TR/12/82 November 1982

Solitary and Travelling Waves in a Rod

by

R. Saxton

<u>Solitary and Travelling Waves in a Rod</u>

We are concerned with the existence and partial analysis of longitudinal travelling waves in a thin rod, with a later application to a growth condition in incompressible elasticity. We consider only solutions $u(x, t) = \phi(z), z - ct$, which have the form of steady travelling waves for the equation of motion, which is derived most simply by means of Hamilton's principle (cp. [4],[5],[6]) applied to the functional

$$\int_0^T (k - I) dt. \tag{1}$$

Here, the total kinetic energy is given by

$$K(t) \quad \tfrac{1}{2} \int_{-\infty}^{\infty} [u_t^2(x,t) + \beta_t^2(u_x(x,t))] \; dt, \tag{2}$$

where $\beta(.)$ is a measure of the lateral deformation undergone during any elongation or contraction of the rod which is assumed to remain straight at all times. $u(x, t)$ is the longitudinal displacement at time t of a material point at x measured from some chosen point along the rod, which is now taken to be of infinite length. We have taken the density in the current and reference configurations to be unity. In terms of u_x, by adopting the limit of incompressibility (Poisson's ratio $\nu \to \tfrac{1}{2}$ in the special case of small deformation) β is given here by

$$\beta(u_x) = \frac{1}{(1 + u_x)} {}^{\frac{1}{2}} - 1, \tag{3}$$

although we need not assume this. To avoid material inversion or

flattening in general, it is only necessary that u_x, $\beta > -1$. I, the total strain energy, is of the form

$$I(t) = \int_{-\infty}^{\infty} W(u_x(x,t), \beta(u_x(x,t)))\, dx \tag{4}$$

where W(. , .) is the strain energy function.

On carrying out the variations, we obtain from (1) the Euler equation

$$u_{tt} - \left(\beta' \beta'' u_{xt}^2 + \beta'^2 u_{xtt}\right)_x - \left(W' + W^{`} \beta'\right)_x = 0, \tag{5}$$

where subscripts denote differentiation with respect to x and t , and superscripts ' , ` denote differentiation with respect to dependent variables u_x and β respectively.

We now consider the substitution in (5) of

$$u(x,t) = \phi(z),\ z = x - ct,\ c \in IR, \tag{6}$$

giving

$$c^2 \frac{d^2\phi}{dz^2} - \frac{d}{dz}\left(\beta'^2\left(\frac{d\phi}{dz}\right).c^2 \frac{d^3\phi}{dz^3}\right)$$

$$- \frac{d}{dz}\left(\beta'\left(\frac{d\phi}{dz}\right).\beta''\left(\frac{d\phi}{dz}\right).c^2\left(\frac{d^2\phi}{dz^2}\right)^2 + W'\left(\frac{d\phi}{dz}, \beta\frac{d\phi}{dz}\right)\right)$$

$$+ W'\left(\frac{d\phi}{dz}, \beta\left(\frac{d\phi}{dz}\right)\right).\beta'\left(\frac{d\phi}{dz}\right)\Bigg) = 0. \tag{7}$$

On further substituting $\psi = \frac{d\phi}{dz}$, $x = c\beta'(\psi)\frac{d\psi}{dz}$ and integrating (7) with respect to z, we obtain

$$-c^2\psi + c\beta'(\psi)\frac{dX}{dz} + W' + W^{`}\beta'(\psi) = 0 \tag{8}$$

where we have omitted the constant of integration.

At some point z_0 the initial data for (8) is given by

$$\psi(z_0) = \psi_0, \frac{d\psi}{dz}(z_0) = \psi_1, x(z_0) = c\beta'(\psi_0)\psi_1, \tag{9}$$

and on multiplying (8) by $\frac{d\psi}{dz}$ and integrating with respect to z ,

we have the energy equation $-\frac{c^2}{z}\psi^2 + \frac{1}{2}X^2 + W(\psi, \beta(\psi)) = E,$ (10)

where E is a constant depending only on the values of ψ_0, ψ_1 and c .

Our analysis of equation (7) is in the phase plane.

(For a related discussion in another problem, c.p. Ball [1], Calderer [2]).

We therefore use (8) and (9) to obtain the equivalent first order system

$$\frac{d}{dz}\underset{\sim}{x}(z) = \underset{\sim}{m}(\psi, X) \tag{11}$$

where

$$\underset{\sim}{x} = \binom{\psi}{X} \quad , \tag{12}$$

$$\underset{\sim}{m} :]-1, \infty[\times \mathbb{R} \to \mathbb{R}^2 \quad , \tag{13}$$

$$m(\psi, X) = \frac{1}{c\beta'(\psi)}\begin{pmatrix} X \\ c^2\psi - w' - w`\beta` \end{pmatrix} \quad , \tag{14}$$

and

$$x(z_0) = \binom{\psi_0}{X_0} = \begin{pmatrix} \psi_0 \\ c\beta'(\psi_0)\psi_1 \end{pmatrix} \tag{15}$$

We now present a result on local existence of trajectories $\underset{\sim}{x}$:

<u>Proposition 1</u> Assume $\underset{\sim}{m}(\psi, x)$ is locally Lipschitz continuous

on its domain of definition. Then the initial value problem (12) - (15) has a unique maximally defined solution

$$\underset{\sim}{x}(z) \in C^1(I) \times C^1(I) \text{ where } I =]z_0 - \alpha, z_0 + \alpha[$$

for some $\alpha \in \mathbb{R}$. The solution $\underset{\sim}{x}(z)$ satisfies

$$x(z_0) = \binom{\psi_0}{X_0}.$$

Proof For $m(\psi,X)$ to be locally Lipschitz, it is enough for B (.), W'(.,.) and W` to be locally Lipschitz continuous in each variable, and for β (.) to be sign definite. This implies we require the existence of some $0<\Gamma<\infty, \Gamma=\Gamma(\in,R)$ such that for any

$$(\psi_1, X_1), W`(\psi_2, X_2) \in]-1+\in, R[\times]-R, R[, \in, R>0,$$
$$\left|W'(\psi_1, X_1) - W`(\psi_2, X_2)\right| \leq \Gamma(\in, R)\left|\psi_1-\psi_2\right| + \left|X_1 - X_2\right| \quad etc.,$$

and $\alpha < 0$ such that $\left|\beta'\right| > \alpha^{\frac{1}{2}}$.

We may then apply standard results in ordinary differential equations (e.g. Friedrichs [3]) to obtain local existence and uniqueness through $\binom{\psi_0}{X_0}$.

Referring to equation (10) we define

$$V(\psi) = -\frac{c^2}{2}\psi^2 + W(\psi, \beta(\psi)) \tag{16}$$

and denote

$$\sigma'(\psi) = c^2 + \frac{d^2V}{d\psi^2}$$
$$\equiv W'' + 2w``\beta' + W`\beta''. \tag{17}$$

We may characterize the equilibrium points for (11) - (14) by the following proposition.

Proposition 2 Let c be given. Then the equilibrium solutions $X(z)=0, \psi(z)=\overline{\psi}, z\in\mathbb{R}$ of (11) are the zeros of the function $c^2\psi - \sigma(\psi)$ where $\sigma(\psi) = c^2\psi + \frac{dv}{d\psi}$.

Further, for $\overline{\psi}$ satisfying $c^2 < \sigma'(\overline{\psi}), (\overline{\psi},0)$ is a centre and $v(\overline{\psi})$ is a relative minimum. When $\overline{\psi}$ satisfies $,c^2 < \sigma'(\overline{\psi}), (\overline{\psi},0)$ is a saddle point and $v(\overline{\psi})$ is a relative maximum.

Proof Follows from (14) and its linearization

$$m(\psi, X) \simeq \frac{1}{c\beta},(\overline{\psi})\begin{pmatrix} 0 & 1 \\ c^2 - \sigma(\overline{\psi}) & 0 \end{pmatrix}\begin{pmatrix} \psi\ \psi \\ X \end{pmatrix}$$

In a neighbourhod of $(\overline{\psi},0)$

Remark In addition to the above cases we may also have $c^2 = \sigma'(\overline{\psi})$ which corresponds to a point of inflection of V.

We now assume that for each, c , there exists a finite number n of Zeros

$$\{\overline{\psi}_i, 0 \le i \le n\}\ \overline{\Psi}_i \le \overline{\Psi}_{i+1} \tag{18}$$

to the function $c^2\psi = \sigma'(\psi)$.

Further we consider two possible hypotheses for σ :-

(H1) As $\psi \to +\infty$, $\dfrac{\sigma(\psi)}{\psi} \sim 1$;

(H2) As $\psi \to +\infty$, $\dfrac{\sigma(\psi)}{\psi} \sim \psi^B$,

where B > 0. (HI) and (H2) are compatible with the Lipschitz assumptions made earlier. We use these conditions to investigate the region $\psi \ge 0$ of the phase plane and obtain the next two results.

Proposition 3 Let $\psi \ge 0$ throughout, the let (ψ_0, ψ_1) be chosen so that

$$E(\psi_0, \psi_1) \ne V(\psi_i), 0 \le i \le n \ .$$

Then a) when $c^2 \le 1$ and either (HI) or (H2) hold, the trajectory through (ψ_0, ψ_1) is periodic with period

$$\tau = \sqrt{2}\int_{J_0}^{J_1} \frac{c\beta(\psi)d\psi}{(E - V(\psi))^{\frac{1}{2}}} \tag{19}$$

where $J_0 < J_1$ and $(J_0, 0), (J_1, 0)$ are the intersections of the trajectory with the axis X = 0.

b) When $c^2 > 1$ and (H2) holds, we again have the result of a).

c) When $c^2 > 1$ and (HI) holds, no periodic orbits exist through any point $(\psi, 0)$ with $\psi > \overline{\psi}_n$.

Proof From (10) and (18), and the Theorem of Poincaré-Bendixon.

Corollary When (ψ_0,ψ_1) are chosen so that

$$E(\psi_0,\psi_1) = V\left(\overline{\psi}_i\right) \quad \text{for some of } i = 0,1,\ldots,n$$

then either A) (ψ_0,ψ_1) is a centre, or when $c^2 < 1$ and (HI) and (H2) hold or $c^2 > 1$ and (H2) holds, then

B) i) for $\psi_1 \geq 0$ we have

$$\lim_{z \to +\infty} \psi(z) = \overline{\psi}_i, \quad \lim_{z \to +\infty} x(z) = 0$$

where $\psi_i \leq \psi_0$ is the closest saddlepoint to the left of $(\psi_0, 0)$

ii) for $\psi_1 \leq 0$

$$\lim_{z \to +\infty} \psi(z) = \overline{\psi}_j, \quad \lim_{z \to +\infty} x(z) = 0$$

where $\psi_j \geq \psi_0$ is the closest saddle point to the right of $(\psi_0, 0)$.

Remark The case B) above is that which provides for the existence of solitary waves. Note that since centres and saddle points alternate (we have excluded the case $c^2 = \sigma'(\overline{\psi})$), given the same ψ_0 in B i) and ii) above implies $\overline{\psi}_j = \psi_{i+2}$.

We conclude by looking at a case for $-1 < \psi \leq 0$, with $\beta(\psi)$ given by (3).

Hypothesis (H3) For $p \in]-1,\infty[$ as $q \to +\infty$

$$w(p,q) \sim \delta_q{}^{\gamma}, \gamma \in \mathbb{R}, \delta > 0.$$

Proposition 4 Under hypothesis (H3), a sufficient condition preventing flattening of the incompressible rod is $\gamma > 6$.

Proof With B given by (3), (10) becomes

$$-\,{}^{c^2}\!/_2\, \psi^2(1+\psi)^2 + {}^{c^2}\!/_8\, \psi'^2 + w(\psi,(1+\psi)^{-\frac{1}{2}}(1+\psi)^3 = E.(1+\psi)^2 \qquad (20)$$

and so as $\psi \to -1$

$$c^2/_8\ \psi'^2 \sim (1+3)^3(E+c^2) - \delta(1+\psi)^{3-\lambda/2}$$

and the result follows since $\delta > 0$ and it is impossible for

$$u_x = \psi = -1.$$

References

[1] J. Ball, Discontinuous Equilibrium Solutions and Cavitation in Nonlinear Elasticity, Preprint.

[2] C. Calderer. The Dynamic Behaviour of Nonlinear Elastic Spherical Shells, J. Elasticity, to appear.

[3] K.O. Friedrichs, "Advanced Ordinary Differential Equations", Gordon and Breach Science Publishers, Inc., New York (1965).

[4] W. A. Green, "Dispersion Relations for Elastic Waves in a Bar", in Progress in Solid Mechanics I, Ed. Sneddon and Hill.

[5] W. Jaunzemis, "Continuum Mechanics", New York MacMillan (1967).

[6] A. E. H. Love, Theory of Elasticity (4th ed.) 278, Cambridge University Press (1927).

XB 2356454 7