TR/88 May 1979

A CLASS OF PIECEWISE CUBIC

INTERPOLATORY POLYNOMIALS

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w9260352

ABSTRACT

A new class of C^1 piecewise—cubic interpolatory polynomials is defined, by generalizing the definition of cubic X-splines given recently by Clenshaw and Negus (1978). It is shown that this new class contains a number of interpolatory functions which present practical advantages, when compared with the conventional cubic spline.

1. Introduction

This paper is concerned with the problem of piecewise-cubic polynomial interpolation where, unlike the cubic spline, the interpolatory function does not necessarily possess a continuous second derivative. More specifically, given the points

$$a = x_0 < x_1 <\ldots < x_k = b \tag{1.1}$$

and the corresponding set of values $y_i=y(x_i)$; $i = 0,l,\ldots,k$, we consider the problem of constructing a piecewise-cubic polynomial $s \varepsilon C^1 [a,b]$, with knots x_i ; $i = 0,1 ,\ldots ,k$, such that

$$s(x_i) = y_i \; ; \; i = 0,l,\ldots,k.$$

One class of such interpolatory functions is the class of X-splines considered recently by Clenshaw and Negus (1978), who show that There are certain practical advantages in allowing discontinuities in $S^{(2)}$ In particular, they derive an X-spline which reduces considerably the computational effort involved in constructing the interpolatory function, whilst retaining the same order of convergence as the conventional cubic spline.

In the present paper we show that the conditions used by Clenshaw and Negus (1978) for defining an X-spline are unnecessarily restrictive and we extend their definition to a much wider class of piecewise-cubic polynomials. We show that any X-spline s of this new class leads to $O(h^4)$ convergence uniformly on [a,b], that the magnitude of the jump discontinuities of $s^{(2)}$ and $s^{(3)}$ at the interior knots are respectively $0(h^2)$ and $0(h)$ and that the class contains several interpolatory functions which are better, in terms of accuracy, smoothness and ease of computation, than those considered by Clenshaw and Negus (1978).

2. Interpolatory Piecewise-Cubic Polynomials

Given the set of values $y_i = y(x_i)$, $x = 0,1,\dots,k$, where x_i are the points (1.1), let H be the piecewise - cubic Hermite polynomial which is such that

$$H(x_i) = y_i \text{ and } H^{(1}(x_i) = y^{(1)}_i \quad i = 0,1,\dots,k .$$

Then, if $y \varepsilon C^4[a,b]$, the following optimal error bound holds

$$\| H - y \| \le \frac{h^4}{384} \| y(4) \| \qquad (2.1)$$

In (2.1), $\| \bullet \|$ denotes the uniform norm on [a,b] and $h = \max_{1 \le i \le k} h_i$,

Where

$$h_i = x_i - x_{i-1} ; \; i = 1,2,\dots,k ,$$

see e.g. Birkhoff and Priver (1967).

Definition 1 . Let s be the piecewise—cubic polynomial obtained from H by replacing the derivatives $y_i^{(1)}$. ; $i = 0,l,\dots,k$, respectively by suitable approximations m_i ; $i = 0,l,\dots,k$. Then, s will be called a piecewise-cubic polynomial (P.CF.) with derivatives m_i ; $i=0,l, ,k$

It follows at once from the definition that

$$s(x) = y_{i-1} + m_{i-1}(x-x_{i-1}) + s[x_{i-1}, x_{i-1}, x_i](x-x_{i-1})^2$$

$$+ s[x_{i-1}, x_{i-1}, x_i, x_i](x-x_{i-1})^2(x-x_i) ,$$

$$x \,\varepsilon\, [x_{i-1}, x_i]; \; i = 1,2,\dots,k, \qquad (2.2)$$

where $s(x_i)= y_i$, $s^{(1)}(x_i)= m_i$ and, with the usual notation for divided differences,

$$\left.\begin{aligned} & s[x_{i-1}, x_{i-1}, x_i] = \{ (y_i - y_{i-1}) / h_i - m_{i-1} \} / h_i \\ \text{and} \quad & s[x_{i-1}, x_{i-1}, x_i, x_i] = \{ m_{i-1} - 2(y_i - y_{i-1}) / h_i + m_i \} / h_i^2 ; \\ & \qquad i = 1,2,\dots,k. \end{aligned}\right\} \qquad (2.3)$$

The following theorem can be established easily by using (2.1),

Theorem 1 . Let s be a P.C.P. with derivatives m_i; i=0,1,...,k.
If y ε $C^4[a,b]$ then, for x c[x_{i-1},x_i];i = 1,2,...,k,

$$|\ s(x) - y(x)\ | \leq \frac{h}{4} \max\{|\ m_{i-1} - y_{i-1}^{(1)}\ |, |\ m_i - y_i^{(1)}\ |\} + \frac{h^4}{384} \|\ y^{(4)}\ \| . \quad (2.4)$$

The theorem shows that the best order of approximation that can be achieved by an interpolatory P.C.P.s is

$$\|\ s-y\ \| = 0(h^4),$$

and that this order is obtained only if the derivatives m_i are such that

$$mi - y_i^{(1)} = o(h^n);\ i=0,1,...,k, \quad (2.5)$$

with n ≥ 3.

Clearly a P.C.P. s is continuous and possesses a continuous first derivative. In general however s $^{(2)}$ has a jump disontinuity at each interior knot- Using (2.2) and (2.3) it can be shown easily that the jump discontinuities of $s^{(2)}$ and s $^{(3)}$ at the interior knots are respectively,

$$d^{(2)}_i = s^{(2)}(x_i+)-s^{(2)}(x_i-)$$
$$= \frac{2}{h_i\beta_i}\{-\beta_i m_{i-1} - 2m_i - \gamma_i m_{i+1} + 3\beta_i Y_i + 3\gamma_i Y_{i+1}\};$$

$$i=1,2,...,k-l, \quad (2.6)$$

and

$$d^{(3)}_i = s^{(3)}(x_i+)-s^{(3)}(x_i-)$$

$$\frac{6}{h_i^2\beta_i^2}\{-\beta_i^2 m_{i-1} + (1-2\beta_i)m_i + \gamma_i^2 m_{i+1} + 2\beta_i^2 Y_i - 2\gamma_i^2 Y_{i+1}\};$$

$$i=1, 2,\ldots, k-1, \quad (2.7)$$

where

$$\beta_i = h_{i+1}/(h_i+h_{i+1}),\ \gamma=1-\beta_i \text{ and } Y_i - (y_i-y_{i-1})/h_i\,. \qquad (2.8)$$

Theorem 2. Let s be a P.C.P. with derivatives m_i; i = 0,l,..,k, and denote the jump discontinuities of $s^{(2)}$ and $s^{(3)}$, at an interior knot x_i, respectively by $d^{(2)}{}_I$ and $d^{(3)}{}_i$. If y e $C^5[a,b]$ then, for some $\xi_i \varepsilon [x_{i-1}, x_{i-1}]$,

$$d_i^{(2)} = \frac{2}{h_i\beta_i}\{-\beta_i(m_{i-1} - y_{i-1}^{(1)} - 2(m_i - y_i^{(1)} - \gamma_i(m_{i+1} - y_{i+1}^{(1)}\}$$
$$+ \frac{1}{12}(h_i^2 - h_{i+1}^2)y_i^{(4)} - \frac{1}{30}(h_i^3 + h_{i+1}^3)y^{(5)}(\xi_i);$$

$$i = 1,2,..,k\text{-}l\,, \qquad (2.9)$$

and, for some. $\eta_i \in [x_{i-1}, x_{i+1}]$

$$d_i^{(3)} = \frac{6}{h_i\beta_i}\{-\beta_i(m_{i-1} - y_{i-1}^{(1)} + 1 - 2\beta_i(m_i - y_i^{(1)} + \gamma_i^2(m_{i+1} - y_{i+1}^{(1)}\}$$
$$+ \tfrac{1}{2}(h_i + h_{i+1})\,y^{(4)}(\eta_i);$$

$$i = 1,2,...,k\text{-}l \qquad (2.10)$$

Proof. Equation(2.9)follows from (2.6),by using the result

$$3\beta_{iy}Y_i + 3\gamma_i Y_{i+1} - \beta_i y_{i-1}^{(1)} - 2y_i^{(1)} - \gamma_i y_{i+1}^{(1)}$$
$$= \frac{1}{24}h_i h_{i+1}(h_i - h_{i+1})y_i^{(4)} - \frac{1}{60}h_i h_{i+1}(h_i^2 + h_{i+1}^2 - h_i h_{i+1})y^{(5)}(\xi_i);$$

$$i = 1, 2, \ldots, k\text{-}1 \qquad 2.11)$$

This result is due to Kershaw (1972: 193) and is established by using Peano's method for finding remainders.

Similarly, equaiton (2.10) follows from (2.7), by using the result

$$2\beta_i^2 Y_i - 2\gamma_i^2 Y_{i+1} - \beta_i^2 y_{i-1}^{(1)} + (1 - 2\beta_i)y_i^{(1)} + \gamma_i^2 y_{i=1}^{(1)} = \frac{1}{12}h_i^2 h_{i+1})y^{(4)}(\eta_i);$$

$$i = 1,, 2, \ldots, k\text{-}1,$$

which is also established by the use of Peano's method.

Using (2.9)and (2.10)we obtain at once the following bounds on the magnitudes of $d_i^{(2)}$ and $d_i^{(3)}$,

$$|\, d_i^{(2)} \,| \le \frac{6(h_i + h_{i+1})}{h_i h_{i+1}} \max_i |\, m_i - y_i^{(1)} \,| + \frac{1}{12} \,|\, h_i^2 - h_{i+1}^2 \,|\, \left\| y^{(4)} \right\| + \frac{1}{30} \left(h_1^3 + h_{i+1}^3\right) \left\| y^{(5)} \right\|$$

$$i=1,2,\ldots\ldots,k-1, \qquad (2.12)$$

$$|\, d_i^{(3)} \,| \le 12 \max\left\{ \frac{1}{h_i^2}, \frac{1}{h_{i+1}^2} \right\} \max_i |\, m_i - y_i^{(1)} \,| + \frac{1}{2}(h_i + h_{i+1}) \| \, y^{(4)} \, \|;$$

$$i=1,2,\ldots,k-1. \qquad (2.13)$$

Theorem 2 shows that the magnitudes of $d^{(2)}{}_i$ and $d^{(3)}{}_i$, like the order of convergence of s, depend only on the quality of the approximations m_i. More specifically if the conditions (2.5) which lead to $0(h^4)$ convergence uniformly on [a,b] hold then, in general,

$$d_i^{(2)} = 0(h^2) \text{ and } d_i^{(3)} = 0(h) .$$

Furthermore, if the knots are equally spaced with $h_i = h$ and in (2.5) $n \ge 4$ then $d^{(2)}{}_i = 0(h^3)$.

Clenshaw and Negus (1978) determine the derivatives m_i of a P.C.P. s by imposing the conditions

$$d_i^{(2)} = \frac{c_i h_{i+1}}{3} d_i^{(3)}; \qquad i=1,2,\ldots,K-1, \qquad (2.14)$$

where the c. are given numbers. These conditions lead to the relations

$$A_i m_{i-1} + B_i m_i + C_i m_{i+1} = D_i Y_i + E_i Y_{i+1}; \quad i=1,2,\ldots,k-l, \qquad (2.15)$$

where

$$\left.\begin{array}{c} A_i = \beta_i(1 - \beta_i - c_i\beta_i),\ B_i = \{2 + c_i - 2\beta_i(1 + c_i),\ C_i = \gamma_i^2(1 + c_i) \\ D_i = \beta_i(3 - 3\beta_i - 2\beta_i c_i),\ E_i = \gamma_i^2(3 + 2c_i), \end{array}\right\} \tag{2.16}$$

and β_i, γ_i, Y_i are given by (2.8). The equations (2. 15) together with the two end conditions

$$mo = y_o^{(1)} \text{ and } m_k = y_k^{(1)}, \tag{2.17}$$

are then used to determine the derivatives m. of a P.C.P. s which Clenshaw and Negus call an "X—spline with parameters c_i". A sufficient condition for the unique existence of an X-spline s is that

$$0 \leq 1 + c_i \leq \beta_i^{-1};\ i = 1,2,....,\ k - 1 \tag{2.18}$$

If (2.18) holds then, it can be shown that,

$$m_i - y_i^{(1)} = 0(h^3).$$

The X-splines of particular interest are those which correspond respectively to the following three choices of the parameters c_i :

i) $\underline{c_i = 0;\ i = 1,2,..\ .,k\text{-}1}$.

This is the only choice of the c_i for which an x-spline s is twice continuously differentiable in [a, b]. Thus, in this case, s coincides with the conventional cubic spline and the equations (2.15) become the well—known consistency relations for cubic spline interpolation

ii) $\underline{c_i - (h_i - h_i + 1)/2h_{i+1};\ i = 1,2,....,\ k - 1}$

This is the only choice of the c. for which the derivatives m_i of the x-spline s are such that

$$m_i - y_i^{(1)} = 0(h^4); \qquad i = 0,1, ...,k .$$

For this reason s is called by Clenshaw and Negus the "optimal interpolating X-spline". Clearly, when the knots are all equally spaced the optimal X-spline coincides with the conventional cubic spline.

(iii) $c_i = -1,\ i - 1,2,\ldots\ k\text{-}1$.

In this case the equations (2.15) become

$$\beta_i m_{i-1} + m_i = \beta_i (3-\beta_i) Y_i + \gamma_i^2 Y_{i+1}\ ;\ i = 1,2,\ldots,k\text{-}1\ . \tag{2.19}$$

Thus, the derivatives of the X-spline s are determined by solving a lower triangular linear system and, for this reason, s is called the "x-spline which minimizes computational labour".

Clenshaw and Negus (1978: chap.2.) claim that the conditions (2.14), with suitable $c_i \neq 0$, can restrain the magnitude of $d_i^{(3)}$ and thus produce X-splines with smaller third derivative jump discontinuities than those of the conventional spline. However, this claim is not justified by Theorem 2 of the present paper which shows that, for the conventional spline and for any other X-spline, $d_i^{(3)}$, is always 0(h). In fact, the results of the present section indicate that the conditions (2.14) are unnecessarily restrictive and suggest the generalization of the x-spline definition given in the following section.

3. A new definition for x-splines

Let p_i denote the cubic polynomial interpolating the function y at the points x_i, x_{i+1} , x_{1+2} and x_{i+3} and define the quadratic polynomials q_i ; i = 0, 1 ,..., k - 2, by

$$q_i = p_i^{(1)};\ i=0,1,\ldots,k\text{-}3, \text{ and } q_{k-2} = q_{k-3} = p_{k-3}^{(1)}. \tag{3.1}$$

The motivation for the new definition of x-splines emerges from the observation that the jump discontinuities (2.6) and (2.7) can be written respectively as

$$\left.\begin{aligned} d_i^{(2)} &= \frac{2}{h_i\beta_i}\{\beta_i(q_{i-1}(x_{i-1})-m_{i-1})+2(q_{i-1}(x_i)=m_i) \\ &\quad +\gamma_i(q_{i-1}(x_{i+1})-m_{i+1})\}, \\ \text{And} \qquad d_i^{(3)} &= \frac{6}{h_i^2\beta_i^2}\{\beta_i^2(q_{i-1}(x_{i-1})-m_{i-1})(1-2\beta_i)(q_{i-1}(x_i)-m_i) \\ &\quad -\gamma_i^2(q_{i-1}(x_{i+1})-m_{i+1}\}; \\ & i=1,2,\dots,k-1 \end{aligned}\right\} \qquad (3.2)$$

This implies that the equations (2.15) can be written as

$$A_i\, m_{i-1} + B_i\, m_i + C_i\, m_{i+1} = A_i\, q_{i+1}(x_{i-1}) + B_i q_{i-1}(x_i) + (x_i) + C_i\, q_{i-1}(x_{i+1});$$

$$i = 1,2,\dots,K-1,$$

and suggests the following definition which extends the class of x-splines considered by Clenshaw and Negus (1978) to a wider class of P.C.P.

<u>Definiton 2</u> . Let a_i , b_i ; $i = 1,2,\dots,k—1$ be $2k—2$ real numbers . Then, a P.C.P. s whose derivatives in. ; i=0,l, . .,k, satisfy the relations

$$\left.\begin{aligned} & m_o = y_o^{(1)}, \\ & a_i m_{i-1} + m_i + b_i m_{i+1} = a_i q_{i-1}(x_{i-1}) + q_{i-1}(x_i) + b_i q_{i-1}(x_{i+1}); \\ & \qquad\qquad i = 1,2,\dots,k-1, \\ & m_k = y_k^{(1)}, \end{aligned}\right\} \qquad (3.3)$$

where the polynomials q_i are defined by (3.1), will be called an x-spline with parameters a_i, b_i ; $i = 1,2,\dots,k-1$ **.**

Clearly, this new definition contains the x-splines considered by Clenshaw and Negus (1978) as the special case

$$a_i = A_i/B_i\ ,\ b_i = C_i/B_i\ ;\ i - 1,2,\ldots,k\text{-}l\ , \qquad (3.4)$$

where the A., B. are given by (2.16).

By Definition 2, the derivatives m_i ; i = 1,2, . . .,k-1 of an X-spline s are determined by solving the (k-l)x(k—1)tri-diagonal linear system defined by (3.3).The matrix of coefficients A of this linear system is strictly diagonally dominant, and therefore non-singular, if

$$|a_i| + |b_i| < 1\ ;\ i = 1,2..,k\text{-}1. \qquad (3.5)$$

Thus, a sufficient condition for the unique existence of s is that its parameters satisfy (3.5.) . If this condition holds then, using a result of Lucas (1974= 576),

$$|\ A^{-1}\ \|_\infty < V, \qquad (3.6)$$

where $v \geq 1$ is such that

$$|\ a_i\ | + |b_i| + 1/V < 1\ ;\ i - 1,2 \ldots\ldots\ldots\ldots k\text{-}1\ .$$

We consider now the effect that the parameters a_i , b_i have on the quality of the X-spline approximation. For this, we let

$$\delta_i = a_i)\, q_{i-1}(x_{i-1}) - y_{i+1}^{(1)}\} + \{q_{i-1}(x_i) - y_i^{(1)}\} + b_i\{q_{i-1}(x_{i+1}) - y_{i+1}^{(1)}\}\ ;$$

$$i=1,2,\ldots,k\text{-}l, \qquad (3.7)$$

and assume that the parameters of the X-spline satisfy (3.5). Then, from (3.3),

$$|\ m_i - y_i^{(1)}| \leq v \max_i |\ \delta_i\ |\ ;\quad i = 1,2\ ,\ldots,k-1\ . \qquad (3.8)$$

Also, by Taylor series expansion about the point xi , we find that,

if y ε C^6 [a,b] then,

$$\delta i = \frac{1}{4!} \, Fi \, y_i^{(4)} + \frac{1}{5!} G_i \, y_i^{(5)} + 0(h^5) \; ; \quad i = 1,2,\ldots, k-1 \, , \tag{3.9}$$

where

$$\left.\begin{aligned} F_i &= a_i h_i (h_i + h_{i+1} + h_{i+2}) - h_i h_{i+1} (h_{i+1} + h_{i+2}) \\ &\qquad + b_i h_{i+1} h_{i+2} (h_i + h_{i+1}) \; , \\ G_i &= a_i \; \{ h_i \; (h_i + h_{i+1})((2h_{i+1} + h_{i+2}) \; (h_{i+1} + h_{i+2}) - h_i \; (2h_i + h_{i+2})) \} \\ &\qquad - h_i h_{i+1} \{ h_i h_{i+1} + (h_{i+1} + h_{i+2} - h_i) \; (2h_{i+1} + h_{i+2}) \} \\ &\qquad + b_i \{ h_i + 1 h_{i+2} \; (h_i + h_{i+1}) \; (3h_{i+1} + h_{i+2} - h_i) \} \; ; \\ &\qquad\qquad i = 1,2,\ldots, k-1 \, , \end{aligned}\right\} \tag{3.10}$$

and, in (3.10),

$$h_{k+1} = -(h_{k-2} + h_{k-1} + h_k). \tag{3.11}$$

When the knots are equally spaced with $h_i = h$; i = 1,2,...,k-l, then (3.9) simplifies considerably and, if y ε C^7[a,b], it gives

$$\begin{aligned} \delta_i &= \frac{1}{12} (3\tilde{a}_i - 1 + \tilde{b}_i \} h^3 \, y_i^{(4)} + \frac{1}{60} \{ 3\tilde{a}_i - 2 + 3\tilde{b}_i \} h^4 \, y_i^{(5)} \\ &\qquad + \frac{1}{720} \{ 23\tilde{a}_i - 2 + 17\tilde{b}_i \} h^5 \, y_i^{(6)} + o(h^6) \; ; \\ &\qquad\qquad i = 1,2,\ldots,k-1 \, , \end{aligned} \tag{3.12}$$

where

$$\left.\begin{aligned} &\tilde{a}i = ai, \quad \tilde{b}i = bi \; ; \quad i = 1,2 ,\ldots, k-2 \, , \\ &\text{and} \\ &\tilde{a}_{k-1} = b_{k-1} \, , \quad \tilde{b}_{k-1} = a_{k-1} \, . \end{aligned}\right\} \tag{3.13}$$

The results (3.8) and (3.9)-(3.10) show that any x-spline s, whose parameters satisfy (3.5), is such that

$$m_i - y_i^{(1)} = 0(h^n);\ i = 1,2,..., k - 1 \qquad (3.14)$$

where, in general, n = 3. However, if the parameters of s are such that $F_i = 0$; i = l,2,...,k-l then n = 4, and if $F_i=G_i=0$; i = 1,2,. .. ,k-1 , then n = 5.

The remainder of this paper is concerned with examining the quality of six particular x-splines. These are the conventional cubic spline, the two x-splines of Clenshaw and Negus (1978) which we discussed briefly in Chapter 2, and three new x-splines of special interest that emerge from Definition 2. In particular, for each of these X-splines, we consider the case of equally spaced knots and, by using (3.12), (3.8), (2.4), (2.12) and (2,13), we derive bounds on $E = \| s - y \|$, $| d_i^{(2)} |$ and $| d_i^{(3)} |$.

4. x-splines of special interest

(I) x-spline s_1 with parameters

$$a_i = \beta_i/2\ ,\ b_i = \gamma_i/2\ ;\quad i=1,2\ldots,k-1\ , \qquad (4.1)$$

where, as before, $\beta i -= h_{i+1}/(h_i+h_{i+1})$ and $\gamma_i = 1-\beta_i$.

The values (4.1) are the only choice of parameters for which $d_i^{(2)} = 0$; i = 1,2,. .. ,k-1 . Thus, s_I is the well-known conventional cubic spline.

When the knots are unequally spaced then, in general, $F_i \neq 0$ and the derivatives m_i of s_I satisfy (3.14) with n=3. However, if the knots are equally spaced then

$$a_i = b_i = 1/4\ ;\ i = 1,2,\ldots,k-1$$

and, by using in this case (2.11) instead of (3.12),

$$\delta i_i = \frac{-h^4}{120} y^{(5)}(\xi\xi_i)\ ; \quad i = 1,2,\ldots,k-1 .$$

Thus, since v = 2,

$$| m_i - y_i^{(1)} | \le \frac{h^4}{60} \| y^{(5)} \| ; \quad i = 1,2,\ldots,k-1 ,$$

and hence,

$$E \le \frac{h^4}{384} \| y^{(4)} \| + \frac{h^5}{240} y^{(5)} \| , \tag{4.2}$$

and

$$\left.\begin{aligned} & d_i^{(2)} = o , \\ & | d_i^{(3)} | \le h \| y^{(4)} + \frac{h^2}{5} \| y^{(5)} \| ; \quad i = 1,2,\ldots,k-1 \end{aligned}\right\} . \tag{4.3}$$

(II) x-spline s_{II} with parameters

$$ai = \beta_i^2 , \;\; bi = \gamma_i^2 ; \quad i = 1,2,\ldots,k-1 . \tag{4.4}$$

The jump discontinuities of $s_{11}^{(2)}$ and $s_{11}^{(3)}$ satisfy (2.14) with

$c_i = (h_i—h_{i+1})/2h_{i+1}$, and (4.4) is the only choice of parameters of the form (3.4) for which $F_i = 0$; 1 = 1 ,2, .. . ,k-1 . It follows that s_{11} is the only P.C.P., in the class considered by Clenshaw and Negus (1978), whose derivatives m. satisfy (3.14) with n > 3.

When the knots are equally spaced then s_{II} coincides with the conventional cubic spline s_I .

(III) x-spline s_{III} with parameters

$$a_i = \beta_i , \; b_i = 0 ; \; i - 1 ,2,\ldots,k-1 . \tag{4.5}$$

The jump discontinuities of $s_{11}^{(2)}$ and $s_{11}^{(3)}$ satisfy (2.14) with

$c_i = -1$, and the values (4.5) reduce the three-term recurrence relation in (3.3) to the two-term recurrence relation (2.19). For this reason, of all the x-splines contained in the definition of Clenshaw and Negus (1978), the construction of s_{III} involves the least computational effort.

When the knots are equally spaced then,

$$a_i = 1/2,\ b_i = 0\ ;\ i=1,2,\ldots,k-1,$$

and, since $\nu = 2$,

$$|\, m_i - y_i^{(1)} \,| \le \frac{h^3}{12} \| y^{(4)} \| + \frac{h^4}{60} \| y^{(5)} \| + 0(h^5)\,.$$

Hence,

$$E \le \frac{9h^4}{384} \| y^{(4)} \| + \frac{h^5}{240} y^{(5)} \| + 0(h^6)\,, \tag{4.6}$$

and

$$\left.\begin{aligned} |\, d_i^{(3)} \,| &\le \frac{2h^2}{3} \| y^{(4)} \| + \frac{h^3}{15} \| y^{(5)} \| + 0(h^4)\,, \\ |\, d_i^{(3)} \,| &\le 2h \| y^{(4)} \| + \frac{h^2}{5} \| y^{(5)} \| + 0(h^3)\,; \\ & i = 1,2,\ldots,k-1\,, \end{aligned}\right\} \tag{4.7}$$

where, in this case, (4,7) is obtained by using (2.13) for the bound on $|\, d_i^{(2)} \,|$ and the relation $d_i^{(20} = -hd_i^{(3)}/3$ instead of (2.12), for the bound on $|\, d_i^{(2)} \,|$.

(IV) x-spline s_{IV} with parameters

$$a_i = b_i = 0;\ i - 1\ ,2,\ldots,k-1\,. \tag{4,8}$$

The derivatives of s_{IV} are given explicitly by

$$m_i = q_{i-1}(x_i);\ i=1,2,\ldots,k-1.$$

Thus, s_{IV} is the x-spline of least computational effort.

In this case,

$$m_i - y_i^{(1)} = q_{i-1}(xi) - y_i^{(1)} = -\frac{1}{24} h_i h_{i+1} (h_{i+1} + h_{i+2}) y^{(4)}(\xi_i) ;$$

$$1 = 1, 2, \ldots, k - 1 ,$$

where $\xi_i \in [x_{i-1}, x_{i+2}]$; $i = 1,2,..,k-2$, $\xi_{k-1} \in [x_{k-3}, x_k]$ and h_{k+1} is given by (3.11). Hence, if the knots are equally spaced,

$$E \le \frac{9h^4}{384} \| y^{(4)} \tag{4.9}$$

and

$$\left.\begin{aligned} &| d_i^{(2)} | \le h^2 \| y^{(4)} \| + \frac{h^3}{15} \| y^{(5)} \| , \\ &| d_i^{(3)} | \le 2h \| y^{(4)} \| ; \quad i = 1,2,\ldots,k-1 \end{aligned}\right\} \tag{4.10}$$

(V) X-spline s_v with parameters,

$$\left.\begin{aligned} &a_i = \frac{h_{i+1}(h_{i+1} + h_{i+2})}{(h_i + h_{i+1})(h_i + h_{i+1} + h_{i+2})} , \quad b_i = 0 ; \quad i = 1,2,\ldots,k-2 , \\ &a_{k-1} = 0 , \ b_{k-1} = \frac{h_{k-1}(h_{k-1} + h_{k-2})}{(h_{k-1} + h_k)(h_{k-2} + h_{k-1} + h_k)} . \end{aligned}\right\} \tag{4.11}$$

The construction of s_v involves the same computational effort as that of s_{III}. However, since the parameters (4.11) are such that $F_i = 0$; $1=1,2,...,k—1$, the derivatives m_i of s_v satisfy (3.14) with $n = 4$.

When the knots are equally spaced then

$$\left.\begin{aligned} &ai = 1/3, \ b_i = 0 ; \quad i = 1,2,\ldots,k-2 , \\ &a_{k-1} = 0, \quad b_{k-1} = 1/3 , \end{aligned}\right\}$$

and, since $\upsilon = 3/2$

$$| mi - y_i^{(1)} | \le \frac{h^4}{40} \| y^{(5)} \| + \frac{17}{1440} h^5 \| y^{(6)} \| + 0(h^6) ;$$
$$i = 1,2,\ldots,k-1 .$$

Hence,

$$E \le \frac{h^4}{384} \| y(4) \| + \frac{h^5}{160} \| y^{(5)} \| + 0(h^6) , \tag{4.12}$$

$$\left.\begin{aligned} & | d_i^{(2)} | \le \frac{11}{30} h^3 \| y^{(5)} \| + \frac{17}{120} h^4 \| y^{(6)} \| + 0(h^5) , \\ \text{and} \\ & | d_i^{(3)} | \le h \| y^{(4)} \| + \frac{3}{10} h^2 \| y^{(5)} \| + 0(h^3) ; \\ & \qquad\qquad i = 1,2,\dots,k-1 . \end{aligned}\right\} \tag{4.13}$$

(VI) X-spline s_{VI} with parameters

$$\left.\begin{aligned} & ai = \frac{h_{i+1}^2 (h_{i+1} + h_{i+2})}{(h_i + h_{i+1} + h_{i+2}) (h_i + h_{i+1})^2} , \quad bi = \frac{h_i^2 (hi+1+hi+2)}{h_{i+2} (h_i + h_{i+1})^2} ; \\ & \qquad\qquad i = 1,2,\dots k-1 , \end{aligned}\right\} \tag{4.14}$$

where h_{k+1} is given by(3.11).

This is the x-spline of highest accuracy, in the sense that (4.14) is the only choice of parameters for which $F_i = G_i = 0$; $i = 1, 2,..,k$-l. This implies that the derivatives m_i of s_{VI} satisfy (3.14) with n = 5. It should be observed that, in this case, the conditions (3.5) which ensure the unique existence of s_{VI} are satisfied only if

$$(h_i+h_{i+1}) (h_i-h_{i+2}) < 2h_{i+2}(h_{i+1}+h_{i+2}) ;$$

$$i = 1, 2 , \dots , k-1 .$$

When the knots are equally spaced then,

$$\left.\begin{aligned} & a_i = 1/6, \quad b_i = 1/2 ; \quad i = 1,2,\dots,k-2 , \\ & a_{k-1} = 1/2, \ b_{k-1} = 1/6 , \end{aligned}\right\}$$

and, since v = 3,

$$| m_i - y_i^{(1)} | \le \frac{31}{370} h^5 \| y^{(6)} \| + 0(h^6) .$$

Hence,

$$E \le \frac{h^4}{384}\| y^{(4)} \| + \frac{31}{2880} h^6 \| y^{(6)} \| + 0(h^7) , \tag{4.15}$$

an

$$\left.\begin{aligned} | S_i^{(2)} | &\le \frac{h^3}{15}\| y^{(5)} \| + \frac{31}{60} h^4 \| y^{(6)} \| 0(h^5) , \\ | S_i^{(3)} | &\le h \| y^{(4)} \| + \frac{31}{60} h^3 \| y^{(6)} \| + 0(h^4) ; \\ & \qquad i = 1,2,\ldots,k-1 . \end{aligned}\right\} \tag{4.16}$$

5. Numerical results and discussion

In Tables 1 - 3 we present numerical results obtained by taking

y(x) = exp(x),

$$x_i - i/20 \ ; \ I - 0, 1,\ldots, 20 \ , \tag{5.1}$$

and constructing each of the six x-splines considered in Section 4. The results listed are values of the absolute error | s(x)-y(x) | , computed at various points between the knots, and values of the jump discontinuities $d_i^{(2)}$ and $d_i^{(3)}$, computed at a selection of interior

knots. The results of Tables 4 - 6 are obtained, in a similar manner, by using the same y and the unequally spaced knots

$$x_i =- i^2/8^2 \ ; \ 1 = 0, 1,\ldots,8 \ , \tag{5.2}$$

In each table, the results corresponding to the x-spline s_I are listed in column (I), those corresponding to S_{II} in column (II), etc.

The use of any of the X-splines $s_{II'}$ $s_{III'}$ $s_{IV'}$ $S_{V'}$ or $s_{VI'}$. in preference to the conventional cubic spline S_I, can be justified only if it leads to increased accuracy or to a reduction of the computational labour. The numerical results of this section indicate that no significant improvement in accuracy is achieved by the so called optimal x-spline S_{II} of Clenshaw and Negus (1973), or indeed by the more 'accurate' s_{VI} . It follows that the only x-splines of real practical importance, for the interpolation of smooth functions, are those whose construction involves less computational effort than the construction of s_I . In particular the two new x-splines s_{Iv} and S_V are of special interest. The x-spline S_{IV} involves the least possible computational effort, and it is of practical interest because of its simplicity. The x-spline S_V has small discontinuities in the second derivative and produces, with less computational effort, results of comparable accuracy to those obtained by the conventional spline.

By Definition 2, the construction of an x-spline requires knowledge of $y^{(1)}$ at the two endpoints x_0 , x_k and, in an interpolation problem, this information is not usually available. However, by using techniques similar to those of Behforooz and Papamichael (1979 a,b), the end conditions

$$m_o = y_o^{(1)}, \quad mk = y_k^{(1)}, \tag{5.3}$$

can be replaced by conditions which use only the available function values of y at the knots whilst retaining the order of the x-spline approximation. For example, instead of (5- 3) , the following end conditions can be used respectively for the construction of S_{IV} and Sv :

$$m_o = q_o(X_o),\ m_k = q_{k-2}(x_k), \tag{5.4}$$

and

$$\left.\begin{aligned} &m_o+\alpha_o m_1 = q_o(x_o)+\alpha_o q_o(x), \\ &\alpha_k m_{k-1}+m_k = \alpha_k q_{k-2}(x_{k-1})+q_{k-2}(x_k), \end{aligned}\right\} \tag{5.5}$$

where, in (5.5),

$$\alpha_j = u_j/(u_j-1) + v_j/\{v_j-1-(v_j-u_j)^2\} \; ; \; j = 0,k$$

with

$$u_o = (h_1+h_2)/h_1, v_o = (h_1+h_2+h_3)/h_1,$$

and

$$u_k = (h_{k-1}+h_k)/h_k, \quad v_k = (h_{k-2}+h_{k-1}+h_k)/h_k \; ;$$

see Behforooz and Papamichael (1979 b).

Table 1

Values of | s(x) - y(x)|. (Knots as in 5.1)

X	(I)	(III)	(IV)	(V)	(VI)
0.01	$.674x10^{-8}$	$.155x10^{-7}$	$.111x10^{-7}$	$.664x10^{-8}$	$.682x10^{-8}$
0.02	$.151x10^{-7}$	$.414x10^{-7}$	.383x10-^{7}	$.148X10^{-7}$	$.154X10^{-7}$
0.09	.705x10-^{8}	.177xl0-^{7}	$.501x10^{-7}$	$.688x10^{-8}$	$.721xl0^{-8}$
0.22	$.189x10^{-7}$	$.117x10^{-7}$	$.467x10^{-7}$	$.190x10^{-7}$	$.188x10^{-7}$
0.36	$.990x10^{-8}$	$.139x10^{-7}$	$.808x10^{-7}$	$.102x10^{-7}$	$.967x10^{-8}$
0.62	$.281x10^{-7}$	$.143x10^{-7}$	$.697x10^{-7}$	$.283X10^{-7}$	$.280x10^{-7}$
0.93	$.374x10^{-7}$	$.617x10^{-7}$	$.353x10^{-6}$	$.369x10^{-7}$	$.378x10^{-7}$
0.96	$.184x10^{-7}$	$.402x10^{-7}$	$.151x10^{-6}$	$.193x10^{-7}$	.177x10-^{7}
0.99	$.179x10^{-7}$	$.327x10^{-8}$	$.245x10^{-7}$	$.182x10^{-7}$	$.177x10^{-7}$

Table 2

Values of $d_{(i)}^{(2)}$. (Knots as in 5.1)

	(I)	(III)	(IV)	(V)	(VI)
x_1	—	-.996x10"	$.225x10^{-2}$	$.121x10^{-4}$	$-.889x10^{-5}$
x_4	—	$-.102X10^{-2}$	$.311x10^{-2}$	$.132x10^{-4}$	$-.103X10^{-4}$
X_7	—	$-.120x10^{-2}$	$.361x10^{-2}$	$.154x10^{-4}$	$-.120x10^{-4}$
X_{10}	—	$-.140x10^{-2}$	$.419x10^{-2}$	$.178x10^{-4}$	$-.139x10^{-4}$
X_{13}	—	$-.162x10^{-2}$	$.487x10^{-2}$	$.207x10^{-4}$	$-.162x10^{-4}$
x_{16}	—	-.189xl0~^{2}	$.566x10^{-2}$	$.241x10^{-4}$	$-.189x10^{-4}$
X_{19}	—	$-.181x10^{-2}$	$-.320x10^{-2}$	$.290x10^{-4}$	$-.210x10^{-4}$

Table 3

Values of $d_{(i)}^{(3)}$. (Knots as in 5.1)

	(I)	(III)	(IV)	(V)	(VI)
x_1	$.525x10^{-1}$	$.597x10^{-1}$	$.244x10^{-1}$	$.524x10^{-1}$	$.526x10^{-1}$
X_4	$.611X10^{-1}$	$.613xl0^{-1}$	$.580x10^{-1}$	$.611xl0^{-1}$	$.611xl0^{-1}$
x_7	$.710x10^{-1}$	$.723x10^{-1}$	$.673x10^{-1}$	$.709x10^{-1}$	$.710x10^{-1}$
x_{10}	.824x10-	$.838x10^{-1}$	$.782x10^{-1}$	$.824x10^{-1}$	$.824x10^{-1}$
x_{13}	$.958x10^{-1}$	$.974x10^{-1}$	$.909x10^{-1}$	$.958x10^{-1}$	$.958x10^{-1}$
x_{16}	.111	.113	.106	.111	.111
x_{19}	.130	.109	.192	.130	.129

Table 4

Values of |s(x)-y(x)|. (Knots as in (5.2)

X	(I)	(II)	(III)	(IV)	(V)	(VI)
0.01	$.512xl0^{-9}$	$.131xl0^{-9}$	$.125x10^{-8}$	$.908x10^{-8}$	$.104x10^{-9}$	$.125x10^{-9}$
0.05	$.287x10^{-8}$	$.763x10^{-8}$	$.558x10^{-7}$	$.196x10^{-6}$	$.611xl0^{-8}$	$.826x10^{-8}$
0.1	$.804xl0^{-7}$	$.104x10^{-6}$	$.290x10^{-6}$	$.586x10^{-6}$	$.956x10^{-7}$	$.105x10^{-6}$
0.17	$.297x10^{-6}$	$.277x10^{-6}$	$.269x10^{-6}$	$.391x10^{-6}$	$.272x10^{-6}$	$.283x10^{-6}$
0.35	$.589x10^{-6}$	$.860x10^{-6}$	$.373x10^{-5}$	$.921x10^{-5}$	$.702x10^{-6}$	$.931xl0^{-6}$
0.5	$.272x10^{-5}$	$.301x10^{-5}$	$.933x10^{-5}$	$.192x10^{-4}$	$.248x10^{-5}$	$.325x10^{-5}$
0.6	$.325x10^{-5}$	$.308x10^{-5}$	$.163x10^{-5}$	$.422x10^{-4}$	$.354x10^{-5}$	$.298x10^{-5}$
0.8	$.721x10^{-5}$	$.566x10^{-5}$	$.118x10^{-4}$	$.328x10^{-4}$	$.654x10^{-5}$	$.480x10^{-5}$
0.9	$.207x10^{-4}$	$.192x10^{-4}$	$.218x10^{-5}$	$.184X10^{-4}$	$.201x10^{-4}$	$.184x10^{-4}$

Table 5

Values of $d_{(i)}^{(3)}$.   (Knots as in 5.2)

	(I)	(II)	(III)	(IV)	(V)	(VI)
x_1	_	$-.166x10^{-3}$	$-.646x10^{-3}$	$.255x10^{-2}$	$-.152x10^{-3}$	$-.169x10^{-3}$
x_2	—	$-.348x10^{-3}$	$-.209x10^{-2}$	$.693x10^{-2}$	$-.288x10^{-3}$	$-.369x10^{-3}$
x_3	-	$-.565x10^{-3}$	$-.457x10^{-2}$	$.148x10^{-1}$	$-.394x10^{-3}$	$-.630x10^{-3}$
x_4	-	$-.839x10^{-3}$	$-.862x10^{-2}$	$.275x10^{-1}$	$-.442x10^{-3}$	$-.103x10^{-2}$
x_5	-	$-.121x10_{-2}$	$-.151X10^{-1}$	$.480x10^{-1}$	$-.373x10^{-3}$	$-.160x\ 10^{-2}$
x_6	-	$-.171x10^{-2}$	$-.254x10^{-1}$	$.427x10^{-1}$	$-.121x10^{-3}$	$-.260x10^{-2}$
x_7	-	$-.249x10^{-2}$	$-.315X10^{-1}$	$-.470x10^{-1}$	$-.941x10^{-3}$	$-.387x10^{-2}$

Table 6

Values of $d_{(i)}^{(3)}$. (Knots as in 5.2)

	(I)	(II)	(III)	(IV)	(V)	(VI)
x_1	$.335x10^{-1}$	$.319x10^{-1}$	$.413x10^{-1}$	$.325x10^{-1}$	$.315x10^{-1}$	$.322x10^{-1}$
x_2	$.663x10^{-1}$	$.668x10^{-1}$	$.801xl0^{-1}$	$.271x10^{-1}$	$.662x10^{-1}$	$.670x10^{-1}$
x_3	.108	.109	.125	$.541x10^{-1}$	.107	.109
x_4	.160	.161	.184	$.871x10^{-1}$	.160	.162
x_5	.233	.233	.263	.128	.229	.234
x_6	.324	.329	.376	.745	.327	.332
X_7	.484	.479	.403	.601	.484	.476

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