TR/07/86 June 1986

Plane wave diffraction by a rational wedge

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Abstract

In this paper new expressions for the acoustic field produced when a plane wave source of sound is diffracted by a soft, hard or mixed soft/hard wedge whose angle can be expressed as a rational multiple of π are given. The solution is expressed in terms of geometrical acoustic source terms and real integrals which represent the diffracted field. The expressions are in a form which allows easy calculation of the acoustic field. Uniformly valid expressions for the far field are also given for all values of the angular variable. The general result obtained includes as special cases, Sommerfeld' s solution for diffraction by a half plane, Reiche's result for the diffraction by a right angled wedge, and a new representation for the solution of the problem of diffraction by a mixed soft/hard half plane.

1. Introduction

The exact solution of the problem of diffraction by a soft or hard wedge of any angle in the two dimensional case of plane acoustic wave incidence is due to Macdonald (1902). The solution was given in the form of a complex contour integral, which was obtained by summing the Fourier series representation of the Green's function. For the special case of a wedge which becomes a half plane, Macdonald showed how the contour integral could be reduced to an elegant form involving real integrals corresponding to Sommerfeld's solution (1896). Sommerfeld obtained the solution for the half plane problem by using the relatively new concept of Riemann surfaces and multivalued functions. Sommerfeld (1901), p.38, indicated briefly how, by his method, the solution in the form of a complex integral could be obtained for a wedge of angle $p\pi / q$ (p, q being positive integers); a Riemann surface of p sheets being required. Sommerfeld also pointed out that when the angle of the wedge is an irrational multiple of π, a Riemann surface of an infinite number of sheets can be employed in the same way. Wiegrefe (1912) gave a solution for the problem of the diffraction of plane waves by a wedge of any angle by this method. His solution was in the form of a complex integral on which he applied asymptotic methods to obtain the far field approximation in the form of geometrical acoustic terms and the cylindrical diffracted wave radiating from the edge of the wedge. Since then wedge problems have been dealt with in a similar manner, that is, the solution has been expressed explicitly in terms of the geometrical acoustic terms and the diffracted field for the situation where the observation point is well removed from the edge of the wedge. Expressions which express the solution exactly in the form of geometrical acoustic terms and real integrals representing the diffracted field are not known except for two special wedge angles. These are the half plane problem, Sommerfeld (1896) and the right angled wedge problem, Reiche (1912). The advantage of this representation is that the solution is readily interpreted physically, and is valid for all positions of the observation point. This representation also admits uniform asymptotic solutions for the total field (valid for all observation angles) to be obtained simply.

The Sommerfeld technique, which is a combination of the physical method of images and the mathematical theory of Riemann surfaces, has been considered abstruse, because a solution is derived heuristically, using the method of images in various Riemann sheets. Indeed Reiche (1912), who gives a solution for the diffraction of a plane wave by a right angled wedge (in the form of geometrical acoustic terms and real integrals which represent the diffracted field) thanks Sommerfeld for supplying the appropriate 'ansatz' to obtain the solution! Carslaw (1920), was an early convert to Sommerfeld’s method, but later gave up using the idea of Riemann surfaces, and instead used the more modern approach of using periodic Green's functions. However, Sommerfeld's method gave exact solutions which made apparent the geometrical acoustic and diffracted wave contributions for the half plane and the right angle wedge. Clearly in the hands of Sommerfeld the technique was very powerful. Sommerfeld did not, however, give an explicit solution in the same form for the rational wedge, although he does seem to have been aware of the qualitative form the solution would take, see Frank and Von Mises (1943) p.853. It is conjectured that the reason for this is because his method would involve constructing and manipulating rather complicated trigonometrical identities.

In this paper we derive the explicit solution for the problem of diffraction by a rational wedge of angle $p\pi / q$, The solution is expressed in terms of geometrical acoustic and real integrals representing the diffracted field. Our approach is to avoid the Sommerfeld use of Riemann surfaces and simply use the periodic Green's function for an arbitrary angle wedge. Then we consider the special case of a rational wedge. It is then shown, by means of some trigonometrical identities, how the complex contour integral can, in this case, be reduced to source terms and real integrals which are convenient for calculations. We remark that recently there has been much work done on uniform asymptotics for the wedge, see Ciarkowski et al (1984). The results presented here offer an alternative approach, in that a wedge of any angle can be approximated to any order of accuracy by a rational wedge of angle $p\pi / q$, and the real integrals obtained in this paper can be asymptotically evaluated without difficulty. Finally the present results offer an infinite number of exact wedge problem solutions which can be used for comparison with various approximate techniques.

In section 2 we shall give the known periodic Green's function for a plane wave source and a wedge of arbitrary angle. The Green's function is in the form of a complex contour integral. Some of the important properties of the Green's function are stated, and appropriate expressions for the solution of the problems of diffraction by a soft, hard, or one face soft one face hard, problem are given for arbitrary angle in terms of this Green's function. In section 3 we shall consider in detail the special case of evaluating the complex contour integral representation of the Green's function for a wedge whose angle can be expressed as a rational multiple of π. In section 4 we shall give expressions for the Green's function for special cases of wedge angles. Finally in section 5 we shall give solutions to some specific problems in diffraction theory which are special cases of the more general result obtained in section 4. The first problem is the classical solution of Sommerfeld for diffraction by a soft or hard half plane. The second is Rieche's (1912) solution for diffraction by a right angle wedge. The solution obtained here agrees with Reiche's result, and is more compact. The last solution is a new result for the problem of diffraction by a soft/hard half plane, see Rawlins (1975).

In order not to disrupt the flow of the arguments in the main body of the paper, various proofs of results needed have been placed in appendixes at the end of the paper.

2. Periodic Green's function for a wedge.

The periodic Green's function $G_\alpha(r,\theta,\theta_0;k)$ for a two dimensional wedge situated in the space $0 < r < \infty$, $2\pi - \alpha \le \theta \le 2\pi$, see fig 1. where (r, θ) are cylindrical polar coordinates has been shown by Carslaw (1920) to be given by

$$G_\alpha(r,\theta,\theta_0;k)=\frac{1}{2\alpha i}\int_c e^{ikr\cos\zeta}\frac{\sin\zeta/\alpha}{\cos\zeta/\alpha-\cos(\theta-\theta_0)/\alpha}d\zeta \tag{1}$$

where the contour of integration c is such that the starting point is given by $i\infty + c_1$ and the termination point is given by $i\infty + c_2$ where $-\pi < c_1 < 0$, $\pi < c_2 < 2\pi$, see fig 2.

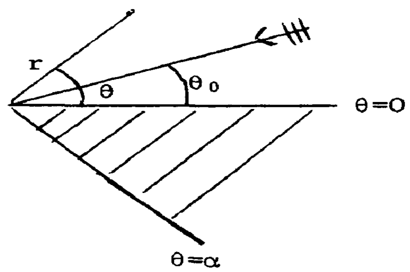

fig 1.

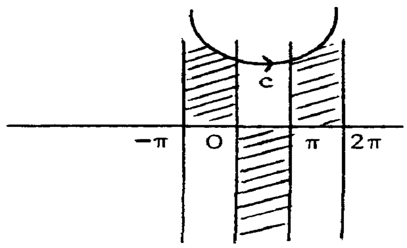

fig 2.

It has also been shown by Carslaw that $G_\alpha(r, \theta, \theta_0; k)$ has the following properties:

$$\left.\begin{aligned}
&\text{(i)}\quad (\nabla^2+k^2)G_\alpha=0, \text{ where } \nabla^2\equiv\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2};\\
&\text{(ii)}\quad G_\alpha(r,\theta,\theta_0;k)=G_\alpha(r,\theta+2\alpha;k);\\
&\text{(iii)}\quad G_\alpha(r,\theta,\theta_0;k) \text{ is finite and continuous;}\\
&\text{(iv)}\quad G_\alpha(r,\theta,\theta_0;k)\sim e^{ikr\cos(\theta-\theta_0)} \quad \text{as } r\to\infty \ |\theta-\theta_0|<\pi\\
&\qquad\qquad \sim 0 \qquad\qquad \text{as } r\to\infty \ |\theta-\theta_0|>\pi
\end{aligned}\right\} \tag{2}$$

The Green's function given above enables one to derive solutions to various diffraction problems in wedge shaped regions. To be specific we shall discuss acoustic waves. The solution u_h or u_s of the problem of a plane wave:

$$u_0 = e^{ikr\cos(\theta-\theta_0)} \tag{3}$$

(with harmonic time variation e^{iwt} assumed, but not shown explicitly, in the rest of the paper) diffracted by a rigid wedge ($\partial u_h / \partial\theta = 0$ for $\theta = 0$ and $\theta = \alpha$) or a soft wedge ($u_s = 0$ for $\theta = 0$ and $\theta = \alpha$) is given by

$$u_h = G_\alpha(r,\theta,\theta_0;k) + G_\alpha(r,\theta,-\theta_0;k), \tag{4}$$

or

$$u_s = G_\alpha(r,\theta,\theta_0;k) - G_\alpha(r,\theta,-\theta_0;k), \tag{5}$$

respectively.

The solution $u_{h/s}$ of the problem of a plane wave (3) diffracted by a wedge whose face $\theta = 0$ is rigid ($\partial u_{h/s}/\partial\theta = 0$) and whose face $\theta = \alpha$ is soft ($u_{h/s} = 0$) is given by

$$u_{h/s} = G_{2\alpha}(r,\theta,\theta_0;k) + G_{2\alpha}(r,\theta,-\theta_0;k) - G_{2\alpha}(r,\theta,2\alpha-\theta_0;k) - G_{2\alpha}(r,\theta,-2\alpha+\theta_0;k) \tag{6}$$

3. Plane wave Green's function for a rational wedge.

If the wedge angle α is a rational multiple of π i.e. $p\pi/q$ where p and q are integers the plane wave Green's function (1) becomes

$$G_{\frac{p\pi}{q}}(r, \theta, \theta_0; k) = \frac{1}{2\pi ip}\int_c e^{ikr\cos\zeta} \frac{q\sin(\zeta q/p)}{\cos(\zeta q/p) - \cos((\theta-\theta_0)q/p)} d\zeta \tag{7}$$

where the contour of integration is as shown in fig 2. By using the identity

$$\frac{q\sin(\zeta q/p)}{\cos(\zeta q/p) - \cos((\theta-\theta_0)q/p)} = \sum_{m=0}^{q-1} \frac{\sin(\zeta/p)}{\cos(\zeta/p) - \cos\left(\frac{\theta-\theta_0}{p} + \frac{2\pi m}{q}\right)},$$

see appendix A, we can write (7) in the form

$$G_{\frac{p\pi}{q}}(r,\theta,\theta_0;k) = \sum_{m=0}^{q-1} I_p(kr, \theta-\theta_0+2\pi mp/q), \tag{8}$$

Where $$I_p(kr,\psi) = \frac{1}{2\pi i}\int_c e^{ikr\cos\zeta} \frac{\frac{1}{p}\sin(\zeta/p)}{\cos(\zeta/p)-\cos(\psi/p)} d\zeta. \tag{9}$$

The identity

$$\frac{\frac{1}{p}\sin(\zeta/p)}{\cos(\zeta/p)-\cos(\psi/p)} = \frac{a_{p-1}(\psi)\sin(\zeta/p)+\sum_{n=0}^{p-2} a_n(\psi)\left\{\sin(\zeta(p-n)/p)-\sin\left(\zeta\frac{(p-2-n)}{p}\right)\right\}}{(\cos\zeta-\cos\psi)}$$

where $a_n(\psi)=\sin((n+1)\psi/p)/\{p\sin(\psi/p)\}$,

derived in appendix B, enables us to rewrite the integral $I_p(kr,\psi)$ given by expression (3) in the form

$$I_p(kr,\psi) = a_{p-1}(\psi)\frac{1}{2\pi i}\int e^{ikr\cos\zeta}\frac{\sin(\zeta/p)}{(\cos\zeta-\cos\psi)}d\zeta$$

$$+\sum_{n=0}^{p-2} a_n(\psi)\frac{1}{2\pi i}\int_c e^{ikr\cos\zeta}\frac{\{\sin(\zeta(p-n)/p)-\sin(\zeta(p-2-n)/p)\}}{(\cos\zeta-\cos\psi)}d\zeta. \tag{10}$$

Multiplying both sides of the equation (10) by $e^{-ikr\cos\psi}$ and then differentiating the resulting expression with respect to kr gives

$$\frac{\partial}{\partial(kr)}\left[e^{-ikr\cos\psi}I_p(kr,\psi)\right]=e^{-ikr\cos\zeta}a_{p-1}(\psi)\frac{1}{2\pi}\int_c e^{ikr\cos\zeta}\sin(\zeta/p)d\zeta$$

$$+e^{-ikr\cos\zeta}\sum_{n=0}^{p-2}a_n(\psi)\frac{1}{2\pi}\int_c e^{ikr\cos\zeta}\{\sin(\zeta(p-n))-\sin(\zeta(p-2-n)/p)\}d\zeta,$$

differentiation under the integral sign being permissible since the resulting expressions are uniformly convergent with respect to kr.

We now use the easily proved result, Erdelyi (1953),(pp.19-21 (23))

$$\frac{1}{2\pi}\int_c e^{ikr\cos\zeta}\sin\nu\zeta d\zeta = \frac{e^{-i\nu\pi/2}}{2}\sin\nu\pi H_\nu^{(2)}(kr), \tag{11}$$

where $H_\nu^{(2)}(kr)$ is the Hankel function of the second kind.

Hence

$$\frac{\partial}{\partial(kr)}\left[e^{-ikr\cos\psi}I_p(kr,\psi)\right] = \frac{1}{2}a_{p-1}(\psi)\sin\left(\frac{\pi}{p}\right)e^{-ikr\cos\psi}H_{\frac{1}{p}}^{(2)}(kr)$$

$$+\frac{1}{2i}\sum_{n=1}^{p-2}\left\{a_n(\psi)\sin\left(\frac{n\pi}{p}\right)e^{in\pi/(2p)-ikr\cos\psi}H_{\frac{(p-n)}{p}}^{(2)}(kr)\right.$$

$$- a_{n-1}(\psi)\sin((n+1)\pi/p)e^{i(n+1)\pi/(2p) - ikr\cos\psi} H^{(2)}_{\frac{(p-1-n)}{p}}(kr) \Big\} . \qquad (12)$$

It is shown in the appendix C that

$$I_p(\infty,\psi) = \sum_N H[\pi - |\psi + 2\pi pN|]\, e^{ikr\cos\zeta}, \qquad (13)$$

where $H[x] = \begin{pmatrix} 1 & x > 0 \\ \frac{1}{2} & x = 0 \\ 0 & x < 0 \end{pmatrix}$ is the Heaviside step function, and summation is over all integer values of N which satisfy the inequality $\pi \geq |\psi + 2\pi pN|$.

Thus integrating equation (12) with respect to kr, and using the result (13) gives

$$I_p(kr,\psi) = \sum_N H[\pi - |\psi + 2\pi pN|]e^{ikr\cos\psi}$$

$$+ \frac{1}{2}e^{ikr\cos\psi}a_{p-1}(\psi)\sin\left(\frac{\pi}{p}\right)e^{-i\pi/(2p)}\int_\infty^{kr} e^{-ix\cos\psi}H^{(2)}_{\frac{1}{p}}(x)dx$$

$$+ \frac{e^{ikr\cos\psi}}{2i}\sum_{n=1}^{p-1}\Big\{a_n(\psi)\sin\left(\frac{n\pi}{p}\right)e^{in\pi/(2p)}\int_\infty^{kr} e^{-ix\cos\psi}H^{(2)}_{\frac{(p-n)}{p}}(x)dx$$

$$- a_{n-1}(\psi)\sin((n+1)\pi/p)e^{i(n+1)\pi/(2p)}\int_\infty^{kr} e^{-ix\cos\psi}H^{(2)}_{\frac{(p-1-n)}{p}}(x)dx\Big\}. \qquad (14)$$

The integrals appearing in the above expression can be shown to converge, see appendix D. The integrals are a generalisation of Schwarz functions, see Luke (1962), chapter 10, where extensive properties and asymptotic results are given.

Substituting the expression (14) into the expression (8) gives

$$G_{\frac{p\pi}{q}}(r,\theta,\theta_0;k) = \sum_{m=0}^{q-1}\sum_N H[\pi - |\theta - \theta_0 + 2\pi mp/q + 2\pi pN|]e^{ikr\cos(\theta-\theta_0+2\pi mp/q)}$$

$$+ \frac{1}{2}\sum_{m=0}^{p-1} e^{ikr\cos(\theta-\theta_0+2\pi mp/q)}a_{p-1}(\theta-\theta_0+2\pi mp/q)\sin\left(\frac{\pi}{p}\right)e^{-i\pi/(2p)},$$

$$\int_\infty^{kr} e^{ix\cos(\theta-\theta_0+2\pi mp/q)}H^{(2)}_{\frac{2}{p}}(x)dx$$

$$+ \frac{1}{2i}\sum_{m=0}^{q-1}\sum_{n=1}^{p-2} e^{ikr\cos(\theta-\theta_0+2\pi mp/q)}\Big\{a_n(\theta-\theta_0+2\pi mp/q)\sin\left(\frac{n\pi}{p}\right)e^{in\pi/(2p)}\Big\}$$

$$\int_\infty^{kr} e^{-ix\cos(\theta-\theta_0+2\pi mp/q)}H^{(2)}_{\frac{(p-n)}{p}}(x)\,dx$$

$$-a_{n-1}(\theta-\theta_0+2\pi mp/q)\sin((n+1)\pi/p)e^{i(n+1)\pi/(2p)}$$

$$\left.\cdot\int_{\infty}^{kr} e^{-ix\cos(\theta-\theta_0+2\pi mp/q)} H^{(2)}_{\frac{(p-1-n)}{p}}(x)dx\right\}, \tag{15}$$

where the summation over N is for those integer values of N which satisfy

$$-\pi<\theta-\theta_0+2\pi mp/q+2\pi pN<\pi. \tag{16}$$

More explicitly we can write:

$$G_{\frac{p\pi}{q}}(r,\theta,\theta_0;k)=\sum_{m=0}^{q-1}\sum_N H\left[\pi-\left|\theta-\theta_0+2\pi mp/q+2\pi pN\right|\right]e^{ikr\cos(\theta-\theta_0+2\pi mp/q)}$$

$$+\frac{1}{2p}\sum_{m=0}^{q-1}e^{ikr\cos(\theta-\theta_0+2\pi mp/q)-i\pi/(2p)}\frac{\sin(\theta-\theta_0+2\pi mp/q)\sin(\pi/p)}{\sin(\theta-\theta_0+2\pi mp/q)/p)}.$$

$$\cdot\int_{\infty}^{kr} e^{-ix\cos(\theta-\theta_0+2\pi mp/q)} H^{(2)}_{\frac{1}{p}}(x)dx$$

$$+\frac{1}{2ip}\sum_{m=0}^{q-1}\sum_{n=1}^{p-2}\frac{e^{ikr\cos(\theta-\theta_0+2\pi mp/q)}}{\sin((\theta-\theta_0+2\pi\pi mp/q)/p}\Bigg\{$$

$$+e^{in\pi/(2p)}\sin((n+1)(\theta-\theta_0+2\pi mp/q)/p)\sin(n\pi/p).$$

$$\cdot\int_{\infty}^{kr} e^{-ix\cos(\theta-\theta_0+2\pi mp/q} H^{(2)}_{\frac{(p-n)}{p}}(x)dx$$

$$-e^{i(n+1)\pi/(2p)}\sin(n(\theta-\theta_0+2\pi mp/q)/p)\sin((n+1)\pi/p).$$

$$\cdot\int_{\infty}^{kr} e^{-ix\cos(\theta-\theta_0+2\pi mp/q)} H^{(2)}_{\frac{(p-1-n)}{p}}(x)dx$$

$$\Bigg\}, \tag{17}$$

where the summation over N is for all integer values of N which can make the argument of the Heaviside step function non negative. Thus the solution $u(r,\theta)$ to the problem of diffraction of the plane wave $u_0(r,\theta)=e^{ikr\cos(\theta-\theta_0)}$ by a soft u_s or hard u_h wedge of open angle $\alpha=p\pi/q$ is given by

$$u_s(r,\theta) = G_{\frac{p\pi}{q}}(r,\theta,\theta_0;k) - G_{\frac{p\pi}{q}}(r,\theta,-\theta_0;k)$$

and (18)

$$u_h(r,\theta) = G_{\frac{p\pi}{q}}(r,\theta,\theta_0;k) + G_{\frac{p\pi}{q}}(r,\theta,-\theta;k)$$

respectively, where $G_{\frac{p\pi}{q}}$ is given by the expression (17). Similarly the solution to the problem of diffraction of the plane wave $u_0(r,\theta) = e^{ikr\cos(\theta-\theta_0)}$ by a soft/hard $u_{s/h}(r,\theta)$ of open angle $\alpha = p\pi/q$ is given by

$$u_{s/h}(r,\theta) = G_{\frac{2p\pi}{q}}(r,\theta,\theta_0;k) + G_{\frac{2p\pi}{q}}(r,\theta,-\theta_0;k)$$
$$- G_{\frac{2p\pi}{q}}(r,\theta,\frac{2p\pi}{q}+\theta_0;k) - G_{\frac{2p\pi}{q}}(r,\theta,\frac{2p\pi}{q}-\theta_0;k) \qquad (19)$$

where $G_{\frac{2p\pi}{q}}(r,\theta,\theta_0;k)$ is given by the expression (17) with p replaced by 2p.

By using the asymptotic results of appendix D we have for $kr\to\infty$:

$$G_{\frac{p\pi}{q}}(r,\theta,\theta_0;k) = \sum_{m=0}^{q-1}\sum_{N} H\left[\pi - \left|\theta-\theta_0 + 2\pi\, mp/q + 2\pi\, pN\right|\right] e^{ikr\cos(\theta-\theta_0+2\pi\, mp/q)}$$
$$+ \frac{e^{\frac{i\pi}{4}}}{\sqrt{\pi}\, p}\sum_{m=0}^{q-1} e^{ikr\cos(\theta-\theta_0+2\pi mp/q)} \frac{\sin(\theta-\theta_0+2\pi\, mp/q)\sin(\pi/p)}{\sin((\theta-\theta_0+2\pi\, mp/q)/p)\,|\cos((\theta-\theta_0+2\pi\, mp/q)/2)|}$$
$$\int_{\infty}^{\sqrt{2kr}\,|\cos((\theta-\theta_0+2\pi\, mp/q)/2)|} e^{-iv^2}\,dv$$

$$+ \frac{e^{\frac{i\pi}{4}}}{\sqrt{\pi}\, p}\cdot\sum_{m=0}^{q-1}\sum_{n=1}^{p-2} \frac{e^{ikr\cos(\theta-\theta_0+2\pi\, mp/q)}}{\sin((\theta-\theta_0+2\pi\, mp/q)/p)|\cos((\theta-\theta_0+2\pi\, mp/q)/2)|}\Bigg\{$$
$$+\sin((n+1)(\theta-\theta_0+2\pi\, mp/q)/p)\sin(n\pi/p)\int_{\infty}^{\sqrt{2kr}\left|\cos((\theta-\theta_0+2\pi\, mp/q)/2)\right|} e^{-iv^2}dv$$
$$-\sin(n(\theta-\theta_0+2\pi\, mp/q)/p)\sin((n+1)\pi/p)\int_{\infty}^{\sqrt{2kr}|\cos((\theta-\theta_0+2\pi\, mp/q)/2)|} e^{iv^2}dv\Bigg\}$$
$$+0((kr)^{-3/2}). \qquad (20)$$

This expression is uniformly valid up to order of $(kr)^{-3/2}$. The integrals are related to the Fresnel integral.

4. Special cases of wedge angles

P = 1

$$G_{\frac{\pi}{q}}(r,\theta,\theta_0;k) = \sum_{m=0}^{q-0}\sum_N H\left[\pi - \left|\theta-\theta_0 + 2\pi m/q + 2\pi N\right|\right] e^{ikr\cos(\theta-\theta_0+2\pi m/q)} \tag{21}$$

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q = 1

$$\begin{aligned} G_{p\pi}(r,\theta,\theta_0;k) = &\sum_N H[\pi - |\theta-\theta_0 + 2\pi pN|]\, e^{ikr\cos(\theta-\theta_0)} \\ &+ \frac{1}{2p} e^{ikr\cos(\theta-\theta_0) - i\pi/(2p)} \frac{\sin(\theta-\theta_0)\sin(\pi/p)}{\sin((\theta-\theta_0)/p)} . \\ &\cdot \int_\infty^{kr} e^{-ix\cos(\theta-\theta_0)} H^{(2)}_{\frac{1}{p}}(x)dx \\ &+ \frac{1}{2ip}\sum_{n=1}^{p-2} \frac{e^{ikr\cos(\theta-\theta_0)}}{\sin((\theta-\theta_0)/p)} \Big\{ \sin((n+1)(\theta-\theta_0)/p)\sin(n\pi/p)\, e^{in\pi/(2p)} . \\ &\cdot \int_\infty^{kr} e^{-ix\cos(\theta-\theta_0)} H^{(2)}_{\frac{(p-n)}{p}}(x)dx \\ &- \sin(n(\theta-\theta_0)/p)\sin((n+1)\pi/p)\, e^{i(n+1)\pi/(2p)} . \\ &\cdot \int_\infty^{kr} e^{-ix\cos(\theta-\theta_0)} H^{(2)}_{\frac{(p-1-n)}{p}}(x)dx \Big\} . \end{aligned} \tag{22}$$

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P = 2

$$\begin{aligned} G_{\frac{2\pi}{q}}(r,\theta,\theta_0;k) = &\sum_{m=0}^{q-1}\sum_N H\left[\pi - \left|\theta-\theta_0 + 4\pi m/q + 4\pi N\right|\right] e^{ikr\cos(\theta-\theta_0+4\pi m/q)} \\ &+ \frac{1}{4}\sum_{m=0}^{q-1} e^{ikr\cos(\theta-\theta_0+4\pi m/q - i\pi/4} \frac{\sin(\theta-\theta_0+4\pi m/q)}{\sin((\theta-\theta_0+4\pi m/q)/2)} . \\ &. \int_\infty^{kr} e^{-ix\cos(\theta-\theta_0+4\pi m/q)} H^{(2)}_{\frac{1}{2}}(x)dx . \end{aligned}$$

which on using the result $H^{(2)}_{\frac{1}{2}}(x) = i\left(\frac{2}{\pi x}\right)^{\frac{1}{2}} e^{-ix}$ becomes

$$G_{\frac{2\pi}{q}}(r,\theta,\theta_0;k) = \sum_{m=0}^{q-1} \sum_N H[\pi - |\theta-\theta_0 + 4\pi m/q + 4\pi N|] e^{ikr\cos(\theta-\theta_0+4\pi m/q)}$$
$$+ \frac{1}{2\sqrt{2}\,\pi} \sum_{m=0}^{q-1} e^{ikr\cos(\theta-\theta_0+4\pi m/q)+i\pi/4} \frac{\sin(\theta-\theta_0+4\pi m/q)}{\sin((\theta-\theta_0+4\pi m/q)/2)} .$$
$$\cdot \int_{\infty}^{kr} e^{ix2\cos^2((\theta-\theta_0+4\pi m/q)/2)} \frac{dx}{\sqrt{x}}$$
$$= \sum_{m=0}^{q-1} \sum_N H[\pi - |\theta-\theta_0 + 4\pi m/q + 4\pi N|] e^{ikr\cos(\theta-\theta_0+4\pi m/q)}$$
$$+ \frac{1}{\sqrt{\pi}} \sum_{m=0}^{q-1} e^{ikr\cos(\theta-\theta_0+4\pi m/q)+i\pi/4} \operatorname{sgn}\left[\cos(\theta-\theta_0+4\pi m/q)/2\right].$$
$$\cdot \int_{\infty}^{\sqrt{2kr}|\cos(\theta-\theta_0+4\pi m/q)/2|} e^{-iv^2} dv , \qquad (23)$$

where $\operatorname{sgn}[x] = \begin{Bmatrix} = 1 & x > 0 \\ = 0 & x = 0 \\ = -1 & x < 0 \end{Bmatrix}$ is the signum function.

We remark in passing that the integral in the last expression can be expressed in terms of the Fresnel Integral.

5. Some specific problems in diffraction theory
Sommerfeld's solution for a half plane.

In terms of the Green's function the solution for the problem of diffraction of the plane wave $u_0(r,\theta) = e^{ikr\cos(\theta-\theta_0)}$ by a soft, or hard, half plane is given by

$$\begin{aligned} u_h(r,\theta) &= G_{2\pi}(r,\theta,\theta_0;k) + G_{2\pi}(r,\theta,-\theta_0;k) , \\ u_s(r,\theta) &= G_{2\pi}(r,\theta,\theta_0;k) - G_{2\pi}(r,\theta,-\theta_0;k). \end{aligned} \qquad (24)$$

Now from (23) with q =1 we get

$$G_{2\pi}(r,\theta,\theta_0;k) = \sum_N H[\pi - |\theta-\theta_0 + 4\pi N|] e^{ikr\cos(\theta-\theta_0)}$$
$$+ \frac{1}{\sqrt{\pi}} e^{ikr\cos(\theta-\theta_0)+i\pi/4} \operatorname{sgn}[\cos(\theta-\theta_0)/2] \int_{\infty}^{\sqrt{2kr}|\cos(\theta-\theta_0)/2|} e^{-iv^2} dv;$$

and since $0<\theta_0<2\pi$, $0<\theta<2\pi$ then $-2\pi<\theta-\theta_0<2\pi$.

The argument of the Heaviside step function can only be positive if N = 0. Hence

$$G_{2\pi}(r,\theta,\theta_0;k) = H[\pi-|\theta-\theta_0|]e^{ikr\cos(\theta-\theta_0)} + \frac{1}{\sqrt{\pi}} e^{ikr\cos(\theta-\theta_0)+i\pi/4} \; .$$

$$. \; \mathrm{sgn}[\cos((\theta-\theta_0)/2)]\int_{\infty}^{\sqrt{2kr}|\cos(\theta-\theta_0)/2)|} e^{-iv^2}\,dv$$

$$= H[\cos((\theta-\theta_0)/2)]e^{ikr\cos(\theta-\theta_0)} + \frac{1}{\sqrt{\pi}} e^{ikr\cos(\theta-\theta_0)+i\pi/4} \; .$$

$$\mathrm{sgn}[\cos((\theta-\theta_0))/2] \int_{\infty}^{\sqrt{2kr}\,|\cos((\theta-\theta_0)/2)|} e^{-iv^2}\,dv.$$

If $\cos((\theta-\theta_0)/2)>0$ then

$$G_{2\pi}(r,\theta,\theta_0;k) = e^{ikr\cos(\theta-\theta_0)} + \frac{1}{\sqrt{\pi}} e^{ikr\cos(\theta-\theta_0)+i\pi/4} \; .$$

$$. \int_{\infty}^{\sqrt{2kr}\cos((\theta-\theta_0)/2)} e^{-iv^2}\,dv$$

$$= \frac{e^{ikr\cos(\theta-\theta_0)+i\pi/4}}{\sqrt{\pi}} \left\{ \int_{-\infty}^{\infty} + \int_{\infty}^{\sqrt{2kr}\cos((\theta-\theta_0)/2)} \right\} e^{-iv^2}\,dv \; .$$

$$G_{2\pi}(r,\theta,\theta_0;k) = \frac{e^{i\pi/4}}{\sqrt{\pi}} e^{ikr\cos(\theta-\theta_0)} \int_{-\infty}^{\sqrt{2kr}\cos((\theta-\theta_0)/2)} e^{-iv^2}\,dv.$$

If $\cos((\theta-\theta_0)/2)<0$ then

$$G_{2\pi}(r,\theta,\theta_0;k) = -\frac{e^{i\pi/4}}{\sqrt{\pi}} e^{ikr\cos(\theta-\theta_0)} \int_{\infty}^{-\sqrt{2kr}\cos((\theta-\theta_0)/2)} e^{-iv^2}\,dv \; ,$$

$$= \frac{e^{i\pi/4}}{\sqrt{\pi}} e^{ikr\cos(\theta-\theta_0)} \int_{-\infty}^{\sqrt{2kr}\cos((\theta-\theta_0)/2} e^{-iv^2}\,dv \; .$$

Hence for any sign of $\cos((\theta-\theta_0)/2)$ we have

$$G_{2\pi}(r,\theta,\theta_0;k) = \frac{e^{i\pi/4}}{\sqrt{\pi}} e^{ikr\cos(\theta-\theta_0)} \int_{-\infty}^{\sqrt{2kr}\cos((\theta-\theta_0)/2)} e^{-iv^2}\,dv \; .$$

The expression for $G_{2\pi}(r,\theta,-\theta_0;k)$ can be found in exactly the same way for $0<\theta+\theta_0<4\pi$, i.e.

$$G_{2\pi}(r,\theta,-\theta_0;k) = H[\pi-|\theta+\theta_0|]e^{ikr\cos(\theta+\theta_0)} + H[\pi-|\theta+\theta_0-4\pi|]\ e^{ikr\cos(\theta+\theta_0)}$$

$$+\frac{e^{i\pi/4}}{\sqrt{\pi}}{}^{+ikr\cos(\theta+\theta_0)}\ \mathrm{sgn}\left[\cos\frac{(\theta+\theta_0)}{2}\right]\int_{\infty}^{\sqrt{2kr}\left|\cos\frac{(\theta+\theta_0)}{2}\right|} e^{-iv^2}dv\,,$$

$$= H[\cos((\theta+\theta_0)/2)] + \frac{e^{i\pi/4}}{\sqrt{\pi}}e^{ikr\cos(\theta+\theta_0)}\ \mathrm{sgn}\left[\cos(\frac{(\theta+\theta_0)}{2})\right]\int_{\infty}^{\sqrt{2kr}\left|\cos\frac{(\theta+\theta_0)}{2}\right|} e^{-iv^2}dv$$

$$G_{2\pi}(r,\theta,-\theta_0;k) = \frac{e^{i\pi/4}}{\sqrt{\pi}}e^{ikr\cos(\theta+\theta_0)}\int_{-\infty}^{\sqrt{2kr}\cos((\theta+\theta_0)/2)} e^{-iv^2}dv\,.$$

Hence the solution of the problem of diffraction of a plane wave by a soft or hard half plane is given by

$$u(r,\theta) = \frac{e^{i\pi/4}}{\sqrt{\pi}}e^{ikr\cos(\theta-\theta_0)}\int_{-\infty}^{\sqrt{2kr}\cos((\theta-\theta_0)/2)} e^{-iv^2}dv$$
$$\mp\frac{e^{i\pi/4}}{\sqrt{\pi}}e^{ikr\cos((\theta+\theta_0)/2)}\int_{-\infty}^{\sqrt{2kr}\cos((\theta+\theta_0)/2)} e^{-iv^2}dv\,, \qquad (25)$$

where the upper sign is for the soft half plane and the lower sign is for the hard half plane. This result agrees with that of Sommerfeld’s.

Reiche's solution for a right angle wedge.

In terms of Green's function the solution for the problem of diffraction of the plane wave $u_0(r,\theta) = e^{ikr\cos(\theta-\theta_0)}$ by a soft, or hard wedge of open angle $\alpha = \frac{3\pi}{2}$ is given by

$$u_{s_h}(r,\theta) = G_{\frac{3\pi}{2}}(r,\theta,\theta_0;k) \mp G_{\frac{3\pi}{2}}(r,\theta,-\theta_0;k) \qquad (26)$$

$G_{\frac{3\pi}{2}}(r,\theta,\theta_0;k)$ is given, from (17) after some simplification, by

$$G_{\frac{3\pi}{2}}(r,\theta,\theta_0;k) = \sum_N H[\pi-|\theta-\theta_0+6\pi N|]e^{ikr\cos(\theta-\theta_0)}$$

$$+\sum_N H[\pi-|\theta-\theta_0+(3+6N)\pi|]e^{-ikr\cos(\theta-\theta_0)}$$

$$+\frac{e^{ikr\cos(\theta-\theta_0)-i\pi/6}}{2\sqrt{3}}\left(\frac{\sin(\theta-\theta_0)}{2\sin((\theta-\theta_0)/3)}-1\right)\int_{\infty}^{kr}e^{-ix\cos(\theta-\theta_0)}H^{(2)}_{\frac{1}{3}}(x)dx$$

$$+\frac{e^{-ikr\cos(\theta-\theta_0)-i\pi/6}}{2\sqrt{3}}\left(\frac{\sin(\theta-\theta_0)}{2\sin((\theta-\theta_0)/3)}-1\right)\int_{\infty}^{kr}e^{ix\cos(\theta-\theta_0)}H^{(2)}_{\frac{1}{3}}(x)dx$$

$$+\frac{e^{ikr\cos(\theta-\theta_0)-i\pi/3}}{2\sqrt{3}}\cos((\theta-\theta_0)/3)\int_{\infty}^{kr}e^{-ix\cos(\theta-\theta_0)}H^{(2)}_{\frac{2}{3}}(x)dx$$

$$-\frac{e^{-ikr\cos(\theta-\theta_0)-i\pi/3}}{2\sqrt{3}}\cos((\theta-\theta_0)/3)\int_{\infty}^{kr}e^{ix\cos(\theta-\theta_0)}H^{(2)}_{\frac{2}{3}}(x)dx\ .$$

Now using the trigonometric identity $\frac{\sin\psi}{2\sin\psi/3}-1=\cos\frac{2\psi}{3}$, and the fact that for $-\frac{3\pi}{2}<\theta-\theta_0<\frac{3\pi}{2}, -\pi<\theta-\theta_0+6\pi N<\pi$ will only be satisfied by N=0, and $-\pi<\theta-\theta_0+(3+6N)\pi<\pi$ is not satisfied by any N, then

$$G_{\frac{3\pi}{2}}(r,\theta,\theta_0;k)=H[\pi-|\theta-\theta_0|]e^{ikr\cos(\theta-\theta_0)}+$$

$$+\frac{e^{-i\pi/6-ikr\cos(\theta-\theta_0)}}{2\sqrt{3}}\cos(2(\theta-\theta_0)/3)\int_{\infty}^{kr}e^{ix\cos(\theta-\theta_0)}H^{(2)}_{\frac{1}{3}}(x)dx$$

$$+\frac{e^{-i\pi/6-ikr\cos(\theta-\theta_0)}}{2\sqrt{3}}\cos(2(\theta-\theta_0)/3)\int_{\infty}^{kr}e^{ix\cos(\theta-\theta_0)}H^{(2)}_{\frac{1}{3}}(x)dx$$

$$+\frac{e^{-i\pi/3+ikr\cos(\theta-\theta_0)}}{2\sqrt{3}}\cos((\theta-\theta_0)/3)\int_{\infty}^{kr}e^{-ix\cos(\theta-\theta_0)}H^{(2)}_{\frac{2}{3}}(x)dx$$

$$-\frac{e^{-i\pi/3-ikr\cos(\theta-\theta_0)}}{2\sqrt{3}}\cos((\theta-\theta_0)/3)\int_{\infty}^{kr}e^{ix\cos(\theta-\theta_0)}H^{(2)}_{\frac{2}{3}}(x)dx \qquad (27)$$

Similarly for $0<\theta+\theta_0<3\pi$ can be shown that

$$G_{\frac{3\pi}{2}}(r,\theta,-\theta_0;k) = H[\pi-|\theta+\theta_0|]e^{ikr\cos(\theta+\theta_0)} + H[\pi-|\theta+\theta_0+3\pi|]e^{-ikr\cos(\theta-\theta_0)}$$

$$+\frac{e^{-i\pi/6+ikr\cos(\theta+\theta_0)}}{2\sqrt{3}}\cos(2(\theta+\theta_0)/3)\int_\infty^{kr} e^{-ix\cos(\theta+\theta_0)}H^{(2)}_{\frac{1}{3}}(x)dx$$

$$+\frac{e^{-i\pi/6-ikr\cos(\theta+\theta_0)}}{2\sqrt{3}}\cos(2(\theta+\theta_0)/3)\int_\infty^{kr} e^{ix\cos(\theta+\theta_0)}H^{(2)}_{\frac{1}{3}}(x)dx$$

$$+\frac{e^{-i\pi/3+ikr\cos(\theta+\theta_0)}}{2\sqrt{3}}\cos((\theta+\theta_0)/3)\int_\infty^{kr} e^{-ix\cos(\theta+\theta_0)}H^{(2)}_{\frac{2}{3}}(x)dx$$

$$-\frac{e^{-i\pi/3-ikr\cos(\theta+\theta_0)}}{2\sqrt{3}}\cos((\theta+\theta_0)/3)\int_\infty^{kr} e^{ix\cos(\theta+\theta_0)}H^{(2)}_{\frac{2}{3}}(x)dx \qquad (28)$$

If we now substitute (27) and (28) into (26) for the soft case, we get precisely the result obtained by Reiche. However, our result is more compact because of the use of the Heaviside step functions.

Diffraction by a hard/soft half plane.

In terms of the Green's function, the solution for the problem of diffraction of the plane wave $u_0(r,\theta)=e^{ikr\cos(\theta-\theta_0)}$ by a hard/soft half plane is given by

$$u_{h/s}(r,\theta) = G_{4\pi}(r,\theta,\theta_0;k)+G_{4\pi}(r,\theta,-\theta_0;k)-G_{4\pi}(r,\theta,4\pi-\theta_0;k)-G_{4\pi}(r,\theta,-4\pi+\theta_0;k). \qquad (29)$$

Now using (22) with p=4 we get

$$G_{4\pi}(r,\theta,\theta_0;k) = \sum_N H[\pi-|\theta-\theta_0+8\pi N|]e^{ikr\cos(\theta-\theta_0)}$$

$$+\frac{e^{-i\pi/8}}{8\sqrt{2}}.e^{ikr\cos(\theta-\theta_0)}\frac{\sin(\theta-\theta_0)}{\sin((\theta-\theta_0)/4)}\int_\infty^{kr} e^{-ix\cos(\theta-\theta_0)}H^{(2)}_{\frac{1}{4}}(x)dx$$

$$+\frac{e^{ikr\cos(\theta-\theta_0)}}{8i}.\frac{1}{\sin((\theta-\theta_0)/4)}\left\{\frac{\sin((\theta-\theta_0)/2)e^{i\pi/8}}{\sqrt{2}}\int_\infty^{kr} e^{-ix\cos(\theta-\theta_0)}H^{(2)}_{\frac{3}{4}}(x)dx\right.$$

$$- \sin((\theta-\theta_0)/4)e^{i\pi/4}\int_{\infty}^{kr} e^{-ix\cos(\theta-\theta_0)}H^{(2)}_{\frac{1}{2}}(x)dx$$

$$+ \frac{e^{ikr\cos(\theta-\theta_0)}}{8i}.\frac{1}{\sin((\theta-\theta_0)/4)}\Big\{\sin(3(\theta-\theta_0)/4)e^{i\pi/4}\int_{\infty}^{kr} e^{-ix\cos(\theta-\theta_0)}H^{(2)}_{\frac{1}{2}}(x)dx$$

$$- \frac{\sin((\theta-\theta_0)/2)e^{3i\pi/8}}{\sqrt{2}}\int_{\infty}^{kr} e^{-ix\cos(\theta-\theta_0)}H^{(2)}_{\frac{1}{4}}(x)dx\Big\}.$$

For $-2\pi<\theta-\theta_0<2\pi$ the only value of N which satisfies $-\pi<\theta-\theta_0+8\pi N<\pi$ is N=0. Hence

$$G_{4\pi}(r,\theta,\theta_0;k) = H[\pi-|\theta-\theta_0|]e^{ikr\cos(\theta-\theta_0)}$$

$$+ \frac{e^{-i\pi/4+ikr\cos(\theta-\theta_0)}}{8}\left(\frac{\sin(3(\theta-\theta_0)/4)}{\sin((\theta-\theta_0)/4)}-1\right)\int_{\infty}^{kr} e^{-ix\cos(\theta-\theta_0)}H^{(2)}_{\frac{1}{2}}(x)dx$$

$$+ \frac{e^{ikr\cos(\theta-\theta_0)}}{4i}\frac{\cos((\theta-\theta_0)/4)}{\sqrt{2}}\Big\{e^{i\pi/8}\int_{\infty}^{kr} e^{-ix\cos(\theta-\theta_0)}H^{(2)}_{\frac{3}{4}}(x)dx-\left(1-2\cos\left(\frac{(\theta-\theta_0)}{2}\right)\right)e^{i\frac{3}{8}}$$

$$.\int_{\infty}^{kr} e^{-ix\cos(\theta-\theta_0)}H^{(2)}_{\frac{1}{4}}(x)dx\Big\},$$

where we have used the trigonometric identity $\sin\psi/\sin(\psi/4) = 4\cos\psi/4\cos\psi/2$. If we also use the identity $\sin 3\psi in 3\psi/-1 = 2\cos^2\psi$

we have

$$G_{4\pi}(r,\theta,\theta_0;k) = H[\pi-|\theta-\theta_0|]e^{ikr\cos(\theta-\theta_0)}$$

$$+ \frac{e^{-i\pi/4+ikr\cos(\theta-\theta_0)}}{4}\cos((\theta-\theta_0)/2)\int_{\infty}^{kr} e^{-ix\cos(\theta-\theta_0)}H^{(2)}_{\frac{1}{2}}(x)dx$$

$$+ \frac{e^{-i\pi/4+ikr\cos(\theta-\theta_0)}}{4\sqrt{2}}\cos((\theta-\theta_0)/4)\Big\{\ .$$

$$.\int_{\infty}^{kr} e^{-ix\cos(\theta-\theta_0)}\left(e^{-i\pi/8}H^{(2)}_{\frac{3}{4}}(x)-e^{i\pi/8}(1-2\cos((\theta-\theta_0)/2)H^{(2)}_{\frac{1}{4}}(x)\right)dx\Big\}. \quad (30)$$

By using the result

$$\int_{\infty}^{kr} e^{-ix\cos(\theta-\theta_0)} H^{(2)}_{\frac{3}{4}}(x)dx = -\int_{\infty}^{kr} \frac{e^{-ix\cos(\theta-\theta_0)}}{x^{\frac{1}{4}}} \frac{d}{dx}[x^{\frac{1}{4}} H^{(2)}_{-\frac{1}{4}}(x)],$$

$$= -e^{-i\pi/4-ikr\cos(\theta-\theta_0)} H^{(2)}_{\frac{1}{4}}(kr) - \frac{e^{-\pi/4}}{4} \int_{\infty}^{kr} e^{-ix\cos(\theta-\theta_0)} H^{(2)}_{\frac{1}{4}}(x) \frac{dx}{x}$$

$$- e^{i\pi/4} \cos(\theta-\theta_0) \int_{\infty}^{kr} e^{-ikr\cos(\theta-\theta_0)} H^{(2)}_{\frac{1}{4}}(x)dx$$

we have the alternative representation

$$G_{4\pi}(r,\theta,\theta_0;k) = H[\pi - |\theta-\theta_0|] e^{ikr\cos(\theta-\theta_0)} + \frac{e^{-i\pi/4+ikr\cos(\theta-\theta_0)}}{4} \cos((\theta-\theta_0)/2).$$

$$\int_{\infty}^{kr} e^{-ix\cos(\theta-\theta_0)} H^{(2)}_{\frac{1}{2}}(x)dx$$

$$- \frac{e^{-i5\pi/8}}{4\sqrt{2}} H^{(2)}_{\frac{1}{4}}(kr)\cos((\theta-\theta_0)/4)$$

$$- \frac{e^{ikr\cos(\theta-\theta_0)-i\pi/8}}{4\sqrt{2}} \cos((\theta-\theta_0)/4).$$

$$\cdot \int_{\infty}^{kr} e^{-ix\cos(\theta-\theta_0)} \left\{1 - 2\cos((\theta-\theta_0)/2 + \cos(\theta-\theta_0) + \frac{1}{4ix}\right\} H^{(2)}_{\frac{1}{4}}(x)dx.$$

In an analgous manner it is not difficult to show that

$$G_{4\pi}(r,\theta,-\theta_0;k) = H[\pi - |\theta+\theta_0|] e^{ikr\cos(\theta+\theta_0)}$$

$$+ \frac{e^{-\pi/4}}{4} \cos((\theta+\theta_0)/2 \int_{\infty}^{kr} e^{-ix\cos(\theta+\theta_0)} H^{(2)}_{\frac{1}{2}}(x)dx$$

$$+ \frac{e^{-i\pi/4+ikr\cos(\theta+\theta_0)}}{4\sqrt{2}} \cos((\theta+\theta_0)/4).$$

$$\cdot \left\{ \int_{\infty}^{kr} e^{-ix\cos(\theta+\theta_0)} \left(e^{-i\pi\pi 8} H^{(2)}_{\frac{3}{4}}(x) - e^{i\pi/8}(1 - 2\cos((\theta-\theta_0)/2)) H^{(2)}_{\frac{1}{4}}(x) \right) dx \right\}; \quad (31)$$

$$G_{4\pi}(r,\theta,4\pi-\theta_0;k) = H[\pi-|\theta+\theta_0-4\pi|]e^{ikr\cos(\theta+\theta_0)}$$

$$+\frac{e^{-\frac{i\pi}{4}+ikr\cos(\theta+\theta_0)}}{4}\cos((\theta+\theta_0)/2)\int_{\infty}^{kr} e^{-ix\cos(\theta+\theta_0)}H^{(2)}_{\frac{1}{2}}(x)dx$$

$$-\frac{e^{-i\pi\pi/+ikr\cos(\theta+\theta_0)}}{4\sqrt{2}}\cos((\theta+\theta_0)/4).$$

$$.\left\{\int_{\infty}^{kr} e^{-ix\cos(\theta+\theta_0)}\left(e^{-\frac{i\pi}{8}}H^{(2)}_{\frac{3}{4}}(x)-e^{i\pi/8}(1-2\cos((\theta+\theta_0)/2)H^{(2)}_{\frac{1}{4}}(x)dx\right)\right\}; \quad (32)$$

$$G_{4\pi}(r,\theta,-4\pi+\theta_0;k) = \frac{e^{-\frac{i\pi}{4}+ikr\cos(\theta-\theta_0)}}{4}\cos((\theta-\theta_0)/2)\int_{\infty}^{kr} e^{-ix\cos(\theta-\theta_0)}H^{(2)}_{\frac{1}{2}}(x)dx$$

$$-\frac{e^{-\frac{i\pi}{4}+ikr\cos(\theta-\theta_0)}}{4\sqrt{2}}\cos((\theta-\theta_0)/4).$$

$$.\left\{\int_{\infty}^{kr} e^{-ix\cos(\theta-\theta_0)}\left(e^{-\frac{i\pi}{8}}H^{(2)}_{\frac{3}{4}}(x)-e^{\frac{i\pi}{8}}(1-2\cos((\theta-\theta_0)/2))H^{(2)}_{\frac{1}{4}}(x)\right)dx.\right. \quad (33)$$

Substituting the expressions (30) to (33) into (29) gives

$$u_{h/s}(r,\theta) = H[\pi-|\theta-\theta_0|]e^{ikr\cos(\theta-\theta_0)} + H[\pi-|\theta+\theta_0|]e^{ikr\cos(\theta+\theta_0)}$$

$$-H[\pi-|\theta+\theta_0-4\pi|]e^{ikr\cos(\theta+\theta_0)}$$

$$+\frac{e^{i\pi/4+ikr\cos(\theta-\theta_0)}}{2\sqrt{2}}\cos((\theta-\theta_0)/4)$$

$$.\left\{\int_{\infty}^{kr} e^{-ix\cos(\theta-\theta_0)}\left(e^{-\frac{i\pi}{8}}H^{(2)}_{\frac{3}{4}}(x)-(1-2\cos((\theta-\theta_0)/2))e^{\frac{i\pi}{8}}H^{(2)}_{\frac{1}{4}}(x)\right)dx\right\}$$

$$+\frac{e^{-\frac{i\pi}{4}+ikr\cos(\theta+\theta_0)}}{2\sqrt{2}}\cos((\theta+\theta_0)/4).$$

$$.\left\{\int_{\infty}^{kr} e^{-ix\cos(\theta+\theta_0)}\left(e^{-\frac{i\pi}{8}}H^{(2)}_{\frac{3}{4}}(x)-(1-2\cos((\theta+\theta_0)/2))e^{\frac{i\pi}{8}}H^{(2)}_{\frac{1}{4}}(x)\right)dx\right\}. \quad (34)$$

As a check on the result (34) we see that it satisfies the reduced wave equation $(\nabla^2+k^2)U_{h/s}=0$ and the boundary conditions $\partial u_{h/s}/\partial\theta=0$ at $\theta=0$, and $U_{h/s}=0$ at $\theta=2\pi$. By using the asymptotic result in appendix D it is also not difficult to show that as $kr \to \infty$

$$u_{h/s}(r,\theta) = H[\pi - |\theta - \theta_0|]e^{ikr\cos(\theta-\theta_0)} + H[\pi - |\theta + \theta_0|]e^{ikr\cos(\theta+\theta_0)}$$

$$- H[\pi - |\theta + \theta_0 - 4\pi|]e^{ikr\cos(\theta+\theta_0)}$$

$$+ \sqrt{\frac{2}{\pi}}\cos\left(\frac{(\theta-\theta_0)}{4}\right)\text{sgn}\left[\cos\left(\frac{(\theta-\theta_0)}{2}\right)\right]e^{\frac{i\pi}{4}+ikr\cos(\theta-\theta_0)}\int_{\infty}^{\sqrt{2kr}\,|\cos\frac{(\theta-\theta_0)}{2}|} e^{-iv^2}\,dv$$

$$+ \sqrt{\frac{2}{\pi}}\cos\left(\frac{(\theta-\theta_0)}{4}\right)\text{sgn}\left[\cos\left(\frac{(\theta-\theta_0)}{2}\right)\right]e^{\frac{i\pi}{4}+ikr\cos(\theta-\theta_0)}\int_{\infty}^{\sqrt{2kr}|\cos\frac{(\theta-\theta_0)}{2}|} e^{-iv^2}\,dv\,; \qquad (35)$$

and provided $\theta \pm \theta_0 \neq \pi$ this can be written

$$u_{h/s}(r,\theta) = H[\pi - |\theta - \theta_0|]e^{ikr\cos(\theta-\theta_0)} + H[\pi - |\theta + \theta_0|]e^{ikr\cos(\theta+\theta_0)}$$

$$- H[\pi - |\theta + \theta_0 - 4\pi|]e^{ikr\cos(\theta+\theta_0)}$$

$$- \frac{e^{-ikr}}{\sqrt{kr}}.\frac{e^{-i\pi/4}}{\sqrt{\pi}}.\frac{1}{2}\left(\frac{\cos((\theta-\theta_0)/4}{\cos((\theta-\theta_0)/2} + \frac{\cos((\theta+\theta_0)/4}{\cos((\theta+\theta_0)/2)}\right). \qquad (36)$$

And as $kr \to 0$

$$u_{h/s}(r,\theta) = e^{i\pi/8}\frac{2^{\frac{5}{4}}}{\pi}\gamma(3/4)\left\{\cos\frac{(\theta-\theta_0)}{4} + \cos\frac{(\theta+\theta_0)}{4}\right\}.(kr)^{\frac{1}{4}} + 0(kr)^{\frac{3}{4}})$$

$$= e^{i\pi/8}\frac{4}{\pi}\sqrt{2}\,\gamma(3/4)\cos\frac{\theta}{4}\cos\frac{\theta_0}{4}\left(\frac{kr}{2}\right)^{\frac{1}{4}} + 0(kr)^{\frac{3}{4}})$$

and by means of the duplication formula for the Gamma function it can be shown that $\Gamma(1/4)\Gamma(3/4) = \sqrt{2\pi}$ so that

$$U_{h/s}(r,\theta) = e^{i\pi/8}\frac{8}{\Gamma\left(\frac{1}{4}\right)}.\cos\frac{\theta}{4}\cos\frac{\theta_0}{4}\left(\frac{kr}{2}\right)^{\frac{1}{4}} + 0((kr)^{\frac{3}{4}}). \qquad (37)$$

Thus our solution satisfies all the criterion of Peters and Stoker's [1954] uniqueness theorem, so the expression (34) is indeed the solution of the stated boundary value problem.

Appendix A.

Here we prove the identity

$$\frac{q\sin qy}{\cos qy-\cos qx}=\sum_{m=0}^{q-1}\frac{\sin y}{\cos y-\cos(x+2\pi m/q)},\ q\ge 1. \qquad (A.1)$$

Proof.

From Bromwich (1964), p211, we have

$$\prod_{m=0}^{q-1}\left\{\cos y-\cos\left(x+\frac{2\pi m}{q}\right)\right\}=2^{1-q}(\cos qy-\cos qx). \qquad (A.2)$$

Taking logarithms of both sides of (A.2) and differentiating the resulting expression with respect to y gives the identity (A. 1).

Appendix B.

Here we prove the identity

$$\frac{\frac{1}{p}\sin(y/p)}{\cos(y/p)-\cos(x/p)}=\frac{a_{p-1}(x)\sin(y/p)+\sum_{n=0}^{p-2}a_n(x)\{\sin(y(p-n)/p)-\sin(y(p-2-n)/p)\}}{\cos y-\cos x},\ p\ge 1$$

(B.1)

where $a_n(x)=\dfrac{\sin((n+1)x/p)}{p\sin(x/p)}$, and the summation term vanishes for p=1.

Proof.

We can write the expression (A.2) in the form

$$\cos y-\cos x=2^{p-1}\prod_{n=0}^{p-1}\{\cos(y/p)-\cos((x+2\pi n)/p\}$$

or
$$\frac{1}{\cos(y/p)-\cos(x/p)}=\frac{2^{p-1}\prod_{n=1}^{p-1}\{\cos(y/p)-\cos((x+2\pi n)/p)\}}{\cos y-\cos x},\ p\ge 1 \qquad (B.2)$$

where the product is taken as unity for p=1. We can also write

$$2^{p-1}\prod_{n=1}^{p-1}\{\cos(y/p)-\cos((x+2\pi n)/p\}=z^{1-p}\prod_{n=1}^{p-1}\left\{z^2-2z\cos((x+2\pi n)/p)+1\right\}, \qquad (B.3)$$

where $z=e^{iy/p}$.

But it can be shown (see Lemma B at end of this appendix) that

$$\frac{1}{p}\prod_{n=1}^{p=1}\left\{z^2-2z\cos((x+2\pi n)/p)+1\right\}=\sum_{n=0}^{2p-2}a_n(x)z^n, \qquad (B.4)$$

where $a_n(x) = \dfrac{\sin((n+1)x/p)}{p\sin(x/p)}$, $n = 0,1,\ldots,(p-2),(p-1)$,

$$= a_{2p-2-n}(x),\ n = p,(p+1),\ldots,(2p-3),(2p-2).$$

Hence from (B.2), (B.3) and (B.4) we have

$$\frac{\frac{1}{p}\sin(y/p)}{\cos(y/p)-\cos(x/p)} = \frac{\sum_{n=0}^{2p-2} a_n(x)e^{iy(n+1-p)/p}\left\{e^{iy/p}-e^{-iy/p}\right\}}{2i(\cos y-\cos x)},$$

$$= \frac{\sum_{n=0}^{p-2} a_n(x)\{(e^{iy(p-n)/p}-e^{-iy(p-n)/p})-(e^{iy(p-2-n)/p}-e^{-iy(p-2-n)/p})\}+a_{p-1}(x)(e^{iy/p}-e^{-iy/p})}{2i(\cos y-\cos x)},$$

$$= \frac{\sum_{n=0}^{p-2} a_n(x)\{\sin(y(p-n)/p)-\sin(y(p-2-n)/p)\}+a_{p-1}(x)\sin(y/p)}{(\cos y-\cos x)}$$

which completes the proof of the identity (B.1)

Lemma B.

$$\frac{1}{p}\prod_{n=1}^{p-1}\left\{z^2-2\cos((x+2\pi n)z/p)+1\right\} = \sum_{n=0}^{2p-2} a_n(x)z^n$$

where $$a_n(x) = \frac{\sin((n+1)x/p)}{p\sin(x/p)},\ n = 0,1,\ldots,(p-2),(p-1),$$

$$= a_{2p-2-n}(x),\ n = p,(p+1),\ldots,(2p-3)(2p-2).$$

Proof

We assume that

$$\frac{1}{p}\left(\frac{z^{2p}-2z^p\cos x+1}{z^2-2z\cos\left(\frac{x}{p}\right)+1}\right)\left(\equiv \frac{1}{p}\prod_{n=1}^{p-1}\{z^2-2z\cos\left(\frac{x+2\pi n}{p}\right)+1\}\right) = \sum_{n=0}^{2p-2} a_n(x)z^n,\ p \ge 1.$$

Then $$a_n(x) = \frac{1}{2\pi ip}\oint_c\left(\frac{z^{2p}-2z^p\cos x+1}{z^2-2z\cos(x/p)+1}\right)\frac{dz}{z^{n+1}}$$

where c encloses z=0.

Let $z = e^{it}$ then

$$a_n(x) = \frac{1}{2\pi p}\int_0^{2\pi}\left(\frac{\cos pt-\cos x}{\cos t-\cos(x/p)}\right)e^{i(p-n-1)t}dt,$$

$$= \frac{1}{2\pi p}\int_0^{2\pi} \frac{(\cos pt\cos(p-n-1)t - \cos x\cos(p-n-1)t)dt}{\cos t - \cos(x/p)}$$

$$+ \frac{i}{2\pi p}\int_0^{2\pi} \frac{(\cos pt\sin(p-n-1)t - \cos x\sin(p-n-1)t)dt}{\cos t - \cos(x/p)}$$

Since $a_n(x)$ is real, one would expect (or demand) that the second integral in the above expression should vanish. That the second integral of the above expression does vanish follows (analytically) from the fact that this integral can be expressed in integrals of the form

$$\int_0^{2\pi} \frac{\sin mt}{\cos t - \cos(x/p)} dt, \qquad \text{m an integer,}$$

and since

$$\frac{1}{\cos t - \cos(x/p)} = \frac{2}{\sin(x/p)} \sum_{n=1}^{\infty} \sin\frac{nx}{p}\cos nt,$$

and

$$\int_0^{2\pi} \cos nt \sin mt dt = 0, \text{for all } m, n,$$

then

$$\int_0^{2\pi} \frac{\sin mt \quad dt}{\cos t - \cos(x/p)} \equiv 0.$$

Thus

$$a_n(x) = \frac{1}{4\pi p}\int_0^{2\pi} \frac{\{\cos(2p-n-1)t + \cos(n+1)t - 2\cos x\cos(p-n-1)t\}dt}{\cos t - \cos(x/p)}.$$

But

$$\int_0^{2\pi} \frac{\cos mt}{\cos t - \cos(x/p)} dt = \frac{2}{\sin(x/p)} \sum_{n=1}^{\infty} \sin\frac{nx}{p} \int_0^{2\pi} \cos mt \cos nt dt,$$

$$= 2\pi \frac{\sin m(x/p)}{\sin(x/p)},$$

since

$$\int_0^{2\pi} \cos mt \cos nt dt = \begin{Bmatrix} \pi & m = n \\ 0 & m \neq n \end{Bmatrix}.$$

Thus

$$a_n(x) = \frac{1}{2}\frac{\{\sin((2p-n-1)x/p) + \sin((n+1)x/p - 2\cos x\sin((p-n-1)x/p)\}}{p\sin(x/p)},$$

$$= \frac{\sin x \cos((p-n-1)x/p) - \cos x \sin((p-n-1)x/p)}{p \sin(x/p)}$$

$$= \frac{\sin((n+1)x/p)}{p \sin(x/p)}, \quad n = 0,1,...(p-2),(p-1)\ ,$$

$$a_{2p-2-n} = a_n \qquad n = 0,1,....(p-2),(p-1).$$

Appendix C.

Here we prove that

$$I_p(\infty,\psi) = \sum_N H[\pi - |\psi + 2\pi pN|] e^{ikr\cos\psi}$$

where $H[x] \begin{cases} =1 & x>0 \\ =\frac{1}{2} & x=0 \\ =0 & x<0 \end{cases}$ is the Heaviside step function.

Proof

$$I_p(\infty,\psi) = \lim_{kr\to\infty} \frac{1}{2\pi i} \int_c e^{ikr\cos\zeta} \frac{\frac{1}{p}\sin(\zeta/p)}{\cos(\zeta/p) - \cos(\psi/p)} d\zeta .$$

We distort the path of integration c so that it takes the form c' shown in the figure (3) below

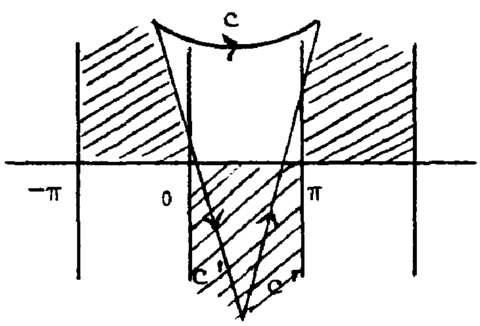

fig 3

Clearly the path c' lies in the shaded region where the integral is uniformly convergent. We have distorted the contour c so that it crosses the line $0<\text{Re}\zeta<\pi$ and therefore if any of the zeros of $\cos(\zeta/p)-\cos(\psi/p)$, viz $\zeta=\pm(\psi+2\pi pN)$ where N is integer, are crossed then they will give rise to pole contributions.

Thus

$$I_p(\infty,\psi) = \sum_N H[\pi - |\psi + 2\pi pN|]\, e^{ikr\cos(\psi+2\pi pN)}$$

$$+ \lim_{kr\to\infty} \frac{1}{2\pi i}\int_c e^{ikr\cos\zeta} \frac{\frac{1}{p}\sin(\zeta/p)}{\cos(\zeta/p) - \cos(\psi/p)}\, d\zeta$$

where summation of N is for those integer values of N which satisfy the inequality $-\pi < \psi + 2\pi pN < \pi$. We can nowfurther distort c' to take up the paths of steepest descent through $\zeta = 0$ and $\zeta = \pi$. Then an application of the method of steepest descent shows that the resulting integral is $0((kr)^{-\frac{1}{2}})$.

Hence

$$I_p(\infty,\psi) = \sum_N H[\pi - |\psi + 2\pi\, p\, N|]\, e^{ikr\cos\psi} .$$

<u>Appendix D.</u>

Here we prove that the integral

$$\int_\infty^{kr} e^{-ix\cos\psi} H_\nu^{(2)}(x)dx \tag{D.1}$$

converges for all $kr \geq 0, -1 < \nu < 1$.

<u>Proof</u>

Clearly it is sufficient to prove the convergence of the integral

$$I = \int_0^\infty e^{-ix\cos\psi} H_\nu^{(2)}(x)dx. \tag{D.2}$$

This integral can be evaluated exactly as follows.

We use the Hankel function representation Watson (1944) (p.180 (11))

$$H_\nu^{(2)}(x) = -\frac{e^{\frac{i\nu\pi}{2}}}{\pi i}\int_{-\infty}^{\infty} e^{-ix\cosh t - \nu t} dt\ ,\ \ x > 0, -1 < \mathrm{Re}\,\nu < 1,$$

in (D.2) and interchange the order of integration giving

$$I = \frac{e^{i(\nu+1)\frac{\pi}{2}}}{\pi}\int_{-\infty}^{\infty} e^{-\nu t}\left\{\int_0^\infty e^{-ix(\cosh t + \cos\psi)} dx\right\} dt,$$

$$= \frac{e^{\frac{i\nu\pi}{2}}}{\pi}\int_{-\infty}^{\infty} \frac{e^{-\nu t}}{\cosh t + \cos\psi} dt.$$

Now let $e^{-t} = y$ then

$$I = 2\frac{e^{i\nu\pi/2}}{\pi}\int_0^{\infty}\frac{y^{\nu}dy}{1+2y\cos\psi+y^2},$$

$$= \frac{2e^{\frac{i\nu\pi}{2}}}{\sin\nu\pi}\cdot\frac{\sin\psi\nu}{\sin\psi}, \quad \begin{matrix}-\pi<\psi<\pi,\\ -1<\nu<1,\end{matrix} \qquad \text{(D.3)}$$

see Erdelyi (1954) (p.309 (12)).

<u>Some asymptotic results.</u>

As $kr\to\infty$ we can replace $H_{\nu}^{(2)}(x)$ in (D.1) by its asymptotic form.

If
$$I = \int_{\infty}^{kr} e^{-ix\cos\psi}H_{\nu}^{(2)}(x)dx\ , \ \nu>0\ ,$$

then
$$I = kr\int_{\infty}^{1} e^{-ikr\cos\psi}H_{\nu}^{(2)}(kru)du\ .$$

By using the asymptotic form for $H_{\upsilon}^{(2)}$ (kru) we have

$$= kr\int_{\infty}^{1} e^{-ikru\cos\psi}\sqrt{\frac{2}{\pi kru}}e^{-i(kru-\frac{\nu\pi}{2}-\pi/4)}du+0((kr)^{-3/2})\ ,$$

as $kr\to\infty$,

$$= \sqrt{\frac{2kr}{\pi}}e^{i(\frac{\nu}{2}+\frac{1}{4})\pi}\int_{\infty}^{1} e^{-ikru(1+\cos\psi)}\frac{du}{\sqrt{u}}+0((kr)^{-3/2})\ ,$$

$$= \frac{\sqrt{2kr}}{\sqrt{\pi}}e^{i\nu\frac{\pi}{2}}e^{i\pi/4}\int_{\infty}^{1} e^{-i2kr\cos^2(\psi/2)u}\frac{du}{\sqrt{u}}+0((kr)^{-3/2})\ ).$$

Let $\sqrt{2kr}\left|\cos\frac{\psi}{2}\right|u^{\frac{1}{2}} = v$ then

$$J = \frac{2}{\sqrt{\pi}}\frac{e^{i\nu\frac{\pi}{2}+i\pi/4}}{|\cos\psi/2|}\int_{\infty}^{\sqrt{2kr}|\cos\psi/2|} e^{-iv^2}dv+0((kr)^{-3/2})\ ), \qquad \text{(D.4)}$$

$kr\to\infty,\ \cos\psi/2\neq 0.$

$$\sim -\frac{e^{\frac{i\nu\pi}{2}-i\pi/4}e^{-ikr(1+\cos\psi)}}{\sqrt{\pi}(\cos\psi/2)^2\sqrt{2kr}}+0((kr)^{-3/2})\ ), \qquad \text{(D.5)}$$

$kr\to\infty,\cos\psi/2\neq 0.$

The full asymptotic expansion is given by Luke (1962) (p.244 (7)).

For kr small we write

$$J = \int_{\infty}^{kr} e^{-ix\cos\psi} H_\nu^{(2)}(x)dx = \int_{\infty}^{0} e^{-ix\cos\psi} H_\nu^{(2)}(x)dx - \int_{kr}^{0} e^{-ix\cos\psi} H_\nu^{(2)}(x)dx$$

and using (D.3) we have

$$J = \int_0^{kr} e^{-ix\cos} H_\nu^{(2)}(x)dx - \frac{2e^{i\nu\pi/2}}{\sin\nu\pi} . \frac{\sin\nu\psi}{\sin\psi} .$$

Now using the small argument asymptotic form for the Hankef function $(\nu > 0)$ gives

$$J = i2 \frac{\nu\Gamma(\nu)}{\pi} \int_0^{kr} e^{-ix\cos\psi} x^{-\nu} dx - \frac{2e^{i\nu\pi/2}}{\sin\nu\pi} . \frac{\sin\nu\psi}{\sin\psi} + 0(x^\nu) ,$$

$$= i \frac{2^{\nu}\Gamma(\nu)}{\pi(1-\nu)} (kr)^{1-\nu} - \frac{2e^{i\nu\pi/2}}{\sin\nu\pi} . \frac{\sin\nu\psi}{\sin\psi} + 0((kr)^\nu) . \qquad (D.6)$$

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