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BLENDING FUNCTION INTERPOLATION TO BOUNDARY DATA ON TRIANGLES

by

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1. Introduction

The purpose of this paper is to introduce a rational twelve parameter C^1 interpolant over a triangulation. This involves the discretization of the bicubic blending function interpolant in Barnhill, Birkhoff and Gordon [1]. The interpolant could be useful for computer aided geometric design and for finite element analysis of fourth order elliptic boundary value problems, for example, the biharmonic problem. Known C^1 polynomial interpolants include the twenty-one and eighteen parameter interpolants which can be found in Mitchell [8].

Blending function interpolation was devised to interpolate to functions. The report by Coons [5] initiated the subject and considerable subsequent work has been done by Gordon [7]. Birkhoff and Gordon [4] characterized certain interpolants to functions defined around the boundary of a square. Barnhill, Birkhoff, and Gordon [1] extended this to interpolation to functions defined around the boundary of a triangle.

For the triangle T with vertices at (0,0), (1,0), and (0,1), bicubic interpolants were devised in [1] for functions and their derivatives normal to the sides of the triangle. A simple case is interpolation to functions defined around the boundary and this is discussed in Section 2 as a motivation for the more general case. The bicubic case is discussed in Section 3.

Blending function interpolants are Boolean sums $P_i \oplus P_j$ of univariate interpolation operators P_i and P_j, $i \neq j$. The associated remainder is of multiplicative type in that formally, if I is the identity operator, then

$$I - (P_i \oplus P_j) = I - (P_i + P_j - P_i P_j) = R_i R_j \qquad (1.1)$$

where $R_K = I - P_k$, $k = i, j$. However, this formal multiplicative property of the univariate remainders is misleading for triangles as the projectors do not commute, and the correct precision set cannot be inferred directly from (1.1). The lack of commutativity is mentioned in [1]. Theorem 4.3 in this paper gives a precise precision statement for the general case. The importance of the precision set is that it implies the order of convergence of the interpolant. If an interpolant has polynomial precision n-1 in two variables, then the remainder is $0(h^n)$, where h is a generic mesh parameter length More information about this order of convergence is given in Barnhill and Gregory [2] and in Barnhill and Mansfield [3].

Section 5 is concerned with the twelve parameter discretization of the bicubic blending function interpolant given in Section 3. The interpolation properties and the precision set are given. The remainder is $0(h^3)$.

2. Bilinear Blending Function Interpolation

Let F(x,y) be a function defined on the triangle T with vertices at (0,0), (1,0), and (0,1). We consider the idempotent linear operators (projectors) P_1, P_2, and P_3 defined by :

$$P_1(F) = \left(\frac{1-x}{1-y}\right)F(y,y) + \left(\frac{x-y}{1-y}\right)F(1,y) \qquad (2.1)$$

$$P_2(F) = \left(\frac{x - y}{x}\right) F(x,0) + \left(\frac{y}{x}\right) F(x,x) \ , \qquad (2.2)$$

$$P_3(F) = \left(\frac{1 - x}{1 - x = y}\right) \ F(x - y,0) + \left(\frac{y}{1,1 - x + y}\right) F(1,1 - x + y). \qquad (2.3)$$

$P_1(F)$, $P_2(F)$, and $P_3(F)$ are the linear interpolants to $F(x,y)$ along the lines through (x,y) parallel to the sides $Y = 0$, $x = 1$, and $x = y$ respectively, see Figures 1 and 2.

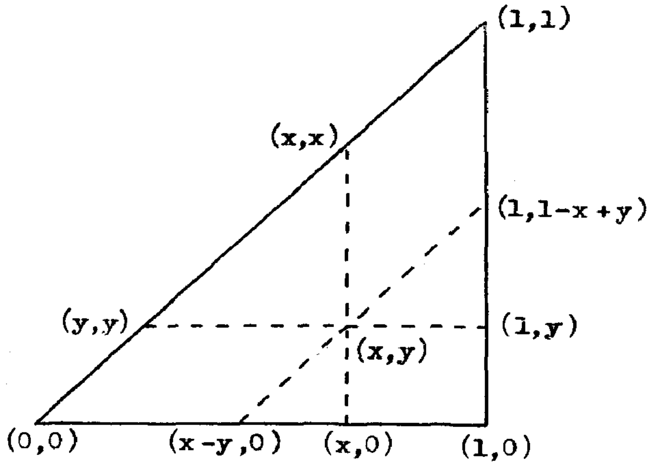

Figure 1

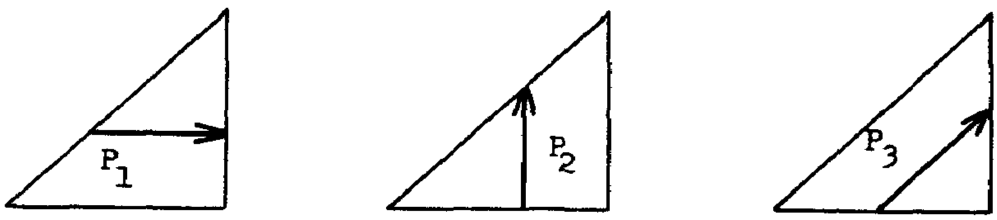

Figure 2

We consider Boolean aums of these operators :

Theorem 2.1. Let $F(x,y)$ be defined on the boundary ∂T of the triangle T. Then the Boolean sums

$$(P_i \oplus P_j)(F) \equiv P_i(F) + P_J(F) - P_iP_j(F), \; i \neq j; \; i, j = 1,2,3, \tag{2.4}$$

interpolate to $F(x,y)$ on ∂T.

Proof : Consider $P_1 \oplus P_2 \equiv P_1 + P_2 - P_1 P_2$. Now $P_1 \oplus P_2 = P_1 + (I - P_1) P_2$ and on the sides $x = 1$ and $x = y$ $P_1 = I$. Hence $I - P_1 = \theta$, the null operator, and thus $(I - P_1)P_2 = \theta$ on these sides. Also $P_1 \oplus P_2 = P_2 - P_1(I - P_2)$ and, on the side $y = 0$, $P_2 = I$ and $I - P_2 = \theta$. Hence $P_1(I - P_2)(F) = 0$, since this is the linear interpolant between Zero function values at (0,0) and (1,0). The interpolation property thus follows. The arguments for the other Boolean sums are analogous. Q.E.D

The above proof motivates the definition of the Boolean sum interpolant.

We consider in more detail the Boolean sum interpolant

$$(P_1 \oplus P_2)(F) = \left(\frac{x-y}{1-y}\right) F(1,y) + \left(\frac{x-y}{x}\right) F(x,0) + \left(\frac{y}{x}\right) F(x,x)$$

$$- \left(\frac{x-y}{1-y}\right)\left[(1-y)\, F(1,0) + y\, F(1,1)\right] . \tag{2.5}$$

Precision

In order to determine the precision set of interpolant (2,5) we require the following lemma :

Lemma 2.1. Let f be a function of one variable and g be

a function of two variables. Then

$$P_1[f(y)\ g(x,y)] = f(y)P_1[g(x,y)] \ , \qquad (2.6)$$

$$P_2[f(x)\ g(x,y)\ ] = f(x)P_2[g(x,y)] \ , \qquad (2.7)$$

$$P_3(f(y-x)\ g(x,y) = f(y-x)\ P_3[g(x,y)] \ , \qquad (2.8)$$

i.e. the projectors are homogeneous in functions of the variable perpendicular to the direction of the projector.

Proof : Each projector contains function evaluations which are homogeneous in functions of the variable perpendicular to the direction of the projector and proof of the lemma thus follows. For example, P_3, contains F(x-y,0 and F(1,l-x + y) and when F(x,y) = f(y-x) g(x,y), F(x-y,0) = f(y-x) g(x-y,0) and F(1,l-x + y) = f(y-x) g(1,l-x+y) . Q.E.D.

Theorem 2.2. The precision set of the bilinear Boolean sum interpolant $(P_1 \oplus P_2)(F)$, (2.5), is y^2 and $x^i\ j^j$, $i = 0,1,2,\ldots;\ j = 0,1.$

Proof : The precision of P_1 and P_2 is $P_1(1) = 1$, $P_1(x) = x$, $P_2(1) = 1$, and $P_2(y) = y$. Thus

$$(P_1 \oplus P_2)\ (x^iy^j) = P_1(x^iy^j) + P_2(x^ij^j) - P_1p_2(x^iy^j)$$

$$= P_1(x^iy^j) + x^iP_2(y^j) - P_1(x^iP_2(y^j)), \text{ for all } i \text{ from Lemma 2.1,}$$

$$= P_1(x^iy^j) + x^iy^j - P_1(x^iy^j) \quad \text{for} \quad j = 0,1.$$

$$= x^iy^j \text{ for all I and for } j = 0,1.$$

Also, since $P_1(y^2) = y^2$ from Lemma 2.1 and, from (2.2), $P_2(y^2) = xy$, a linear function in x, we have

$$(P_1 \oplus P_2)(y^2) = P_1(y^2) + P_2(y^2) - P_1P_2(y^2)$$
$$= y^2 + xy - P_1(xy)$$
$$= y^2$$

where P_1-(xy) = xy follows from application of Lemma 2.1 in the variable y and the linear precision of P_1, in the variable x. Q.E.D.

Analogous results hold for the other Boolean sum interpolants. see Section 4, where this theorem is generalised for (2n - 1) st degree two point Taylor interpolation projectors.

In [1] a "trilinear" blending function interpolant $Q^*(F)$ was considered which contains all three projectors and has the advantage of having certain symmetry in its variables. This interpolation operator has a representation

$$Q^* = \frac{1}{2}(P_i \oplus P_j + P_i \oplus P_k), \quad i \neq j \neq k \neq i, \tag{2.9}$$

and so has the quadratic precision x^iy^j, $0 \leq i + j \leq 2$, which is the intersection of the two precision sets of $P_i \oplus P_j$ and $P_i \oplus p_k$.

3. Bicubic Blending Function Interpolation

The cubic interpolation projectors P_1, P_2 and P_3

which interpolate to the function and its first derivatives on ∂T are defined by :

$$P_1(F) = \frac{(x-1)^2(2x-3y+1)}{(1-y)^3}F(y,y) + \frac{(x-1)^2(y-x)}{(1-y)^2}F_{1,0}(y,y)$$

$$+ \frac{(x-y)^2(3-2x-y)}{(1-y)^3}f(1,y) + \frac{(x-y)^2(x-1)}{(1-y)^2}F_{1,0}(1,y) \quad , \qquad (3.1)$$

$$P_2(F) = \frac{(y-x)^2(x+2y)}{x^3}F(x,0) + \frac{(y-x)^2 y}{x^2}F_{0,1}(x,0)$$

$$+ \frac{y^2(3x-2y)}{x^3} F(x,x) + \frac{y^2(y-x)}{x^2} F_{0,1}(x,x) \quad , \qquad (3.2)$$

$$P_3(F) = \frac{(1-x)^2(3y-x+1)}{(1-x+y)^3} F(x-y,0) + \frac{(x-1)^2}{(1-x+y)^2}\left[F_{10}(x-y,0)+F_{0,1}(x-y,0)\right]$$

$$+ \frac{y^2(3-3x+y)}{(1-x+y)^3}F(1,1-x+y) + \frac{y^2(x-1)}{(1-x+y)^2}\left[F_{1,0}(1,1-x+y)+F_{0,1}(1,1-x+y)\right]$$

$$(3.3)$$

We consider the bicubic Boolean sum

$(P_1 \oplus P_2)(F) \;=$

$$\frac{(x-y)^2(-2x-y+3)}{(1-y)^3}\ \Big\{F(1,y) - F(1,0)(y-1)^2(2y+1) - F_{0,1}(1,0)(y-1)^2 y$$

$$- F(1,1)y^2(-2y+3) - F_{0,1}(1,1)y^2(y-1)\ \Big\}$$

$$+\ \frac{(x-y)^2(x-1)}{(1-y)^2}\ \Big\{F_{1,0}(1,y) - F_{1,0}(1,0)\ (y-1)^2(2y+1) - \left[\frac{\partial}{\partial x}\ F_{0,1}(x,0)\ \right]_{x=1}(y-1)^2 y$$

$$-\ [F_{1,0}(1,1)]\ +\ F_{0,1}(1,1)\ ]\ \ y^2(-2y+3) - \left[\frac{\partial}{\partial x}F_{0,1}(x,x)\right]_{x=1} y^2(y-1)$$

$$- F(1,0)6y^2(1-y) - F_{0,1}(1,0)2y^2(1-y) - F(1,1)6y^2(y-1)$$

$$- F_{0,1}(1,1)Y^2(-2y+1)\ \Big\}$$

$$+\ \ \frac{(y-x)^2(x+2y)}{x^3}f(x,0) + \frac{(y-x)^2 y}{x^2}F_{0,1}(x,0) + \frac{y^2(3x-2y)}{x^3}F(x,x)$$

$$+\ \ \frac{y^2(y-x)}{x^2}\ \ F_{0,1}(x,x)\ . \qquad (3.4)$$

Interpolation Properties

The interpolatory properties of the Boolean sum operator can be derived formally in a manner similar to that in Theorem 2.1, with the assumption of sufficient continuity on the derivatives of F. In order to obtain weaker sufficient conditions for interpolation we must consider the actual Boolean sum interpolant.

<u>Theirem 3.1.</u> Let $F(x,y)$, $F_{0,1}(\eta,\eta), F_{0,1}(x,0), F_{1,0}(\eta,\eta), F_{1,0}(1,y)$ be defined on ∂T and assume $\frac{\partial}{\partial\eta} F(\eta,\eta) \left[\frac{\partial}{\partial\eta} F_{0,1}(\eta,\eta)\right]_{\eta=1}$,

and $F_{1,1}(1,0)$ exist on ∂T. Then the Boolean sum $(P_1 \oplus P_2(F)$ interpolates to F on ∂T, to $\frac{\partial F}{\partial x}$ on the sides x = y and x= 1 of ∂T, and to $\frac{\partial F}{\partial y}$ on the sides x=y and y=0 of ∂ T.

<u>Proof</u>; The interpolation of the Boolean sum to the function on ∂T follows from (3.4). To prove the interpolation properties for the derivative we differentiate (3.4) and, after some cancellation, eventually obtain the following:

$$\left[\frac{\partial}{\partial y}\left(p_1 \oplus P_2\right)(F)\right]_{y=0} = F_{0,1}(x,0) + x^3\left\{\left[\frac{\partial}{\partial y} F_{1,0}(1,y)\right]_{y=0} - \left[\frac{\partial}{\partial x} F_{0,1}(x,0)\right]_{x=1}\right\}$$

$$\left[\frac{\partial}{\partial y}\left(P_1 \oplus P_2\right)(F)\right]_{x=y=\eta} = F_{0,1}(\eta,\eta)$$

$$\left[\frac{\partial}{\partial x}\left(P_1 \oplus P_2\right)(F)\right]_{x=1} = F_{1,0}(1,y)$$

$$+ \left\{\left[\frac{\partial}{\partial x} F_{0,1}(x,0)\right]_{x=1} - \left[\frac{\partial}{\partial x} F_{0,1}(x,0)\right]_{x=1}\right\}(y-1)^2\, y$$

$$+ \left\{\left[\frac{\partial}{\partial x} F_{0,1}(x,x)\right]_{x=1} - \left[\frac{\partial}{\partial x} F_{0,1}(x,x)\right]_{x=1}\right\} y^2(y-1)$$

$$\left[\frac{\partial}{\partial x}(P_1 \oplus P_2)(F)\right]_{x=y=\eta} = \frac{\partial}{\partial\eta} F(\eta,\eta)$$

Thus, with the assumption that the partial derivatives in these equations exist and also the existence of $F_{1,0}(\eta,\eta)$, we obtain the derivative interpolation properties.

Corollary . The Boolean sum interpolates any order of derivative on ∂T tangential to the side if this derivative exists.

This follows from Interpolation to F on ∂T.

Removable Singularities

The Boolean sum interpolant and its first order partial derivatives have removable singularities at (0,0) and (0,1). Consider, for example, $\frac{\partial}{\partial x}(P_1 \oplus P_2)(F)$ at $(x,y) = (0,0)$.

Differentiating (3.4) with respect to x we find the singular terms involve only the following :

$$\lim_{(x,\ y) = (0,0)} \left[\frac{\partial}{\partial x}(P_1 \oplus P_2)(F) \right]$$

$$= \lim_{(x,y)=(0,0)} \left[\frac{(y-x)^2(x+2y)}{x^3} F_{1,0}(x,0) + \frac{y^2(3x-2y)}{x^3} F_{1,0}(x,x) \right.$$

$$\left. + \frac{y^2(y-x)}{x^4}\{ -6\ F(x,0) + 6F(x,x) - 2xF_{0,1}(x,0) - 4x\,F_{0,1}(x,x) \} \right] \quad (3.5)$$

Taylor expansions give the following:

$$F(x,x) = F(x,0) + x\ F_{0,1}(x,0) + \int_0^x \tilde{x} F_{0,2}(x,\tilde{x})\,d\tilde{x} \quad , \tag{3.6}$$

$$F_{0,1}(x,x) = F_{0,1}(x,0) + \int_0^x F_{0,2}\ (x\tilde{x})\ d\tilde{x} \quad , \tag{3.7}$$

$$F_{1,0}(x,x) = F_{1,0}(x,0) + \int_0^x F_{1,1}(x,\tilde{x})\ d\tilde{x} \ , \tag{3.8}$$

Substitution of these expansions in (3.5) gives

$$\lim_{(x,y)\,=\,(0,0)} \left[\frac{\partial}{\partial x} \left(P_1 \oplus P_2\right)(F) \right]$$

$$= \lim_{(x,y)=(0,0)} \Big[F_{1.0}(x,0) + \frac{y^2(3x-2y)}{x^3} \int_0^x F_{1,1}(x,\tilde{x})d\tilde{x}$$

$$+ \frac{y^2(y-x)}{x^4} \left\{ 6 \int_0^X \tilde{x} F_{0,2}\ (x,\tilde{x})d\tilde{x} - 4x \int_0^x F_{0,2}(x,\tilde{x})d\tilde{x} \right\} \Big]$$

(3.9)

Consider the first integral term. From Holder's inequality,

$$\left| \int_0^x F_{1,1}(x,\tilde{x})d\tilde{x} \right| \le x^{\frac{1}{p}} \left[\int_0^x | F_{1,1}(x,\tilde{x}) |^{p'}\ d\tilde{x} \right]^{1/p'}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $p,\ p' \ge 1$. Now

$$\lim_{(x,y)\,=\,(0,0)} \left[\frac{y^2(3x-2y)}{x^3} x^{1/p} \right] = 0 \text{ for } p < \infty.$$

Similar consideration of the other integral terms in (3.9) gives the result that

$$\lim_{(x,y)=(0,0)} \left[\frac{\partial}{\partial x} \left(P_1 \oplus P_2\right) (F) \right] = F_{1,0}(0,0) \ , \qquad (3.10)$$

provided that $F_{1,1}$ and $F_{0,2}$ are in $L_{p'}(T), 1 < P' \leq \infty$.

The other singularities are treated similarly and so the Boolean sum interpolant (3.3) interpolates to the function and its first partial derivatives at the vertices of the triangle T.

Precision

The precision of the bicubic Boolean sum interpolant $\left(P_1 \oplus P_2\right)(F)$ is

$$x^i y^j, \quad i = 0,1,....; \quad j = 0, 1, 2, 3, \text{ and } y^4, \ xy^4, \ y^5 \ .$$

This is a corollary of Theorem 4.1 in next Section.

4. Blending function Interpolation of nth Order,

The (2n-1)st degree interpolation projectors P_1, P_2, and P_3 which interpolate to the function and its first n-1 derivatives on ∂T are defined by the two point Taylor interpolation polynomials, see Davis [6,p.37]. For example, the projector P_2 is defined by the following:

$$P_2(F) = y^n \sum_{k=0}^{n-1} B_k(x)(y-x)^{(k)} + (y-x)^n \sum_{k=0}^{n-1} A_k(x)\, y^{(k)} \qquad (4.1)$$

where $A_k(x) = \frac{\partial^k}{\partial y^k}\left[\frac{F(x,y)}{(y-x)^n}\right]_{y=0}$, $B_k(x) = \frac{\partial^k}{\partial y^k}\left[\frac{F(x,y)}{(y)^n}\right]_{y=x}$,

and $(y-x)^{(k)} = \frac{1}{k!}(y-x)^k$ etc.

P_2 (F) Interpolates to the following:

$$F(x,0),\ F_{0,1}(x,0),\ \dots,\ F_{0,n-1}(x,0),$$
$$F(x,x),\ F_{0,1}(x,x),\ \dots,\ F_{0,n-1}(x,x),$$

Boolean sums of -these projectors interpolate to the function and to its first n-1 derivatives taken in directions parallel to the sides of the triangle.

Precision

The following is a generalisation of Lemma 2.1 on homogeneity,

Lemma 4.1. With the P_i as defined above, equations (2.6)-(2.8) are valid.

Proof ; Each projector contains function and partial derivative evaluations which are homogeneous in functions or the variable perpendicular to the direction of the projector.
The argument for P_3 involves the following:

$$\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial x}\right) f(x-y)\ g(x,y) = f(x-y)\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial x}\right) g(x,y)$$
$$+ g(x,y)\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial x}\right) f(x-y),$$

the last directional derivative being zero. Q.E.D.

Theorem 4.1 The precision set for $P_1 \oplus P_2$ is the following :

$$x^i y^j, \quad i = 0,1,\dots ; \quad j = 0,1,\dots , \ 2n-1, \text{ and}$$

$$2n \le i + j \le 3n-1 \text{ for } 2n \le j \le 3n-1 .$$

This is the shaded region of Figure 3. where the nodes (i, j) are the powers of $x^i y^j$.

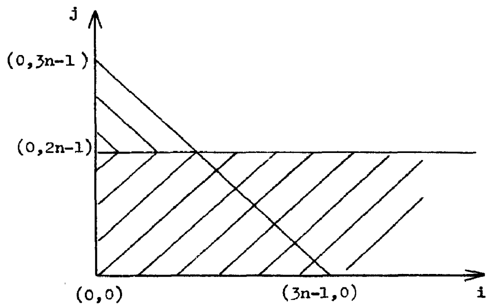

Figure 3.

Proof : 1. We consider $x^i \, y^j$, $0 \le j \le 2n-1; \ 0 \le i$.

$$\begin{aligned}(P_1 \oplus P_2)(x^i, y^i) &= P_1 (x^i, y^i) + P_2 (x^i y^j) - P_1 P_2 (x^i y^j) \\ &= P_1 (x^i y^j) + x^i P_2 (y^j) - P_1 \left[x^i \ P_2 (y^j) \right],\end{aligned}$$

from Lemma 4.1. For $0 \le j \le 2n-1$, $P_2 (y^j) = y^j$, so that

$$(P_1 \oplus P_2) (x^i y^j) = x^i y^j.$$

2. We consider $x^i \, y^j$ and $2n \le j \le 3n\text{-}1$ and $2n \le i+j \le 3n\text{-}1$.

$$(P_1 \oplus P_2)\,(x^i y^j) = y^j P_1(x^i) + x^i P_2(y^j) - P_1[\,x^i P_2(y^j)\,].$$

Since $0 \le i \le n-1$, $P_1(x^i) = x^i$. From (4.1) for $P_2\,(y^j)$, the $A_k = 0$ and $B_k = x^{(j-n-k)}(j-n)!$ so that

$$P_2\,(y^j) = y^n \sum_{k=0}^{n-1} x^{(j-n-k)}(j-n)!\,(y-x)^{(k)}, \quad 2n \le j \le 3n-1-i.$$

Thus $P_2(y^j)$ is a polynomial which, in the variable x, is of degree $\le 2n-1-i$. Hence the homogeneity and precision of P_1 imply that $(P_1 \oplus P_2)\,(x^i y^j) = x^i y^j$ Q.E.D.

When P_3, is involved, the argument is somewhat different:

<u>Theorem 4.2</u> P_3, $\oplus$ P_2 has the same precision set as P_1, $\oplus$ P_2.

<u>Proof</u>: 1. Precision for $x^i y^j$, $0 \le j < 2n-1$, $0 \le i$ follows from the proof of Theorem 4.1 since only the homogeneity and precision of the projector P_2 are involved.

2. We consider $x^i (y-x)^j$ for $2n \le j \le 3n-1$ and $2n \le i + j \le 3n-1$. These monomials, together with those for which we already have precision, span the same precision set as in Theorem 4.1 for $P_1 \oplus P_2$. By the homogeneity of P_2, $(P_3 \oplus P_2)\left[x^i (y-x)^j\right] = (y-x)^j P_3(x^i) + x^i P_2\left[(y-x)^j\right]$

$$- P_3\left\{x^i P_2\left[(y-x)^j\right]\right\}$$

Since $0 \le i \le n-1$, $P_3\,(x^i) = x^i$. From (4.1) for $P_2\left[(y-x)^j\right]$, the $B_k = 0$ and

$$A_k = (j-n)!\,(-x)^{(j-n-k)}, \qquad 1 \le j-n-k \le 2n-1-i$$

so that $P_2\left[(y-x)^j\right] = (y-x)^n \sum_{k=0}^{n-1} (j-n)!\,(-x)^{(j-n-k)} y^{(k)}$.

Homogeneity of P_3 in the variable y-x and precision of P_3, of degree 2n-1 implies the conclusion. We note that the remainder of P_3 involves the derivative $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^{2n}$ .Q.E.D.

We state a general theorem that covers both Theorems 4.1 and 4.2. We consider the Boolean sum $P_i \oplus P_j$, $i \neq j$, $1 \leq i, j \leq 3$. P_i and P_j are taken along the lines $\xi \equiv$ constant and $\eta \equiv$ constant, respectively. (Thus, e.g., ξ is the variable perpendicular to the direction of P_1.)

Theorem 4.3 $P_i \oplus P_j$ has the following precision set:

$\xi^i \eta^j$, $i = 0,1,\ldots; j = 0,1\ldots, 2n-1$, and

$2n \leq i + j \leq 3n-1$ for $2n \leq j \leq 3n-1$.

Table I contains ξ and η for the six possibilities of $P_i \oplus P_j$, $i \neq j$, $1 \leq i, j \leq 3$.

i	j	ξ	η
1	2	x	y
1	3	y-x	y
2	1	y	x
2	3	y-x	x
3	1	y	y-x
3	2	x	y-x

Table I

5. A Twelve Parameter C^1 Interpolant

We derive an interpolant on the triangle T in terms of the twelve point functionals F, $F_{1,0}$, $F_{0,1}$, and $F_{1,1}$ at the three vertices. The interpolant has quadratic precision interpolates to a function and to a normal derivative function on each side of the triangle T which are uniquely defined by the point functionals on that side. Over a certain type of triangulation of a polygon Ω, the interpolant defines a piecewise interpolant which is in $C^1(\Omega)$ (n), (Theorem 5.1).

The cardinal basis functions for the cubic two point Taylor interpolants on [0,1l are the following:

$$\phi_1(X) = (X-1)^2(2X+1) \quad , \quad \phi_2(X) = (X-1)^2X ,$$

$$\phi_3(x) \equiv \phi_1(1-X) = X^2(-2X+3), \quad \phi_4(X) \equiv -\phi_2(1-X) = X^2(X-1). \tag{5.1}$$

Let $\tilde{F}(x,y)$ satisfy the following :

$$\tilde{F}(x,0) = \phi_1(x)\,F(0,0) + \phi_2(x)\,F_{1,0}(0,0) + \phi_3(x)\,F(1,0) + \phi_4(x)\,F_{1,0}(1,0) \tag{5.2}$$

$$\tilde{F}_{0,1}(x,0) = \phi_1(x)\,F_{0,1}(0,0) + \phi_2(x)\,F_{1,1}(0,0) + \phi_3(x)\,F_{0,1}(1,0) + \phi_4(x)\,F_{1,1}(1,0) \tag{5.3}$$

$$\tilde{F}(1,y) = \phi_1(y)\,F(1,0) + \phi_2(y)\,F_{0,1}(1,0) + \phi_3(y)\,F(1,1) + \phi_4(y)\,F_{0,1}(1,1) \tag{5.4}$$

$$\tilde{F}_{1,0}(1,y) = \phi_1(y)\,F_{1,0}(1,0) + \phi_2(y)\,F_{1,1}(1,0) + \phi_3(y)F_{(1,0)}(1,1) + \phi_4(y)F_{0,1}(1,1) \tag{5.5}$$

$$\tilde{F}(\eta,\eta) = \phi_1(\eta)\, F(0,0) + \phi_2(\eta)[F_{1,0}(0,0) + F_{0,1}(0,0)]$$

$$+ \phi_3(\eta)\, F(1,1) + \phi_4(\eta)[F_{1,0}(1,1) + F_{0,1}(1,1)] \qquad (5.6)$$

$$\frac{\partial}{\partial \nu} \tilde{F}(\eta,\eta) = (1-\eta)\,[-F_{1,0}(0,0) + F_{0,1}(0,0)]$$

$$+ \eta[-F_{1,0}(1,1) + F_{0,1}(1,1)] \qquad (5.7)$$

where $\frac{\partial}{\partial \nu} = -\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ represents a derivative in the direction of the outward normal on the side x = y. Equations (5.2) - (5.6) are cubic two point Taylor interpolants along the sides of* the triangle and (5.7) is a linear interpolant. The tangential derivatives of $\tilde{F}(x,y)$ along the sides are defined by the equations for $\tilde{F}(x,y)$ along the sides, as noted in the Corollary to Theorem 3.1. The function $\tilde{F}$ satisfies the hypotheses of Theorem 3.1. We take the Boolean sura $(P_1 \oplus P_2)(\tilde{F})$ and from equation (3.4), equations (5.2) - (5.7), and

$\tilde{F}_{0,1}(x,x) = \frac{1}{2}\left[\frac{\partial}{\partial x}\tilde{F}(x,x) + \frac{\partial}{\partial \nu}\tilde{F}(x,x)\right]$ we obtain the following interpolant :

$$U(x,y) = (P_1 \oplus P_2)(\widetilde{F}(x,y))$$

$$= \frac{y^2(x-y)^2(x-1)}{(y-1)}\Big\{ F_{1,1}(1,1) + \tfrac{3}{2}F_{0,1}(1,1) - 3F(0,0) - \tfrac{3}{2}F_{0,1}(0,0)$$

$$- \tfrac{1}{2}F_{0,1}(0,0) - 3F(1,1) - \tfrac{3}{2}F_{1,0}(1,1) + 6F(1,0) + 2F_{0,1}(1,0)\Big\}$$

$$+ \frac{(y-x)^2(x+2y)}{x^3}\{\phi_1(x)F(0,0) + \phi_2(x)F_{1,0}(0,0) + \phi_3(x)F(1,0)$$
$$+ \phi_4(x)F_{1,0}(1,0)\}$$

$$+ \frac{(y-x)^2 y}{x^2}\{\phi_1(x)F(0,0) + \phi_2(x)[F_{1,1}(0,0) + \phi_3(x)F_{0,1}(1,0)$$
$$+ \phi_4(x)F_{1,1}(1,0)\}$$

$$+ \frac{y^2(3x-2y)}{x^3}\{\phi_1(x)F(0,0) + \phi_2(x)[F_{1,0}(0,0) + F_{0,1}(0,0)]$$
$$+ \phi_3(x)F(1,1) + \phi_4(x)[F_{1,0}(1,1) + F_{0,1}(1,1)]\}$$

$$+ \frac{y^2(y-x)}{2x^2}\{\phi'_1(x)F(0,0) + \phi'_2(x)[F_{1,0}(0,0) + F_{0,1}(0,0)]$$
$$+ \phi'_3(x)F(1,1) + \phi'_4(x)[F_{1,0}(1,1) + F_{0,1}(1,1)]$$
$$+ (1-x)[-F_{1,0}(0,0) + F_{0,1}(0,0)]$$
$$+ x[-F_{1,0}(1,1) + F_{0,1}(1,1)\} \qquad (5.8)$$

The interpolant (5.8) has the function and normal derivatives defined by equations (5.2) - (5.7) on the sides of the triangle T. This follows from Theorem 3.1. Because of the way (5.2) - (5.7) were defined, U interpolates to F, $F_{1,0}$, $F_{0,1}$, at the three vertices.

Theorem 5.1. Let Ω be a polygonal region which can be subdivided into a system of right triangles with the perpendicular sides parallel to the co-ordinate axes. Then, with a suitable change of variable to the triangle T, the bicubic Boolean sum interpolant $(P_1 \oplus P_2)\ ,\tilde{F}$ (5.8), on each of the right triangles, defines a C (Ω) piecewise Interpolation function

Proof: consider an internal side in O which will be common to two triangles: On this side the interpolant, its tangential derivative and its normal derivative are continuous and uniquely defined by the same functions for eaoh of the two triangles. Consider a common vertex: The interpolants on the triangles with this common vertex have the same function and partial derivative values "there, namely F, $F_{1,0}$ and $F_{0,1}$. The continuity C^1 (Ω) thus follows.

Precision

The linear interpolant (5.7) to the normal derivative has quadratic precision in F. Since this is the least accurate interpolant, we expect the quintic precision of the bicubic Boolean sum of the continuous function to be reduced to quadratic precisio for the discretized Boolean sum interpolant.

Theorem 5.2. The 12-parameter C^1 interpolant (5.8) has quadratic precision and is also exact for $(x + y)^3$.

Proof : Consider, for example, $F(x,y) = y^2$, then

$$U(x,y) = [\, x^2(-2x+3) + 2x^2(x\text{-}l)\,] \frac{y^2(3x-2y)}{x^3} + [6x(-x+1) + 2x\,(3x\text{-}2) + 2x] \frac{y^2(y-x)}{2x^2}$$

$$= \frac{y^2}{x}\,[3x - 2y + 2y - 2x] = y^2 .$$

The other functions are treated similarly. Q. E. D.

Corollary 5.1 The remainder of the 12-parameter interpolant is $0(h^3)$.

6. Nine Parameter Interpolants

Consideration of the continuity at the vertex of a triangle, in the region Q, implies that at least the function and two independent directional derivatives must be specified there for a piecewise interpolant to be $C^1(\Omega)$. Thus these nine parameters are the minimum possible conditions needed to define a piecewise $C^1(\Omega)$ interpolant. We make the observation that defining $F_{1,1} = 0$ at the vertices in (5.8) defines a nine parameter interpolant and does not affect the precision for the monomials l, x, y, x^2 and y^2. Similar modifications can be made to many other interpolation formulae, for example, the eighteen parameter interpolant which can be found in Mitchell [8], based on the point functionals $F_{i,j},\ 0<i+j<2$, at the three vertices of the triangle, can be reduced to a nine parameter family with linear precision by defining $F_{2,0} = F_{1,1} = F_{0,2} = 0$ at the three vertices. Unfortunately, for finite element applications to 4th order elliptic equations, we require at least quadratic precision for theoretical convergence.

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