 0$ and $yj = p-1$, i.e., $i \in S_0$ and $j \in S_{p-1}$. Since, for every integer r, $0 \leq r \leq p-1$, $\underline{y}$ has at least one component equal to r(modp), the existence of the sets $S_0, S_1, ..., S_{p-1}$, of Definition 1 is established and the proof of the theorem is complete.

A vector $\underline{y}$ with the above properties is said to be an ordering vector for the matrix A. Clearly the actual numerical value of a component of $\underline{y}$ is not important. Only the difference between any two components is significant and thus, if an ordering vector has M distinct components these will always be taken to be the integers 0,1,2,..., M-1.

An ordering vector for a p-cyclic matrix is the vector $y = \{y_1\}_{i=1}^{N}$, where

$$y_1 = k \text{ if } i \,\square\!, S_k \ ,$$

and S_k, $k = 0,1, ..., p-1$, are the sets of definition 1. In fact, p ordering vectors $y^{(n)} = \{y_1{}^{(n)}\}_{i=1}^{N}$, $n = 1,2, ..., p$, each with p distinct components may be obtained by setting

$$y_i^{(n)} \equiv (k+n-1)(\mathrm{mod}\, p) \text{ if } i \in S_k.$$

In general, for a given p-cyclic matrix there also exist ordering vectors with M > p distinct components. Thus, for the 3 - cyclic matrix A_1, of Ex.1, four ordering vectors are :

$\underline{y}^{(1)} = \{2,02,1,1\}$, $\underline{y}^{(2)} = \{0,1,0,2,2,\}$, $\underline{y}^{(3)} = \{1,2,1,0,0\}$,

each with three distinct components and also the vector

$$\underline{y}^{(4)} = \{0,1,3,2,2\} \ ,$$

which has four distinct components. An ordering vector $\underline{y} = \{y_i\}$ with M > p distinct components can always be transformed into a vector $\underline{y}' = \{y'_i\}$ with p distinct components by setting

$$y'_i \equiv yi \ (\mathrm{mod}\, p) \ .$$

For example, by setting

$$y'_i \equiv y_i^{(4)} \pmod 3$$

the vector $\underline{y}^{(4)}$, considered above, is transformed into the vector

$$\underline{y}' = \{0,1,0,2,2\} = \underline{y}^{(2)}.$$

Conversely, a vector $\underline{y} = \{ y_i]$ with p distinct components can sometimes be transformed into a vector with more than p distinct components by replacing some of the y_i by $y_i + p$. For example if the component $y_3^{(2)} = 0$, of $\underline{y}^{(2)}$,, is replaced by 3 the vector $y^{(4)}$ is obtained. If however, the components $y_3^{(3)} = 1$ and $y_5^{(3)} = 0$, of $\underline{y}^{(3)}$, are replaced respectively by 4 and 3 the resulting vector $\{1,2,4,0,3\}$ is not an ordering vector.

We define a normal form of a p-cyclic matrix as follows.

<u>Definition 2</u>. The p-oyclic matrix A, partitioned by π, is said to be in a normal form if there exists an ordering vector $y = \{y_i\}_{i=1}^N$ such that if $A_{ij} \neq \underset{\sim}{0}$ and $j > i$ then $y_j - y_i = p - 1$ and if $A_{ij} \neq \underset{\sim}{0}$ and $i > j$ then $y_i - y_j = 1$.

It is clear that the ordering vector of definition 2 has its components arranged in ascending order of magnitude. Thus, the p-cyclic matrix A is in a normal form if a row vector $\underline{y} = \{y_i\}_{i=1}^N$, with its components arranged in ascending order of magnitude, is an ordering vector for A.

Let A be a p-cyclic matrix which has a vector $y = \{y_i\}_{i=1}^N$

as an ordering vector and let P denote an nxn permutation matrix which permutes the entries of A by blocks. The following theorem shows that A can always be transformed into a normal form PAP^T.

Theorem 2. The matrix PAP^T, obtained by arranging the rows and columns of blocks of A with increasing y_i is in a normal form.

Proof : The vector $\underline{y}'$, obtained by arranging the components of $\underline{y}$ in ascending order of magnitude, is an ordering vector for $PAP^{T.}$

If P_{ij} denotes the nxn permutation matrix such that P_{ij} A is A with its i and j rows of blocks interchanged, then the matrix P of Theorem 2 say be determined as follows :

P is set equal to the unit matrix I and when the components y_i and y_j of $\underline{y}$ are interchanged P is premultiplied by P_{ij}. The final matrix P obtained in this way is the required permutation matrix.

Clearly, P is not unique and the number of normal forms PAP^T, associated with an ordering vector $\underline{y}$, equals the number of ways that $\underline{y}$ can be transformed into $\underline{y}'$.

Example 2. Consider the 3-oyolic matrix A_1 , of Example 1. A permutation matrix P, which gives a normal form PA_1P^T associated with the ordering vector $\underline{y}^{(1)} = \{2,0,2,1\ ,1\ ]$ is determined as follows:

$\underline{y}^{(1)} = \{2,0,2,1,1\}$: $P = I$,

$(2,0,2,1,1\} \rightarrow \{0,2,2,1,1\}$: $P = P_{12}$,

$\{0,2,2,1,1] \rightarrow \{0,1,2,1,2\}$: $P = P_{25}\ P_{1\,2}$,

$\{0,l,2,1,2\} \rightarrow \{0,1,1,2,2\}\ \underline{y}^{(1)'}$: $P = P_{34}\,P_{25}\,P_{12}$.

i.e. $p = p_{34}\ p_{25}\ p_{12}$.

The theorem which follows establishes the block structure for a normal form of a p-cyclic matrix.

<u>Theorem 3.</u> The p-cyclic matrix A, partitioned by π, is in a normal form of a p-cyclic matrix if and only if it has the block form

$$\begin{bmatrix} D_1 & & & B_{1p} & & & \\ B_{21} & D_2 & & & B_{2,p+1} & & \\ & B_{32} & D_3 & & & \bullet & \\ & & \bullet & \bullet & & & B_{M-p+1,M} \\ & & & \bullet & \bullet & & \\ & & & & \bullet & \bullet & \\ & & & & & B_{M,M-1} & D_M \end{bmatrix}, \quad M \geqq p, \tag{2}$$

where all the D_i, are square block diagonal matrices, whose diagonal blocks are submatrices A_{ij}. of A.

<u>Proof</u>: Assume that A is in a normal form of a p-cyclic matrix and that the vector $y = \{y_i\}_{i=1}^{(N)}$, with $M \geqq p$ distinct components arranged in ascending order of magnitude, is an ordering vector for A. Assume further that n_i of the components of $\underline{y}$ are equal to $i-1$, $i = 1,2,\ldots,M$, and let

$$L_0 = 0, \quad L_r = \sum_{i=1}^{r} n_i, \quad r = 1,2,\ldots, M,$$

so that

$$y_i = k-1 \text{ for } i = (L_{k-1}+1)(1)L_k.$$

The vector $\underset{-}{y}$ is such that if $A_{ij} \neq \underset{\sim}{0}$ and $j > i$,

$y_j - y_I = p - 1$ and if A_{ij} □, $\underset{\sim}{0}$ and $i > j$, $y_i - y_j = 1$. It follows that In the n_k row blocks of A, lying between the row blocks L_{k-1} and L_k+1, apart from the diagonal entries $A_{ii}, i = (L_{k-1}+1)\ (1)\ L_k$, the only other non-null entries A_{ij} occur in the positions defined by

$$\left.\begin{array}{l} i = (L_{k-1}+1)\ (1)\, L_k \\ j = (L_{k-1}+1)\ (1)\, L_{k-1} \end{array}\right\} \quad K > 1, \tag{3}$$

and

$$\left.\begin{array}{l} i = (L_{k-1}+1)\ (1)\, L_k \\ j = (L_{k+p-2}+1)\ (1)\, L_{k+p-1} \end{array}\right\} \quad k < M - p + 2\,. \tag{4}$$

Non-null off-diagonal entries occur, for k = 1 in positions defined by (4) only, and for $k \geq M - p + 2$ in positions defined by (3) only. This establishes the fact that A has the block form (2). We have also shown that each D_i is a block diagonal matrix consisting of n_i rows and columns of blocks.

Conversely, suppose that A has the block form (2) where each D, is a block diagonal matrix consisting of n. rows and columns of blocks. To show that A is p-cyclic and in a normal form it is sufficient to exhibit an appropriate ordering vector. The vector

$$\{0,0\ \ldots,0,1,1,\ \ldots,1,2,2,..,2,3\ ,\ldots\ldots, M-1\},$$

with n_i of its components equal to i - 1 is such a vector and this completes the proof of the theorem.

Assume that the matrix A is p-cyclic and has $\underline{y}$ as an ordering vector. The proof of Theorem 3 shows that the block structure of a normal form PAP^T, associated with $\underline{y}$, is completely determined by the number of distinct components in $\underline{y}$. Thus, PAP^T has the block form (2) where,

(i) M is equal to the number of distinct components in $\underline{y}$,

(ii) each D_i is a block diagonal matrix consisting of n_i rows and columns of blocks, where n_i is the number of components in $\underline{y}$ equal to i - 1. (We point out that although the D_i's are block diagonal they are not necessarily true diagonal matrices).

Example 3. Consider the 3-cyclio matrix A_1 of Example 1. The matrices (5), (6), (7) and (8), which follow, are normal forms of A_1 associated respectively with the ordering vectors $\underline{y}^{(1)} = \{2,0,2,1,1\}$, $\underline{y}^{(2)} = \{0,1,0,2,2\}$, $\underline{y}^{(3)} = \{1,2,1,0,0\}$ and $\underline{y}^{(4)}$ $\{0,1,3,2,2\}$. (In what follows the subscripts i, j of an element a_{ij} always refer to the position of this element in the original matrix A_1.)

$$P_{34}\, P_{25}\, P_{12}\, A_1\, P_{12}\, P_{25}\, P_{34} = \left[\begin{array}{c|cc|cc} a_{22} & 0 & 0 & a_{23} & a_{21} \\ \hline a_{52} & a_{55} & 0 & 0 & 0 \\ a_{42} & 0 & a_{44} & 0 & 0 \\ \hline 0 & a_{35} & a_{34} & a_{33} & 0 \\ 0 & a_{15} & a_{14} & 0 & a_{11} \end{array}\right], \quad (5)$$

$$P_{23}\, A_1\, P_{23} = \left[\begin{array}{cc|c|cc} a_{11} & 0 & 0 & a_{14} & a_{15} \\ 0 & a_{33} & 0 & a_{34} & 0_{35} \\ \hline a_{21} & a_{23} & a_{22} & 0 & 0 \\ \hline 0 & 0 & a_{42} & a_{44} & 0 \\ 0 & 0 & a_{52} & 0 & a_{55} \end{array}\right], \quad (6)$$

$$P_{14}\, P_{25}\, A_1\, P_{14}\, P_{25} = \left[\begin{array}{cc|cc|c} a_{44} & 0 & 0 & 0 & a_{42} \\ 0 & a_{55} & 0 & 0 & a_{52} \\ \hline a_{34} & a_{35} & a_{33} & 0 & 0 \\ a_{14} & a_{15} & 0 & a_{11} & 0 \\ \hline 0 & 0 & a_{23} & a_{21} & a_{22} \end{array}\right], \quad (7)$$

$$P_{35} A_1 P_{35} = \left[\begin{array}{cc|cc|c} a_{11} & 0 & a_{15} & a_{14} & 0 \\ \hline a_{21} & a_{22} & 0 & 0 & a_{23} \\ \hline 0 & a_{52} & a_{55} & 0 & 0 \\ 0 & a_{42} & 0 & a_{44} & 0 \\ \hline 0 & 0 & a_{35} & a_{34} & a_{33} \end{array}\right], \qquad (8)$$

Appropriate ordering vectors for the matrices (5), (6), (7) and (8) are respectively the vectors,

$\underline{y}^{(1)'} = \{0,1,1,2,2\}$, $\underline{y}^{(2)'} = \{0,0,1,2,2\}$, $\underline{y}^{(3)'} = \{0,0,1,1,2\}$

and $\underline{y}^{(4)'} = \{0,1,2,2,3\}$.

From Theorems 2 and 3 it follows that a definition of a p-cyclic matrix, equivalent to Definition 1 is :

Definition 3. The $n\times n$ matrix A is p-cyclic, relative to the partitioning π, if there exists an $n\times n$ permutation matrix P, which permutes the entries of A by blocks, such that PAP^T has the block form (2).

When p = 2, the above definition is the one given by Forsythe and Wasow (1960) for 2- cyclic (property(A)) matrices.

For any p-cyclic matrix A , an appropriate ordering vector is the vector $\underline{y} = \{y_i\}$

where,

$$y_i = i \quad \text{if} \quad i \in S_i\,, \quad i = 0,1,2,..,p-1,$$

and S_i are the sets of Definition 1 . A normal form PAP^T associated with the above vector has the block form,

$$\begin{bmatrix} D_1 & & & & & & B_{1p} \\ B_{21} & D_2 & & & & & \\ & B_{32} & D_3 & & & & \\ & & \cdot & \cdot & & & \\ & & & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \\ & & & & & B_{p,p-1} & D_p \end{bmatrix}, \qquad (9)$$

(i,e*, form (2) with M = p), and thus another definition of a p-cyclic matrix is ;

<u>Definition 4.</u> The matrix A is p-cyclic, relative to the partitioning π, if there exists a permutation matrix P, which permutes the entries of A by blocks, such that PAP^T has the block form (9).

This is how Varga (1962) defines a p-cyclic matrix. The block form (9) is usually referred to as the normal form of a p-cyclic matrix. However, our definition of a normal form does not impose the restriction M=p on (2) and is thus more get

We note that it is always possible to transform a p-cyclic matrix A into the block form

$$\left[\begin{array}{c|c|c|c|c|c|c}
D_1 & B_{12} & & & & & \\ \hline
 & D_2 & B_{23} & & & & \\ \hline
 & & \cdot & \cdot & & & \\ \hline
B_{p1} & & & \cdot & \cdot & & \\ \hline
 & B_{p+1,2} & & & \cdot & \cdot & \\ \hline
 & & \cdot & & & D_{M-1} & B_{M-1,M} \\ \hline
 & & & B_{M,M-p+1} & & & D_M
\end{array}\right], \quad M \geqq p, \qquad (10)$$

where, as before, the D_i's are block diagonal matrices. Thus, if a vector $\underline{y}$ with $M \geqq p$ distinct components is an ordering vector for A, the above form is obtained by arranging the rows and columns of blocks of A with decreasing y_i. When $p = 2$, forms (2) and (10) are identical and give the well known tri-diagonal representation of a properly (A) matrix.

3. <u>More general consistently ordered block forms.</u>

Square matrices which have the block form,

$$\left[\begin{array}{c|c|c|c|c|c|c|c}
D_1 & & & & B_{1,q+1} & & & \\ \hline
 & D_2 & & & & B_{2,q+2} & & \\ \hline
 & & \cdot & & & & \cdot & \\ \hline
 & & & \cdot & & & & B_{M-q,M} \\ \hline
B_{r+1,1} & & & & \cdot & & & \\ \hline
 & B_{r+2,2} & & & & \cdot & & \\ \hline
 & & \cdot & & & & \cdot & \\ \hline
 & & & B_{M,M-r} & & & & D_M
\end{array}\right], \qquad (11)$$

where all the D_i are non-singular block diagonal matrices, are consistently ordered by the definitions of Broyden (1964, 1968) and Verner and Bernal (1968, see Theorem III, p. 218) . It is known that such matrices are also $(q + r)/d$-cyclic, where

$$d = (q , r)$$

is the highest common factor of q and r. We now show that their cyclicity can easily be established by constructing appropriate ordering vectors.

Let $q^* = q/d$, $r^* = r/d$ and examine separately the following two cases, in which we assume without loss of generality that $M \geq q + r$.

<u>Case (i):</u> $(r,q^* + r^*) = 1$, i.e. r and q^*+r^* relatively prime.

Consider the vector $y = \{y_i\}_{i=1}^{M}$ where ,

$$\left.\begin{aligned} y_i \equiv \frac{(i-1)}{r} \ (\mathrm{mod}\,(q^* + r^*)) \ , \\ i = 1,2,\ldots\ldots,M. \end{aligned}\right\} \qquad (12)$$

Since $(r,q^* + r^*) = 1$ there always exists an integral solution x to the congruence

$$rx \equiv (i-1) \ (\mathrm{mod}\,(q^* + r^*)) \ ,$$

(see for ex. Birkhoff and MacLane (1962), p24), and the vector $\underline{y}$ has M components of which $q^*+ r^*$ are distinct and equal to $0,1,\ldots\ldots, q^* + r^* -1$. Further, it can easily be shown that,

$$q^*+r^*-1 \equiv -1 \ (\mathrm{mod}\,(q^* + r^*)) \equiv \frac{q}{r} \ (\mathrm{mod}\,(q^* + r^*)).$$

Since a matrix in the form (11) is such that if $B_{ij} \neq \underset{\sim}{0}$ then $j - i$ is equal to q or $-r$, it follows that the vector $\underline{y}$, as defined by (12), is such that if $B_{ij} \neq \underset{\sim}{0}$ then $y_j - y_i$ is equal to -1 or $q^* + r^* - 1$. Thus, with $p = q^* + r^*$, $\underline{y}$ satisfies the conditions of Theorem 1, showing that the matrix is $(q^* + r^*)$ - cyclic.

<u>Case (ii)</u> : $(r,q^* + r^*) \neq 1$.

Express r^* as,

$$r^* = kd_1 \ ,$$

where $(k,d) = 1$ and $(d_1 ,d) \neq 1$, or $d_1 = 1$, and consider the vector $\underline{y} = \{y_i\}_{i=1}^{M}$, whose components are generated as follows:

$$\left.\begin{aligned} \alpha_i &\equiv \frac{(i-1)}{k} \ (\mathrm{mod}(q+r)) \ , \\ \beta_i &\equiv \frac{\alpha_i}{d} \ (\mathrm{mod}\ (q+r+1)) \ , \\ \gamma_i &\equiv \frac{\beta_i}{d_1} \ (\mathrm{mod}\ (q^*+r^*)) \ , \\ & \qquad i = 1,2 \ldots\ldots \ M. \end{aligned}\right\} \tag{13}$$

Since (k,d) = 1 and q + r = (q* + r*)d it follows that (k,q + r) = 1 and that the vector $\underline{\alpha} = \{\alpha_i\}_{i=1}^{M}$ has M components of which q + r are distinct and equal to 0,1 , …., q + r - 1. Further, since

$$q + r - d_1 d \equiv - d_1 d \equiv d_1 d \ (\mathrm{mod}(q+r)) \equiv \frac{q}{k} \ (\mathrm{mod}(q+r)),$$

the components of $\underline{\alpha}$ are such that if $B_{ij} \not\equiv \underset{\sim}{0}$ and $i \not\equiv j$ then $\alpha_j - \alpha_i$ is equal to $-d_1 d$ or $q + r - d_1 d$. Also, since (d, q + r + 1) = 1 and $q + r - d_1 d = (q^* + r^* - d_1) d$, the components of the vector $\underline{\beta} = \{\beta_i\}_{i=1}^{M}$, are integers, in the range 0 to q + r, such that if $B_{ij} \not\equiv \underset{\sim}{0}$ and $i \neq j$ then $\beta_j - \beta_i$ is equal to $-d_1$ or $q^* + r^* - d_1$. Finally, since $(d_1, q^* + r^* - d_1) = 1$, the last formula in (13) is identical to formula (12) with r, q* and (i - 1) replaced respectively by d_1, $q^* + r^* - d_1$ and β_i. It follows that the vector $\underline{y}$ has M components, of which q* + r* are distinct and equal to 0,1, …, q* + r*- 1, such that if $B_{ij} \not\equiv \underset{\sim}{0}$ and $i \neq j$ then $y_j - y_i$ is equal to -1 or

$q^* + r^*-1$. This shows that the matrix is $(q^*+ r^*)$ -cyclic.

We note that the block forms (2) and (10) are respectively the particular cases $q = q = p - 1$, $r = r^*= 1$ and $q = q^* = 1$, $r = r^* = p-1$ of (11). When $p = M$ formula (12) generates the ordering vectors $[0,1,\ldots,p - 2, p – 1\}$ and $[p -1, p -2, \ldots, 1,0\}$ for (2) and (10) respectively. We also note that when $d_1 = 1$, i. e, when $(r^* ,d) = 1$, the vector $\underline{\beta} = \{\beta_i\}_{i=1}^{M}$, generated by the second formula in (13), satisfies all the properties of Theorem 1. Thus, in this special case, $\underline{\beta}$ is an appropriate ordering vector for (11). However, the values of the components β_i range from 0 to $q + r$ and the third formula "normalises" the β_i's so that they take the values $0,1, \ldots, q^* + r^*-1$.

Example 4. We illustrate the use of formulae (12) and (13) by considering the following three cases of form (11).

Case (i): $q = 5$, $r = 3$, $M = 10$. (8-cyclic).

By formula (12),

$$y_i \ \square\,\text{ß}\frac{(i-1)}{3}\ (\mathrm{mod}\,8)\ \ \square\,\text{ß}3(i-1)\,\mathrm{mod}\,8).$$

$i = 1,2,\ldots\ldots, 9,10.$

Thus, an ordering vector in this case is,

$\underline{y} = \{0,3,6,1,4,7,2,5,0,3\}$.

Case (ii). $q = 10$, $r - 6$, $M =16$. (8 - cyclic).

In this case $d = 2$, $q^* = 5$ $r^*= k = 3$ and $d_1.= 1$.

By formula (13),

$$\alpha_i \equiv \frac{(i-1)}{3} (\text{mod } 16) \equiv 11\,(i-1)(\text{mod } 16),$$

$$i = 1,2,\ \ldots\ldots,\ 15,\ 16,$$

i. e.

$$\underline{\alpha} = \{0,11,6,1,12,7,2,13,8,3,14,9,4,15,10,5\}\ ,$$

$$\beta_i \equiv \frac{\alpha_i}{2}\ (\text{mod } 17) \equiv 9\alpha_i\ (\text{mod } 17)\ ,$$

$$i = 1,2,\ \ldots,\ 15,16,$$

i. e.

$$\underline{\beta} = \{0,14,3,9,6,12,1,15,4,10,7,13,2,16,5,11\}$$

and

$$y_i \equiv \beta_i\ (\text{mod } 8)\ ,$$

$$i = 1,2,\ \ldots\ldots,15,\ 16,$$

i.e.

$$\underline{y} = \{\ 0,6,3,1,6,4,1,7,4,2,7,5,2,0,5,3\}\ .$$

Case (iii). $q = 30$, $r = 24$, $M = 54$. (9-cyclic)

In this case $d = 6$, $q^* = 5$, $r^* = 4$, $k = 1$ and $d_1 = 4$.

By formula (13),

$$\alpha_i \equiv (i-1)(\text{mod } 54),$$

$$i = 1,2,\ \ldots,\ 53,54,$$

i.e.

$$\alpha_1 = 0, \alpha_2 = 1,\ldots,\alpha_{24} = 23, \alpha_{25} = 24, \alpha_{26} = 25,\ldots,\alpha_{31} = 30, \alpha_{32} = 31,\ldots\alpha_{54} = 53,$$

$$\beta_i \equiv \frac{\alpha_i}{6}\ (\text{mod } 55) \equiv 46\alpha_i\ (\text{mod } 55),$$

$$i = 1,2,\ldots,53,\ 54,$$

i.e.

$\beta_1 = 0,\ \beta_2 = 46, \dots, \beta_{24} = 13,\ \beta_{25} = 4, \beta_{26} = 50, \dots, \beta_{31} = 5\ \ \beta_{32} = 51, \dots, \beta_{54} = 18$ and

$$y_i \equiv \frac{\beta_i}{4} \pmod 9 \equiv 7\beta_i \pmod 9$$

$$i = 1,2, \dots, 53,54,$$

i.e.

$y_1 = 0,\ y_2 = 7, \dots, y_{24} = 1,\ y_{25} = 1, y_{26} = 8, \dots, y_{31} = 8,\ y_{32} = 6, \dots, y_{54} = 0.$

A p-cyelic matrix can always be transformed, by a permutation similarity transformation, into form (11). For the case r = 1 this is established by Theorems 2 and 3. We now show that this property is more general and holds for any r which is relatively prime to $q = p - r$.

Assume that the matrix A is p-cyclic, relative to the partitioning π let $\underline{y} = \{y_i\}_{i=1}^{N}$ be an ordering vector for A and define the vector,

$$\left.\begin{array}{l} \underline{\delta} = \{\delta_i\}_{i=1}^{N} \\ \delta_i \equiv r y_i \pmod p,\ i = 1,2,\dots, N, \end{array}\right\} \tag{14}$$

where $p = q + r$ and $(q,r) = 1$. If $A_{ij} \neq \underset{\sim}{0}$ and $i \neq j$ then $y_j - y_i$ is equal to -1 or p - 1, Since,

$$r(p-1) \equiv -r \pmod p \equiv q \pmod p,$$

it follows that the vector δ is such that if $A_{ij} \neq \underset{\sim}{0}$, $i \neq j$, then $\delta_j - \delta_i$ is equal to q or - r. By proving a result similar to Theorem 3 it is easy to show that the matriic PAP^T, obtained by arranging the rows and columns of blocks of A with increasing δ_i., has the form (11). The permutation matrix P may of course be determined by the technique of Example 2.

Example 5. The matrix,

$$A_4 = \begin{bmatrix} a_{11} & 0 & 0 & 0 & a_{15} & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & a_{26} \\ 0 & a_{32} & a_{33} & 0 & 0 & 0 \\ 0 & 0 & a_{43} & a_{44} & 0 & 0 \\ 0 & 0 & 0 & a_{54} & a_{55} & 0 \\ 0 & 0 & 0 & 0 & a_{65} & a_{66} \end{bmatrix}$$

partitioned so that its diagonal submatrices are 1x1, is 5-cyclic and in a normal form An ordering vector for this matrix is the vector

$$\underline{y} = (0,1,2,3,4.5) \quad ,$$

and with r = 2 formula (12) generates the vector,

$$\underline{\delta} = (0,2,4,1,3,0) \; .$$

A permutation matrix P, such that PA_4P^T is A_4, with its rows and columns of blocks arranged with ascending δ_i, is

$$P = P_{43} \;\; P_{23} \;\; P_{36}.$$

The transformed matrix is then

$$PA_4P^T = \left[\begin{array}{cc|c|c|c|c} a_{11} & 0 & 0 & 0 & a_{15} & 0 \\ 0 & a_{66} & 0 & 0 & a_{65} & 0 \\ \hline 0 & 0 & a_{44} & 0 & 0 & a_{43} \\ \hline a_{21} & a_{26} & 0 & a_{22} & 0 & 0 \\ \hline 0 & 0 & a_{54} & 0 & a_{55} & 0 \\ \hline 0 & 0 & 0 & a_{32} & 0 & a_{33} \end{array}\right] ,$$

and has the form (11) with r = 2 and q.= 5 - 2 = 3.

From the above it is clear that the vector $\underline{\delta}$, as defined by (14), may be thought of as a generalized ordering vector and that block form (11) is a generalized normal form of a p-cyclic matrix. It has however been proved by Nichols and Fox (1969) that the optimum convergence rate for the SOR method is obtained when the matrix has the form (11) with r = 1 and q = p-1. (i.e., when the matrix has the form (2)). Hence, in practice one would transform a p-cyclic matrix into form (2) and this could be done by using an ordering vector y satisfying the conditions of Theorem 1, In particular if A has the form (11) and r □ 1 then formula (12) or (13) immediately gives the required $\underline{y}$.

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