Replication of Wiener-transformable stochastic processes with application to financial markets with memory

Elena Boguslavskaya, Yuliya Mishura, and Georgiy Shevchenko

Abstract We investigate Wiener-transformable markets, where the driving process is given by an adapted transformation of a Wiener process. This includes processes with long memory, like fractional Brownian motion and related processes, and, in general, Gaussian processes satisfying certain regularity conditions on their covariance functions. Our choice of markets is motivated by the well-known phenomena of the so-called "constant" and "variable depth" memory observed in real world price processes, for which fractional and multifractional models are the most adequate descriptions. Motivated by integral representation results in general Gaussian setting, we study the conditions under which random variables can be represented as pathwise integrals with respect to the driving process. From financial point of view, it means that we give the conditions of replication of contingent claims on such markets. As an application of our results, we consider the utility maximization problem in our specific setting. Note that the markets under consideration can be both arbitrage and arbitrage-free, and moreover, we give the representation results in terms of bounded strategies.

Key words: Wiener-transformable process; fractional Brownian motion; long memory; pathwise integral; martingale representation; utility maximization

Elena Boguslavskaya
Department of Mathematics, Brunel University London, Uxbridge UB8 3PH, UK
e-mail: elena@boguslavsky.net

Yuliya Mishura
Department of Probability Theory, Statistics and Actuarial Mathematics, Taras Shevchenko National University of Kyiv, 64, Volodymyrs'ka St., 01601 Kyiv, Ukraine
e-mail: myus@univ.kiev.ua

Georgiy Shevchenko
Department of Probability Theory, Statistics and Actuarial Mathematics, Taras Shevchenko National University of Kyiv, 64, Volodymyrs'ka St., 01601 Kyiv, Ukraine
e-mail: zhora@univ.kiev.ua

1 Introduction

Consider a general continuous time market model with one risky asset. For simplicity, we will work with discounted values. Let the stochastic process $\{X_t, t \in [0,T]\}$ model the discounted price of risky asset. Then the discounted final value of a self-financing portfolio is given by a stochastic integral

$$V^{\psi}(T) = V^{\psi}(0) + \int_0^T \psi(t)dX(t), \tag{1}$$

where an adapted process ψ is the quantity of risky asset in the portfolio. Loosely speaking, the self-financing assumption means that no capital is withdrawn or added to the portfolio; for precise definition and general overview of financial market models with continuous time we refer a reader to [4, 12].

Formula (1) raises several important questions of financial modeling, we will focus here on the following two.

- *Replication*: identifying random variables (i.e. discounted contingent claims), which can be represented as final capitals of some self-financing portfolios. In other words, one looks at integral representations

$$\xi = \int_0^T \psi(t)dX(t) \tag{2}$$

  with adapted integrand ψ; the initial value may be subtracted from ξ, so we can assume that it is zero.
- *Utility maximization*: maximizing the expected utility of final capital over some set of admissible self-financing portfolios.

An important issue is the meaning of stochastic integral in (1) or (2). When the process X is a semimartingale, it can be understood as Itô integral. In this case (1) is a kind of Itô representation, see e.g. [11] for an extensive coverage of this topic. When the Itô integral is understood in some extended sense, then the integral representation may exist under very mild assumptions and may be non-unique. For example, if $X = W$, a Wiener process, and ψ satisfies $\int_0^T \psi_s^2 ds < \infty$ a.s., then, as it was shown by [7], any random variable can be represented as a final value of some self-financing portfolio for any value of initial capital.

However, empirical studies suggest that financial markets often exhibit long-range dependence (in contrast to stochastic volatility that can be both smooth and rough, i.e., can demonstrate both long-and short-range dependence). The standard model for the phenomenon of long-range dependence is the fractional Brownian motion with Hurst index $H > 1/2$. It is not a semimartingale, so the usual Itô integration theory is not available. The standard approach now is to define the stochastic integral in such models as a pathwise integral, namely, one usually considers the fractional integral, see [2, 23].

The models based on the fractional Brownian motion usually admit arbitrage possibilities, i.e. there self-financing portfolios ψ such that $V_{\psi}(0) \leq 0$, $V_{\psi}(T) \geq 0$

almost surely, and $V_\psi(T) > 0$ with positive probability. In the fractional Black–Scholes model, where $X_t = X_0 \exp\{at + bB_t^H\}$, and B^H is a fractional Brownian motion with $H > 1/2$, the existence of arbitrage was shown in [19]. Specifically, the strategy constructed there was of a "doubling" type, blowing the portfolio in the case of negative values; thus the potential intermediate losses could be arbitrarily large. It is worth to mention that such arbitrage exists even in the classical Black–Scholes model: the aforementioned result by Dudley allows gaining any positive final value of capital from initial zero by using a similar "doubling" strategy. For this reason, one usually restricts the class of admissible strategies by imposing a lower bound on the running value:

$$V^\psi(t) \geq -a, \quad t \in (0, T), \tag{3}$$

which in particular disallows the "doubling" strategies. However, in the fractional Black–Scholes model, the arbitrage exists even in the class of strategies satisfying (3), as was shown in [6].

There are several ways to exclude arbitrage in the fractional Brownian model. One possibility is to restrict the class of admissible strategies. For example, in [6] the absence of arbitrage is proved under further restriction that interval between subsequent trades is bounded from below (i.e. high frequency trading is prohibited). Another possibility is to add to the fractional Brownian motion an independent Wiener process, thus getting the so-called mixed fractional Brownian motion $M^H = B^H + W$. The absence in such mixed models was addressed in [1, 5]. In [1], it was shown that there is no arbitrage in the class of self-financing strategies $\gamma_t = f(t, M^H, t)$ of Markov type, depending only on the current value of the stock. In [5], it was shown that for $H \in (3/4, 1)$ the distribution of mixed fractional Brownian motion on a finite interval is equivalent to that of Wiener process. As a result, in such models there is no arbitrage strategies satisfying the non-doubling assumption (3). A more detailed exposition concerning arbitrage in models based on fractional Brownian motion is given in [3].

The replication question, i.e. the question when a random variable can be represented as a pathwise (fractional) integral in the models with long memory was studied in many articles, even in the case where arbitrage opportunities are present. The first results were established in [16], where it was shown that a random variable ξ has representation (2) with respect to fractional Brownian motion if it is a final value of some Hölder continuous adapted process. The assumption of Hölder continuity might seem too restrictive at the first glance. However, the article [16] gives numerous examples of random variables satisfying this assumption.

The results of [16] were extended in [22], where similar results were shown for a wide class of Gaussian integrators. The article [15] extended them even further and studied when a combination of Hölder continuity of integrator and small ball estimates lead to existence of representation (2).

For the mixed fractional Brownian motion, the question of replication was considered in [22]. The authors defined the integral with respect to fractional Brownian motion in pathwise sense and that with respect to Wiener process in the extended

Itô sense and shown, similarly to the result of [7], that any random variable has representation (2).

It is worth to mention that the representations constructed in [16, 15, 22] involve integrands of "doubling" type, so in particular they do not satisfy the admissibility assumption (3).

Our starting point for this article was to see what contingent claims are representable as final values of some Hölder continuous adapted processes. It turned out that the situation is quite transparent whenever the Gaussian integrator generates the same flow of sigma-fields as the Wiener process. As a result, we came up with the concept of Wiener-transformable financial market, which turned out to be a fruitful idea, as a lot of models of financial markets are Wiener-transformable. We consider many examples of such models in our paper. Moreover, the novelty of the present results is that we prove representation theorems that, in financial interpretation, are equivalent to the possibility of hedging of contingent claims, in the class of *bounded* strategies. While even with such strategies the non-doubling assumption (3) may fail, the boundedness seems a feasible admissibility assumption.

More specifically, in the present paper we study a replication and the utility maximization problems for a broad class of asset prices processes, which are obtained by certain adapted transformation of a Wiener process; we call such processes *Wiener-transformable* and provide several examples. We concentrate mainly on non-semimartingale markets because the semimartingale markets have been studied thoroughly in the literature. Moreover, the novelty of the present results is that we prove representation theorems that, in financial interpretation, are equivalent to the possibility of hedging of contingent claims, in the class of bounded strategies. We would like to draw the attention of the reader once again to the fact that the possibility of representation means that we have arbitrage possibility in the considered class of strategies and they may be limited, although in a narrower and more familiar class of strategies the market can be arbitrage-free. Therefore, our results demonstrate rather subtle differences in the properties of markets in different classes of strategies.

The article is organized as follows. In Section 2, we recall basics of pathwise integrations in the fractional sense. In Section 3, we prove a new representation result, establishing an existence of integral representation with bounded integrand, which is of particular importance in financial applications. We also define the main object of study, Wiener-transformable markets, and provide several examples. Section 4 is devoted to application of representation results to the utility maximization problems.

2 Elements of fractional calculus

As announced in the introduction, the integral with respect to Wiener-transformable processes will be defined in pathwise sense, as fractional integral. Here we present the basic facts on fractional integration; for more details see [20, 23]. Consider functions $f, g : [0, T] \to \mathbb{R}$, and let $[a, b] \subset [0, T]$. For $\alpha \in (0, 1)$ define Riemann-Liouville

fractional derivatives on finite interval $[a,b]$

$$\left(\mathscr{D}_{a+}^{\alpha}f\right)(x) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha\int_{a}^{x}\frac{f(x)-f(u)}{(x-u)^{1+\alpha}}du\right)1_{(a,b)}(x),$$

$$\left(\mathscr{D}_{b-}^{\alpha}g\right)(x) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{g(x)}{(b-x)^{\alpha}} + \alpha\int_{x}^{b}\frac{g(x)-g(u)}{(u-x)^{1+\alpha}}du\right)1_{(a,b)}(x). \quad (4)$$

Assuming that $\mathscr{D}_{a+}^{\alpha}f \in L_1[a,b]$, $\mathscr{D}_{b-}^{1-\alpha}g_{b-} \in L_{\infty}[a,b]$, where $g_{b-}(x) = g(x) - g(b)$, the generalized Lebesgue–Stieltjes integral is defined as

$$\int_{a}^{b}f(x)dg(x) = \int_{a}^{b}\left(\mathscr{D}_{a+}^{\alpha}f\right)(x)\left(\mathscr{D}_{b-}^{1-\alpha}g_{b-}\right)(x)dx.$$

Let function g be θ-Hölder continuous, $g \in C^{\theta}[a,b]$ with $\theta \in (\frac{1}{2},1)$, i.e.

$$\sup_{t,s\in[0,T],t\neq s}\frac{|g(t)-g(s)|}{|t-s|^{\theta}} < \infty.$$

In order to integrate w.r.t. function g and to find an upper bound of the integral, fix some $\alpha \in (1-\theta,1/2)$ and introduce the following norm:

$$\|f\|_{\alpha,[a,b]} = \int_{a}^{b}\left(\frac{|f(s)|}{(s-a)^{\alpha}} + \int_{a}^{s}\frac{|f(s)-f(z)|}{(s-z)^{1+\alpha}}dz\right)ds.$$

For simplicity we abbreviate $\|\cdot\|_{\alpha,t} = \|\cdot\|_{\alpha,[0,t]}$. Denote

$$\Lambda_{\alpha}(g) := \sup_{0\leq s < t \leq T}|\mathscr{D}_{t-}^{1-\alpha}g_{t-}(s)|.$$

In view of Hölder continuity, $\Lambda_{\alpha}(g) < \infty$.

Then for any $t \in (0,T]$ and for any f with $\|f\|_{\alpha,t} < \infty$, the integral $\int_{0}^{t}f(s)dg(s)$ is well defined as a generalized Lebesgue–Stieltjes integral, and the following bound is evident:

$$\left|\int_{0}^{t}f(s)dg(s)\right| \leq \Lambda_{\alpha}(g)\|f\|_{\alpha,t}. \quad (5)$$

It is well known that in the case if f is β-Hölder continuous, $f \in C^{\beta}[a,b]$, with $\beta + \theta > 1$, the generalized Lebesgue–Stieltjes integral $\int_{a}^{b}f(x)dg(x)$ exists, equals to the limit of Riemann sums and admits bound (5) for any $\alpha \in (1-\theta,\beta\wedge1/2)$.

3 Representation results for Gaussian and Wiener-transformable processes

Let throughout the paper $(\Omega, \mathscr{F}, \mathbf{P})$ be a complete probability space supporting all stochastic processes mentioned below. Let also $\mathbb{F} = \{\mathscr{F}_t, t \in [0,T]\}$ be a filtration satisfying standard assumptions. In what follows, the adaptedness of a process $X = \{X(t), t \in [0,T]\}$ will be understood with respect to $\mathbb{F}$, i.e. X will be called adapted if for any $t \in [0,T]$, $X(t)$ is $\mathscr{F}_t$-measurable.

We start with representation results, which supplement those of [15].

Consider a continuous centered Gaussian process G with incremental variance of G satisfying the following two-sided power bounds for some $H \in (1/2,1)$.

(A)There exist $C_1, C_2 > 0$ such that for any $s, t \in [0,T]$

$$C_1 |t-s|^{2H} \le \mathbf{E} |G(t) - G(s)|^2 \le C_2 |t-s|^{2H}. \tag{6}$$

Assume additionally that the increments of G are positively correlated. More exactly, let the following condition hold

(B)For any $0 \le s_1 \le t_1 \le s_2 \le t_2 \le T$

$$\mathbf{E}\left(G(t_1) - G(s_1)\right)\left(G(t_2) - G(s_2)\right) \ge 0.$$

A process satisfying (6) is often referred to as a *quasi-helix*.

Note that the right inequality in (6) implies that

$$\sup_{t,s\in[0,T]} \frac{|G(t) - G(s)|}{|t-s|^H |\log(t-s)|^{1/2}} < \infty \tag{7}$$

almost surely (see e.g. p. 220 in [14]).

We will need the following small deviation estimate for sum of squares of Gaussian random variables, see e.g. [13].

Lemma 1. *Let $\{\xi_i\}_{i=1,\dots,n}$ be jointly Gaussian centered random variables. For all x such that $0 < x < \sum_{i=1}^n \mathbf{E}\xi_i^2$, it holds*

$$\mathbf{P}\left(\sum_{i=1}^n \xi_i^2 \le x\right) \le \exp\left\{ -\frac{\left(\sum_{i=1}^n \mathbf{E}\xi_i^2 - x\right)^2}{\sum_{i,j=1}^n (\mathbf{E}\xi_i\xi_j)^2} \right\}.$$

Theorem 1. *Let a centered Gaussian process G satisfy (A) and (B) and ξ be a random variable such that there exists an adapted r-Hölder continuous process Z with $Z(T) = \xi$. There exists a bounded adapted process ψ, such that $\|\psi\|_{\alpha,T} < \infty$ for some $\alpha \in (1-H,1)$ and ξ admits the representation*

$$\xi = \int_0^T \psi(s) dG(s), \tag{8}$$

almost surely.

Remark 1. A similar result was proved in [15], Theorem 4.1, which assumed (6) with different exponents in the right-hand side and in the left-hand side of the inequality. Having equal exponents allowed us to establish existence of a *bounded* integrand ψ, thus extending previous results.

Proof. To construct an integrand, we modify ideas of [15] and [21]. Throughout the proof, C will denote a generic constant, while $C(\omega)$, a random constant; their values may change between lines.

Choose some $\alpha \in \left(1 - H, (r + 1 - H) \wedge \frac{1}{2}\right)$.

We start with the construction of ψ. First take some $\theta \in (0,1)$, put $t_n = T - \theta^n$, $n \geq 1$, and let $\Delta_n = t_{n+1} - t_n$. It is easy to see that

$$T - t_n \leq C\Delta_n. \tag{9}$$

Denote for brevity $\xi_n = Z(t_n)$. Then by Assumption 1, $|\xi_n - \xi_{n+1}| \leq C(\omega)\theta^{rn}$. Therefore, there exists some $N_0 = N_0(\omega)$ such that

$$|\xi_n - \xi_{n+1}| \leq n\theta^{rn} \tag{10}$$

for all $n \geq N_0(\omega)$.

We construct the integrand ψ inductively between the points $\{t_n, n \geq 1\}$. First let $\psi(t) = 0$, $t \in [0, t_1]$. Assuming that we have already constructed $\psi(t)$ on $[0, t_n)$, define $V(t) = \int_0^t \psi(s)dG(s), t \in [0, t_n]$.

Consider some cases.

<u>Case I.</u> $V(t_n) \neq \xi_{n-1}$. By Lemma 4.1 in [15], there exists an adapted process $\{\phi_n(t), t \in [t_n, t_{n+1}]\}$, bounded on $[t_n, t]$ for any $t \in (t_n, t_{n+1})$ and such that $\int_{t_n}^t \phi_n(s)dG(s) \to +\infty$ as $t \to t_{n+1}-$. Define a stopping time

$$\tau_n = \inf\left\{t \geq t_n : \int_{t_n}^t \phi_n(s)dG(s) \geq |\xi_n - V_{t_n}|\right\},$$

and set

$$\psi(t) = \phi_n(t)\,\mathrm{sign}\left(\xi_n - V(t_n)\right)\mathbf{1}_{[t_n, \tau_n]}(t), t \in [t_n, t_{n+1}).$$

It is obvious that $\int_{t_n}^{t_{n+1}} \psi(s)dG(s) = \xi_n - V(t_n)$ and $V(t_{n+1}) = \xi_n$.

<u>Case II.</u> $V(t_n) = \xi_{n-1}$. We consider a uniform partition $s_{n,k} = t_n + k\delta_n, k = 1, \ldots, n$ of $[t_n, t_{n+1}]$ with a mesh $\delta_n = \Delta_n/n$ and an auxiliary function

$$\phi_n(t) = a_n \sum_{k=0}^{n-1} \left(G(t) - G(s_{n,k})\right)\mathbf{1}_{[s_{n,k}, s_{n,k+1})}(t),$$

where $a_n = n^{-2}\theta^{(\alpha - H - 1)n}$. Since ϕ_n is piecewise Hölder continuous of order up to H, by the change of variables formula (Theorem 4.3.1 in [23])

$$\int_{t_n}^{t_{n+1}} \phi_n(t)dG(t) = a_n \sum_{k=0}^{n-1} \big(G(s_{n,k+1}) - G(s_{n,k})\big)^2.$$

Define a stopping time

$$\sigma_n = \inf\left\{t \ge t_n : \int_{t_n}^{t} \phi_n(s)dG(s) \ge |\xi_n - \xi_{n-1}|\right\} \wedge t_{n+1},$$

and set

$$\psi(t) = \mathrm{sign}(\xi_n - \xi_{n-1})\phi_n(t)\mathbf{1}_{[t_n,\sigma_n]}(t), t \in [t_n, t_{n+1}).$$

Now we want to ensure that, almost surely, $V(t_n) = \xi_{n-1}$ for all n large enough. By construction, Case I is always succeeded by Case II. So we need to ensure that $\sigma_n < t_{n+1}$ for all n large enough, equivalently, that

$$a_n \sum_{k=0}^{n-1} \big(G(s_{n,k+1}) - G(s_{n,k})\big)^2 > |\xi_n - \xi_{n-1}|.$$

Thanks to (10), it is enough to ensure that

$$\sum_{k=0}^{n-1} \big(G(s_{n,k+1}) - G(s_{n,k})\big)^2 > a_n^{-1} n\theta^{rn} = n^2\theta^{(r+H+1-\alpha)n}$$

for all n large enough. Define $\xi_k = G(s_{n,k+1}) - G(s_{n,k})$, $k = 0, \ldots, n-1$. Thanks to our choice of α, $r + H + 1 - \alpha > 2H$, so $n^2\theta^{(r+H+1-\alpha)n} < C_1 n^{1-2H}\theta^{2Hn}$ for all n large enough. Therefore, in view of (6),

$$\sum_{k=0}^{n-1} \mathbf{E}\xi_k^2 \ge C_1 n\delta_n^{2H} = C_1 n^{1-2H}\theta^{2Hn} > n^2\theta^{(r+H+1-\alpha)n},$$

so we can use Lemma 1. Using (A) and (B), estimate

$$\sum_{i,j=0}^{n-1} \big(\mathbf{E}\xi_i\xi_j\big)^2 \le \max_{0 \le i,j \le n-1} \mathbf{E}\xi_i\xi_j \sum_{i,j=0}^{n-1} \mathbf{E}\xi_i\xi_j$$

$$\le C_1\delta_n^{2H}\mathbf{E}\Big(\sum_{i=0}^{n-1} \xi_i\Big)^2 = C_1\delta_n^{2H}\mathbf{E}\big(G(t_{n+1}) - G(t_n)\big)^2$$

$$\le C_1^2\delta_n^{2H}\Delta_n^{2H} \le C_1^2 n^{-2H}\Delta^{4H} = C_1^2 n^{-2H}\theta^{4Hn}.$$

Hence, by Lemma 1,

$$\mathbf{P}\left(\sum_{k=0}^{n-1}\left(G(s_{n,k+1})-G(s_{n,k})\right)^2 \leq n^2\theta^{(r+H+1-\alpha)n}\right)$$

$$\leq \exp\left\{-\frac{\left(C_1 n^{1-2H}\theta^{2Hn}-n^2\theta^{(r+H+1-\alpha)n}\right)^2}{C_1^2 n^{-2H}\theta^{4Hn}}\right\} \leq \exp\left\{-Cn^{2-2H}\right\}.$$

Therefore, by the Borel–Cantelli lemma, almost surely there exists some $N_1(\omega) \geq N_0(\omega)$ such that for all $n \geq N_1(\omega)$

$$\sum_{k=0}^{n-1}\left(G(s_{n,k+1})-G(s_{n,k})\right)^2 > n^2\theta^{(r+H+1-\alpha)n},$$

so, as it was explained above, we have $V(t_n) = \xi_{n-1}$, $n \geq N_1(\omega)$.

Since all functions ϕ_n are bounded, we have that ψ is bounded on $[0, t_N]$ for any $N \geq 1$. Further, thanks to (7), for $t \in [t_n, t_{n+1}]$ with $n \geq N_1(\omega)$,

$$|\psi(s)| \leq C(\omega)a_n\delta_n^H|\log\delta_n|^{1/2} \leq C(\omega)n^{-2}\theta^{(\alpha-H-1)n}n^{-H}\theta^{Hn}n^{1/2}$$
$$= C(\omega)n^{\alpha-H-3/2}\theta^{(\alpha-1)n}. \tag{11}$$

Therefore, ψ is bounded (moreover, $\psi(t) \to 0$, $t \to T-$).

Further, by construction, $\|\psi\|_{\alpha,t_N} < \infty$ for any $N \geq 1$. Moreover, $|V(t) - \xi_{N-1}| \leq |\xi_N - \xi_{N-1}|$, $t \in [t_N, t_{N+1}]$. Thus, it remains to to verify that $\|\psi\|_{\alpha,[t_N,1]} < \infty$ and $\int_{t_N}^1 \psi(s)dG(s) \to 0$, $N \to \infty$, which would follow from $\|\psi\|_{\alpha,[t_N,1]} \to 0$, $N \to \infty$.

Let $N \geq N_1(\omega)$. Write

$$\|\psi\|_{\alpha,[t_N,T]} = \sum_{n=N}^{\infty}\int_{t_n}^{t_{n+1}}\left(\frac{|\psi(s)|}{(s-t_N)^\alpha} + \int_{t_N}^s\frac{|\psi(s)-\psi(u)|}{|s-u|^{1+\alpha}}du\right)ds.$$

Thanks to (11),

$$\int_{t_n}^{t_{n+1}}\frac{|\psi(s)|}{(s-t_N)^\alpha}ds \leq C(\omega)\Delta_n^{1-\alpha}n^{\alpha-H-3/2}\theta^{(\alpha-1)n} = C(\omega)n^{\alpha-H-3/2}.$$

Further,

$$\int_{t_N}^{t_{n+1}}\int_{t_n}^s\frac{|\psi(s)-\psi(u)|}{|s-u|^{1+\alpha}}du\,ds$$

$$= \sum_{k=1}^n\int_{s_{n,k-1}}^{s_{n,k}}\left(\int_{t_N}^{t_n}+\int_{t_n}^{s_{n,k-1}}+\int_{s_{n,k-1}}^s\right)\frac{|\psi(s)-\psi(u)|}{|s-u|^{1+\alpha}}du\,ds =: I_1 + I_2 + I_3.$$

Start with I_1, observing that ψ vanishes on $(\sigma_n, t_{n+1}]$:

$$I_1 \leq \int_{t_n}^{t_{n+1}} \sum_{j=N}^{n} \int_{t_{j-1}}^{t_j} \frac{|\psi(s)| + |\psi(u)|}{|s-u|^{1+\alpha}} du\, ds$$

$$\leq C(\omega) n^{\alpha-H-3/2} \theta^{(\alpha-1)n} \int_{t_n}^{t_{n+1}} (s-t_n)^{-\alpha} ds$$

$$+ C(\omega) \sum_{j=N}^{n-1} j^{\alpha-H-3/2} \theta^{(\alpha-1)j} \int_{t_n}^{t_{n+1}} (s-t_{j+1})^{-\alpha} ds$$

$$\leq C(\omega) n^{\alpha-H-3/2} \theta^{(\alpha-1)n} \Delta_n^{1-\alpha} + C(\omega) \sum_{j=N}^{n-1} j^{\alpha-H-3/2} \theta^{(\alpha-1)j} \Delta_n^{1-\alpha}$$

$$= C(\omega) n^{\alpha-H-3/2} + C(\omega) \sum_{j=N}^{n-1} j^{\alpha-H-3/2} \theta^{(\alpha-1)(j-n)}.$$

Similarly,

$$I_2 \leq C(\omega) n^{\alpha-H-3/2} \theta^{(\alpha-1)n} \sum_{k=1}^{n} \int_{s_{n,k-1}}^{s_{n,k}} \int_{t_n}^{s_{n,k-1}} |s-u|^{-1-\alpha} du\, ds$$

$$\leq C(\omega) n^{\alpha-H-3/2} \theta^{(\alpha-1)n} \sum_{k=1}^{n} \int_{s_{n,k-1}}^{s_{n,k}} (s-s_{n,k-1})^{-\alpha} ds$$

$$\leq C(\omega) n^{\alpha-H-3/2} \theta^{(\alpha-1)n} n \delta_n^{1-\alpha} = C(\omega) n^{2\alpha-H-3/2}.$$

Finally, assuming that $\sigma_n \in [s_{n,l-1}, s_{n,l})$,

$$I_3 \leq C(\omega) \sum_{k=1}^{l-1} \int_{s_{n,k-1}}^{s_{n,k}} \int_{s_{n,k-1}}^{s} a_n \frac{(s-u)^H |\log(s-u)|^{1/2}}{(s-u)^{1+\alpha}} du\, ds$$

$$+ \int_{s_{n,l-1}}^{\sigma_n} \int_{s_{n,l-1}}^{s} \frac{|\psi(s) - \psi(u)|}{|s-u|^{1+\alpha}} du\, ds + \int_{\sigma_n}^{s_{n,l}} \int_{s_{n,l-1}}^{\sigma_n} \frac{|\psi(s) - \psi(u)|}{|s-u|^{1+\alpha}} du\, ds$$

$$\leq C(\omega) a_n \sum_{k=1}^{n} \int_{s_{n,k-1}}^{s_{n,k}} (s-s_{n,k-1})^{H-\alpha} |\log(s-s_{n,k-1})|^{1/2} ds$$

$$+ C(\omega) n^{\alpha-H-3/2} \theta^{(\alpha-1)n} \int_{\sigma_n}^{s_{n,l}} \int_{s_{n,l-1}}^{\sigma_n} \frac{1}{|s-u|^{1+\alpha}} du\, ds$$

$$\leq C(\omega) a_n n \delta_n^{H+1-\alpha} |\log \delta_n|^{1/2} + C(\omega) n^{\alpha-H-3/2} \theta^{(\alpha-1)n} \delta_n^{-\alpha}$$

$$= C(\omega) n^{\alpha-H-3/2} + C(\omega) n^{2\alpha-H-3/2} \leq C(\omega) n^{2\alpha-H-3/2}.$$

Gathering all estimates we get

$$\int_{t_N}^{1} |D_{t_N+}^{\alpha}(\psi)(s)| ds \le C(\omega) \sum_{n=N}^{\infty} \left(n^{2\alpha-H-3/2} + \sum_{j=N}^{n-1} j^{\alpha-H-3/2} \theta^{(\alpha-1)(j-n)} \right)$$

$$\le C(\omega) \left(N^{2\alpha-H-1/2} + \sum_{j=N}^{\infty} j^{\alpha-H-3/2} \sum_{n=j+1}^{\infty} \theta^{(1-\alpha)(n-j)} \right)$$

$$\le C(\omega) N^{2\alpha-H-1/2},$$

which implies that $\|\psi\|_{\alpha,[t_N,T]} \to 0$, $N \to \infty$, finishing the proof.

Now we turn to the main object of this article.

Definition 1. A Gaussian process $G = \{G(t), t \in \mathbb{R}^+\}$ is called m-Wiener-transformable if there exists m-dimensional Wiener process $W = \{W(t), t \in \mathbb{R}^+\}$ such that G and W generate the same filtration, i.e. for any $t \in \mathbb{R}^+$

$$\mathscr{F}_t^G = \mathscr{F}_t^W.$$

We say that G is m-Wiener-transformable to W (evidently, process W can be non-unique.)

Remark 2. (i) In the case when $m = 1$ we say that the process G is Wiener-transformable.

(ii)Being Gaussian so having moments of any order, m-Wiener-transformable process admits at each time $t \in \mathbb{R}^+$ the martingale representation $G(t) = \mathbf{E}(G(0)) + \sum_{i=1}^{m} \int_0^t K_i(t,s) dW_i(s)$, where $K_i(t,s)$ is $\mathscr{F}_s^W$-adapted for any $0 \le s \le t$ and $\int_0^t \mathbf{E}(K_i(t,s))^2 ds < \infty$ for any $t \in \mathbb{R}^+$.

Now let the random variable ξ be $\mathscr{F}_T^W$-measurable, $\mathbf{E}\xi^2 < \infty$. Then in view of martingale representation theorem, ξ can be represented as

$$\xi = \mathbf{E}\xi + \int_0^T \vartheta(t) dW(t), \tag{12}$$

where ϑ is an adapted process with $\int_0^T \mathbf{E}\vartheta(t)^2 dt < \infty$.

As it was explained in introduction, we are interested when ξ can be represented in the form

$$\xi = \int_0^T \psi(s) dG(s),$$

where the integrand is adapted, and the integral is understood in the pathwise sense.

Theorem 2. *Let the following conditions hold.*

(i) *Gaussian process G satisfies condition (A) and (B).*
(ii)*Stochastic process ϑ in representation (12) satisfies*

$$\int_0^T |\vartheta(s)|^{2p} ds < \infty \tag{13}$$

a.s. with some $p > 1$.

Then there exists a bounded adapted process ψ such that $\|\psi\|_{\alpha,T} < \infty$ for some $\alpha \in \left(1 - H, \frac{1}{2}\right)$ and ξ admits the representation

$$\xi = \int_0^T \psi(s)dG(s),$$

almost surely.

Remark 3. As it was mentioned in [15], it is sufficient to require the properties (A) and (B) to hold on some subinterval $[T - \delta, T]$. Similarly, it is enough to require in (ii) that $\int_{T-\delta}^T |\vartheta(t)|^{2p}dt < \infty$ almost surely.

First we prove a simple result establishing Hölder continuity of Itô integral.

Lemma 2. *Let $\vartheta = \{\vartheta(t), t \in [0, T]\}$ be a real-valued progressively measurable process such that for some $p \in (1, +\infty]$*

$$\int_0^T |\vartheta(s)|^{2p}ds < \infty$$

a.s. Then the stochastic integral $\int_0^t \vartheta(s)dW(s)$ is Hölder continuous of any order up to $\frac{1}{2} - \frac{1}{2p}$.

Proof. First note that if there exist non-random positive constants a, C such that for any $s, t \in [0, T]$ with $s < t$

$$\int_s^t \vartheta^2(u)du \leq C(t - s)^a,$$

then $\int_0^t \vartheta(s)dW(s)$ is Hölder continuous of any order up to $a/2$. Indeed, in this case by the Burkholder inequality, for any $r > 1$ and $s, t \in [0, T]$ with $s < t$

$$\mathbf{E}\left|\int_s^t \vartheta(u)dW(u)\right|^r \leq C_r \mathbf{E}\left(\int_s^t \vartheta^2(u)du\right)^{r/2} \leq C(t - s)^{ar/2},$$

so by the Kolmogorov–Chentsov theorem, $\int_0^t \vartheta(s)dW(s)$ is Hölder continuous of order $\frac{1}{r}\left(\frac{ar}{2} - 1\right) = \frac{a}{2} - \frac{1}{2r}$. Since r can be arbitrarily large, we deduce the claim.

Now let for $n \geq 1$, $\vartheta_n(t) = \vartheta(t)\mathbf{1}_{\int_0^t |\vartheta(s)|^{2p}ds \leq n}$, $t \in [0, T]$. By the Hölder inequality, for any $s, t \in [0, T]$ with $s < t$

$$\int_s^t \vartheta_n^2(u)du \leq (t - s)^{1-1/p}\left(\int_s^t |\vartheta(u)|^{2p}du\right)^{1/p} \leq n^{1/p}(t - s)^{1-1/p}.$$

Therefore, by the above claim, $\int_0^t \vartheta_n(s)dW(s)$ is a.s. Hölder continuous of any order up to $\frac{1}{2} - \frac{1}{2p}$. However, ϑ_n coincides with ϑ on $\Omega_n = \{\int_0^T |\vartheta(t)|^{2p}dt \leq n\}$. Consequently, $\int_0^t \vartheta_n(s)dW(s)$ is a.s. Hölder continuous of any order up to $\frac{1}{2} - \frac{1}{2p}$ on Ω_n. Since $\mathbf{P}(\bigcup_{n\geq 1} \Omega_n) = 1$, we arrive at the statement of the lemma.

Proof of Theorem 2. Define

$$Z(t) = \mathbf{E}\xi + \int_0^t \vartheta(s)dW(s).$$

This is an adapted process with $Z(T) = \xi$, moreover, it follows from Lemma 2 that Z is Hölder continuous of any order up to $\frac{1}{2} - \frac{1}{2p}$. Thus, the statement follows from Theorem 1.

In the case where one looks at improper representation, no assumptions on ξ are needed.

Theorem 3. *(Improper representation theorem) Assume that an adapted Gaussian process $G = \{G(t),\ t \in [0,T]\}$ satisfies conditions $(A),(B)$. Then for any random variable ξ there exists an adapted process ψ that $\|\psi\|_{\alpha,t} < \infty$ for some $\alpha \in \left(1 - H, \frac{1}{2}\right)$ and any $t \in [0,T)$ and ξ admits the representation*

$$\xi = \lim_{t \to T-} \int_0^t \psi(s)dG(s),$$

almost surely.

Proof. The proof is exactly the same as for Theorem 4.2 in [22], so we just sketch the main idea.

Consider an increasing sequence of points $\{t_n, n \geq 1\}$ in $[0,T)$ such that $t_n \to T$, $n \to \infty$, and let $\{\xi_n, n \geq 1\}$ be a sequence of random variables such that ξ_n is $\mathcal{F}_{t_n}$-measurable for each $n \geq 1$, and $\xi_n \to \xi$, $n \to \infty$, a.s. Set for convenience $\xi_0 = 0$. Similarly to Case I in Theorem 1, for each $n \geq 1$, there exists an adapted process $\{\phi_n(t), t \in [t_n, t_{n+1}]\}$, such that $\int_{t_n}^t \phi_n(s)dG(s) \to +\infty$ as $t \to t_{n+1}-$. For $n \geq 1$, define a stopping time

$$\tau_n = \inf\left\{t \geq t_n : \int_{t_n}^t \phi_n(s)dG(s) \geq |\xi_n - \xi_{n-1}|\right\}$$

and set

$$\psi(t) = \phi_n(t)\,\mathrm{sign}\left(\xi_n - \xi_{n-1}\right)\mathbf{1}_{[t_n,\tau_n]}(t),\, t \in [t_n, t_{n+1}).$$

Then for any $n \geq 1$, we have $\int_0^{t_{n+1}} \psi(s)dG(s) = \xi_n$ and $\int_0^t \psi(s)dG(s)$ lies between ξ_{n-1} and ξ_n for $t \in [t_{n-1}, t_n]$. Consequently, $\int_0^t \psi(s)dG(s) \to \xi$, $t \to T-$, a.s., as required.

Further we give several examples of Wiener-transformable Gaussian processes satisfying conditions (A) and (B) (for more detail and proofs see, e.g. [15]) and formulate the corresponding representation results.

3.1 Fractional Brownian motion

Fractional Brownian motion B^H with Hurst parameter $H \in (0,1)$ is a centered Gaussian process with the covariance

$$\mathbf{E}B^H(t)B^H(s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right);$$

an extensive treatment of fractional Brownian motion is given in [17]. For $H = \frac{1}{2}$, fractional Brownian motion is a Wiener process; for $H \neq \frac{1}{2}$ it is Wiener-transformable to the Wiener process W via relations

$$B^H(t) = \int_0^t K^H(t,s)dW(s) \tag{14}$$

and

$$W(t) = \int_0^t k^H(t,s)dB^H(s), \tag{15}$$

see e.g. [18].

   Fractional Brownian motion with index $H \in (0,1)$ satisfies condition (A) and satisfies condition (B) if $H \in (\frac{1}{2},1)$.

   Therefore, a random variable satisfying (13) with any $p > 1$ admits the representation (8).

3.2 Fractional Ornstein–Uhlenbeck process

Let $H \in (\frac{1}{2},1)$. Then the fractional Ornstein–Uhlenbeck process $Y = \{Y(t), t \geq 0\}$, involving fractional Brownian component and satisfying the equation

$$Y(t) = Y_0 + \int_0^t (b - aY(s))ds + \sigma B^H(t),$$

where $a,b \in \mathbb{R}$ and $\sigma > 0$, is Wiener-transformable to the same Wiener process as the underlying fBm B^H.

   Consider a fractional Ornstein–Uhlenbeck process of the simplified form

$$Y(t) = Y_0 + a\int_0^t Y(s)ds + B^H(t), \ t \geq 0.$$

It satisfies condition (A); if $a > 0$, it satisfies condition (B) as well.

   As it was mentioned in [15], the representation theorem is valid for a fractional Ornstein-Uhlenbeck process with a negative drift coefficient too. Indeed, we can annihilate the drift of the fractional Ornstein-Uhlenbeck process with the help of Girsanov theorem, transforming a fractional Ornstein-Uhlenbeck process with negative drift to a fractional Brownian motion $\widetilde{B}^H$. Then, assuming (13), we represent

the random variable ξ as $\xi = \int_0^T \psi(s)d\widetilde{B}^H(s)$ on the new probability space. Finally, we return to the original probability space. Due to the pathwise nature of integral, its value is not changed upon changes of measure.

3.3 Subfractional Brownian motion

Subfractional Brownian motion with index H, that is a centered Gaussian process $G^H = \{G^H(t), t \geq 0\}$ with covariance function

$$\mathbf{E}G^H(t)G^H(s) = t^{2H} + s^{2H} - \frac{1}{2}\left(|t+s|^{2H} + |t-s|^{2H}\right),$$

satisfies condition (A) and condition (B) for $H \in (\frac{1}{2}, 1)$.

3.4 Bifractional Brownian motion

Bifractional Brownian motion with indices $A \in (0,1)$ and $K \in (0,1)$, that is a centered Gaussian process with covariance function

$$\mathbf{E}G^{A,K}(t)G^{A,K}(s) = \frac{1}{2^K}\left(\left(t^{2A} + s^{2A}\right)^K - |t-s|^{2AK}\right),$$

satisfies condition (A) with $H = AK$ and satisfies condition (B) for $AK > \frac{1}{2}$.

3.5 Geometric Brownian motion

Geometric Brownian motion involving the Wiener component and having the form

$$S = \{S(t) = S(0)\exp\{\mu t + \sigma W(t)\}, \ t \geq 0\},$$

with $S(0) > 0$, $\mu \in \mathbb{R}$, $\sigma > 0$, is Wiener-transformable to the underlying Wiener process W. However, it does not satisfy the assumptions of Theorem 2. One should appeal here to the standard semimartingale tools, like the martingale representation theorem.

3.6 Linear combination of fractional Brownian motions

Consider a collection of Hurst indices $\frac{1}{2} \le H_1 < H_2 < \ldots < H_m < 1$ and independent fractional Brownian motions with corresponding Hurst indices H_i, $1 \le i \le m$. Then the linear combination $\sum_{i=1}^m a_i B^{H_i}$ is m-Wiener-transformable to the Wiener process $W = (W_1, \ldots, W_m)$, where W_i is such Wiener process to which fractional Brownian motion B^{H_i} is Wiener-transformable. In particular, the mixed fractional Brownian motion $M^H = W + B^H$, introduced in [5], is 2-Wiener-transformable.

The linear combination $\sum_{i=1}^m a_i B^{H_i}$ satisfies condition (A) with $H = H_1$, and condition (B) whenever $H_1 > 1/2$.

We note that in the case of mixed fractional Brownian motion, the existence of representation (8) cannot be derived from Theorem 2, as we have $H = \frac{1}{2}$ in this case. By slightly different methods, it was established in [22] that arbitrary $\mathscr{F}_T$-measurable random variable ξ admits the representation

$$\xi = \int_0^T \psi(s) d\big(B^H(s) + W(s)\big),$$

where the integral with respect to B^H is understood, as here, in the pathwise sense, the integral with respect to W, in the extended Itô sense. In contrast to Theorem 1, we can not for the moment establish this result for the bounded strategies. Therefore, it would be interesting to study which random variables have representations with bounded ψ in the mixed model.

3.7 Volterra process

Consider Volterra integral transform of Wiener process, that is the process of the form $G(t) = \int_0^t K(t,s) dW(s)$ with non-random kernel $K(t,\cdot) \in L_2[0,t]$ for $t \in [0,T]$. Let the constant $r \in [0, 1/2)$ be fixed. Let the following conditions hold.

$(B1)$ The kernel K is non-negative on $[0,T]^2$ and for any $s \in [0,T]$ $K(\cdot,s)$ is non-decreasing in the first argument;

$(B2)$ There exist constants $D_i > 0, i = 2,3$ and $H \in (1/2, 1)$ such that

$$|K(t_2,s) - K(t_1,s)| \le D_2 |t_2 - t_1|^H s^{-r}, \quad s, t_1, t_2 \in [0,T]$$

and

$$K(t,s) \le D_3 (t-s)^{H-1/2} s^{-r};$$

and at least one of the following conditions

$(B3,a)$ There exist constant $D_1 > 0$ such that

$$D_1 |t_2 - t_1|^H s^{-r} \le |K(t_2,s) - K(t_1,s)|, \quad s, t_1, t_2 \in [0,T];$$

$(B3,b)$ There exist constant $D_1 > 0$ such that

$$K(t,s) \geq D_1(t-s)^{H-1/2}s^{-r}, \quad s,t \in [0,T].$$

Then the Gaussian process $G(t) = \int_0^t K(t,s)dW(s)$, satisfies condition (A), (B) on any subinterval $[T-\delta, T]$ with $\delta \in (0,1)$.

4 Expected utility maximization in Wiener-transformable markets

4.1 Expected utility maximization for unrestricted capital profiles

Consider the problem of maximizing the expected utility. Our goal is to characterize the optimal asset profiles in the framework of the markets with risky assets involving Gaussian processes satisfying conditions of Theorem 2. We follow the general approach described in [9] and [12], but apply its interpretation from [10]. We fix $T > 0$ and from now on consider $\mathscr{F}_T^W$-measurable random variables. Let the utility function $u : \mathbb{R} \to \mathbb{R}$ be strictly increasing and strictly concave, $L^0(\Omega, \mathscr{F}_T^W, \mathbf{P})$ be the set of all $\mathscr{F}_T^W$-measurable random variables, and let the set of admissible capital profiles coincide with $L^0(\Omega, \mathscr{F}_T^W, \mathbf{P})$. Let $\mathbf{P}^*$ be a probability measure on $(\Omega, \mathscr{F}_T^W)$, which is equivalent to $\mathbf{P}$, and denote $\varphi(T) = \frac{d\mathbf{P}^*}{d\mathbf{P}}$. The budget constraint is given by $\mathbf{E}_{\mathbf{P}^*}(X) = w$, where $w > 0$ is some number that can be in some cases, but not obligatory, interpreted as the initial wealth. Thus the budget set is defined as

$$\mathscr{B} = \left\{ X \in L^0\left(\Omega, \mathscr{F}_T^W, \mathbf{P}\right) \cap L^1\left(\Omega, \mathscr{F}_T^W, \mathbf{P}^*\right) | \mathbf{E}_{\mathbf{P}^*}(X) = w \right\}.$$

The problem is to find such $X^* \in \mathscr{B}$, for which $\mathbf{E}(u(X^*)) = \max_{X \in \mathscr{B}} \mathbf{E}(u(X))$. Consider the inverse function $I(x) = (u'(x))^{-1}$.

Theorem 4 ([10, Theorem 3.34]). *Let the following condition hold: Strictly increasing and strictly concave utility function $u : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, bounded from above and*

$$\lim_{x \downarrow -\infty} u'(x) = +\infty.$$

Then the solution of this maximization problem has a form

$$X^* = I(c\varphi(T)),$$

under additional assumption that $\mathbf{E}_{\mathbf{P}^*}(X^*) = w$.

To connect the solution of maximization problem with specific W-transformable Gaussian process describing the price process, we consider the following items.

1. Consider random variable $\varphi(T)$, $\varphi(T) > 0$ a.s. and let $\mathbf{E}(\varphi(T)) = 1$. Being the terminal value of a positive martingale $\varphi = \{\varphi_t = \mathbf{E}(\varphi(T)|\mathscr{F}_t^W), t \in [0,T]\}$, $\varphi(T)$ admits the following representation

$$\varphi(T) = \exp\left\{\int_0^T \vartheta(s)dW_s - \frac{1}{2}\int_0^T \vartheta^2(s)ds\right\}, \tag{16}$$

where ϑ is a real-valued progressively measurable process for which

$$\mathbf{P}\left\{\int_0^T \vartheta^2(s)ds < \infty\right\} = 1.$$

Assume that ϑ satisfies (13). Then $\varphi(T)$ is a terminal value of a Hölder continuous process of order $\frac{1}{2} - \frac{1}{2p}$.

2. Consider W-transformable Gaussian process $G = \{G(t), t \in [0,T]\}$ satisfying conditions (A) and (B), and introduce the set

$$\mathscr{B}_w^G = \left\{\psi\colon [0,T] \times \Omega \to \mathbb{R} \;\middle|\; \psi \text{ is bounded } \mathscr{F}_t^W\text{-adapted, there exists a generalized}\right.$$

$$\left.\text{Lebesgue-Stieltjes integral } \int_0^T \psi(s)dG(s), \text{ and } \mathbf{E}\left(\varphi(T)\int_0^T \psi(s)dG(s)\right) = w\right\}.$$

Theorem 5. *Let the following conditions hold*

(i) Gaussian process G satisfies condition (A) and (B).
(ii) Function $I(x), x \in \mathbb{R}$ is Hölder continuous.
(iii) Stochastic process ϑ in representation (16) satisfies (13) with some $p > 1$.
(iv) There exists $c \in \mathbb{R}$ such that $\mathbf{E}(\varphi(T)I(c\varphi(T))) = w$.

Then the random variable $X^* = I(c\varphi(T))$ admits the representation

$$X^* = \int_0^T \overline{\psi}(s)dG(s), \tag{17}$$

with some $\overline{\psi} \in \mathscr{B}_w^G$, and

$$\mathbf{E}(u(X^*)) = \max_{\psi \in \mathscr{B}_w^G} \mathbf{E}\left(u\left(\int_0^T \psi(s)dG(s)\right)\right). \tag{18}$$

Proof. From Lemma 2 we have that for any $c \in \mathbb{R}$ the random variable $\xi = I(c\varphi(T))$ is the final value of a Hölder continuous process

$$U(t) = I(c\varphi(t)) = I\left(c\exp\left\{\int_0^t \vartheta(s)dW(s) - \frac{1}{2}\int_0^t \vartheta^2(s)ds\right\}\right).$$

and the Hölder exponent exceeds ρ. Together with (i)–(iii) this allows to apply Theorem 2 to obtain the existence of representation (17). Assume now that (18) is not valid, and there exists $\psi_0 \in \mathscr{B}_w^G$ such that $\mathbf{E}\left(\varphi(T)\int_0^T \psi_0(s)dG(s)\right) = w$, and $\mathbf{E}u\left(\int_0^T \psi_0(s)dG(s)\right) > \mathbf{E}u(X^*)$. But in this case $\int_0^T \psi_0(s)dG(s)$ belongs to $\mathscr{B}$, and we get a contradiction with Theorem 4.

Remark 4. Assuming only (i) and (iv), one can show in a similar way, but using Theorem 3 instead of Theorem 2 that

$$\mathbf{E}(u(X^*)) = \sup_{\psi \in \mathcal{B}_w^G} \mathbf{E}\left(u\left(\int_0^T \psi(s)dG(s)\right)\right).$$

However, the existence of a maximizer is not guaranteed in this case.

Example 1. Let $u(x) = 1 - e^{-\beta x}$ be an exponential utility function with constant absolute risk aversion $\beta > 0$. In this case $I(x) = -\frac{1}{\beta}\log(\frac{x}{\beta})$. Assume that

$$\varphi(T) = \exp\left\{\int_0^T \vartheta(s)dW(s) - \frac{1}{2}\int_0^T \vartheta^2(s)ds\right\}$$

is chosen in such a way that

$$\mathbf{E}\left(\varphi(T)|\log\varphi(T)|\right)$$
$$= \mathbf{E}\left(\exp\left\{\int_0^T \vartheta(s)dW(s) - \frac{1}{2}\int_0^T \vartheta^2(s)ds\right\}\right. \tag{19}$$
$$\left. \times \left|\int_0^T \vartheta(s)dW(s) - \frac{1}{2}\int_0^T \vartheta^2(s)ds\right|\right) < \infty.$$

Then, according to Example 3.35 from [10], the optimal profile can be written as

$$X^* = -\frac{1}{\beta}\left(\int_0^T \vartheta(s)dW(s) - \frac{1}{2}\int_0^T \vartheta^2(s)ds\right) + w + \frac{1}{\beta}H(\mathbf{P}^*|\mathbf{P}), \tag{20}$$

where $H(\mathbf{P}^*|\mathbf{P}) = \mathbf{E}\left(\varphi(T)\log\varphi(T)\right)$, condition (19) supplies that $H(\mathbf{P}^*|\mathbf{P})$ exists, and the maximal value of the expected utility is

$$\mathbf{E}(u(X^*)) = 1 - \exp\{-\beta w - H(\mathbf{P}^*|\mathbf{P})\}.$$

Let $\varphi(T)$ be chosen in such a way that the corresponding process ϑ satisfies the assumption of Lemma 2. Also, let W-transformable process G satisfy conditions (A) and (B) of Theorem 4, and ϑ satisfy (13) with $p > 1$. Then we can conclude directly from representation (20) that conditions of Theorem 4 hold. Therefore, the optimal profile X^* admits the representation $X^* = \int_0^T \psi(s)dG(s)$.

Remark 5. Similarly, under the same conditions as above, we can conclude that for any constant $d \in \mathbb{R}$ there exists ψ_d such that $X^* = d + \int_0^T \psi_d(s)dG(s)$. Therefore, we can start from any initial value of the capital and achieve the desirable wealth. In this sense, w is not necessarily the initial wealth as it is often assumed in the semimartingale framework, but is rather a budget constraint in the generalized sense.

Remark 6. In the case when W-transformable Gaussian process G is a semimartingale, we can use Girsanov's theorem in order to get the representation, similar to (17). Indeed, let, for example, G be a Gaussian process of the form $G(t) =$

$\int_0^t \mu(s)ds + \int_0^t a(s)dW(s)$, $|\mu(s)| \le \mu$, $a(s) > a > 0$ are non-random measurable functions, and ξ is $\mathscr{F}_T^W$-measurable random variable, $\mathbf{E}(\xi^2) < \infty$. Then we transform G into $\widetilde{G} = \int_0^{\cdot} a(s)d\widetilde{W}(s)$, with the help of equivalent probability measure $\widetilde{\mathbf{P}}$ having Radon–Nikodym derivative

$$\frac{d\widetilde{\mathbf{P}}}{d\mathbf{P}} = \exp\left\{-\int_0^T \frac{\mu(s)}{a(s)}dW(s) - \frac{1}{2}\int_0^T \left(\frac{\mu(s)}{a(s)}\right)^2 ds\right\}.$$

With respect to this measure $\mathbf{E}_{\widetilde{\mathbf{P}}}|X^*| < \infty$, and we get the following representation

$$X^* = \mathbf{E}_{\widetilde{\mathbf{P}}}(X^*) + \int_0^T \psi(s)d\widetilde{W}_s = \mathbf{E}_{\widetilde{\mathbf{P}}}(X^*) + \int_0^T \frac{\psi(s)}{a(s)}d\widetilde{G}(s) \qquad (21)$$

$$= \mathbf{E}_{\widetilde{\mathbf{P}}}(X^*) + \int_0^T \frac{\psi(s)}{a(s)}dG(s) = \mathbf{E}_{\widetilde{\mathbf{P}}}(X^*) + \int_0^T \psi(s)\mu(s)ds + \int_0^T \psi(s)dW(s). \quad (22)$$

Representations (17) and (21) have the following distinction: (17) "starts" from 0 (but can start from any other constant) while (21) "starts" exactly from $\mathbf{E}_{\widetilde{\mathbf{P}}}(X^*)$.

As we can see, the solution of the utility maximization problem for W–transformable process depends on the process in indirect way, through the random variable $\varphi(T)$ such that $\mathbf{E}\varphi(T) = 1$, $\varphi(T) > 0$ a.s. Also, this solution depends on whether or not we can choose the appropriate value of c, but this is more or less a technical issue. Let us return to the choice of $\varphi(T)$. In the case of the semimartingale market, $\varphi(T)$ can be reasonably chosen as the likelihood ratio of some martingale measure, and the choice is unique in the case of the complete market. The non-semimartingale market can contain some hidden semimartingale structure. To illustrate this, consider two examples.

Example 2. Let the market consist of bond B and stock S,

$$B(t) = e^{rt}, \; S(t) = \exp\left\{\mu t + \sigma B_t^H\right\},$$

$r \ge 0$, $\mu \in \mathbb{R}$, $\sigma > 0$, $H > \frac{1}{2}$. The discounted price process has a form $Y(t) = \exp\left\{(\mu - r)t + \sigma B_t^H\right\}$. It is well-known that such market admits an arbitrage, but even in these circumstances the utility maximization problem makes sense. Well, how to choose $\varphi(T)$? There are at least two natural approaches.

1. Note that for $H > \frac{1}{2}$ the kernel K^H from (14) has a form

$$K^H(t,s) = C(H)s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{1}{2}}(u-s)^{H-\frac{3}{2}}du,$$

and representation (15) has a form

$$W(t) = (C(H))^{-1}\int_0^t s^{\frac{1}{2}-H}K^*(t,s)dB_s^H,$$

where

$$K^*(t,s) = \left(t^{H-\frac{1}{2}}(t-s)^{\frac{1}{2}-H} \right.$$

$$- \left(H - \frac{1}{2} \right) \int_s^t u^{H-\frac{3}{2}}(u-s)^{\frac{1}{2}-H} du \right) \frac{1}{\Gamma\left(\frac{3}{2}-H\right)}.$$

Therefore,

$$(C(H))^{-1} \int_0^t s^{\frac{1}{2}-H} K^*(t,s) d\left((\mu-r)s + \sigma B_s^H \right)$$

$$= \sigma W(t) + \frac{\mu-r}{C(H)} \int_0^t s^{\frac{1}{2}-H} K^*(t,s) ds$$

$$= \sigma W(t) + \frac{\mu-r}{C(H)\Gamma\left(\frac{3}{2}-H\right)} \int_0^t \left(s^{\frac{1}{2}-H} t^{H-\frac{1}{2}}(t-s)^{\frac{1}{2}-H} \right.$$

$$- \left(H - \frac{1}{2} \right) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}}(u-s)^{\frac{1}{2}-H} du \right) ds$$

$$= \sigma W_t + \frac{\mu-r}{C(H)\Gamma(\frac{3}{2}-H)} \frac{\Gamma^2(\frac{3}{2}-H)}{(\frac{3}{2}-H)\Gamma(2-2H)} t^{\frac{3}{2}-H}$$

$$= \sigma W_t + (\mu-r)C_1(H) t^{\frac{3}{2}-H},$$

where

$$C_1(H) = \left(\frac{3}{2} - H \right)^{-1} \left(\frac{\Gamma(\frac{3}{2}-H)}{2H\Gamma(2-2H)\Gamma(H+\frac{1}{2})} \right)^{\frac{1}{2}}.$$

In this sense we say that the model involves a hidden semimartingale structure. Consider a virtual semimartingale asset

$$\hat{Y}(t) = \exp\left\{ (C(H))^{-1} \int_0^t s^{\frac{1}{2}-H} K^*(t,s) d\log Y(s) \right\}$$

$$= \exp\left\{ \sigma W_t + (\mu-r)C(H) t^{\frac{3}{2}-H} \right\}.$$

We see that measure $\mathbf{P}^*$ such that

$$\frac{d\mathbf{P}^*}{d\mathbf{P}} = \exp\left\{ -\int_0^T \left(\frac{(\mu-r)C_2(H)}{\sigma} s^{\frac{1}{2}-H} + \frac{\sigma}{2} \right) dW_s \right.$$

$$\left. -\frac{1}{2} \int_0^T \left(\frac{(\mu-r)C_2(H)}{\sigma} s^{\frac{1}{2}-H} + \frac{\sigma}{2} \right)^2 ds \right\}, \tag{23}$$

where $C_2(H) = C_1(H) \left(\frac{3}{2} - H \right)$, reduces $\hat{Y}(t)$ to the martingale of the form $\exp\left\{ \sigma W_t - \frac{\sigma^2}{2} t \right\}$. Therefore, we can put $\varphi(T) = \frac{d\mathbf{P}^*}{d\mathbf{P}}$ from (23). Regarding the Hölder property, $\vartheta(s) = s^{\frac{1}{2}-H}$ satisfies (13) with some $p > 1$ for any $H \in (\frac{1}{2}, 1)$. Therefore, for the utility function $u(x) = 1 - e^{-\alpha x}$ we have

$$X^* = \frac{1}{\alpha}\left(\int_0^T \varsigma(s)dW_s - \frac{1}{2}\int_0^T \varsigma_s^2 ds\right) + W + \frac{1}{2}H(\mathbf{P}^*|\mathbf{P}),$$

where $\varsigma(s) = \frac{(\mu-r)C_2(H)}{\sigma}s^{\frac{1}{2}-H} + \frac{\sigma}{2}$, and $|H(\mathbf{P}^*|\mathbf{P})| < \infty$.

2. It was proved in [8] that the fractional Brownian motion B^H is the limit in $L_p(\Omega,\mathscr{F},\mathbf{P})$ for any $p > 0$ of the process

$$B^{H,\varepsilon}(t) = \int_0^t K(s+\varepsilon,s)dW(s) + \int_0^t \psi_\varepsilon(s)ds,$$

where W is he underlying Wiener process, i.e. $B^H(t) = \int_0^t K(t,s)dW(s)$, where

$$K(t,s) = C_H s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{1}{2}}(u-s)^{H-\frac{3}{2}}du,$$

$$\psi_\varepsilon(s) = \int_0^s \partial_1 K(s+\varepsilon,u)dW_u,$$

$$\partial_1 K(t,s) = \frac{\partial K(t,s)}{\partial t} = C_H s^{\frac{1}{2}-H}t^{H-\frac{1}{2}}(t-s)^{H-\frac{3}{2}}.$$

Consider prelimit market with discounted risky asset price Y^ε of the form

$$Y^\varepsilon(t) = \exp\left\{(\mu-r)t + \sigma\int_0^t \psi_\varepsilon(s)ds + \sigma\int_0^t K(s+\varepsilon,s)dW_s\right\}.$$

This financial market is arbitrage-free and complete, and the unique martingale measure has the Radon-Nikodym derivative

$$\varphi_\varepsilon(T) = \exp\left\{-\int_0^T \zeta_\varepsilon(t)dW_t - \frac{1}{2}\int_0^T \zeta_\varepsilon^2(t)dt\right\},$$

where

$$\zeta_\varepsilon(t) = \frac{\mu-r+\sigma\psi_\varepsilon(t)}{\sigma K(t+\varepsilon,t)} + \frac{1}{2}\sigma K(t+\varepsilon,t).$$

Note that $K(t+\varepsilon,t) \to 0$ as $\varepsilon \to 0$. Furthermore, $\rho_t = \frac{\mu-r+\sigma\psi_\varepsilon(t)}{\sigma K(t+\varepsilon,t)}$ is a Gaussian process with $\mathbf{E}\rho_t = 0$ and

$$\mathbf{var}\,\zeta_\varepsilon(t) = \int_0^t \left(\frac{\partial_1 K(t+\varepsilon,u)}{K(t+\varepsilon,t)}\right)^2 du$$

$$= \int_0^t \left(\frac{u^{1/2-H}(t+\varepsilon)^{H-1/2}(t+\varepsilon-u)^{H-3/2}}{t^{1/2-H}\int_t^{t+\varepsilon}v^{H-1/2}(v-t)^{H-3/2}}\right)^2 du$$

$$\geq \varepsilon^{1-2H}\int_0^t (t+\varepsilon-u)^{2H-3}du = \frac{\varepsilon^{1-2H}t}{2-2H}\left(\varepsilon^{2H-2} - (t+\varepsilon)^{2H-2}\right) \to \infty.$$

Therefore, we can not get a reasonable limit of $\varphi_\varepsilon(T)$ as $\varepsilon \to 0$. Thus one should use this approach with great caution.

4.2 Expected utility maximization for restricted capital profiles

Consider now the case when the utility function u is defined on some interval (a, ∞). Assume for technical simplicity that $a = 0$. Therefore, in this case the set $\mathscr{B}_0$ of admissible capital profiles has a form

$$\mathscr{B}_0 = \left\{ X \in L^0(\Omega, \mathscr{F}, \mathbf{P}) : X \geq 0 \text{ a.s. and } \mathbf{E}(\varphi(T)X) = w \right\}.$$

Assume that the utility function u is continuously differentiable on $(0, \infty)$, introduce $\pi_1 = \lim_{x \uparrow \infty} u'(x) \geq 0$, $\pi_2 = u'(0+) = \lim_{x \downarrow 0} u'(x) \leq +\infty$, and define $I^+ : (\pi_1, \pi_2) \longrightarrow (0, \infty)$ as the continuous, bijective function, inverse to u' on (π_1, π_2).

Extend I^+ to the whole half-axis $[0, \infty]$ by setting

$$I^+(y) = \begin{cases} +\infty, & y \leq \pi_1 \\ 0, & y \geq \pi_2. \end{cases}$$

Theorem 6 ([10], Theorem 3.39). *Let the random variable $X^* \in \mathscr{B}_0$ have a form $X^* = I^+(c\varphi(T))$ for such constant $c > 0$ that $\mathbf{E}(\varphi(T)I^+(c\varphi(T))) = w$. If $\mathbf{E}u(X^*) < \infty$ then*

$$\mathbf{E}(u(X^*)) = \max_{X \in \mathscr{B}_0} \mathbf{E}(u(X)),$$

and this maximizer is unique.

From here we deduce the corresponding result on the solution of utility maximization problem similarly to Theorem 5. Define, as before,

$$\mathscr{B}_w^G = \left\{ \psi \colon [0, T] \times \Omega \to \mathbb{R} \; \middle| \; \psi \text{ is bounded } \mathscr{F}_t^W\text{-adapted, there exists a generalized} \right.$$

Lebesgue-Stieltjes integral $\int_0^T \psi(s)dG(s) \geq 0$, and $\left. \mathbf{E}\left(\varphi(T) \int_0^T \psi(s)dG(s) \right) = w \right\}$.

Theorem 7. *Let the following conditions hold*

(i) Gaussian process G satisfies conditions (A) and (B).
(ii) Function $I^+(x), x \in \mathbb{R}$ is Hölder continuous.
(iii) Stochastic process ϑ in representation (16) satisfies (13) with some $p > 1$.
(iv) There exists $c \in \mathbb{R}$ such that $\mathbf{E}(\varphi(T)I^+(c\varphi(T))) = w$.

Then the random variable $X^* = I^+(c\varphi(T))$ admits the representation

$$X^* = \int_0^T \overline{\psi}(s)dG(s),$$

with some $\overline{\psi} \in \widetilde{\mathscr{B}}_w^G$. If $\mathbf{E}u(X^*) < \infty$, the X^* is the solution to expected utility maximization problem: and

$$\mathbf{E}(u(X^*)) = \max_{\psi \in \tilde{\mathscr{B}}_w^G} \mathbf{E}\left(u\left(\int_0^T \psi(s)dG(s)\right)\right).$$

Example 3. Consider the case of CARA utility function u. Let first $u(x) = \frac{x^\gamma}{\gamma}, x > 0,$ $\gamma \in (0,1)$. Then, according to [10, Example 3.43],

$$I^+(c\varphi(T)) = c^{-\frac{1}{1-\gamma}}(\varphi(T))^{-\frac{1}{1-\gamma}}.$$

If $d := \mathbf{E}(\varphi(T))^{-\frac{\gamma}{1-\gamma}} < \infty$ then unique optimal profile is given by $X^* = \frac{w}{d}(\varphi(T))^{-\frac{1}{1-\gamma}}$, and the maximal value of the expected utility is equal to

$$\mathbf{E}(u(X^*)) = \frac{1}{\gamma}w^\gamma d^{1-\gamma}.$$

As it was mentioned,

$$\varphi = \varphi(T) = \exp\left\{\int_0^T \vartheta(s)dW(s) - \frac{1}{2}\int_0^T \vartheta^2(s)ds,\right\} \tag{24}$$

thus

$$(\varphi(T))^{-\frac{1}{1-\gamma}} = \exp\left\{-\frac{1}{1-\gamma}\int_0^T \vartheta(s)dW(s) + \frac{1}{2(1-\gamma)}\int_0^T \vartheta^2(s)ds\right\}.$$

Therefore, we get the following result.

Theorem 8. *Let the process ϑ in the representation* (24) *satisfy* (13)*, and*

$$\mathbf{E}\exp\left\{-\frac{\gamma}{1-\gamma}\int_0^T \vartheta(s)dW_s + \frac{\gamma}{2(1-\gamma)}\int_0^T \vartheta_s^2 ds\right\} < \infty.$$

Let the process G satisfy the same conditions as in Theorem 5. Then $X^* = \int\limits_0^T \psi(s)dG(s)$.

In the case where $u(x) = \log x$, we have $\gamma = 0$ and $X^* = \frac{w}{\varphi(T)}$. Assuming that the relative entropy $H(\mathbf{P}|\mathbf{P}^*) = \mathbf{E}(\frac{1}{\varphi(T)}\log\varphi(T))$ is finite, we get that

$$\mathbf{E}(\log X^*) = \log w + H(\mathbf{P}|\mathbf{P}^*).$$

Conclusion

We have studied a broad class of non-semimartingale financial market models, where the random drivers are Wiener-transformable Gaussian random processes, i.e. some adapted transformations of a Wiener process. Under assumptions that the incremental variance of the process satisfies two-sided power bounds, we have given sufficient conditions for random variables to admit integral representations with bounded adapted integrand; these representations are models for bounded replicating strategies. It turned out that these representation results can be applied to solve utility maximization problems in non-semimartingale market models.

References

1. Androshchuk, T., Mishura, Y.: Mixed Brownian–fractional Brownian model: absence of arbitrage and related topics. Stochastics **78**(5), 281–300 (2006)
2. Bender, C., Sottinen, T., Valkeila, E.: Pricing by hedging and no-arbitrage beyond semimartingales. Finance Stoch. **12**(4), 441–468 (2008)
3. Bender, C., Sottinen, T., Valkeila, E.: Fractional processes as models in stochastic finance. In: Advanced mathematical methods for finance, pp. 75–103. Springer, Heidelberg (2011)
4. Björk, T.: Arbitrage theory in continuous time, 2nd ed. edn. Oxford University Press, Oxford (2004)
5. Cheridito, P.: Mixed fractional Brownian motion. Bernoulli **7**(6), 913–934 (2001)
6. Cheridito, P.: Arbitrage in fractional Brownian motion models. Finance Stoch. **7**(4), 533–553 (2003)
7. Dudley, R.M.: Wiener functionals as Itô integrals. Ann. Probability **5**(1), 140–141 (1977)
8. Dung, N.T.: Semimartingale approximation of fractional Brownian motion and its applications. Comput. Math. Appl. **61**(7), 1844–1854 (2011)
9. Ekeland, I., Temam, R.: Convex analysis and variational problems. North-Holland Publishing Co., Amsterdam-Oxford; American Elsevier Publishing Co., Inc., New York (1976). Translated from the French, Studies in Mathematics and its Applications, Vol. 1
10. Föllmer, H., Schied, A.: Stochastic finance, extended edn. Walter de Gruyter & Co., Berlin (2011). An introduction in discrete time
11. Karatzas, I., Shreve, S.E.: Brownian motion and stochastic calculus, *Graduate Texts in Mathematics*, vol. 113, second edn. Springer-Verlag, New York (1991)
12. Karatzas, I., Shreve, S.E.: Methods of mathematical finance, *Applications of Mathematics (New York)*, vol. 39. Springer-Verlag, New York (1998)
13. Li, W.V., Shao, Q.M.: Gaussian processes: inequalities, small ball probabilities and applications. Handbook of Statistics **19**, 533–597 (2001)
14. Lifshits, M.A.: Gaussian random functions, *Mathematics and its Applications*, vol. 322. Kluwer Academic Publishers, Dordrecht (1995)
15. Mishura, Y., Shevchenko, G.: Small ball properties and representation results. Stochastic Process. Appl. **127**(1), 20–36 (2017)

16. Mishura, Y., Shevchenko, G., Valkeila, E.: Random variables as pathwise integrals with respect to fractional Brownian motion. Stochastic Process. Appl. **123**(6), 2353–2369 (2013)
17. Mishura, Y.S.: Stochastic calculus for fractional Brownian motion and related processes, *Lecture Notes in Mathematics*, vol. 1929. Springer-Verlag, Berlin (2008)
18. Norros, I., Valkeila, E., Virtamo, J.: An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions. Bernoulli **5**(4), 571–587 (1999)
19. Rogers, L.C.G.: Arbitrage with fractional Brownian motion. Math. Finance **7**(1), 95–105 (1997)
20. Samko, S.G., Ross, B.: Integration and differentiation to a variable fractional order. Integral Transform. Spec. Funct. **1**(4), 277–300 (1993)
21. Shalaiko, T., Shevchenko, G.: Integral representation with respect to fractional brownian motion under a log-hölder assumption. Modern Stochastics: Theory and Applications **2**(3), 219–232 (2015)
22. Shevchenko, G., Viitasaari, L.: Adapted integral representations of random variables. International Journal of Modern Physics: Conference Series **36**, 16 (2015). Article ID 1560004
23. Zähle, M.: On the link between fractional and stochastic calculus. In: Stochastic dynamics (Bremen, 1997), pp. 305–325. Springer, New York (1999)