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CONFORMING FINITE ELEMENT METHODS FOR THE CLAMPED PLATE PROBLEM

by

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ABSTRACT

Finite element methods for solving biharmonic boundary value problems are considered. The particular problem discussed is that of a clamped thin plate. This problem is reformulated in a weak, form in the Sobolev space W_2^2 Techniques for setting up conforming trial

Functions are utilized in a Galerkin technique to produce finite element solutions. The shortcomings of various trial function formulations are discussed, and a macro—element approach to local mesh refinement using rectangular elements is given.

1• Introduction

In this paper we consider the problem of the clamped plate. Here the function u=u(x,y), which at any point (x,y) is the transverse displacement.of the plate from its equilibrium position, satisfies

$$\Delta^2[u(x,y)] = f(x,y),\ (x,y) \in \Omega, (1)$$

$$u(x,y)=\frac{\partial u(x,y)}{\partial n}=0,\ (x,y) \in \partial\Omega, (2)$$

Where Ω is a simply connected open bounded domain with boundary $\partial\Omega$, and $\partial/\partial n$ denotes the derivative in the direction of the outward normal, to the boundary. It is assumed that the function f satisfies all required continuity conditions, and that the boundary $\partial\Omega$ satisfies certain smoothness conditions (for example a restricted cone condition, see Agmon [1] .) Problem (1)—(2) is called the first biharmonic problem.

The solution of problem (1)-(2) is the function which minimizes the functional

$$I[v] \equiv \iint_{\Omega} \left\{ \left(\frac{\partial^2 v}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 v}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 v}{\partial y^2}\right)^2 - 2fv \right\} dx\, dy \qquad (3)$$

over the space $\overset{o}{W}{}^{2}_{2}(\Omega)$. For equation (3) $W^{2}_{2}(\Omega) . \equiv H^2(\Omega)$

is the Sobolev space of functions which together with their first and second generalized derivatives are in $L_2(\Omega)$, and $\overset{o}{W}{}^{2}_{2}(\Omega)$. is the subspace of $W^{2}_{2}(\Omega)$ functions of which also satisfy the homogeneous boundary conditions (2). The technique of minimizing a functional to solve the problem (1)-(2) is a variational technique.

An alternative approach is to form a <u>weak</u> problem associated with (1) — (2) by multiplying (1) by a test function $v \,\varepsilon\, \overset{\circ}{W}{}^{2}_{2}(\Omega)$ and integrating over Ω. Thus

$$(\Delta^2 u, v) = (f, v) \qquad \forall\, v \,\varepsilon\, \overset{\circ}{W}{}^{2}_{2}(\Omega) \; . \quad \textbf{(4)}$$

After integration by parts and use of the boundary conditions, equation (4) becomes

$$a_1(u, v) = (f, v) \qquad \forall\, c \varepsilon\, \overset{\circ}{W}{}^{2}_{2}(\Omega) \; ,$$

where

$$a_1(u, v) \equiv \iint \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial y^2}\right) dx\ dy \; . \qquad \textbf{(5)}$$

In (5) the bilinear form a_1 (u,v) is a <u>Dirichlet form</u> associated with the biharmonic operator Δ^2 However, the Dirichlet form is not unique. Following Agmon [1] p.96, we use the identity

$$\Delta^2 \equiv \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) + \mathbf{4} \frac{\partial^2}{\partial x \partial y} \left(\frac{\partial^2}{\partial x \partial y}\right)$$

and obtain from (4) the weak formulation

$$a_2(u, v) = (f, v) \qquad \forall\, v \,\varepsilon\, \overset{\circ}{W}{}^{2}_{2}(\Omega),$$

where

$$a_2(u, v) \equiv \iint_\Omega \left\{ \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}\right) \left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2}\right) + 4\, \frac{\partial^2 u}{\partial x \partial y}\, \frac{\partial^2 v}{\partial x \partial y} \right\} dx\ dy \; . \; (6)$$

Infinitely many Dirichlet forms

$$a_t(u, v) = t\, a_1(u, v) + (1 - t)\; a_2(u, v) \;,$$

for real t, can be obtained from (5) and (6) In particular choice of t ½ leads to the form

$$a(u, v) \equiv \iint_{\Omega} \left\{ \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial y^2} \right\} dx\, dy \,, \qquad (7)$$

Note that I[v] = a(v,v) -2(f,v). We thus finally have the following weak form of the problem (1)-(2): find $u \in \overset{o}{W}{}_2^2(\Omega)$. such that

$$a_2(u, v) = (f, v) \qquad\qquad \forall\, v\, \varepsilon\, \overset{o}{W}{}_2^2(\Omega). \quad (8)$$

Under sufficient conditions of smoothness of f and $\partial\,\Omega$ the bilinear form a(u,v) is $\overset{o}{W}{}_2^2(\Omega)$. <u>elliptic</u> and

<u>continuous</u>; that is there exist constants $\rho > 0$, $\gamma > 0$ such that respectively

$$a(v, v) \geqq \rho \|v\|^2_{\overset{0}{w}{}_2^2(\Omega)} \,, \qquad \forall\, v\, \varepsilon\, \overset{0}{w}{}_2^2(\Omega), \qquad (9)$$

and

$$|\, a(u, v) | \leqq \gamma \| u \|_{\overset{0}{w}{}_2^2} \;\; \| v \|_{\overset{0}{w}{}_2^2} \,, \forall\, u, v\, \varepsilon\, \overset{0}{w}{}_2^2(\Omega) \qquad (10)$$

The finite element approximation U(x,y) to the problem (1)-(2) is derived via the weak formulation (8). for this the region Ω is partitioned into non-overlapping elements (usually triangles or rectangles) so that there are m nodes $Z_1, Z_2 \ldots\ldots Z_m$ in

$\bar{\Omega} \equiv \Omega \cup \partial\Omega$ Some of these nodes may coincide, and thus the concept of multiple nodes is allowed. In particular, at element vertices on the boundary, those nodes associated with essential boundary conditions are not included in the set $\{z_i\}_{i=1}^{m}$.. There are k nodes in any one element.

Consider the interpolant $\tilde{u}(x, y)$, $(x, y)\ \varepsilon\ \overline{\Omega}$, which for any element takes the values of u, and some or all of the derivatives of u of order $\leq$ p, at the k nodes in the element. Let the interpolant in each element have the form

$$\tilde{u}(x, y)\big|_e = \sum_{i=1}^{k} (D_i u)_i\, \phi_i(x, y), \tag{11}$$

where $(D_i.u)_i$ are partial derivatives of u with respect to x and y of order less than or equal to p evaluated at the points (x_i, y_i), and the ϕ_i (x,y) are the <u>cardinal basis functions</u> of the (Hermite) interpolation. The approximating function U(x,y) derived with the finite element method has, in each element, the from

$$U(x, y)\big|_e = \sum_{i=1}^{k} (D_i U)_i\, \phi_i(x, y), \tag{12}$$

here the $(D_i U_i)$. are derivatives of U as in (11).

The cardinal basis functions are local to each element, but, taken over the totality of elements of Ω they together form the linearly independent set of functions $\{B_i(x, y)\}_{i=1}^{m}$ These B_i ' s are the basis functions of the finite element method. Each B_i is associated with a single node z_i,i = 1,2,...,m, and is non-zero only in those elements which have z_i as a node.

Further,

$$D_j B_i (z_j) = \delta_{ij} , I, j= 1,2,\ldots\ldots,m,$$

and for any node z_i, which belongs to an element involving part of the boundary $\partial\Omega$, we demand that the associated $B_i(x,y)$ satisfies the essential boundary conditions (2) at nodal points on the boundary. If the $ø_i$'s, which in each element are polynomials, are chosen so that

$$B_i (x,y) \ \varepsilon \ \overset{0}{w}{}_2^2(\Omega) \ , \ i = 1,2,....,m,$$

then the set $\{B_i (x,y) \}_{i=1}^{m}$ spans an m-dimensional piecewise polynomial space S^h which is a subspace of $\overset{o}{W}{}_2^2(\Omega)$.

A discrete formulation of the weak problem (8) is**;**
find U ε S^h such that

$$a (U,V) = (f,v) \ \forall \ v \ \varepsilon \ S^h . \tag{14}$$

In particular U can be calculated by setting

$$U(x,y)=\sum_{i=1}^{m} (D_i U)_i \ B_i (x,y), i=1,2,.....,m, \tag{15}$$

and solving

$$a(U,B_j.) = (f, B_j), \qquad j=1,2,.......m,$$

that is

$$\sum_{i=1}^{m} (D_i U)_i \ a(B_i, B_j), = (f, B_j), \qquad j = 1,2,\ldots m. \tag{16}$$

Equations (16) are known as the global stiffness equations of the finite element method.

If follows from (9) that

$$\rho \| u - U \|^2_{\overset{o}{W}{}^2_2} \leqq a(u - U, u - U) \leqq a(u - U, u - V) \quad \forall\, v \,\varepsilon\, S^h ,$$

since from (8) and (14) we have

$$a(u - U, Z) = \mathbf{0} \qquad \forall\, z \,\varepsilon\, S^h \subset \overset{o}{W}{}^2_2(\Omega) \ .$$

Thus with (10)

$$\rho \| u - U \|^2_{\overset{o}{W}{}^2_2} \leqq \gamma \| u - U \|_{\overset{o}{W}{}^2_2} \| u - V \|_{\overset{o}{W}{}^2_2}$$

and so

$$\| u - U \|_{\overset{o}{W}{}^2_2(\Omega)} \leqq \frac{\gamma}{\rho} \| u - V \|_{\overset{o}{W}{}^2_2(\Omega)} \qquad \forall\, v \,\varepsilon\, S^h \tag{17}$$

The inequality (17) holds in particular when $V \in S^h$ is the interpolant ũ to u mentioned previously , and the problem of bounding the finite element error thus becomes one of interpolation theory. Many bounds for the errors in two dimensional interpolation have been derived, especially in triangles and rectangles. We therefore limit consideration here to the cases where in the problem (1)-(2) the boundary $\partial\Omega$, is either polygonal or rectangular in shape. It is further assumed that the smoothness of $\partial\Omega$ is such that inequalities (9) and (10) hold.

When $\partial\Omega$ in polygonal, the region Ω is split into triangular elements having generic length h. In this case if the piecewise polynomial space $S^h \subset \overset{\circ}{W}{}_2^2(\Omega)$ consists of functions which in each clement are complete polynomials of degree q , so that the interpolant can be written

$$\tilde{u}(x, y)\,|_e = \sum_{i+j=0}^{q} a_{ij}\, x^i y^j ,$$

the bounds then have the form

$$\|u-\tilde{u}\|_{\overset{\circ}{W}{}_2^2(\Omega)} \leq \left\{\sum \| D^{q+1} u\|^2_{L^2(\Omega)}\right\}^{\frac{1}{2}} , \qquad (18)$$

where

$$|\,u\,|_{q+1} \equiv \left\{\sum \| D^{q+1} u\|^2_{L^2(\Omega)}\right\}^{\frac{1}{2}} ,$$

the summation being over the q + 2 derivatives

$$D^{q+1} u = \frac{\partial^{q+1} u}{\partial x^{\alpha 1}\,\partial y^{\alpha 2}} , \qquad \alpha 1 + \alpha 2 = q + 1 .$$

For the derivation of bounds of this type see, for example Zlamal [15], Bramble and Zlamal [5] Bramble and Hilbert [4], and Ciarlet and Raviart [7].

When the region has a rectangular boundary, it is partitioned into rectangular elements. For this Birkhoff, Schultz and Varga [3] show that, if in each element the interpolant $\tilde{u}$ has the form

$$\tilde{u}(x,y)|_e = \sum_{i=0}^{2s-1} \sum_{j=0}^{2s-1} a_{ij} x^i y^j \ ,$$

then

$$\|u-\tilde{u}\|_{\overset{o}{W}{}_2^2(\Omega)} \leq K_2 h^{2s-2} |u|_{2s} \ , \qquad \textbf{(19)}$$

where h is the length of the longer side of the rectangle.

The use of (17) together with (18) or (19) gives bounds for the error in the finite element approximation.

2. <u>Conforming Elements</u>

The error hounds of Section 1 have been derived through the use of the inequality (17), which in turn is dependent on the condition that $S^h \subset \overset{o}{W}{}_2^2(\Omega)$. This inclusion is fundamental to the whole analysis, and is known as the <u>conforming</u> <u>condition</u>. The sufficient condition that functions U(x,y) belong to $\overset{o}{W}{}_2^2(\Omega)$ is that they belong to $C^1(\overline{\Omega})$ and satisfy the essential boundary conditions (2) on $\partial\Omega$. In constructing the basis functions $\{B_i(x,y)\}_{i=1}^m$ we seek to satisfy these simpler conditions and consider only <u>conforming elements</u>.

<u>Triangles</u>

Perhaps the best known trial function $U \varepsilon C^1(\overline{\Omega})$ for the case of a triangle partition is that which in each element is the complete quintic polynomial in x and y

$$U(x, y)\Big|_e = \sum_{i+j=0}^{5} a_{ij}\, x^i y^j \,, \qquad (20)$$

having 21 degrees of freedom; see Zlamal [15] This is uniquely determined in the triangle by prescribing at each of the three vertices the six values $U, U_x, U_y, U_{xx}, U_{xy}, U_{yy}$, and at each of the mid—points of the sides the values of the normal derivatives $\frac{\partial U}{\partial n}$.

If the vertices are treated as nodes of multiplicity 6, this means that in the notation of (12) the orders of the derivatives $(D_i U)_i$ at the vertices are 0, 1 or *2,* whilst at the mid-points the $(D_i U)_i$. are of order 1. The quintic cardinal basis functions $ø_i$ -(x,y) in each triangle are chosen so that the B_i (x,y) satisfy (13). Thus we have $U \;\varepsilon\; C^1(\overline{\Omega})$ If in addition the essential boundary conditions (2) are imposed at the boundary nodes, the desired inclusion is obtained. It is seen from (18) that use of this piecewise quintic trial function leads to an $O(h^4)$ error bound on the finite element error, when it is assumed that the sixth order derivatives of u are all in $L_2(\Omega)$

Clearly a disadvantage of this technique is that the order in of the linear system (16) is likely to be large on account of the six fold nodes at the element vertices. Thus methods have been sought which reduce the total number of nodes whilst keeping conformity. One effective way of removing three of the twenty—one degrees of freedom in each element of the above is to demand that for $U|_e$ in (20) the normal derivative along each side of the triangle be a cubic. The nodal

values at the vertices of the triangle determine the cubic along the side, so that continuity of normal derivative across interelement boundaries is maintained. However, in each element the 18 -parameter reduced quintic trial function is no longer a complete quintic,
so that the order of the error bound is reduced. Other approaches for reducing the number of parameters while preserving conformity include that of grouping together terms from complete polynomials and of augmenting cubic polynomials with rational functions, Birkhoff and Mansfield [2]. However, in this latter case S^h is a space of piecewise rational functions and new problems arise in the evaluations of the integrals in (16). Alternatively there is the technique of Clough and Tocher [8] of forming a macro-triangle by splitting each triangle of the partition into three subtriangles, and combining different cubic polynomials in each to form a 12 - parameter trial function in each macro-triangle.

Rectangles

When the region Ω is rectangular, trial functions $U \in C^1(\overline{\Omega})$ can be obtained using rectangular elements by taking U in each rectangle to be the bicubic polynomial

$$U(x,y)\Big|_e = \sum_{i=0}^{3} \sum_{j=0}^{3} a_{ij}\, x^i y^j ,$$

which has 16 degrees of freedom.This is uniquely determin-ed by prescribing at each of the four vertices the
values U, U_x, U_y, U_{xy} . These are the derivatives $(D_i U)_i$

or (12) where each vertex is a node of multiplicity 4. Bicubic basis func tions $ø_i(x,y)$ are again chosen to produce $B_i(x,y)$, $i = 1, 2, \ldots, m$, satisfying (13), and these with the implementation of the essential boundary conditions give the desired inclusion. It is seen from

(19) that this produces an $O(h^2)$ error bound, provided that the fourth derivatives of u are in $L_2(\Omega)$.

Rectangular elements lack the versatility of triangles in that it is difficult to produce a concentration of elements near any point of Ω without also introducing (unnecessarily) extra elements in other parts of Ω. The ability to perform this local mesh refinement is particularly desirable when the boundary $\partial\Omega$ contains a re-entrant corner, the presence of which slows the rate of convergence with decreasing mesh size of the finite element solution of the problem to the exact solution. An example of local mesh refinement with rectangles in the neighbourhood of a point 0 is given in Figure 1. It is seen that the

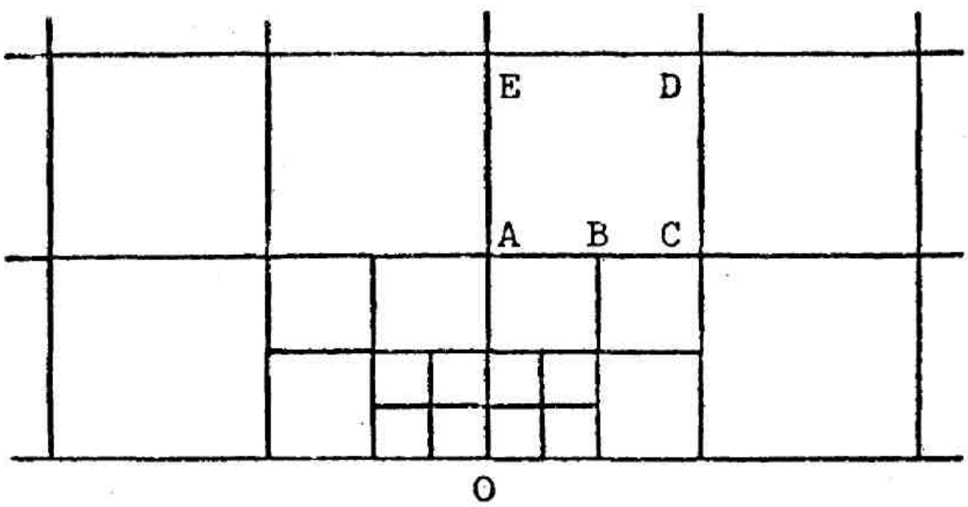

Figure 1

refinement produces mid-side nodes in some elements; e.g. the point B in ABCDE. Clearly special procedures must be adopted to produce, for this case, trial functions which are $C^1(\overline{\Omega})$. One such approach with square elements is due to Gregory and Whiteman [11] who adopt the macro-element approach and split elements of the type ABCDE into two equal parts along the line BB', Figure 2. In each part they use the bicubic interpolant to the values of U, U_x, U_y, U_{xy} at

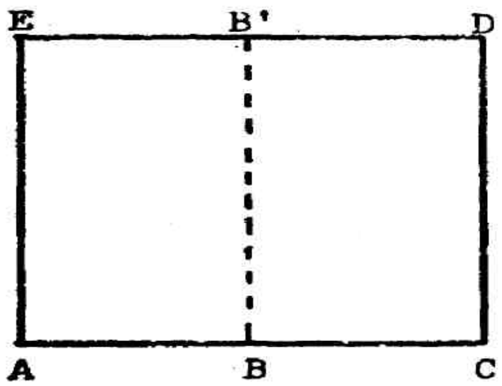

Figure 2

the four vertices, arid then eliminate the values of these four quantities at the point B' using Hermite cubic interpolation with the appropriate values at D and E. The resulting trial function is C^1 over the macro-element ABCDE, and is cubic along each of AB,BC,CD,DE and EA.

This together with use of the standard bicubic, described above, in all "four node elements" produces a trial function which is $C^1(\overline{\Omega})$.

3 Discussion

Applications of finite element methods to plate problems of the type (1)-(2) are without number.

kolar et al [12] consider the case of a clamped square thin plate under the action of a uniform load (f=constant) for which the exact value of the displacement u(x,y) is known. Using complete quintics with triangles and bicubics with rectangles as described above they obtain greater accuracy for a given number of subdivisions of the plate with triangular elements than with squares. This is to be expected on account of the higher degree of local trial function and the higher order error bound in the triangular case.

Gallagher [10], Chapter 12, gives an intensive study of a similar problem for the case of a simply supported plate using triangular and rectangular elements with conforming and nonconforming trial functions. In particular he finds that the results for triangles with the 21- parameter piecewise quintic and the 18-parameter piecewise constrained quintic conforming trial functions are "highly accurate". However, it must be pointed out that the amount of computational effort required in forming the global stiffness equations (16) in this case is considerable. This has to be balanced against accuracy in any computation.

The functional I[v] in (3) and the weak form (8) of the problem (1)-(2) are associated with the potential energy of the plate. All the emphasis in this paper has been on finding global trial functions which satisfy the "C^1 conforming condition" for the space $\overset{\circ}{W}{}_2^2(\Omega)$.

There are of course many other energy formulations, such as the complementary and Riessner energies, and,as has been pointed out, many functionals and weak forms; see for example Ciarlet [6], The possibility of using functionals and weak formulations defined over spaces for which the conforming condition is that the trial functions be in $C^{0}(\overline{\Omega})$ rather than $C^{1}(\overline{\Omega})$ has been considered. An example of this approach is that of Westbrook [13] who uses a perturbed variational principle for the clamped plate which has only a C^{o} conforming condition.

The global stiffness equations (16) can be thought of as difference equations. This view has been taken for the case of Lagrangian approximating functions for Poisson problems by Whiteman [l4]. When, as here, Hermite global approximating functions are used, the equations (16) involve as unknowns not only nodal values of the approximating function U(x,y) but also values of certain derivatives of U at the nodal points. When riewed from the difference point of view, equations (16) thus differ from the usual concept of difference equations. Difference stars of this type have been derived using <u>mehrstellenverfahren</u> by Collatz [9]- In regions which can be partitioned using a <u>regular</u> triangular or rectangular mesh, a considerable saving in computation time can be made by generating the stiffness equations (16) using the difference stars rather than through the usual finite element approach of repeated use of a local stiffness matrix.

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