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EFFECT OF DISCONTINUOUS BOUNDARY CONDITIONS ON FINITE-DIFFERENCE SOLUTIONS

by

J. Barkley Rosser

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0. Preamble. If one is solving a Laplace differential equation by the standard 5-point or 9-point difference approximation, a discontinuity of the boundary values will cause the approximate solution to have a distinctive error in the interior. The amount and nature of these errors is discussed. A short, but excellent, treatment of one case is to be found on pp. 222-224 of Milne [1],

Since the justification of the 9-point difference approximation involves existence of eight derivatives for the solution, whereas the justification of the 5-point difference approximation involves only four, it is generally assumed that the errors for the 9-point difference approximation will be much worse than for the 5-point difference approximation if there are irregularities on the boundary. In fact, on pp. 222-224 of Milne [1] only the 5-point difference approximation is considered.

The fact is that the 9-point difference approximation gives better overall results than the 5-point. As a possible explanation for this, note that the 9-point difference approximation involves an average over the eight points nearest the center, and thus uses more information than the 5-point difference approximation.

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1 . Removable discontinuities. It can happen through a mischance that a boundary value at some point is incorrectly assigned. If the mistake were noticed, and corrected, there would be no discontinuity. However, in fact, one boundary value is assigned too large by an amount ε.

Let $u(x, y)$ be a solution of the difference approximation with zero boundary values except at the one point, and a value of ε there. Whatever errors might have arisen from the solution with no discontinuity, there will be added to them the value of $u(x, y)$. Let us see what this amounts to in a few cases.

By the maximum principle, $0 < u(x, y) < \varepsilon$ at all interior points. Consider first a square, with a value of e at the midpoint of the upper Edge and the value zero elsewhere on the boundary. Let us set up a Grid on the square, composed of squares of side unity. Let

$$\phi(r,s) = e^{\pm s\beta} e^{\pm ir\alpha}. \tag{1.1}$$

If $\phi(r, s)$ is to satisfy the 5-point difference equation $\Delta_5\phi(r, s) = 0$, then we must have

$$0 = (2 - \cos\alpha - \cosh\beta)\,\phi(r, s) . \tag{1. 2}$$

If $\phi(r, s)$ is to satisfy the 9-point difference equation $\Delta_9\phi(r, s) = 0$, then we must have

$$0 = (5 - 2\cos\alpha - 2\cosh\beta - \cos\alpha\cosh\beta)\,\phi(r, s) = 0 . \tag{1. 3}$$

Evidently the same conditions hold if we take any linear combination of the right sides of (1.1), such as

$$\sinh(N - s)\,\beta\,\sin r\alpha .$$

Take

(1.4) $$\beta = 2\log\left\{\sin\frac{\alpha}{2} + \sqrt{1+\sin^2\frac{\alpha}{2}}\right\}$$

Then

$$e^{\beta} = \left\{\sin\frac{\alpha}{2} + \sqrt{1+\sin^2\frac{\alpha}{2}}\right\}^2$$

$$= 1 + 2\sin^2\frac{\alpha}{2} + 2\sin\frac{\alpha}{2}\sqrt{1+\sin^2\frac{\alpha}{2}}\ ,$$

$$e^{-\beta} = 1 + 2\sin^2\frac{\alpha}{2} - 2\sin\frac{\alpha}{2\sqrt{1+\sin^2\frac{\alpha}{2}}}\ ,$$

$$\cosh\beta = 1 + 2\sin^2\frac{\alpha}{2} = 2 - \cos\alpha\ .$$

So with this choice of β, we have $\Delta_5\phi(r, s) = 0$.

<u>Theorem 1.1</u> Any linear combination of

(1.5) $$\left\{\sin\frac{\alpha}{2} + \sqrt{1+\sin^2\frac{\alpha}{2}}\right\}^{\pm 2s} e^{\pm ira}$$

satisfies the 5-point difference approximation for the Laplace operator.

Similarly, take

(1.6) $$\beta = \log\frac{(5-2\cos\alpha) + \sqrt{(5-2\cos\alpha)^2 - (2+\cos\alpha)^2}}{2+\cos\alpha}$$

Then

$$e^{\beta} = \frac{(5-2\cos\alpha) + \sqrt{(5-2\cos\alpha)^2 - (2+\cos\alpha)^2}}{2+\cos\alpha}$$

$$e^{-\beta} = \frac{(5-2\cos\alpha) - \sqrt{(5-2\cos\alpha)^2 - (2+\cos\alpha)^2}}{2+\cos\alpha}$$

$$\cosh\beta = \frac{5-2\cos\alpha}{2+\cos\alpha}\ .$$

So with this choice of ß we have $\Delta_9\Phi(r, s) = 0$.

Theorem 1.2. Any linear combination of

$$\left\{\frac{(5 - 2\cos\alpha) + \sqrt{(5 - 2\cos\alpha)^2 - (2 + \cos\alpha)^2}}{2 + \cos\alpha}\right\}^{\pm s} e^{\pm ir\alpha} \tag{1.7}$$

satisfies the 9-point difference approximation for the Laplace equation.

Theorem 1.3. Let k and Ω be positive integers, with k not an integer multiple of 2Ω. Then for even k

$$\sum_{n=0}^{\Omega-1} \cos\frac{kn\pi}{\Omega} = 0 \quad , \tag{1.8}$$

and for odd k

$$\sum_{n=0}^{\Omega-1} \cos\frac{kn\pi}{\Omega} = 0 \quad , \tag{1.9}$$

Proof. Although the proof is well known, we will repeat it, since it is quite short. We have

$$\sum_{n=0}^{\Omega-1} \frac{kn\pi}{\Omega} = \frac{1}{2\sin\frac{k\pi}{2\Omega}} \sum_{n=0}^{\Omega-1} \left\{\sin\frac{k(2n+1)\pi}{2\Omega} - \sin\frac{k(2n-1)\pi}{2\Omega}\right\}$$
$$= \frac{1}{2\sin\frac{k\pi}{2\Omega}} \left\{\sin\frac{k(2\Omega-1)\pi}{2\Omega} - \sin\frac{(-k\pi}{2\Omega}\right\}$$

From this, our result follows.

Corollary 1. For $0 < k \le \Omega$ and $0 \le j < \Omega$, we have

$$\sum_{n=1}^{\Omega-1} \sin\frac{kn\pi}{\Omega} \sin\frac{in\pi}{\Omega} = \frac{\Omega}{2}\delta_{kj} \, . \tag{1.10}$$

Proof. Write (1.10) as

$$\frac{1}{2}\sum_{n=0}^{\Omega-1}\left\{\cos\frac{(k-j)n\pi}{\Omega}-\cos\frac{(k+j)n\pi}{\Omega}\right\}$$

Corollary 2. For $0 < k < 2\Omega$

$$\sum_{m=0}^{\Omega-1}(-1)^m \sin\frac{(2m+1)k\pi}{2\Omega} = \Omega\delta_{k\Omega} \quad . \tag{1.11}$$

Proof. Replace Ω by 2Ω in Corollary 1, and take $j = \Omega$.

Theorem 1. 4. In the square of side 2Ω in the r-s-plane with the left lower vertex at the origin

$$\frac{\varepsilon}{\Omega}\sum_{m=0}^{\Omega-1}(-1)^m\frac{\sinh s\beta_m}{\sinh 2\Omega\beta_m}\sin\frac{(2m+1)r\pi}{2\Omega} \quad ,$$

with β_m defined by (1.4) using $\alpha = (2m+1)\pi/2\Omega$, satisfies the 5-point difference approximation, is zero for $r = 0$ and $r = 2\Omega$ and $s = 0$, while for $s = 2ft$ it is zero except at $r = \Omega$, where it takes on the value $.\varepsilon$

Proof. By Theorem 1.1,the given expression satisfiesthe5-point difference approximation. Obviously it is zero for $r = 0$ and $r = 2\Omega$ and $s = 0$. For $s = 2\Omega$, it reduces to(1.11), so that the theorem holds there also.

Theorem 1.5. In the square of side 2Ωft in the r-s-plane with the left lower vertex at the origin

$$\frac{\varepsilon}{\Omega}\sum_{m=0}^{\Omega-1}(-1)^m\frac{\sinh s\beta_m}{\sinh 2\Omega\beta_m}\sin\frac{(2m+1)r\pi}{2\Omega} \quad ,$$

with β_m defined by (1. 6) using $\alpha = (2m + 1)\pi/2\Omega$, satisfies the 9-point difference approximation, is zero for $r = 0$ and $r = 2\Omega$ and $s = 0$, while for $s = 2\Omega$ it is zero except at $r = \Omega$, where it takes on the value ε.

The expressions appearing in Theorems 1. 4 and 1. 5 are easily evaluated on a computer. Taking $\Omega = 32$ gave the results in Tables 1 and 2, where the values tabulated are to be multiplied by $\varepsilon X10^{-6}$ There is symmetry about the vertical line $r = 32$, which is why only values for $r \geq 32$ are shown.

There is not much choice between the 5-point and 9-point values. However, in a given horizontal line, the 9-point values have a smaller maximum and are more smoothly graduated, hence preferable.

There is the question how these values would behave if we change the square to a rectangle or move the point of discontinuity to different points on the top. For the 5-point approximation, the value at the grid point just below the ε cannot be less than e/4, since it is one quarter the sum of four non-negative quantities, one of which is ε For the 9-point approximation, the corresponding value cannot be less than $\varepsilon/5$.

To determine the largest possible values for the grid points, we let $\Omega \to \infty$.

<u>Theorem 1,6</u>. In the lower half plane

$$\frac{\varepsilon}{\pi} \int_0^{\pi} \left\{ \sin \frac{x}{2} + \sqrt{1 + \sin^2 \frac{x}{2}} \right\}^{2s} \cos rx \, dx$$

s \ r	32	33	34	35	36	37	38	39	40
64	1000000	0	0	0	0	0	0	0	0
63	363247	136486	60899	31791	18833	12265	8558	6282	4791
6Z	180014	121799	75319	47431	31278	21667	15685	11781	9119
61	113214	9 537 5	71148	51335	37181	27442	20734	160 37	12671
60	82089	75342	62563	49579	38670	30184	23774	18961	15327
59	64458	61340	54183	45747	377 36	30851	25215	20706	17124
58	53064	51377	47081	41490	35677	30267	25532	21522	18186
57	45043	44023	41276	37457	33213	29011	25122	21666	18667
56	39062	38395	36542	3 3849	30706	27441	24279	21351	18716

TABLE 1. Values from Theorem 1-4 multiplied by $10^6/\varepsilon$.
5-point approximation.

S \ R	32	33	34	35	36	37	38	39	40
64	1000000	0	0	0	0	0	0	0	0
63	309268	161804	63350	31677	18587	12107	8466	6228	4757
62	159379	126701	79420	48702	31558	21680	15641	11735	9082
61	105808	95034	73054	52657	37793	27679	20810	16049	12662
60	79063	74358	63123	50395	39251	30513	23939	19036	15357
59	63002	60549	54214	46142	38147	31155	2 5409	20819	17186
58	52258	50823	46949	41641	35924	30 500	25710	21645	18266
57	44550	43639	41114	37487	33345	29170	25265	21779	18750
56	38737	38124	36 393	33825	30767	27541	24385	21445	18793

TABLE 2. Values from Theorem 1. 5 multiplied by $10^6/\varepsilon$.
9-point approximation.

satisfies the 5-point difference approximation, approaches zero as $r \to \infty$ or $r \to -\infty$ or $s \to -\infty$, while for $s = 0$ it is zero except at $r = 0$, where it takes on the value ε.

Proof. By Theorem 1.1 the integrand satisfies the 5-point difference approximation. So it still will do so after being integrated. The rest follows easily.

Theorem 1. 7. In the lower half plane

$$\frac{\varepsilon}{\pi} \int_0^\pi \left\{ \frac{5 - 2 \cos x + \sqrt{(5 - 2 \cos x)^2 - (2 + \cos x)^2}}{2 + \cos x} \right\} \cos rx \, dx$$

satisfies the 9-point difference approximation, approaches zero as $r \to \infty$ or $r \to -\infty$ or $s \to -\infty$, while for $s = 0$ it is zero except at $r = 0$, where it takes the value ε.

The integrals given can easily be evaluated on a computer by standard quadrature formulas. We used (25.4.18) of Abramowitz and Stegun [2] with $h = \pi/2^{10}$. Subsequent checking verified that a much larger value of h would have sufficed. The values, except for a factor, are given in Tables 3 and 4.

The values given in these tables for any grid point are the largest possible for a convex figure with a horizontal tangent at the origin. For consider a figure with part of its top boundary coinciding with the r-axis, including the origin. Around the boundary off the r-axis, positive values will be given by Theorems 1. 6 or 1.7. By the principle of the maximum, if we approximate a harmonic function which has these values

s \ r	0	1	2	3	4	5	6	7	8
0	1000000	0	0	0	0	0	0	0	0
-1	363380	136620	61033	31925	18969	12401	8695	6421	4931
-2	180281	122066	75587	47699	31548	21939	15959	12058	9399
-3	113613	95776	71549	51737	37586	27849	21145	16451	13090
-4	82621	75874	63097	50114	39209	30726	24320	19513	15885
-5	65122	62004	54849	46415	38408	31527	25897	21394	17820
-6	53858	52172	47878	42290	36480	31077	26348	22346	19019
-7	45966	44947	42202	38387	34147	29952	26070	22623	19636
-8	40114	39448	37 597	34907	31770	28512	25359	22441	19818

TABLE 3. Values from Theorem 1.6 multiplied by $10^6/\varepsilon$
5-point approximation.

s \ r	0	1	2	3	4	5	6	7	8
0	1000000	0	0	0	0	0	0	0	0
-1	309401	161937	63484	31811	18723	12243	8603	6366	4897
-2	159646	126968	79688	48970	31828	21952	15915	12012	9362
-3	106208	95434	73455	53059	38198	28086	21220	16464	13081
-4	79595	74890	63657	509 31	39790	31055	24485	19 588	15915
-5	63666	61213	54879	46810	38819	31831	26091	21507	17883
-6	53053	51618	47746	42441	36728	31309	26526	22469	19099
-7	45473	44564	42041	38417	34279	30110	26214	22736	19718
-8	39789	39177	37 448	34883	31831	28612	25465	22535	19894

TABLE 4. Values from Theorem 1.7 multiplied by $10^6/\varepsilon$.
9-point approximation.

off the r-axis and is zero on the r-axis, all interior points will have positive values. We subtract this function from that given by Theorems 1. 6 or 1.7 to get a function which is zero all around the boundary except at the origin, where it is ε.

So the value at the grid point just below ε will always lie somewhere between $\varepsilon/5$ and $2\varepsilon/5$, whether one uses the 5-point or the 9-point formula. There is one exception to this, namely if ε is at a corner. In that case, the value ε would not enter at all into the calculation by means of the 5-point difference approximation; it would give values all identically zero, which is the correct value. However, with the 9-point difference approximation a value of e at a comer will have an effect. We will give especial attention to this later.

Suppose we put e at 2N + 1 consecutive points on the r-axis, from r = -N to r = +N inclusive. The 5-point approximation in the lower half plane can be obtained by shifting the solution of Theorem 1. 6 right a nd left and adding. If we go to a low enough value of s, the values of the sums will be less than ε. Also, if we go far enough to either side, the same will be true. So, by the principle of the maximum, all values in the large rectangle will be less than ε. However, the values at (0, -S) will be the sum of 2N + 1 values from Theorem 1.6 at s = -S from r = -N to r = N inclusive. So the sum of these values is less than ε for each N. It follows that the sum of the values in each row is finite, and less than or equal to ε.

Let S_s be the sum of the elements in the s-th row. Because our values satisfy the 5-point approximation, we readily conclude

(1.12) $$g_s - g_{s+1} + g_{s+2} = 0.$$

We have of course

(1.13) $$g_0 = \varepsilon \cdot$$

The general solution of (1.12) is

$$g_s = A + Bs$$

We have no trouble concluding that

(1.14) $$g_s = \varepsilon$$

In a similar way, we conclude that for the values given in Theorem 1.7 the sum of the values in a given row is ε.

2. Genuine discontinuities of the boundary values. For the 5-point difference approximation, the discussion in Milne [1] was based on the following idea. Let the discontinuity be a jump of amount ε, located at the origin along the r-axis. If we add

(2.1) $$\frac{\varepsilon}{\pi}\left\{\arctan\frac{-s}{r}-\frac{\pi}{2}\right\}$$

with s in the lower half plane) to the given function, it will become continuous. So we can account for the errors caused by the discontinuity by seeing how much (2.1) differs from its 5-point approximation. Except for a factor of $-\frac{1}{2}\varepsilon$ this will be given by solving for the 5-point difference approximation in the lower half plane for the function which is +1 along the positive part of the real axis and -1 along the negative part of the real axis, and comparing it with

$$1 + \frac{2}{\pi} \arctan \frac{s}{r} \tag{2.2}$$

in the lower half plane.

By anti-symmetry, we can take the approximation to the function which is zero on the negative imaginary axis and +1 on the positive real axis, and compare it with

$$- \frac{2}{\pi} \arctan \frac{r}{s} . \tag{2.3}$$

One can do the same for the 9-point difference approximation.

We have expanded on Milne by looking at the 9 - point case, as well as at other cases than the one of infinite extent. In order not to confuse the issue with two different discontinuities at once we consider a square with zero values on three sides, while on top the values proceed linearly from 1 down to zero as one proceeds from left to right.

Lemma 2.1. For odd k

$$\sum_{n=1}^{\Omega-1} \cos \frac{kn\pi}{\Omega} = 0 \tag{2.4}$$

Proof. As k is odd, it cannot be an integer multiple of 2Ω, so that we may appeal to (1.9).

Lemma 2. 2. If $\Omega \geq 2$ then

$$\sum_{n=0}^{\Omega-1} \cos^2 \frac{n\pi}{\Omega} = \frac{\Omega}{2} . \tag{2.5}$$

Proof. We have

$$\sum_{n=0}^{\Omega-1} \cos^2 \frac{n\pi}{\Omega} = \frac{1}{2} \sum_{n=0}^{\Omega-1} \left\{\cos \frac{2n\pi}{\Omega} + 1\right\}$$

As $\Omega \geq 2$, we may appeal to (1.8).

Lemma 2. 3. Let $1 < k < 2\Omega - 1$. Then for odd k

$$\sum_{n=0}^{\Omega-1} \cos \frac{n\pi}{\Omega} \cos \frac{kn\pi}{\Omega} = 0 \quad , \tag{2.6}$$

and for even k

$$\sum_{n=0}^{\Omega-1} \cos \frac{n\pi}{\Omega} \cos \frac{kn\pi}{\Omega} = 1 \quad , \tag{2.7}$$

Proof. We have

$$\sum_{n=0}^{\Omega-1} \cos \frac{n\pi}{\Omega} \cos \frac{kn\pi}{\Omega} = \frac{1}{2} \sum_{n=0}^{\Omega-1} \left\{ \cos \frac{(k+1)n\pi}{\Omega} + \cos \frac{(k-1)n\pi}{\Omega}\right\} \quad .$$

Now use Theorem 1.3.

Corollary. If $1 < k < 2\Omega - 1$ and k is even, then

$$\sum_{n=0}^{\Omega-1} \cos \frac{n\pi}{\Omega} \cos \frac{kn\pi}{\Omega} = 0 \quad ,$$

Theorem 2. 1. Let $0 < r < 2\Omega$. Then

$$\sum_{n=1}^{\Omega-1} \frac{1 + \cos \frac{\pi n}{\Omega}}{\sin \frac{\pi n}{\Omega}} \sin \frac{\pi r n}{\Omega} = \Omega - r \quad . \tag{2.8}$$

Proof. Let

$$\Theta = \sum_{n=1}^{\Omega-1} \frac{1 + \cos \frac{\pi n}{\Omega}}{\sin \frac{\pi n}{\Omega}} \sin \frac{\pi r n}{\Omega} \quad .$$

Case1.Letrbeeven.Recalling $2i \sin \frac{\pi rn}{\Omega}\ e^{\pi rni/\Omega} - e^{-\pi mi/\Omega}$,

and $2i \sin \frac{\pi rn}{\Omega}\ e^{\pi rni/\Omega} - e^{-\pi mi/\Omega}$, we get

$$\Theta = \sum_{n=1}^{\Omega-1} \left\{1 + \cos \frac{\pi n}{\Omega}\right\}\left\{e^{\pi i(r-1)n/\Omega} + e^{\pi i(r-5)n/\Omega} + e^{\pi i(r-5)n/\Omega}\right.$$

$$+ \ . \ . \ . + e^{-\pi i(r-5)n/\Omega} + e^{\pi i(r-3)n/\Omega} + e^{\pi i(r-l)n/\Omega}\}$$

As $e^{\pi i}(r-l)n/\Omega + e^{-\pi i(r-l)n/\Omega} = 2 \cos(r-l)Trn/n$, etc., we get

$$\Theta = 2\sum_{n=1}^{\Omega-1} \left\{1 + \cos \frac{\pi n}{\Omega}\right\}\left\{\cos \frac{\pi n}{\Omega} + \cos \frac{3\pi\pi}{\Omega}\right.$$
$$\left. + \ldots + \cos \frac{(r-3)\pi)}{\Omega} + \cos \frac{(r-1)\pi)}{\Omega} \right\} .$$

By Lemma 2.1, we get

$$\Theta = 2\sum_{n=1}^{\Omega-1} \cos \frac{\pi n}{\Omega} \left\{\cos \frac{\pi n}{\Omega} + \cos \frac{3\pi\pi}{\Omega} + \ldots + \cos\cos \frac{(r-3)\pi)}{\Omega} + \cos \frac{(r-1)\pi)}{\Omega}\right\}$$

$$= -r + 2\sum_{n=0}^{\Omega-1} \cos \frac{\pi n}{\Omega} \left\{\cos \frac{\pi n}{\Omega} + \cos \frac{3\pi\pi}{\Omega}\right.$$

$$\left. + \ldots\ldots + \cos \frac{(r-3)\pi)}{\Omega} + \cos \frac{(r-1)\pi)}{\Omega} \right\}$$

Our theorem now follows by Lemmas 2. 2 and 2.3.

Case 2. Let r be odd. As before, we get

$$\Theta = \sum_{n=1}^{\Omega-1} \{1 + \cos \frac{\pi n}{\Omega}\} \{1 + 2 \cos \frac{2\pi\pi}{\Omega}$$
$$+ 2 \cos \frac{4\pi\pi}{\Omega} + \ldots\ldots + 2 \cos \frac{(r-3)\pi^{n}}{\Omega} + 2 \cos \frac{(r-1)\pi)}{\Omega}\}$$

By Lemma 2.1 and the Corollary to Lemma 2. 3, we get

$$\Theta = \Omega - 1 + 2\sum_{n=1}^{\Omega-1} \{\cos\frac{2\pi\pi}{\Omega} + \cos\frac{4\pi\pi}{2} + \ldots + \cos\frac{(r-3)\pi)}{\Omega} + \cos\frac{(r-1)\pi)}{\Omega}\}$$

$$= \Omega - 1 - 2\frac{r-1}{2} + 2\sum_{n=0}^{\Omega-1}\{\cos\frac{2\pi\pi}{\Omega}$$

$$+ \cos\frac{4\pi\pi}{2} + \ldots + \cos\frac{(r-3)\pi)}{\Omega} + \cos\frac{(r-1)\pi)}{\Omega}$$

Our theorem now follows by Theorem 1.3.

Theorem 2.2. In the square of side Ω in the r-s-plane with the left lower corner at the origin

$$\frac{1}{\Omega}\sum_{n=1}^{\Omega-1}\frac{\sinh s\beta_n}{\sinh \Omega\beta_n}\,\frac{1+\cos\frac{\pi n}{\Omega}}{\sin\frac{\pi n}{\Omega}}\sin\frac{\pi r n}{\Omega}$$

is zero for $r = 0$ and $r = \Omega$ and for $s = 0$, while for $s = \Omega$ it decreases linearly from 1 to 0 as r goes from O to Ω (except at $r = 0$). If $ß_n$ is defined by (1.4) using $\alpha = \pi n/\Omega$, it satisfies the 5-point difference approximation. If $ß_n$ is defined by (1.6) using $\alpha = \pi n/\Omega$ it satisfies the 9-point difference approximation.

It is of course a trivial exercise in Fourier analysis (see Rosser [3]) to find the harmonic function on the square which assumes the same boundary conditions. We have listed 10^7 times the difference (the harmonic function value minus the value given in Theorem 2. 2) for both the 5-point case and the 9-point case, for a square of side 8, in Tables 5 and 6.

s \ r	1	2	3	4	5	6	7
7	0	71749	45364	24887	13661	7320	3224
6	-71749	0	23556	22249	15857	9753	4598
5	-45364	-23556	0	9510	10390	7773	4028
4	-24887	-22249	-9 510	0	4045	4297	2552
3	-13661	-15857	-10390	-4045	0	1471	1161
2	-7320	-9753	-7773	-4297	-1471	0	310
1	-3224	-4598	-4028	-2552	-1161	-310	0

TABLE 5. 10^7 times harmonic minus 5-point.
5-point values from Theorem 2.2.

s \ r	1	2	3	4	5	6	7
7	286462	80419	30653	13893	6969	3550	1519
6	85219	66680	36454	19816	10946	5870	2578
5	31451	37120	28193	1852 5	11445	6548	297 3
4	14095	20108	18651	14260	9746	5937	2789
3	7043	11067	11534	9779	7223	4634	2241
2	3581	5923	6596	5965	4643	3091	1528
1	1532	2598	2994	2803	2248	1529	766

TABLE 6. 10^7 times harmonic minus 9-point.
9-point values from Theorem 2.2.

We have not listed the values for $r = 0$, $r = 8$, $s = 0$, and $s = 8$, since these are naturally zero.

Various things strike us about these two tables. One is the antisymmetry displayed in Table 5. However, that is easily explained. Suppose we reflect the square across the diagonal connecting its upper left corner and lower right corner, and add to the original square. As far as the harmonic function is concerned, we will obviously get

$$\frac{(8 - r)s}{64}, \tag{2.9}$$

since this is harmonic and satisfies the boundary conditions. As far as the 5-point approximation is concerned, we will get the same, and for the same reasons. This explains the antisymmetry.

Actually, the 5-point approximation does not satisfy (2.9) in the upper left hand corner. However, what happens in the upper left hand corner does not enter into the 5-point approximation in any way.

Not so for the 9-point. The 9-point approximation has a 0 in the upper left hand corner. After reflection, the same holds. However, (2.9) equals unity in the upper left hand corner. If we add together the 9-point approximation and its reflection, and then add in the 9-point approximation for the square with unity in the upper left corner and zero at all other boundary points, we will get (2.9) How do we calculate this third approximation? Obviously by subtracting from (2.9) the original 9-point approximation and its reflection. The result, multiplied by 10^7 , is shown in Table 7.

s \ r	1	2	3	4	5	6	7
7	572923	165638	62104	27988	14012	7131	3051
6	165638	133361	73574	39925	22014	11793	5176
5	62104	73574	56387	37177	22979	13144	5967
4	27988	39925	37177	28520	19525	11902	5593
3	14012	22014	22979	19525	14446	9276	4489
2	7131	11793	13144	11902	9276	6181	3057
1	3051	5176	5967	5593	4489	3057	1532

TABLE 7. 9-point approximation for an 8X8 square with a 1 in the upper left corner and zero on the rest of the boundary, multiplied by 10^7 .

s \ r	1	2	3	4	5	6	7
7	0	-2400	-399	-101	-37	-16	-6
6	2400	0	-333	-146	-61	-26	-10
5	399	333	0	-63	-44	-24	-10
4	101	146	63	0	-16	-14	-7
3	37	61	44	16	0	-5	-3
2	16	26	24	14	5	0	-1
1	6	10	10	7	3	1	0

TABLE 8. 10^7 times harmonic minus modified 9-point.

The difficulty is that the formula in Theorem 2. 2 is zero in the upper left hand corner, rather than the limit of the values along the top edge. What we should put in the upper left hand corner is the average of the limits along the left edge and along the top, namely $\frac{1}{2}$. That would amount to adding half the values in Table 7 to the original 9-point approximation. For the difference of this and the harmonic function, we subtract half the values of Table 7 from those of Table 6. The results are shown, multiplied by 10^7, in Table 8.

So Table 8 shows the errors if one uses the appropriately chosen 9-point approximation. The improvement over the 5-point approximation, whose errors are shown in Table 5, is fantastic

There is some merit in comparing our results so far with those of Milne [l]. In T able 5, the errors for s = 0 and r = 8 are zero, of course, since we have gotten to the other boundary. Milne is comparing the 5-point approximation for the infinite quarter plane with values of (2. 3), but reflected about the line x = -y, and with the harmonic function subtracted from the 5-point approximation (instead of vice versa), and with y for -y. So (see his p. 223) when he gets to eight units from the boundary, his values are several units times 10^{-4} , rather than zero. Nevertheless, his values out to four grid points from the point of discontinuity agree with Table 5 to a very few percent.

So in a region of larger size one would expect agreement with Milne's values for quite a distance out. To check this, we computed the equivalents of Tables 5 through 8 for a square of side 64, except that we took only the 8x8 blocks in the four corners. In the upper left hand corner, next to the discontinuity, the agreement with Milne's values (rather, with more accurate ones which we will present) was such than only a very accurate calculation can detect the difference. *In* the other three corners the 5-point approximation and the true solution agreed to better than three significant figures, and often up to four or five significant figures. This supports the customary doctrine that away from a discontinuity one may expect accuracy of the order of h^{-2}.

Suppose that, to the square of order 8 on which Tables 5-6 are based, one adjoins to the left another square of order 8 which is gotten by reflecting about the left side and changing the signs. Inside the 8X16 rectangle one would still have the 5-point and 9-point approximations satisfied. If we multiply by $\frac{1}{2}\varepsilon$, we would approximate to the case of a discontinuity of amount ε along the top edge. So Tables 5 and 6 to the right, and their reflections with signs changed to the left, would indicate the sorts of errors that would be induced. Notice that at the point of discontinuity we have assigned as a value the average of the right and left limits. If some other value were assigned, we would introduce additional errors, as in Section 1.

As shown in Tables 7 and 8, one can greatly reduce the error for the 9-point approximation by doing a little fiddling with the value at the discontinuity. For the 9-point block in which the discontinuity occurs at the upper left hand corner, one should assign

$$\frac{3}{4}\text{(limit from right)} + \frac{1}{4}\text{(limit from left)} .$$

If the discontinuity occurs at the upper right hand corner, one should assign

$$\frac{1}{4}\text{(limit from right)} + \frac{3}{4}\text{(limit from left)} .$$

Then one will get errors like $\frac{1}{2}\varepsilon \times 10^{-7}$ times the values shown in Table 8.

Tables 5-8 inclusive are based on a square of side 8. For larger squares, the results of a discontinuity are more pervasive. As we noted, by the time one gets to a square of side 64, the errors out as far as 8 grid points are practically indistinguishable from those for the infinite case. So we might see what the latter are.

Milne [1] has given these for the 5-point approximation (see p. 223 of Milne [1]). We define

$$(2.10)\quad u(r,s)=\frac{1}{\pi}\int_0^{\pi}\left\{\sin\frac{x}{2}+\sqrt{1+(\sin)\frac{x}{2})^2}\right\}^{2s}\frac{(1+\cos x)\sin\ rx\ dx}{\sin x}$$

(2.11) $U(r,s)=$

$$\frac{1}{\pi}\int_0^{\pi}\left\{\frac{(5-2\cos x\sqrt{(5-2\cos\ x)^2-(2+\cos\ x)^2}}{2+\cos x}\right\}\frac{(1+\cos x)\sin\ rx\ dx}{\sin x}$$

Theorem 2.3. Both $u(r, s)$ and $U(r, s)$ are zero for $r - 0$. For $s = 0$, both are +1 for positive integral values of r and -1 for negative integral values of r. In the lower half plane $u(r, s)$ satisfies the 5-point approximation, and U(r, s) satisfies the 9-point approximation.

Proof. By Theorems 1.1 and 1. 2, we verify that u(r, s) and U(r, s) satisfy the 5-point and 9-point approximations. As u(r, s) and U(r, s) are odd functions of r, it remains to evaluate them for s = 0 and r a positive integer. As

$$1 + \cos x = \frac{1}{2} e^{ix} + 1 + \frac{1}{2} e^{-ix}$$

and

$$\frac{\sin rx}{\sin x} = e^{(r-1)ix} + e^{(r-3)ix} + \ldots + e^{-(r-3)ix} + e^{-(r-1)ix},$$

we get the value 1 by direct integration.

The formulas for u(r, s) and U(r, s) were derived by interpreting the formula of Theorem 2. 2 as a Riemann sum for an integral and taking the limit as $\Omega \to \infty$. I wish to thank Prof. Herbert S. Wilf for this suggestion. The values of u(r, s) and U(r, s) were calculated by using (25.4.18) of Abramowitz and Stegun [2] with $h = \pi/2^{10}$. A check with $h = \pi/2^9$ showed that we could have used a larger value of h.

In Table 9 we have tabulated

$$-10^7\left(\frac{2}{\pi} \arctan \frac{r}{s} + u(r,s)\right).$$

Thus this is an extension of the table on p. 223 of Milne [1]. It would be interesting to compare these with the errors obtained in the upper left hand 8 × 8 block that we computed earlier for the square of side 64.

However, to the accuracy shown, there is no difference! For example, at the eighth grid point down, next to the left edge, the errors were

$$-0.00039\ 46571$$

for the square of side 64 and

$$-0.00039\ 46587$$

for the infinite case.

One can fill out other parts of Table 9 by appealing to the anti-symmetry.

In Table 10 we have given an abridged table of

$$-10^7\left(\frac{2}{\pi}\arctan\frac{r}{s} + U(r,s)\right) \quad .$$

As with Tables 7 and 8, we wish to use a more appropriate 9-point solution. If we form

$$1 - U(r,s) - U(s,r) \ ,$$

we will get the 9-point solution for the case in which we have unity at the origin and zero along both the positive r-axis and the positive s-axis. 10^7 times this is tabulated in Table 11 If we add half this to U(r, s) (representing having $\frac{1}{2}$ at the origin instead of 0) we get the appropriate 9-point approximation. In Table 12 we have tabulated 10 times the difference between this and $-\frac{2}{\pi}$ arctan $\frac{r}{s}$. If desired, one could fill out the rest of Table 10 by adding half the values of Table 11 to the values of Table 12. One can fill out other parts of Tables 11 and 12 by appealing to symmetry and antisymmetry respectively.

S \ r	1	2	3	4	5	6	7	8
-1	0	71800	45563	25370	14600	8929	5790	3947
-2	-71800	0	23797	22989	17406	12469	8915	6474
-3	-45563	-23797	0	10116	12005	10808	8893	7092
-4	-25370	-22989	-10116	0	5118	6846	6869	6203
-5	-14600	-17406	-12005	-5118	0	2917	4214	4549
-6	-8929	-12469	-10808	-6846	-2917	0	1810	2759
-7	-5790	-8915	-8893	-6869	-4214	-1810	0	1197
-8	-3947	-6474	-7092	-6203	-4549	-2759	-1197	0

TABLE 9. $-\,10^7\{\frac{2}{\pi}\arctan\frac{r}{2} + u(r,s)\}$

S \ r	9	10	11	12	13	14	15	16
-1	2803	2059	1555	1203	949	762	621	512
- 2	4800	3635	2808	2209	1765	1431	1176	977
- 3	5613	4456	3565	2880	2350	1937	1613	1354
- 4	5349	4518	3786	3169	2660	2242	1900	1620
- 5	4388	4008	3558	3113	2705	2344	2032	1764
- 6	3137	3176	3030	2797	2533	2269	2020	1794
- 7	1897	2242	2354	2324	2213	2061	1894	1725
- 8	831	1356	1651	1785	1810	1768	1685	1581
- 9	0	600	1002	1249	1381	1431	1427	1387
-10	-600	0	447	760	965	1088	1148	1165
-11	-1002	-447	0	341	590	761	871	933
-12	-1249	-760	-341	0	267	467	610	707
-13	-1381	-965	-590	-267	0	212	375	496
-14	-1431	-1088	-761	-467	-212	0	172	306
-15	-1427	-1148	-871	-610	-375	-172	0	141
-16	-1387	-1165	-933	-707	-496	-306	-141	0

TABLE 9 (cont) $-\,10^7\{\frac{2}{\pi}\arctan\frac{r}{s} + u(r,s)\}$.

s \ r	17	18	19	20	21	22	23	24
- 1	428	361	307	263	228	198	173	153
- 2	820	695	594	511	443	386	339	299
- 3	1147	979	842	729	635	556	490	433
- 4	1389	1198	1039	906	795	700	619	550
- 5	1536	1343	1178	1037	916	813	724	647
- 6	1591	1412	1255	1117	997	892	800	720
- 7	1563	1413	1275	1150	1038	938	849	769
- 8	1469	1355	1244	1139	1041	952	870	795
- 9	1325	1251	1172		1012	937	865	799
-10	1150	1116	1069	1014	956	897	839	783
-11	960	962	945	917	880	839	795	751
-12	767	800	811	806	790	766	737	705
-13	581	638	672	688	691	684	669	650
-14	409	484	536	569	588	596	595	587
-15	253	341	406	454	486	507	517	520
-16	117	212	287	344	388	418	439	451
-17	0	98	179	244	295	334	362	382
-18	-98	0	83	152	209	254	289	316
-19	-179	-83	0	71	131	180	220	252
-20	-244	-152	-71	0	61	113	156	192
-21	-295	-209	-131	-61	0	53	98	137
-22	-334	-254	-180	-113	-53	0	46	86
-23	-362	-289	-220	-156	-98	-46	0	41
-24	-382	-316	-252	-192	-137	-86	-41	0

TABLE 9 (cont). $-10^7\left\{\frac{2}{\pi}\arctan\frac{r}{s}+u(r,s)\right\}$.

s \ r	1	2	3	4	5	6	7
-1	286615	80726	31116	14523	7794	4628	2960
-2	85525	67291	37369	21050	12542	7923	5271
-3	31914	38035	29549	20324	13722	9402	6606
-4	14725	23343	20450	16591	12611	9402	7020
-5	7370	12666	13813	12645	10613	8550	6775
-6	4663	7982	9457	9435	8562	7369	6166
-7	2978	5301	6638	7044	6789	6171	5414

TABLE . $10^7 \{\frac{2}{\pi} \arctan \frac{r}{s} + U(r,s)\}$,

The errors for the 5-point approximation (indicated in Table 9) are far greater than those for the modified 9-point approximation (indicated in Table 12). The difference is rendered even more striking by the fact that Table 9 shows 10^7 times the error, while Table 12 shows 10^9 times the error!

We have looked at squares of side 8 and 64, and at the infinite case, and have found the errors near the discontinuity to be strikingly similar, and only slightly dependent on size. It seems reasonable to assume that the same holds for all sorts of configurations.

As pointed out in Milne [1], if one has a fine mesh and modest requirements for accuracy, it may not be necessary to remove the discontinuity. Tables 8 and 12 show the great advantage of using a 9-point approximation.

3. Discontinuous derivatives Though the boundary values may be continuous, they may have discontinuous derivatives, which causes some trouble. We will look at the case of a square of side 2Ω which is zero on three sides; on the top it rises linearly from zero to $\frac{1}{2}$ at the midpoint and then decreases linearly back to zero. By scaling, this is equivalent to a finer grid on a unit square which is zero on three sides, and on the top rises linearly from zero to $\frac{1}{2}$ at the midpoint and then decreases linearly back to zero.

Lemma 3.1. Let $0 < k < 2\Omega$. Then

$$\sum_{m=0}^{\Omega-1} \cos \frac{(2m+1)k\pi}{2\Omega} = 0 . \tag{3.1}$$

Proof, We have

$$\sum_{m=0}^{\Omega-1} \cos \frac{)2m+1)k\pi}{2\Omega} = \frac{1}{2}\sum_{m=0}^{\Omega-1} \{e^{(2m+1)k\pi i/2\Omega} + e^{-(2m+1)k\pi i/2\Omega}$$

$$= \frac{1}{2}\sum_{m=0}^{\Omega-1} \{e^{(2m+1)k\pi i/2\Omega} + e^{(4\Omega-2m-1)k\pi i/2\Omega}\}$$

$$= \frac{1}{2}\sum_{m=0}^{2\Omega-1} e^{(2m+1)k\pi i/2\Omega} = \frac{1}{2}e^{k\pi i/2\Omega}\sum_{m=0}^{2\Omega-1} e^{mk\pi i/\Omega}$$

$$= \frac{1}{2}e^{k\pi i/2\Omega}\,\frac{1-e^{2k\pi i}}{1-e^{k\pi i/\Omega}} = 0.$$

s \ r	1	2	3	4	5	6	7	8
-1	573229	166251	63029	29248	15664	9290	5938	4017
-2	166251	134582	75405	42393	25208	15905	10572	7341
-3	63029	75405	59099	4077 4	27535	18859	13245	9555
-4	29248	42393	40774	33182	25256	18837	14063	10610
-5	15664	25208	27535	25256	21227	17112	13564	10716
-6	9290	15905	18859	18837	17112	14739	12337	10186
-7	5938	10572	13245	14063	13564	12337	10828	9307
-8	4017	7341	9555	10610	10716	10186	9307	8290

TABLE 11. 10^7 times 9-point with unity in upper left.

s \ r	9	10	11	12	13	14	15	10
	2840	2080	1563	1211	955	765	623	514
- 2	5286	3923	2988	2325	1843	1485	1214	1005
- 3	707 3	5358	4143	326 3	2612	2121	1744	1450
- 4	8119	6308	4974	3979	3224	2644	2192	1836
- 5	8499	6790	547 5	4458	3665	3041	2546	2150
- 6	8371	6884	5682	4716	3939	3312	2804	2389
- 7	7911	6691	5654	4785	4063	3465	2968	2555
- 8	7267	6312	5456	4709	4065	3516	3049	2653
- 9	6550	5830	5149	4527	3973	3485	3059	2691
-10	5830	5305	4779	4277	3812	3391	3014	2679
-11	5149	4779	4384	3989	3608	3252	2925	2628
-12	4527	4277	3989	3684	3379	3084	2805	2546
-13	397 3	3812	3608	3379	3139	2899	2666	2444
-14	3485	3391	3252	3084	2899	2707	2514	2327
-15	3059	3014	2925	2805	2666	2514	2358	2201
-16	2691	2679	2628	2546	2444	2327	2201	2072

TABLE 1l (cont.). 10^7 times 9-Point with unity in upper left.

s \ r	17	18	19	20	21	22	23	24
- 1	429	360	308	264	228	198	174	153
- 2	840	710	60 5	520	450	392	344	30 3
- 3	1219	1033	884	761	660	576	506	446
_ 4	1551	1322	1135	981	853	747	657	581
- 5	1829	1568	1353	1175	1026	901	795	705
- 6	2049	1768	1535	1340	1175	1036	917	816
- 7	2210	1922	1679	1474	1299	1150	1023	913
- 8	2316	2030	1786	1577	1398	1244	1110	995
- 9	2372	2096	1857	1651	1472	1316	1181	1062
-10	2384	2125	1897	1698	1523	1369	1234	1114
-11	2 361	2122	1909	1720	1552	1403	1271	1153
-12	2309	209 3	1897	1721	1563	1421	1293	1179
-13	2236	2043	1866	1704	1557	1423	1302	1193
-14	2147	1978	1819	1673	1538	1413	1300	1196
-15	2048	1901	1761	1630	1507	1393	1288	1191
-16	1943	1817	1695	1578	1468	1364	1267	1177
-17	1836	1728	1622	1520	1422	1328	1240	1157
-18	1728	1637	1547	1457	1371	1287	1207	1132
-19	1622	1547	1470	1392	1316	1242	1171	1102
-20	1520	1457	1392	1326	1260	1195	1131	1069
-21	1422	1371	1316	1260	1203	1146	1089	1034
-22	1328	1287	1242	1195	1146	1096	1046	997
-23	1240	1207	1171	1131	1089	1046	1003	959
-24	1157	1132	1102	1069	1034	997	959	921

TABLE 11 (cont.)- 10^7 times 9-point with unity in upper left.

R / s	1	2	3	4	5	6	7	8
	0	-239987	-39905	-10128	-3795	-1739	-890	-49.1
-2	239987	0	-33308	-14669	-6190	-2907	-1511	-849
-3	3990 5	33308	0	-6335	-4574	-2716	-1593	-962
-4	10128	14669	6335	0	-1726	-1637	-1207	-833
-5	379 5	6190	4574	1726	0	-608	-679	-577
-6	1739	2907	2716	1637	60 8	0	-255	-318
-7	890	1511	1593	1207	679	255	0	-122
-8	491	849	962	833	577	318	122	0

TABLE 12. 10^9 times harmonic minus modified 9-point.

r / s	9	10	11	12	13	14	15	16
- 1	-288	-177	-113	-7 5	-51	-36	-26	-19
- 2	-506	-316	-205	-138	-95	-67	-48	-36
- 3	-602	-390	-261	-179	-126	-90	-66	-49
- 4	-569	-392	-27 3	-194	-140	-103	-77	-58
- 5	-448	-336	-250	-186	-139	-105	-80	-62
- 6	-298	-252	-203	-161	-126	-99	-78	-61
- 7	-164	-165	-148	-127	-105	-86	-70	-57
- 8	-64	-91	-96	-91	-81	-70	-60	-50
- 9	0	-36	-54	-59	-58	-54	-48	-42
-10	36	0	-22	-33	-38	-39	-37	-34
-11	54	22	0	-14	-22	-25	-26	-26
-12	59	33	14	0	-9	-14	-17	-18
-13	58	38	11	9	0	-6	-10	-12
-14	54	39	25	14	6	0	-4	-7
-15	48	37	26	17	10	4	0	-3
-16	42	34	26	18	12	7	3	0

TABLE 12 (cont.). 10 times harmonic minus modified 9-point.

s \ r	17	18	19	20	21	22	23	24
- 1	-14	-11	-8	-6	-5	-4	-3	-3
- 2	-27	-20	-16	-12	-10	-8	-6	-5
- 3	-37	-28	-22	-17	-14	-11	-9	-7
- 4	-44	-34	-27	-21	-17	-14	-11	-9
- 5	-48	-38	-30	-24	-19	-16	-13	-11
- 6	-49	-39	-31	-25	-21	-17	-14	-12
- 7	-46	-38	-31	-25	-21	-17	-15	-12
- 8	-42	-35	-29	-25	-21	-17	-15	-12
- 9	-37	-31	-27	-23	-19	-17	-14	-12
-10	-30	-27	-24	-21	-18	-15	-13	-12
-11	-24	-22	-20	-18	-16	-14	-12	-11
-12	-18	-18	-16	-15	-14	-12	-11	-10
-13	-13	-13	-13	-12	-12	-11	-10	-9
-14	-9	-10	-10	-10	-9	-9	-8	-8
-15	-5	-6	-7	-7	-7	-7	-7	-7
-16	-2	-4	-5	-5	-6	-6	-6	-6
-17	0	-2	-3	-4	-4	-4	-5	-4
-18	2	0	-1	-2	-3	-3	-3	-4
-19	3	1	0	-1	-2	-2	-3	-3
-20	4	2	1	0	-1	-1	-2	-2
-21	4	3	2	1	0	-1	-1	-1
-22	4	3	2	1	1	0	0	-1
-23	5	3	3	2	1	0	0	0
-24	4	4	3	2	1	1	0	0

TABLE 12 (cont.). 10^9 times harmonic minus modified 9-point.

Lemma 3. 2. if $0 < k \le \Omega$ and $0 \le j < \Omega$, then

$$\sum_{m=0}^{\Omega-1} \cos\frac{(2m+1)k\pi}{2\Omega}\cos\frac{(2m+1)j\pi}{2\Omega} = \frac{\Omega}{2}\delta_{jk} . \tag{3.2}$$

Proof. The expression in question equals

$$\frac{1}{2}\sum_{m=0}^{\Omega-1}\left\{\cos\frac{(2m+1)(k+j)\pi}{2\Omega} + \cos\frac{(2m+1)(k-1)\pi}{2\Omega}\right\} .$$

Theorem 3.1. If $0 < |k| < \Omega$, then

$$\sum_{m=0}^{\Omega-1}\frac{\cos\dfrac{(2m+1)k\pi}{2\Omega}}{1-\cos\dfrac{(2m+1)\pi}{2\Omega}} = \Omega(\Omega - |k|). \tag{3.3}$$

Proof. We first note

$$\frac{1}{1-\cos\dfrac{(2m+1)\pi}{2\Omega}} = \frac{(1-i^{2m+1})(1-i-2m-1)}{(1-e^{(2m+1)\pi)\pi i/2\Omega})(1-e^{-(2m+1)\pi)\pi i/2\Omega})} . \tag{3.4}$$

Also

$$\frac{1-i^{2m+1}}{1-e^{(2m+1)\pi)\pi i/2\Omega}} = \frac{1-e^{\Omega(2m+1)\pi)\pi i/2\Omega}}{1-e^{(2m+1)\pi)\pi i/2\Omega}} \tag{3.5}$$

$$=1+e^{(2m+l)\pi i/2\Omega}+e^{2(2m+l)\pi i/2\Omega}+\dots+e^{(\Omega-l)(2m+l)\pi i/2\Omega} .$$

Similarly

$$\frac{1-i^{2m+1}}{1-e^{(2m+1)\pi)\pi i/2\Omega}} \tag{3.6}$$

$$= 1 + e^{-(2m+l)\pi)\pi i/} + e^{-2(2m+l)\pi)\pi i/2\Omega} + \dots + e^{-(\Omega-1)(2m+1)\pi)\pi i/2\Omega}$$

Combining (3.4), (3.5), and (3.6) gives

$$\frac{1}{1 - \cos\frac{(2m+1)\pi}{2\Omega}} = e^{(\Omega - 1)(2m+1)\pi i/2\Omega} + 2e^{(\Omega - 1)(2m+1)\pi i/2\Omega}$$

$$+ \ldots + (\Omega - 1)e^{(2m+1)\pi i/2\Omega} + \Omega + (\Omega - 1)e^{-(2m+1)\pi i/2\Omega}$$

$$+ \ldots + 2e^{(\Omega-2)(2m+1)\pi i/2\Omega} + e^{-(\Omega-1)(2m+1)\pi l/2\Omega}$$

$$= \Omega + 2\Big\{(\Omega - 1)\cos\frac{(2m+1)\pi}{2\Omega} + (\Omega - 2)\cos\frac{2(2m+1)\pi}{2\Omega} + \ldots$$
$$+ 2\cos\frac{(\Omega-2)(2m+1)\pi}{2\Omega} + \cos\frac{(\Omega-1)(2m+1)\pi}{2\Omega}\Big\} .$$

From this, our theorem follows by Lemma 3.2.

<u>Corollary</u>. If $0 \leq r \leq 2\Omega$, then

(3.7) $$\sum_{m=0}^{\Omega-1} \frac{(-1)^m \sin\frac{(2m+1)r\pi}{2\Omega}}{1 - \cos\frac{(2m+1)\pi}{2\Omega}} = \begin{cases} \Omega r & \text{if } 0 \leq r \leq \Omega \\ \Omega(2\Omega - r) & \text{if } \Omega \leq r \leq 2\Omega . \end{cases}$$

<u>Theorem 3.2</u> . In the square of side 2Ω in the r-s-plane with the left lower corner at the origin

$$\frac{1}{2\Omega^2} \sum_{m=1}^{\Omega-1} \frac{\sinh s^{\beta}m}{\sinh 2\Omega\beta_m} \frac{(-1)^m \frac{(2m+1)r\pi}{2\Omega}}{1 - \cos\frac{(2m+1)\pi}{2\Omega}}$$

is zero for $r = 0$ and $r = 2\Omega$ and for $s - 0$, while for $s = 2\Omega$ it increases linearly from 0 to ½ as r goes from 0 to Ω and then decreases linearly back to 0 as r goes from Ω to 2Ω. If 3 is defined by (1.4) using $a = (2m + l)\pi/2\Omega$, it satisfies the 5-point difference approximation. If β_m is defined by (1. 6) using $\alpha = (2m + l)\pi/2\Omega$, it satisfies the 9-point approximation.

s \ r	8	9	10	11	12	13	14	15
15	-669665	-144578	-28244	-6515	-1740	-484	-115	-16
14	-365566	-195760	-78350	-30475	-12466	-5394	-2374	-912
13	-232806	-174120	-99990	-52018	-26469	-13473	-6679	-2771
12	-167881	-143815	-100596	-63013	-37349	-21368	-11513	-5026
11	-130010	-118273	-92531	-65360	-43150	-26939	-15472	-7023
10	-104626	-98053	-81938	-625956	-44568	-29673	-17396	-8369
9	-85870	-81798	-71163	-57242	-42960	-29970	-18735	-8966
8	-71027	-68318	-60966	-50752	-39506	-28498	-18294	-8905

TABLE 13. 10^8 times harmonic minus 5-point approximation for a discontinuous derivative with a 16 X 16 grid.

s \ r	8	9	10	11	12	13	14	15
15	-344365	-157109	-61560	-29890	-16506	-9602	-5419	-2465
14	-168165	-127260	-77328	-45521	-27531	-16787	-9718	-4476
13	-105864	-93375	-69538	-47 844	-31858	-20 569	-12307	-5767
12	-75173	-69920	-57625	-43774	-31459	-21402	-13236	-6316
11	-56501	-53773	-46748	-37748	-28640	-20 312	-12923	-6268
10	-43744	-42124	-37737	-31650	-24917	-18224	-11857	-5829
9	-34403	-33351	-30414	-26126	-21086	-15764	-10429	-5180
8	-27245	-26518	-24451	-21329	-17504	-13287	-8896	-4452

TABLE 14. 10^8 times harmonic minus 9-point approximation for a discontinuous derivative with a 16 x 16 grid.

#1383

s \ r	32	33	34	35	36	37	38	39
63	-167 563	-36288	-7196	-1751	-540	-205	-90	-44
62	-91716	-49260	-19894	-790 5	-3379	-1586	-814	-450
61	-58763	-44087	--25543	-13533	-7130	-3878	-2206	-1315
60	-42849	-368 30	-26019	-16620	-10214	-6260	-3899	-2488
59	-33788	-30854	-24423	-17647	-12139	-8184	-5509	-3741
58	-27940	-26300	-22287	-17490	-13063	-9492	-6316	-4888
57	-23835	-22824	-20190	-16765	-13301	-10245	-77 58	-5830
56	-20784	-20115	-18306	-15815	-13123	-10577	-8364	-6539

TABLE 15. 10^8 times harmonic minus 5-point approximation for a discontinuous derivative with a 64 X 64 grid.

s \ r	32	33	34	35	36	37	38	39
63	-86735	-39803	-15936	-8055	-4764	-3118	-2187	-1612
62	-43066	-328 52	-20409	-12526	-8133	-5599	-4047	-3040
61	-27972	-24867	-18962	-13634	-9783	-7168	-5393	-4162
60	-20745	-19453	-16444	-13095	-10189	-7918	-6213	-4942
59	-16481	-15822	-14137	-12011	-9919	-8097	-6602	-5409
58	-13650	-13269	-12244	-10848	-9 350	-7934	-6686	-5628
57	-11627	-11387	-10721	-9768	-8683	-7592	-6574	-5666
56	-10105	-9943	-9489	-8815	-8014	-7171	-6347	-5581

TABLE 16. 10^8 times harmonic minus 9-point approximation for a discontinuous derivative with a 64 x 64 grid.

We easily define the harmonic function which assumes the same boundary values. We have listed 10 times the difference (the harmonic function value minus the value given in Theorem 3.2) for both the 5-point case and the 9-point case for $\Omega = 8$ in Tables 13 and 14, and for $\Omega = 32$ in Tables 15 and 16. Because of symmetry, values are listed only from the midpoint onward.

For the 16 x 16 grid, the 9-point approximation is markedly superior for most points, as one can see by comparing Tables 13 and 14. From Tables 15 and 16, we see that for the 64 X 64 grid the 9-point approximation is still superior for most points, but less strikingly so.

Comparing the 16 X 16 grid tables with the 64 X 64 grid tables certainly leads one to conjecture that for a given discontinuity of the derivative the error decreases like the first power of the mesh size. We have no idea if this is really so, much less how to prove it. Such evidence as we have is based on too few cases to be more than suggestive.

4. The order of the error. The usual type of estimate of the order of the error goes something like this (see Theorem 3 on p. 218 of Milne [1]). If the region can be enclosed in a circle of radius ρ, and all fourth derivatives of $u(x, y)$ are bounded in absolute value by M, then if one uses a 5-point approximation with a mesh of side h, the error at each nodal point will be less than

$$\frac{Mh^2p^2}{24} \tag{4.1}$$

As this applies to each nodal point, the result certainly fails if the boundary values or their derivatives are discontinuous.

However, one can approach the matter differently. Fix a point inside the region. For meshes for which this is a mesh point, how does the error vary with the mesh size? Suppose one has a discontinuity of the boundary. If the mesh is chosen so that this point is two mesh points interior from the point of discontinuity, the error will inevitably be about 0.0035 times the amount of the discontinuity for the 5-point approximation. Now hold the point fixed, and halve the mesh size. The error will now not exceed 0.0013 times the amount of the discontinuity. This is not a drop of the order of h^2, but is better than h. If we halve again, we have dropped to less than 0.0004 times the amount of the discontinuity. This is still not the order of h^2, but is improving. With yet another halving, the error drops by the order of about a factor of four.

If one has a very fast way to solve the 5-point approximation with a fine mesh, then one can take the mesh fine enough so that for most points of the region, one will get adequate accuracy. For discontinuities of the derivative of a boundary condition, this result is even more so. Only the points near the discontinuity will be badly affected. If one can afford to take the mesh fine enough so that all points of interest are as much as sixteen mesh points from the discontinuity, one will not do too badly.

One could try refining the mesh locally, near the discontinuity. However, it would probably be less trouble to remove the discontinuity, as suggested on pp.221-222 of Milne [1].

As the 9-point approximation is more accurate than the 5-point, the same applies still more strongly to it. Of course, one cannot hope to get accuracy to order six, as is guaranteed for the Laplace equation when everything is very smooth and one uses the 9-point approximation.

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