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Superconvergence Properties of Quintic Interpolatory Splines

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ABSTRACT

Let Q be a quintic spline with equi-spaced knots on [a,b] interpolating a given function y at the knots. The parameters which determine Q are used to construct a piecewise defined polynomial P of degree six. It is shown that P can be used to give at any point of [a,b] better orders of approximation to y and its derivatives than those obtained from Q. It is also shown that the superconvergence properties of the derivatives of Q, at specific points of [a,b], are all simple consequences of the properties of P.

1. Introduction

Let Q be a quintic spline on [a,b] with equally spaced knots

$$x_i = a + ih \; ; \; i = 0,1,...,k, \qquad (1.1)$$

where h= (b-a)/k. Then, $Q \in C^4[a,b]$ and in each of the intervals $[x_{i-1}, x_i]$; i=1,2,...,k, Q is a polynomial of degree five or less. The set of all such quintic splines forms a linear space, of dimension k + 5, which we denote by Sp(5;k). An element Q ∈ Sp (5;k) is said to be a quintic spline interpolant of a function y if

$$Q(x_i) = y_i \; ; \quad i=0,1,...,k, \qquad (1.2)$$

where $y_i = y(x_i)$. Since dim Sp(5;k)=k+5, the conditions (1.2) are not sufficient to determine an interpolatory Q uniquely and four additional linearly independent conditions are always needed for this purpose. These are usually taken to be end conditions, i.e. conditions imposed on Q or its derivatives $Q^{(j)}$; j = 1,2,3,4, near the two end points a and b. As might be expected the choice of end conditions plays a critical role on the quality of the spline approximation.

In what follows we assume that Q is a Sp (5;k) interpolant of a function $y \in C^7[a,b]$ and, for notational simplicity, we use the following abbreviations,

$$m_i = Q^{(1)}(x_i), \quad m_i = Q^{(2)}(x_i), \quad n_i = Q^{(3)}(x_i) \quad \text{and} \quad n_i = Q^{(4)}(x_i) .$$

It is well known that the best order of uniform convergence that can be achieved by Q and its derivatives is

$$\| Q^{(r)}-y^{(r)} \| = 0(h^{6-r}); \; r\text{-}0,1.2.3.4.$$

where || · || denotes the uniform norm on [a,b] . It is also known that for a variety of end conditions

$$m_i - y_i^{(1)} = 0(h^6); \quad i = 0,1,...,k. \qquad (1.3)$$

The class of end conditions for which (1.3) holds includes the conditions

$$m_i \;=\; y_i^{(1)} \quad ; i = 0,1,k\text{-}1,k, \tag{1.4}$$

$$n_i \;=\; y_i^{(3)} \quad ; i = 0,1,k\text{-}1,k \tag{1.5}$$

$$\Delta^s\ N_i = \nabla^s N_{k-i} \;= 0 \;; i = 0,1,\ s = 3,\ 4,\ 5, \tag{1.6}$$

and, for periodic y, the end conditions

$$Q^{(r)}(a) - Q^{(r)}(b);\ r=1,2,3,4 \tag{1.7}$$

of the periodic quintic spline; see e.g. [1] and [2].

The result (1.3) is an example of the improved order of approximation that the derivatives of Q display, under appropriate end conditions, at specific points of [a,b]. Another example is the result

$$n_i - y_i^{(3)} \;=\; 0(h^4); \quad i = 0,1,\ldots,k, \tag{1.8}$$

which holds whenever (1.3) holds. This paper is concerned with such superconvergence properties of Q. More specifically, in Section 3, we use the parameters which determine Q to construct a piecewise defined polynomial of degree six, and we show that the superconvergence properties of Q are all simple consequences of the properties of P. We also show that, under certain conditions, P can be used with very little additional computational effort to compute, at any point $x \in [a,b]$, more accurate approximations to y and its derivatives than those obtained from Q.

The work of the present paper generalizes the results of [3], concerning the superconvergence properties of a cubic interpolatory spline s, to the case of quintic spline interpolation. The cubic spline results were established in [3] by considering the properties of a piecewise quartic derived from the quartic Hermite polynomials matching y at the points x_{i-1} , x_i , x_{i+1} and $y^{(1)}$ at x_{i-1} , x_i , by replacing the derivatives $y^{(1)}_{i-1}$ and $y^{(1)}_{i}$ respectively by $s^{(1)}(x_{i-1})$ and $s^{(1)}(x_i)$ It is important to observe that the results of the present paper cannot be established by direct generalization of the above, i.e. by considering the sextic Hermite polynomials matching y at x_{i-1} , x_i , x_{i+1} and $y^{(1)}$, $y^{(2)}$ at x_{i-1}, x_i . As will become apparent later, the reason for this is that $Q^{(2)}$ does not exhibit superconvergence at the knots. Thus, in Section 3, we derive P by considering a class of certain sixth degree Hermite-Birkhoff polynomials.

2. Preliminary Results

If the values $m_i = Q^{(1)}(x_i)$ and $n_i = Q^{(3)}(x_i)$; $i = 0, 1, \ldots, k$, are known then Q can be determined in each interval $[x_{i-1}, x_i]$ by using the quintic Hermite—Birkhoff polynomial h_i, which is such that

$$\left.\begin{aligned} & h_i(x_j) = y_j;\ j = i-1, i, \\ \text{and}\quad & h_i^{(1)}(x_j) = y_j^{(1)};\ j = i-1, i, \\ & h_i^{(3)}(x_j) = y_j^{(3)};\ j = i-1, i. \end{aligned}\right\} \tag{2.1}$$

This gives

$$Q(x) = \Psi_{1,i}(x)y_{i-1} + \Psi_{2,i}(x)y_i + \Psi_{3,i}(x)m_{i-1} + \Psi_{4,i}(x)m_i + \Psi_{5,i}(x)n_{i-1} + \Psi_{6,i}(x)n_i;$$

$$x \in [x_{i-1}, x_i];\ i = 1,2 \ldots, k, \tag{2.2}$$

where

$$\left.\begin{aligned}
\psi_{1,i}(x) &= \frac{1}{2h^5}(x-x_i)^2\{-2(x-x_i)^3 - 5h(x-x_i)^2 + 5h^3\}; \\
\psi_{2,i}(x) &= \frac{1}{2h^5}(x-x_{i-1})^2\{2(x-x_{i-1})^3 - 5h(x-x_{i-1})^2 + 5h^3\}; \\
\psi_{3,i}(x) &= \frac{1}{4h^4}(x-x_{i-1})(x-x_i)^2\{-2(x-x_i)^2 - 3h(x-x_i) + 3h^2\}, \\
\psi_{4,i}(x) &= \frac{1}{4h^4}(x-x_{i-1})^2(x-x_i)\{-2(x-x_{i-1})^2 + 3h(x-x_{i-1}) + 3h^2\}, \\
\psi_{5,i}(x) &= \frac{1}{48h^2}(x-x_{i-1})^2(x-x_i)^2\{-2(x-x_i) - h\}, \\
\text{and}\quad & \\
\psi_{6,i}(x) &= \frac{1}{48h^2}(x-x_{i-1})^2(x-x_i)^2\{-2(x-x_{i-1}) + h\}.
\end{aligned}\right\} \tag{2.3}$$

Thus,

$$Q^{(2)}(x_i-) = \frac{5}{h^2}(y_{i-1} - y_i) + \frac{1}{2h}(3m_{i-1} + 7m_i) - \frac{1}{24}(n_i + 3n_i);$$

$$i = 1,2,\ldots,k, \tag{2.4}$$

$$Q^{(2)}(x_i+) = \frac{5}{h^2}(y_{i+1} - y_i) + \frac{1}{2h}(3m_{i+1} + 7m_i) + \frac{1}{24}(n_{i+1} - 3n_i);$$

$$i = 0, 1, \ldots, k-1, \tag{2.5}$$

$$Q^{(4)}(x_i-) = \frac{60}{h^4}(y_i - y_{i-i}) - \frac{30}{h^3}\left(m_i + m_{i-1}\right) + \frac{1}{2h}(7n_i + 3n_{i-1});$$
$$i = 1,2,...,k, \qquad (2.6)$$

$$Q^{(4)}(x_i+) = \frac{60}{h^4}(y_i - y_{i+1}) + \frac{30}{h^3}(m_i + m_{i+1}) - \frac{1}{2h}(7n_i + 3n_{i+1});$$
$$i = 0,1,...,k-1, \qquad (2.7)$$

and the continuity of $Q^{(2)}$ and $Q^{(4)}$ implies that

$$12h\{3m_{i-1} + 14m_i + 3m_{i+1},\} - h^3\{n_{i-1} - 6n_i + n_{i+1}\} - 120\{y_{i+1} - y_{i-1}\};$$
$$i = 1,2,...,k-1, \qquad (2.8)$$

and

$$60h\{m_{i-1} + 2m_i + m_{i+1}\} - h^3\{3n_{i-1} + 14n_i + 3n_{i+1}\} = 120\{y_{i+1} - y_{i-1}\};$$
$$i = 1,2,...,k-1. \qquad (2.9)$$

It follows, from (2.8) and (2.9), that

$$m_i = \frac{-h^2}{120}\{n_{i-1} + 18n_i + n_{i-1}\} + \frac{1}{2h}\{y_{i+1} - y_{i-1}\};$$

$$i = 1,2,...,k-1, \qquad (2.10)$$

$$n_i = \frac{-3}{2h^2}\{m_{i-1} + 8m_i + m_{i-1}\} + \frac{15}{2h^3}\{y_{i+1} - y_{i-1}\};$$

$$i = 1,2,...,k-l, \qquad (2.11)$$

and these two equations, in conjunction with (2.8) and (2.9), lead to the consistency relations

$$n_{i-2} + 26n_{i-1} + 66n_i + 26n_{i+1} + n_{i+2} = \frac{60}{h^3}\{-y_{i-2} + 2y_{i-1} - 2y_{i-1} + y_{i+2}\};$$

$$i = 2,3,...,k-3, \qquad (2.12)$$

and

$$m_{i-2} = 26m_{i-1} = 66m_i = 26m_{i+1} + m_{i+2} = \frac{5}{h}\{-y_{i-2} - 10y_{i-1} + 10y_{i+1} + y_{i+2}\};$$
$$i = 2,3,...,k-2. \qquad (2.13)$$

The above quintic spline identities (2.4) — (2.13) can also be derived from the results of [4] - [7].

The parameters m., n.; i=0, 1,...,k needed for the construction of Q by means of (2.2) may be determined as follows. The k+ 1 parameters m. are obtained by solving the linear system consisting of (2.13) and the four equations derived from the end conditions of Q. The remaining parameters n_i; i = 0, 1 , . . .,k are then determined from (2.11) and (2.10).

The following lemma is needed for the analysis in Section 3.

Lemma 2. 1 Let $m_i = Q^{(1)}(x_i)$ and $n_i - Q^{(3)}(x_i)$. If $y \in C^7[a,b]$ then there exist constants c and d, independent of h, such that

$$| n_i - y_i^{(3)} | \le \frac{c}{h^2} \max_{o \le j \le k} | m_j - y_j^{(1)} | + dh^4 \| y^{(7)} \|; \quad i = 0,1,\ldots, k. \qquad (2.14)$$

The result (2.14) is established easily from (2.11) and (2.8) by Taylor series expansion about the point x_i.

3. Improved orders of approximation.

Given the values $y_i = y(x_i)$; $i = 0,1, \ldots, k$, where x_i are the equally spaced points (1.1), let H_i; $i = 1, 2, \ldots, k-1$, denote the sextic Hermite-Birkhoff polynomials which are such that

$$H_i(x_j) = y_j; \quad j = i-1, i, i+1,$$
$$H_i^{(1)}(x_j) = y_j^{(1)}; \quad j = i-1, i,$$
$$H_i^{(3)}(x_j) = y_j^{(3)}; \quad j = i-1, i,$$

Also, let H_k be the sextic Hermite-Birkhoff polynomial matching y at the points x_{k-2}, x_{k-1}, x_k and $y^{(1)}$, $y^{(3)}$ at the points x_{k-1}, x_k. Then, it can be shown that

$$H_i(x) = h_i(x) + \Gamma_i\,\theta_i(x); \; i = 1,2, \ldots, k, \tag{3.1}$$

where $h_i(x)$ is the quintic Hermite-Birkhoff polynomial, of Section 8.4, matching y, $y^{(1)}$ and $y^{(3)}$ at the points x_{i-1}, x_i,

$$\theta_i(x) = (x-x_{i-1})^2 (x-x_i)^2 \{2(x-x_{i-1})^2 - 2h(x-x_{i-1}) - h^2\};$$
$$i = 1,2, \ldots, k, \tag{3.2}$$

and

$$\left.\begin{aligned}
\Gamma_i &= \frac{1}{12h^6}\{y_{i-1} - 2y_i + y_{i+1} + h(y_{i-1}^{(1)} - y_i^{(1)}) \\
&\qquad - \frac{h^3}{12}(y_{i-1}^{(3)} + 5y_i^{(3)})\}; \\
&\qquad\qquad i = 1,2, \ldots, K-1, \\
\Gamma_k &= \frac{1}{12h^6}\{y_{k-2} - 2y_{k-1} + y_k + h(y_{k-1}^{(1)} - y_k^{(1)}) \\
&\qquad + \frac{h^3}{12}(5y_{k-1}^{(3)} + y_k^{(3)})\}
\end{aligned}\right\} \tag{3.3}$$

Also, it can be shown that if $y \in C^7[a,b]$ then, for $x \in [x_{i-1}, x_{i+1}]$,

$$|H_i^{(r)}(x) - y^{(r)}(x)| \le K_r h^{7-r} \|y^{(7)}\| \quad r = 0, 1, \ldots, 6, \tag{3.4}$$

where the K_r are constants independent of h.

Definition 3.1. Let Q be a quintic spline which agrees with the function y at the equally spaced knots (1.1). Denote by p_i; $i = 1,2, \dots ,k$, the sextic polynomials obtained from the H_i. by replacing the derivatives $y_j^{(1)}$ and $y_j^{(3)}$ $j = i-1, i$ in (3.1) respectively by $m. = Q^{(1)}(x_j)$ and $n_j = Q^{(3)}(x_i)$; $j = i-1, i$. Then, the piecewise polynomial

$$p(x) = \begin{cases} p_1(x), & x \in I_1 = [x_0, x_1], \\ p_i(x), & x \in I_i = [x_{i-1}, x_i];\ i = 2, 3, \dots, k, \end{cases} \tag{3.5}$$

will be called the piecewise sextic induced by Q.

It follows at once from the definition that $P \in C^1[a,b]$, and that

$$P(x) = Q(x) + \Delta_i\theta_i(x),\ x \in I_i; \qquad i = 1,2,\dots,k, \tag{3.6}$$

where $\theta_i(x)$ is given by (3.2) and

$$\left.\begin{aligned} \Delta_i &= \frac{1}{12h^6}\{y_{i-1} - 2y_i + y_{i+1} + h(m_{i-1} - m_i) \\ &\qquad - \frac{h^3}{12}(n_{i-1} + 5_{ni})\}; \\ &\qquad i = 1,2, \dots, k-1, \\ \Delta_k &= \frac{1}{12h^6}\{y_{k-2} - 2y_{k-1} + y_k + h(m_{k-1} - m_k) \\ &\qquad + \frac{h^3}{12}(5n_{k-1} + n_k)\}; \end{aligned}\right\} \tag{3.7}$$

Also, by using the identities (2.8) and (2.9), it can be shown that

$$\Delta_k = \Delta_{k-1}, \tag{3.8}$$

and hence that

$$p_{k-1}(x) = p_k(x), \qquad x \in (x_{k-2}, x_k],$$

i.e. the same sextic polynomial is used over the intervals I_{k-1} and I_k.

<u>Theorem 3.1</u>. Let Q be an interpolatory quintic spline which agrees with the function $y \in C^7[a,b]$ at the equally spaced knots (1.1) and satisfies end conditions such that

$$|m_i - y_i^{(1)}| \leq Ah^6; \ i = 0,1, \ldots, k, \tag{3.9}$$

where A is a constant. Let P be the piecevxse sextic induced by Q. Then, there exist constants B_r such that, for $x \in I_i$, i=1,2,...,k,

$$| p^{(r)}(x) - y^{(r)}(x) | \leq B_r h^{7-r}; \quad r = 0,1, \ldots, 6. \tag{3.10}$$

<u>Proof.</u> From (3.1), (3.6) and (3,4)

$$\begin{aligned} |p^{(r)}(x) - y^{(r)}(x)| &\leq |p^{(r)}(x) - H_i^{(r)}(x)| + |H_i^{(r)}(x) - y^{(r)}(x)| \\ &\leq |Q^{(r)}(x) - h_i^{(r)}(x)| + |\Gamma_i - \Delta_i| \max_{x \in I_i} |\theta_i^{(r)}(x)| \\ &\quad + K_r h^{7-r} \| y(7) \|; \end{aligned}$$

$$r = 0,1, \ldots, 6, \ x \in I_i; \ i = 1,2, \ldots, k. \tag{3.11}$$

Also, from (3.3) and (3.7),

$$|\Gamma_i - \Delta_i| \leq \frac{1}{6h^5} \max_{o \leq j \leq k} |m_j - y_j^{(1)}| + \frac{1}{24h^3} \max_{o \leq j \leq k} |n_j - y_j^{(3)}|, \tag{3.12}$$

and from (3.2)

$$\max_{x \in I_i} |\theta_i^{(r)}(x)| = a_r h^{6-r}; \ r = 0,1, \ldots, 6, \tag{3.13}$$

where the a are constants independent of h. The result (3.10) follows from (3.11), by using (3.12), (3.13), (3.9), (2.2) and the result of Lemma 2.1.

The piecewise sextic P, of Definition 2.1 is defined completely by the parameters which determine the quintic spline Q. Thus, under the conditions of Theorem 2.1, P can be used with very little additional computational effort to produce at any point $x \in [a,b]$ more accurate approximations to y and its derivatives than those obtained from Q.

Theorem 3.2 below establishes the superconvergence properties of the derivatives of Q and derives expressions for the derivatives $P^{(r)}(xi)$; r = 2,4,5,6 in terms of the parameters of Q. The theorem generalizes the results of [3, Theor.2] to the case of quintic spline interpolation.

Theorem 3.2. Let Q he an interpolatory quintic spline with equally spaced knots (1-1) matching the values y_i; $i = 0,1, ..., k$ at the knots. Let P be the piecewise sextic induced by Q. Then

$$P^{(r)}(x_{i-1}+ \alpha_r h) = Q^{(r)}(x_{i-1} + \alpha_r h);$$

$$r = 0,1,\ldots 5,\ i = 1,2\ldots k, \tag{3.14}$$

for all choices of the y_i, if and only if

$$\left.\begin{aligned} &\alpha_0 = 0,1,\ \alpha_1 = 0,1/2,1,\ \alpha_2 = \left(1 \pm \sqrt{1-\frac{4}{\sqrt{30}}}\right)/2, \\ &\alpha_3 = 0,1/2,1,\ \alpha_4 = (3 \pm \sqrt{3})/6\ \alpha_5 = 1/2. \end{aligned}\right] \tag{3.15}$$

Also,

$$P^{(2)}(x_i) = \begin{cases} M_Q - h^2(N_o - 2N_1 + N_2)/720 ;\ i = 0, \\ M. - h^2(N_{i-1} - 2N_i + N_{i+1})/720;\ i-1,2....k-1 , \\ M_k - h^2(N_{k-2} - 2N_{k-1} + N_k)/720;\ i=k, \end{cases} \tag{3.16}$$

$$p^{(4)}(x_i) = \begin{cases} (13N_o - 2N_1 + N_2)/12;\quad i = 0, \\ (N_{i-1} + 10N_i + N_{i+1})/12;\quad i = 1,2,\ldots, k-1, \\ (N_{k-2} - 2N_{k-1} + 13N_k) / 12;\ i=k, \end{cases} \tag{3.17}$$

$$p^{(5)}(x_i) = \begin{cases} (-3N + 4N_1 - N_2)/2h;\ i = 0, \\ (N_{i+1} - N_{i-1})/2h;\quad i = 1,2,\ldots,k-1, \\ (N_{k-2} - 4N_{k-1} + 3N_k)/2h;\ i = k, \end{cases} \tag{3.18}$$

$$P^{(6)}(X) = \begin{cases} (N - 2N_1 + N_2)/h^2, \ x \in [X_O, X_1], \\ (N_{i-1} - 2N_i + N_{i+1})/h^2, \ x \in (x_{i-1}, xi]; \ i = 1,2, \ldots, k-1, \\ (N_{k-2} - 2N_{k-1} + N_k)/h^2, \ x \in (xk-1, x_k], \end{cases} \quad (3.19)$$

where $M. = Q^{(2)}(x_i)$ and $N_i = Q^{(4)}(x_i)$.

Proof. Equation (3.6) shows that a necessary and sufficient condition for (3.14) to hold, for all choices of the y_i, is that the points

$$x_{i-1} + \alpha_r h; \ r - 0,1, \ldots, 5; \ i = 1,2, \ldots, k, \quad (3.20)$$

are respectively real zeros in $[x_{i-1}, x_i]$ of the polynomials

$$\frac{d^r\theta_i(x)}{d_x^r} \quad ; \quad r = 0,1, \ldots, 5; \quad i = 1,2, \ldots, k$$

where 9. (x) is given by (3.2) . It can be shown easily that such zeros occur only at the points defined by (3.20) and (33.5). This completes the proof of the first part of the theorem.

For the second part of the theorem, the derivatives of P give

$$\left.\begin{aligned} P^{(2)}(x_0) &= M_0 - 2h_2\Delta_1 \\ P^{(2)}(x_i) &= M_i - 2h^4\Delta_i; \ i=1,2,\ldots.k, \end{aligned}\right\} \quad (3.21)$$

$$\left.\begin{aligned} P^{(4)}(x_0) &= N + 120h^2\Delta_1, \\ P^{(4)}(x_0) &= N_0 + 120h^2\Delta_i; \ i = 1,2,\ldots,k, \end{aligned}\right\} \quad (3.22)$$

$$\left.\begin{aligned} P^{(4)}(x_o) &= (N_1 - N_o)/h - 720h\Delta_1. \\ P^{(5)}(x_i) &= (N_1 - N_{i-1}/h + 720h\Delta_i; \ i = 1,2,\ldots,k, \end{aligned}\right\} \quad (3.23)$$

and

$$P^{(6)}(x) = 1440A., \ x \in I_i; \ i = 1,2, \ldots, k, \quad (3.24)$$

where from (3.7) , by using (2.6) - (2.9), it can be shown that

$$\left.\begin{aligned} \Delta_i &= \frac{1}{1440h^2}\{Ni-1-2N_i + N_{i+1}\}; \\ & \qquad x — 1,2, \ldots, k\ 1, \\ \Delta_k &= \Delta_{k-1}. \end{aligned}\right\} \quad (3.25)$$

The results (3.16) - (3.19) follow by substituting these values of Δi in (3.21) - (3.24).

The results concerning the third, fourth, fifth and sixth derivatives of P are in fact direct consequences of the results of [3, Theor.23. To see this we observe that

$$P^{(2)}(x) = Q^{(2)}(x) + 60\Delta_i(x-x_{i-1})^2 (x-x_i)^2-2h^4 \Delta_i$$
$$= \tilde{P}(x) - 2h^4\Delta_i, \; x \in I_i; \qquad i=1,2,\ldots,k, \qquad (3.26)$$

where

$$\tilde{P}(x) = Q^{(2)}(x) + 60\Delta_i (x-x_{i-1})^2(x-x_i)^2$$
$$= Q^{(2)}(x) + \frac{1}{24h^2}\{N_{i-1} - 2N_i + N_{i+1}\} (x-x_{i-1})^2 (x-x_i)^2 , \quad (3.27)$$

and $Q^{(2)}$ is a cubic spline with knots (1 .1).. Comparison of (3.27) with the results of [3] shows that $\tilde{P}$ can be regarded as the quartic induced by the cubic spline $Q^{(2)}$. Since

$$\{p^{(2)}(x)\}^{(r)} = \tilde{P}^{(r)}(x), \quad = \; 1,2,3,4,$$

the results of Theorem 3.2 concerning the third, fourth, fifth and sixth derivatives of P follow at once from [3, Theor.2.].

From the Definition 3.1 and the fact that $P_k(x) = P_{k-1}(x)$, $X \in (x_{k-2},x_k]$, it follows that

$$P \in C^I(x_0,x_{k-2}) \cup C^\infty(x_{k-2},x_k).$$

Also, from the definition,$P^{(3)}$ is continuous in (x_o , x_k). Let $d_i(r)$; $r = 2,4,5,6$ denote the jump discontxnuitxes of $P^{(r)}$; $r = 2,4,5,6$ at each interior knot x_i ; $i = 1,2, \ldots ,k-2$. Then, using (3.7), (3.26) and the quintic spline identities (2.10) and (2.11), it can be shown that

$$d_i^{(2)} = p^{(2)}(x_i +) - p^{(2)}(x_i -1)$$
$$= - 2h^2(\Delta_{i+1} - \Delta_i)$$
$$= -\frac{1}{1296h}\{m_{i-1} + 9mi + 9_{mi+1} + m_{i+2}$$
$$-\frac{1}{9h}(-33y_{i-1} - 81y_i + 81_{i+1} + 33y_{i+2})\} ;$$

$$i = 1,2 \ldots\ldots\ldots k-2, \qquad (3.28)$$

and that

$$d_i^{(2)} = -\frac{h^2}{60} d_i^{(4)} = \frac{h3}{360} d_i^{(5)} = -\frac{h^4}{720} d_i^{(6)} ; 1,2, ..., \quad k-2. \qquad (3.29)$$

By using (3.25) the result (3.28) can be written as

$$d_i^{(2)} = -\frac{h^2}{720} \Delta^3 \; N_{i-1} ; \quad i = 1,2, ..., k-2 \qquad (3.30)$$

This implies that if P is a sextic induced by a quintic spline with end conditions

$$\Delta^3 N_i = \Delta^3 N_{k-i} = 0; \; i=0,1 \qquad (3.31)$$

then

$$d_i^{(r)} = 0 ; \; i = 1,2, k-3, k-2, \; r = 2,4,5,6,$$

i.e.

$$P \in C^\infty [(x_o, x_3) \cup C^\infty (x_{k-4}, x_k)].$$

Theorem 3.3. Let P be the piecewise sextic induced by a quintlc spline Q, and denote the jump discontinuity of $p^{(r)}$; r = 2,4,5,6 at an interior knot x_i, by $d^{(r)}{}_i$. If $y \in C^7[a,b]$ then

$$|d_i^{(r)}| \le a_r h^{1-r} \max_i |m_i - y_i^{(1)}| + 0(h^{7-r}); \; r = 2,4,5,6, \qquad (3.32)$$

where

$$a_2 = \frac{5}{334}, \; a_4 = \frac{25}{27}, \; a_5 \; \frac{50}{9}, \; a_6 = \frac{100}{9}. \qquad (3.33)$$

Proof. From (3.28), by Taylor series expansion about x_i ,

$$d_i^{(2)} = -\frac{1}{1296h} \{ (mi-1 - y_{i-1}^{(1)}) + 9(m_i - y_i^{(1)}) + 9(m_{i+1} + y_{i+1}^{(1)})$$
$$+(m_{i+2} + y_{i+2}^{(1)})\} + 0(h^5);$$
$$i = 1,2, ... , k\text{-}2. \qquad (3.34)$$

The result (3.32) - (3.33) follows at once from (3.34) and (5.29).

Theorem 3.3 shows that if the end conditions of Q are such that (1 .3) holds then

$$d_i^{(r)} \; 0(h^{7-r}) ; \quad r = 2,4,5,6 \quad . \qquad (3.35)$$

4. Numerical results

Let Q be the quintic spline with knots

$$x_i = 0.05i; \quad i = 0, 1, \dots, 20, \qquad (4.1)$$

which interpolates the function

$$y(x) = \exp(x),$$

at the knots and satisfies the end conditions (3.31) i.e., the conditions

Δ3Ni=

$$\Delta^3 N_i = \Delta^3 N_{20-i} = 0; \; i=1, \qquad (4.2)$$

where $N_i = Q^{(4)}(x_i)$. In Table 4.1 we list values of the absolute errors

$$e^{(r)}(x) = |Q^{(r)}(x) - \exp(x)|; \; r = 0,1, \dots, 5,$$

and

$$E^{(r)}(x) = |P^{(r)}(x) - \exp(x)|; \; r = 0,1, \dots, 6,$$

computed at various points of [0,1] by constructing this Q and the corresponding sextic P induced by Q. In Table 4.2 we list values of the jump discontinuities $d_i^{(2)}$, $d_i^{(4)}$, $d_i^{(5)}$ and $d_i^{(6)}$ of the derivatives $p^{(2)}$, $P^{(4)}$, $p^{(5)}$ and $P^{(6)}$ computed at a selection of interior knots. The results illustrate the improvement in accuracy obtained when P is used instead of Q, and confirm some of the theoretical results contained in Theorems 3.2 and 3.3.

TABLE 4.1

$$e^{(r)}(x) = |Q^{(r)}(x) - \exp(x)|$$

$$E^{(r)}(x) = |P^{(r)}(x) - \exp(x)|$$

	r / x.	0	I	2	3	4	5	6
$e^{(r)}(x)$	0.025	$0.14x1\bar{0}^{11}$	$0.17x1\bar{0}^{10}$	$0.84x1\bar{0}^{8}$	$0.15x1\bar{0}^{6}$	$0.97x1\bar{0}^{4}$	$0.25x1\bar{0}^{3}$	-
$E^{(r)}(x)$		$0.33x1\bar{0}^{12}$	$0.17x1\bar{0}^{10}$	$0.49x1\bar{0}^{9}$	$0.15X1\bar{0}^{6}$	$0.11x1\bar{0}^{4}$	$0.25x1\bar{0}^{3}$	$0.20x1\bar{0}^{1}$
$e^{(r)}(x)$	0.3125	$0.75x1\bar{0}^{12}$	$0.87x1\bar{0}^{10}$	$0.99x1\bar{0}^{9}$	$0.13x1\bar{0}^{5}$	$0.38x1\bar{0}^{4}$	$0.17x1\bar{0}^{1}$	-
$E^{(r)}(x)$		$0.20x1\bar{0}^{13}$	$0.26x1\bar{0}^{11}$	$0.32x1\bar{0}^{9}$	$0.53X1\bar{0}^{7}$	$0.12x1\bar{0}^{5}$	$0.68x1\bar{0}^{3}$	$0.52x1\bar{0}^{1}$
$e^{(r)}(x,$	0.4250	$0.15x1\bar{0}^{11}$	$0.29x1\bar{0}^{11}$	$0.11x1\bar{0}^{7}$	$0.34x1\bar{0}^{7}$	$0.16x1\bar{0}^{3}$	$0.15x1\bar{0}^{3}$	-
$E^{(r)}(x)$		$0.49x1\bar{0}^{13}$	$0.29x1\bar{0}^{11}$	$0.29x1\bar{0}^{9}$	$0.34x1\bar{0}^{7}$	$0.40x1\bar{0}^{5}$	$0.15x1\bar{0}^{3}$	$0.38x1\bar{0}^{1}$
$e^{(r)}(x)$	0.7125	$0.11x1\bar{0}^{11}$	$0.13x1\bar{0}^{9}$	$0.14x1\bar{0}^{8}$	$0.19x1\bar{0}^{5}$	$0.57x1\bar{0}^{4}$	$0.25x1\bar{0}^{1}$	-
$E^{(r)}(x)$		$0.14x1\bar{0}^{13}$	$0.44x1\bar{0}^{11}$	$0.48x1\bar{0}^{9}$	$0.80x1\bar{0}^{7}$	$0.20x1\bar{0}^{5}$	$0.10x1\bar{0}^{2}$	$0.78x1\bar{0}^{1}$
$e^{(r)}(x)$	0.850	$0.13x1\bar{0}^{12}$	$0.92x1\bar{0}^{11}$	$0.20x1\bar{0}^{7}$	$0.69x1\bar{0}^{7}$	$0.49x1\bar{0}^{3}$	$0.58x1\bar{0}^{1}$	-
$E^{(r)}(x)$		$0.13x1\bar{0}^{12}$	$0.92x1\bar{0}^{11}$	$0.19x1\bar{0}^{10}$	$0.69x1\bar{0}^{7}$	$0.34x1\bar{0}^{6}$	$0.50x1\bar{0}^{3}$	$0.13x1\bar{0}^{2}$
$e^{(r)}(x)$	0.9125	$0.12x1\bar{0}^{11}$	$0.I5x1\bar{0}^{9}$	$0.17x1\bar{0}^{8}$	$0.24x1\bar{0}^{5}$	$0.70x1\bar{0}^{4}$	$0.31x1\bar{0}^{1}$	-
$E^{(r)}(x)$		$0.99x1\bar{0}^{13}$	$0.11x1\bar{0}^{10}$	$0.52x1\bar{0}^{9}$	$0.12x1\bar{0}^{6}$	$0.36x1\bar{0}^{5}$	$0.12x1\bar{0}^{2}$	$0.83x1\bar{0}^{1}$
$e^{(r)}(x)$	0.9875	$0.22x1\bar{0}^{11}$	$0.18x1\bar{0}^{9}$	$0.70x1\bar{0}^{8}$	$0.21x1\bar{0}^{5}$	$0.36x1\bar{0}^{4}$	$0.34x1\bar{0}^{1}$	-
$E^{(r)}(x)$		$0.86x1\bar{0}^{12}$	$0.19x1\bar{0}^{10}$	$0.57x1\bar{0}^{8}$	$0.45x1\bar{0}^{6}$	$0.31x1\bar{0}^{4}$	$0.21x1\bar{0}^{2}$	$0.11x10^{\circ}$

TABLE 4.2

$$d_i^{(r)} = p^{(r)}(x_i+) - {}^{(r)}(x_i-)$$

x_i	$d_i^{(2)}$	$d_i^{(4)}$	$d_i^{(5)}$	$d_i^{(6)}$
x_4	$-0.44\text{x}10^{-9}$	$0.11\text{x}10^{-4}$	$-0.13\text{x}10^{-2}$	$0.50\text{x}10^{-1}$
x_7	$-0.64\text{x}10^{-9}$	$0.\ 15\text{x}10^{-4}$	$-0.18\text{x}10^{-2}$	$0.74\text{x}10^{-1}$
x_{10}	$-0.74\text{x}10^{-9}$	$0.18\text{x}10^{-4}$	$-0.21\text{x}10^{-2}$	$0.85\text{x}10^{-1}$
x_{13}	$-0.81\text{x}10^{-9}$	$0.19\text{x}10^{-4}$	$-0.23\text{x}10^{-2}$	$0.93\text{x}10^{-1}$
x_{16}	$-0.\ 15\text{x}10^{-8}$	$0.36\text{x}10^{-4}$	$-0.43\text{x}10^{-2}$	0.17

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