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THE LINEAR DEPENDENCE RELATIONS AND TRUNCATION ERRORS FOR INTERPOLATING EVEN ORDER SPLINES

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Abstract:

Linear dependence relations are derived for even order polynomial splines defined over the points midway between a uniformly spaced set of knots. The leading term of the truncation error is determined and is illustrated by an example.

1. Introduction

The linear dependence relations for odd order polynomial splines have been studied for small orders by a variety of workers, however the general relations for odd polynomial splines on a uniform partition were obtained by Fyfe in [6] and Swartz [11]. The local truncation error of the odd order polynomial splines defined in this way has been analysed by Swartz [11] and Albasiny and Hoskins [3]. Less work has been done with even order polynomial splines defined over points midway between the knots of a uniform partition as in [8] and [9]. Subbotin [10] has demonstrated the existence of such even degree splines with certain boundary conditions and now the linear dependence relations and the local truncation error will be described [7].

2. Linear Dependence Relations

Let s(x) be a spline of degree N interpolating to the function f(x) on $[x_0, x_n]$ and defined over points midway between the knots of a uniform partition, then s(x) satisfies the following conditions:

(a) s(x) is a polynomial of degree at most N in each interval $[x_k - h/2,\ x_k + h/2]$, k = 1, 2, ..., n-1, and in the intervals $[x_0,\ x_0 + h/2]$, $[x_n - h/2,\ x_n]$, $h = (x_n - x_0)/n$ and $x_k = x_0 + kh$, k = 0, 1, 2, ..., n

(b) using the notation $s_k \equiv s(x_k)$ and $f_k \equiv f(x_k)$, $s_k = f_k$,

$k = 0, 1, 2, \ldots, n$

and

(c) $s(x) \in C^{N-1}[x_0, x_n]$.

Although N even is the case of interest, all the formulae below also hold for odd N. It is convenient subsequently to refer to such a spline as a midpoint interpolating polynomial (m.i.p.) spline.

The linear dependence relations satisfied by an m.i.p. spline are

$$\frac{h^N}{N!} \sum_{i=0}^{N} C^{(0)}_{i+1/2,N} \; s^{(p)}_{k+i} = \frac{h^{N-p}}{(N-p)!} \sum_{i=0}^{N} C^{(p)}_{i+1/2,N} \; s_{k+i} \qquad (1)$$

$$p = 0, 1, 2, \ldots, N$$

where

$$C^{(p)}_{i+1/2,N} = (-1)^p \, (N-p)! \; Q^{(p)}_{N+1}(i+1/2)$$

and $Q^{(p)}_{N+1}(x)$ is the pth derivative with respect to x of the B-spline

$$Q_{N+1}(x) = \frac{1}{N!} \nabla^{N+1} x_+^N$$

defined on the integer knots. Proof of the result (1) is exactly analogous to that found in Loscalzo and Talbot [6], Swartz [11], Fyfe [5] and is given in detail in Meek [7].

Several values of the coefficients $C^{(p)}_{i+1/2,N}$ are given in table 1.

N = 1

p \ i	0	1
0	1/2	1/2
1	-1	1

N = 2

p \ i	0	1	2
0	1/4	6/4	1/4
1	-1/2	0	1/2
2	1	-2	1

N = 3

p \ i	0	1	2	3
0	1/8	23/8	23/8	1/8
1	-1/4	-5/4	5/4	1/4
2	1/2	-1/2	-1/2	1/2
3	-1	3	-3	1

N = 4

p \ i	0	1	2	3	4
0	1/16	76/16	230/16	76/16	1/16
1	-1/8	-22/8	0	22/8	1/8
2	1/4	4/4	-10/4	4/4	1/4
3	-1/2	2/2	0	-2/2	1/2
4	1	-4	6	-4	1

Table 1

The coefficients $C^{(p)}_{i+1/2,N}$, i = 0,1, ... , N; p = 0, ., ..., N

For example, a fourth degree m.i.p. spline satisfies the relation (p=2 in table 1.)

$$\frac{1}{16}\,\frac{h^4}{4!}\,(s''_{i+2} + 76s''_{i+1} + 230s''_i + 76s''_{i-1} + s''_{i-2})$$

$$= \frac{1}{4}\,\frac{h^2}{2!}\,(s_{i+2} + 4s_{i+1} - 10s_i + 4s_{i-1} + s_{i-2})\,.$$

Various properties of the coefficients $C^{(p)}_{i+\frac{1}{2},N}$ follow immediately from their definition in terms of B-spline [4].

A generating function for these coefficients is known [7] and is

$$\sum_{i=0}^{N} C^{(p)}_{i+1/2,N} \; z^{i+1/2} = (-1)^{p} \, (1-z)^{N+1} \left(z \frac{d}{dz} \right)^{N-p} \left(\frac{z^{1/2}}{1-z} \right) , \qquad (2)$$

$$p = 0,1,2, \ldots , N .$$

3. Leading term of the Truncation Error

The leading term of the truncation error may be found in the following manner.

Define the coefficients $a_{0,N} = 1$, $a_{2,N}$, $a_{4,N}$ ... by the equation

$$\sinh^{N+1} x = x^{N+1}(a_{0,N} + a_{2,N} x^{2} + \ldots)$$

and for $t = 0, 1, 2, \ldots$ let

$$E^{(p)}_{t,N} = \frac{1}{t!} \sum_{i=0}^{N} C^{(p)}_{i+1/2,N} \; (i-N/2)^{t} \; , \qquad (3)$$

with the indeterminate 0^{0} taken to be unity. Then it follows that

$$E^{(p)}_{t,N} = 0 \qquad \begin{cases} \text{if} \quad t+p \quad \text{is} \quad \text{odd} \\ \text{if} \quad t < p \quad , \end{cases} \qquad (4)$$

and in addition if $q = [(N-p+2)/2]$ then

and

$$\left. \begin{aligned} E^{(p)}_{p+2m,N} &= \frac{(N-p)!}{2^{2m}} \; a2_{2m,N} \; , \quad m = 0,1,\ldots,q-1 \\ E^{(p)}_{p+2q,N} &= \frac{(N-p)!}{2^{2q}} \; a_{2q,N} + (-1)^{N-p} \frac{(2-2^{2q})}{2^{2q}} \frac{B_{2q}}{2q} \end{aligned} \right\} \qquad (5)$$

with B_{2q} a Bernoulli number [1].

The proof is given from equation (3) by rewriting it as first

$$E^{(p)}_{t,N} = \frac{1}{t!} \left(z \frac{d}{dz} \right)^{t} \sum_{i=0}^{N} C^{(p)}_{i+1/2,N} \quad z^{i-N/2} \Bigg|_{z=1}$$

then by using equation (2)

$$E^{(p)}_{t,N} = \frac{(-1)^p}{t!} \left(z\frac{d}{dz}\right)^t \left\{\frac{(1-z)^{N+1}}{z^{(N+1)/2}} \left(z\frac{d}{dz}\right)^{N-p} \left(\frac{z^{1/2}}{1/z}\right)\right\} \Bigg|_{z=1}$$

Now substitution of $z = e^{2x}$ and performing the differentiation N-p times gives

$$E^{(1)}_{t,N} = \frac{1}{2^{t-p}\, t!} \left(\frac{d}{dx}\right)^t \left\{x^p \left(\sum_{i=0}^{q} e_{2i,N}\, x^{2i} + 0\,(x^{2q+2})\right)\right\} \Bigg|_{x=0}$$

where

$$e_{2i,N} = \begin{cases} (N-p)!\;\; a_{2i,N} & \text{if } i = 0,1,\ldots,q-1 \\ (N-p)!\;\; a_{2i,N} + (-1)^{N-p}\frac{B_{2i}}{2i}(2-2^{2i}) & \text{if } i = q \end{cases}$$

Clearly the properties (4) are true and without loss of generality now let $t = p + 2m$, $m = 0, 1, 2, \ldots$ and a

$$E^{(p)}_{p+2m,N} = \frac{1}{2^{2m}} e_{2m,N}$$

and equations (5) follow.

The local truncation error for m.i.p. splines is obtained in a manner similar to [3], is given in detail in [7], and brief summary now follows.

Since the m.i.p. spline satisfies equation (1), the appropriate equation giving the truncation error $e_{p,N,k+N/2}$ is

$$\frac{h^N}{N!} \sum_{i=0}^{N} C^{(0)}_{i+1/2,N}\, f^{(p)}_{k+i} = \frac{h^{N-p}}{(N-p)!} \sum_{i=0}^{N} C^{(p)}_{i+1/2,N}\, f_{k+i} + e_{p,N,k+N/2}. \qquad (6)$$

The left-hand side of equation (6), expanded about the midpoint of the interval $[x_k,\ x_{k+N}]$ by Taylor's theorem, is

$$h^N \left(\sum_{m=0}^{j-1} \frac{h^{2m}}{2^{2m}} a_{2m,N} f_{k+N/2}^{(p+2m)} + \frac{h^{N+2j}}{N!} E_{2j,N}^{(0)} f_{k+N/2}^{(p+2j)} + 0(h^{N+2j+2}) \right) ,$$

where $j = \left[\frac{N+2}{2} \right]$. Expansion of the right-hand side of (6) gives

$$h^N \left(\sum_{m=0}^{q-1} \frac{h^{2m}}{2^{2m}} a_{2m,N} f_{k+N/2}^{(p+2m)} + \frac{h^{N+2q}}{(N-p)!} E_{p+2q}^{(p)} f_{k+N/2}^{(p+2q)} + 0(h^{N+2q+2}) \right)$$

$$+ e_{p,N,k+N/2} ,$$

where $q = \left[\frac{N-p+2}{2} \right]$. Now if $j > q$, then the truncation error is

$$e_{p,N,k+N/2} = (-1)^{N-p+1} \frac{h^{N+2q}}{(N-p)!} \frac{(2-2^{2q})}{2^{2q}} \frac{B_{2q}}{2q} f_{k+N/2}^{(p+2q)} + 0(h^{N+2q+2}), \quad (7)$$

and it is easy to see that $j > q$ for all $p > 1$ and when $p=l$ and N is even. If $p=l$ and N is odd, then $j=q$ and the truncation error is given by

$$e_{1,N,k+N/2} = (-1)^N \frac{h^{2N+1}}{N!} \left(\frac{2-2^{N+1}}{2^{N+1}} \right) B_{N+1} f_{k+N/2}^{(N+2)} + 0(h^{2N+3}) \quad (8)$$

For example, if $f(x)$ has sufficiently many continuous derivatives,

$$\frac{1}{16} \frac{h^4}{4!} (f''_{i+2} + 76f''_{i+1} + 230f''_i + 76f''_{i-1} + f''_{i-2})$$

$$= \frac{1}{4} \frac{h^2}{2!} (f_{i+2} + 4f_{i+1} - 10f_i + 4f_{i-1} + f_{i-2}) - \frac{7}{1920} h^8 f_i^{vi} + 0(h^{10}) .$$

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