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A SIMPLE FINITE - DIFFERENCE MODIFICATION FOR IMPROVING ACCURACY NEAR A CORNER IN HEAT FLOW PROBLEMS

by

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Introduction

Traditional numerical methods of solution of heat flow problems are subject to inaccuracies near sharp corners, where the derivatives of the exact solution may "become unbounded (Jeffreys[1]). Previous methods of overcoming this difficulty (Motz[2], Woods[3], Bell and Crank[4]) have tended to be elaborate and computationally time consuming. The simple procedure presented in this paper dovetails easily into standard finite-difference schemes and is computationally efficient. The results for a model problem compare favourably with those obtained by more sophisticated approaches.

A Model Problem

Consider the problem of heat conduction in the region illustrated in figure 1, where $T(x,y,t)$ represents the temperature at any point in the region, κ is the diffusivity, assumed constant, and the following conditions hold,

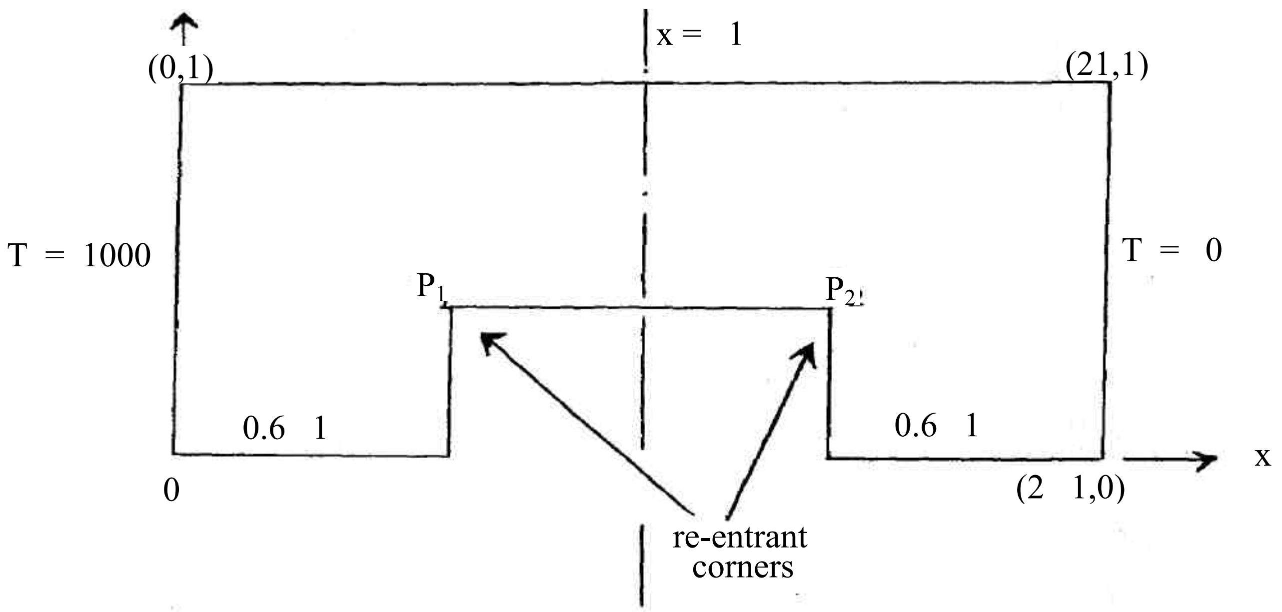
$$\frac{1}{\kappa}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \quad , \qquad \textbf{(1)}$$

$$T(O,y,t) = 1000$$

$$T(2\ell,y,t) = 0$$

and 3T/¶n = 0 on all other boundaries, where $\partial/\partial n$ is the derivative normal to the boundary.

Figure 1. Model Problem.



The initial condition is taken to be

$$T(x,y,\tau) \quad - \quad 10^3 \operatorname{erfc}\{x/2(\kappa\tau)^{\frac{1}{2}}\} \qquad , \qquad (2)$$

where $(\kappa\tau) = 0.0005$. This corresponds to a small-time solution in a plane semi-infinite medium (see Carslaw and Jacger[5]) and it is assumed that the difference between (2) and the solution after such a small-time, τ, when $T(x,y,0) = 0$ is negligible.

The problem possesses a steady—state solution which is anti-symmetric about the line $x = \ell$.

<u>The Finite-Difference Procedure</u>

The region in figure 1 is covered by a square grid of size h where h is chosen such that all boundary lines are mesh lines. Denoting the temperature at a point (i,j) on the grid, after a time nót, by T_{ij}^{n}. , the usual explicit discretisation of equation (1) is

$$T_{ij}^{n+1} = (1 - 4\kappa\delta t/h^2)T_{ij}^{n} + \kappa\delta t(T_{i+1j}^{n} + T_{ij+1}^{n} + T_{ij-1}^{n})/h^2 \ . \qquad (3)$$

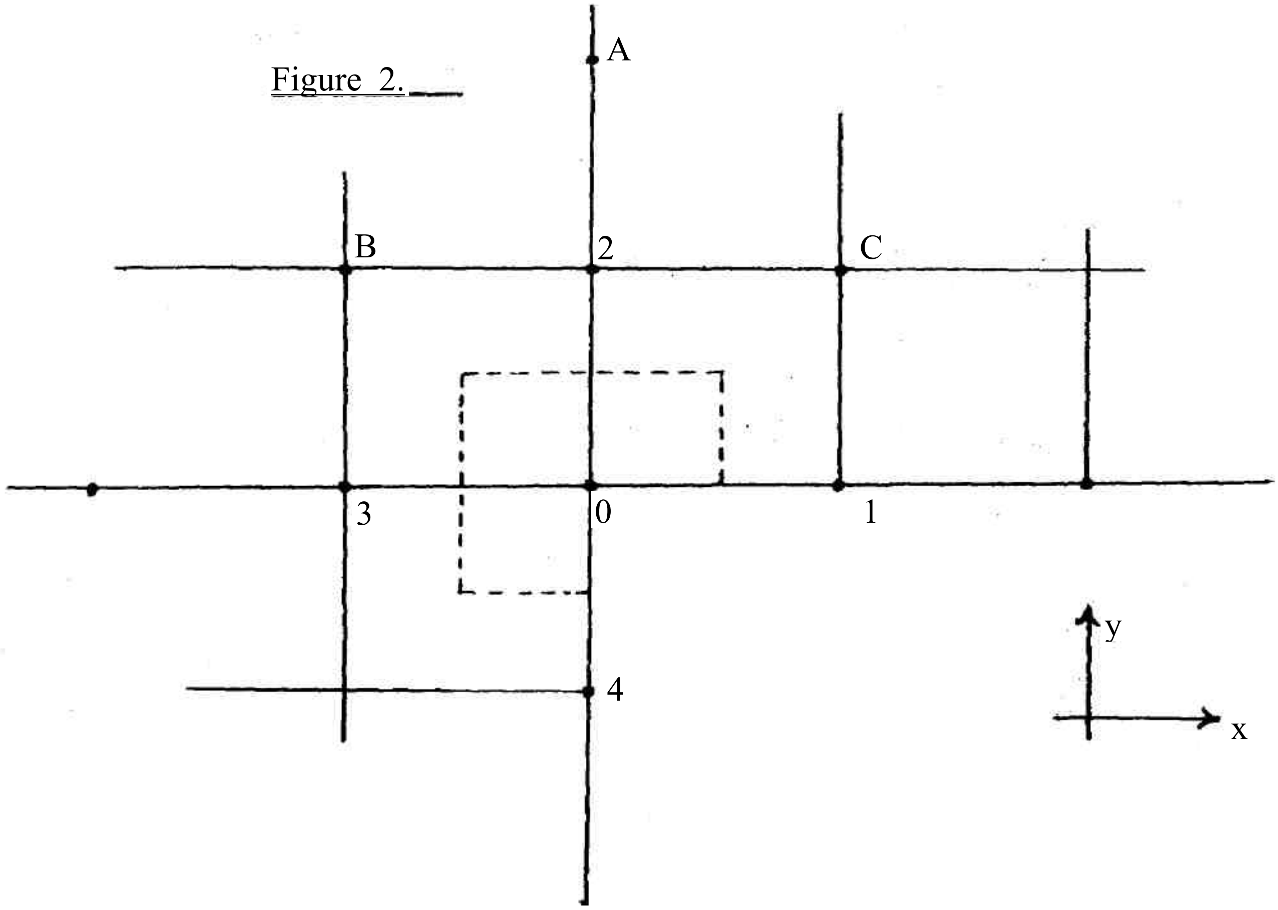
Bell and Crank[4] showed that (3) accounts for the first singular term in the local solution about a re-entrant corner.

In fact, a five-point replacement which satisfies all the singular terms in the local solution and is of order h^2 accuracy is easily seen to be

$$T_0^{n+1} = (1 - 4\kappa\delta t/h^2)T_0^{n} + 2\kappa\delta t(T_1^{n} + 2T_2^{n} + 2T_3^{n}. + T_4^{n})/3h^2 \ , \qquad (4)$$

where $T_0^n, T_1^n, T_2^n, T_3^n$ and T_4^n arc the temperatures at the points indicated in figure 2.

Figure 2.



Bell[6] noticed that the expression given in (4) was exactly the same as that produced by the approximation of a heat-balance relationship (Crank[7]) at the corner. For a corner whose interior angle is a multiple of $\pi/4$, the heat-balance method always produces a replacement that satisfies the singular terms in the local solution.

Thus, a possible improvement to the simple scheme outlined in the previous section is to use the replacement (4), instead of (3), at the points P_1 and P_2. Clearly such a procedure does not entirely eliminate the errors associated with sharp corners. The derivatives of the solution are large though not unbounded in a neighbourhood of each corner and the special replacement (4) merely represents the behaviour at the most critical point in each neighbourhood.

A Method of Modified Replacements

Bell and Crank[4] obtained a local solution about a re-entrant corner, P, in polar coordinates and expressed it as a single power series which can be written as

$$T(r,\theta,t) = a_0(t) + a_1(t)r^{\frac{2}{3}}\cos 2\theta/3 + a_2(t)r^{\frac{4}{3}}\cos 4\theta/3$$
$$+ \; o(r^2) \quad . \qquad (5)$$

In the following method an approximation, based on the first three terms of (5), is used to estimate the radial second derivative of T at points adjacent to a re-entrant corner. The transverse second derivative, which completes the expression on the right hand side of equation (1), is replaced by the usual difference approximation. It is assumed that the transverse second derivative is well behaved in comparison with the second derivative in the radial direction. To illustrate the scheme consider the approximation of (1) at the point marked 2 in figure 2. In this case, the transverse component of (1) is $\partial^2 T/\partial x^2$ and this is approximated in the usual way by

$$\frac{\partial^2 T}{\partial x^2} \simeq (T_B + T_C - 2T_2)/h^2 \qquad ,$$

where B and C are as indicated.
To obtain an approximation for the radial component, $\partial^2 T/\partial y^2$, the form of the solution in the radial direction $\overline{OA}$ is taken to be

$$T^* = \alpha_0 + \alpha_1 y^3 + \alpha_2 y^3 \qquad , \qquad (6)$$

where the α's are unknown functions of time.

Thus ,

$$\left(\frac{\partial^2 T}{\partial y^2}\right)_2 \simeq \left(\frac{\partial^2 T^*}{\partial y^2}\right)_2 = 2(2\alpha_2 h^{\frac{4}{3}} - \alpha_1 h^{\frac{2}{3}})/9h^2 \quad .$$

The a's are evaluated at, each time level in terms of the temperatures T_0, T_2 and T_A . Substituting for the values of α into the above and rearranging gives the modified replacement

$$\left(\frac{\partial^2 T}{\partial y^2}\right)_2 \simeq 2\left(w_A T_A + w_0 T_0 - w_2 T_2\right)/9h^2 \ , \tag{7}$$

where $w_A = 1.5\sqrt[3]{2}\ /(\sqrt[3]{4} - 1)$,

$w_0 = (2+\sqrt[3]{4} - 1.5\sqrt[3]{2})/(\sqrt[3]{4}-1)$,

and $w_2 = w_A + w_0$.

A suitable approximation to equation (l) at the point 2 is, therefore,

$$T_2^{n+1} = \left\{1 - 2\frac{\kappa\delta t(9 + w_2)}{9h^2}\right\} T_2^n + \kappa\delta t\,(9T_B^n + 9T_c^n + 2w_A T_A^n + 2w_0 T_0^n)/9h^2 \quad . \tag{8}$$

Since the solution about O along the other three mesh lines has the same form as (6), the replacement (8), with suitable changes of suffix, can be applied at the other adjacent points (i.e. 1, 3 and 4 in figure 2). Incorporating the above treatment of adjacent points with the use of (4) produces a scheme in which the behaviour of the solution, at points close to a comer, has been represented by a system of modified finite—difference replacements. The combined scheme will be referred to as the Modified Replacement Method (MRM).

The replacement (8) is written in terms of an explicit procedure but the implicit extension is straightforward. It is thought that the implementation of the MRM will not effect the unconditional stability of the implicit finite-difference process.

A suitable explicit algorithm is as follows: 1. let the initial condition be T_{ij}^{0}. and put n = 0 : 2. compute T_{ij}^{n+1}. at all points in the region using equation (3) ; 3. overwrite the values of T_{ij}^{n+1}. at the points to be treated, using (4) at the corner points and (8) at the four points adjacent to each corner; 4. let n become n+1 and return to stage 2, continuing until the required time level is reached.

Results

(a) Time-dependent problem.

The results obtained from the time-dependent Motz procedure developed by Bell and Crank[4] are considered, throughout the following discussion, to be the ideal solution. The values of the parameters defining the model problem are specfied in Table 1.

(i) Solutions were computed corresponding to a small time

solution ($n\kappa\delta t = 0.05$). The results of the MRM scheme were compared with the ideal solution. It was observed that at this initial stage of the time process there was little to choose between them.

(ii) At an intermediary stage ($n\kappa\delta t = 0.1$) Table 1 shows the results of the MRM are in good agreement with those of Bell and Crank[4].

(iii) At large times, ($n\kappa\delta t$ — 0.5), when the solution is tending to steady-state values, the accuracy produced by the MRM is also good. The largest absolute difference between the results obtained and the ideal solution, at any point, is three degrees. Generally, the discrepancy is much less. Thus, by slightly modifying the usual explicit finite-difference process, a considerable improvement in accuracy has been achieved.

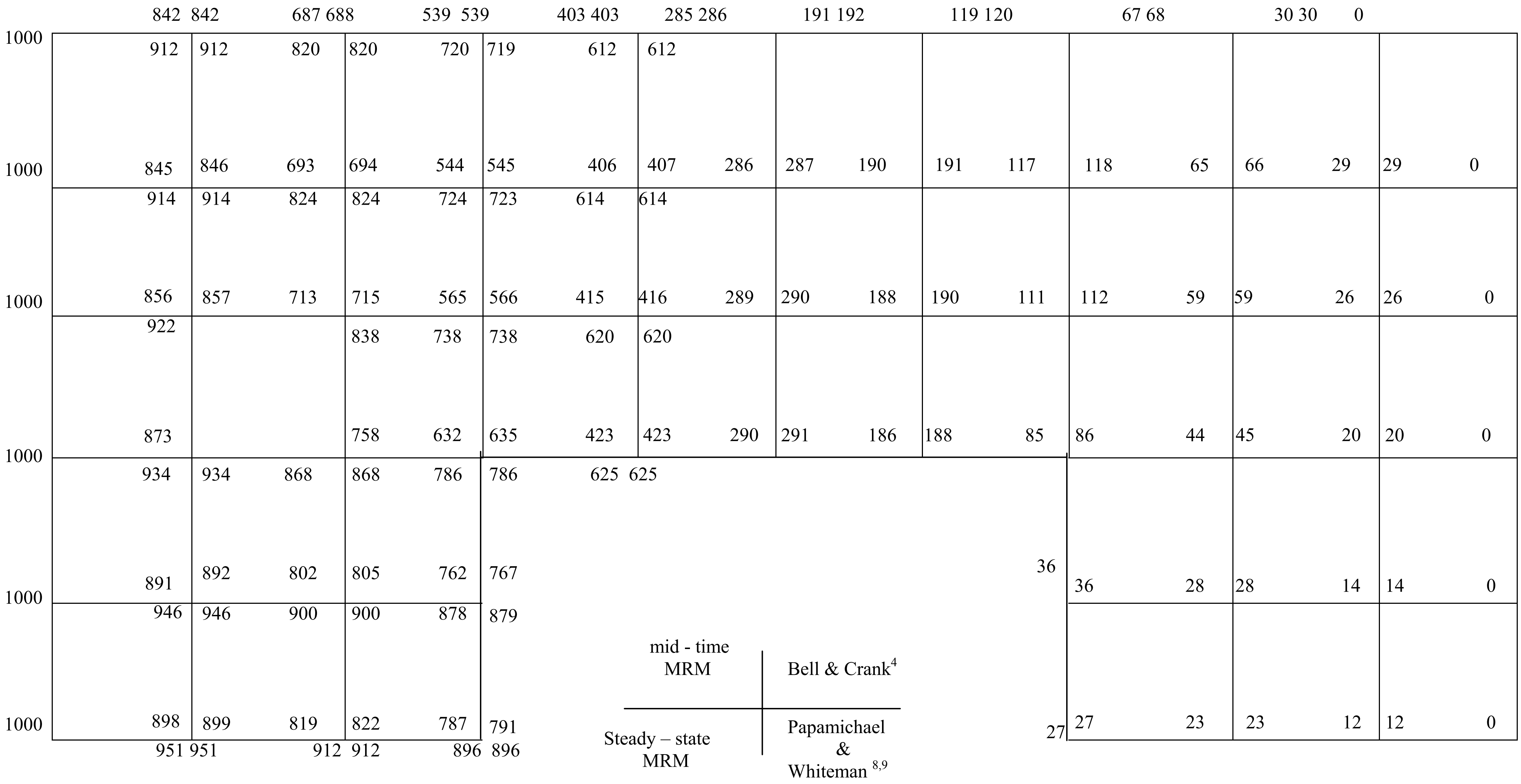
(b) Steady-state problem.

When $\partial T/\partial t = 0$ in the model problem there is anti-symmetry about the line $x = 1$ and it is sufficient to consider only the left-hand half of the region illustrated in figure 1.

After a suitable change of notation, the MRM can be written in terms of a Gauss—Scidel iterative procedure for Laplace's equation. In Table 1 the MRM results are compared with those produced by Papainichael and whiteman[8,9] using a conformal transformation method. The agreement is good.

The computational procedure for the Gauss—Seidel version of the MRM is not as neat as that described earlier for the transient problem, but the computing involved is still quite acceptable. Although the method has been developed in terms of a model problem, with simple boundary conditions, the essence of the approach is applicable to all problems of this type.

TABLE 1. Mid-time and steady-state conditions,



h=0.05 $\kappa\delta t=h^2/6$ $n\kappa\delta t=0.1$ $\ell=0.5$

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