TR/05/83 June1983

STATISTICAL PROCEDURES BASED ON EXPONENTIAL SCORES

by

D.H. Young

CONTENTS

1. INTRODUCTION

Many statistical models have been proposed in which underlying exponential distributions are assumed. Applications of such models occur in a large number of areas, for example in reliability and life-testing investigations and in the study of the pattern of intervals between point events in a series of events when a Poisson process is often postulated. The methods of statistical inference for such models are usually simple to apply but unfortunately are sensitive to departures from the exponential form. In a significance testing situation this leads to a difficulty in interpretation since the true observed significance level may differ appreciably from that calculated on the assumption of an underlying exponential distribution, if the assumption is incorrect.

To overcome this drawback, distribution-free tests have been proposed in which the observations are first ranked and the ranks then replaced by exponential scores which are the expected values of the order statistics in a sample from the standard exponential distribution. This guarantees the validity of the test, whatever the form of the underlying distribution. In addition there is no loss of efficiency in very large samples when the underlying distributions are exponential, and often more generally, when the distributions belong to a Lehmann family which includes the Weibull distributions with common power parameter and hence the exponential distribution as a special case.

In this report, we describe a number of statistical tests based on exponential scores, some new, some well-known, many of which have been proposed and evaluated within the last ten years. The purpose of this is to demonstrate the wide area of application of exponential scores procedures. The procedures which are described deal with goodness of fit tests for the exponential distribution, the comparison of two samples with and without censoring, and the comparison of $k>2$ samples, and finally tests for trend and serial dependence alternatives against a renewal process for the intervals in a series of events. Some of the research findings related to powers of some of the tests are discussed, these findings being based on a series of investigations made by the author over the period 1975-1978.

2. THE EXPONENTIAL SCORES AND SOME PROPERTIES

In this section we present some properties of the exponential scores which will be needed in the later development. Let $X_1, X_2, \ldots, X_n$ represent a random sample of observations from the unit exponential distribution with p.d.f.

$$f(x) = e^{-x} \quad , \qquad x > 0 \tag{2.1}$$

and c.d.f.

$$F(x) = 1 - e^{-x} \; , \qquad x > 0 \tag{2.2}$$

If $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$ denote the ordered observations in the sample, the p.d.f. of $X_{(r)}$ is

$$g_r(x) = n\binom{n-1}{r-1}\{F(x)\}^{r-1}\{1 - F(x)\}^{n-r} f(x)$$

$$= n\binom{n-1}{r-1}(1-e^{-x})^{r-1} e^{-x(n-r+1)}, \qquad x > 0 \tag{2.3}$$

If we set $e_{r,n} = E(X_{(r)})$, we have

$$e_{r,n} = n\binom{n-1}{r-1}\int_0^{\infty} x(1-e^{-x})^{r-1} e^{-x(n-r+1)} dx \tag{2.4}$$

The $\{e_{r,n}\}$ form the set of exponential scores.

In order to find a useful computational form for $e_{r,n}$, it is best to first find the moment generating function of $X_{(r)}$, which we denote by $M_{X_{(r)}}(t)$. We have

$$M_{X_{(r)}}(t) = E(e^{tx_{(r)}})$$

$$= n\binom{n-1}{r-1}\int_0^{\infty} e^{tx}(1-e^{-x})^{r-1} e^{-x(n-r+1)} dx$$

$$= n\binom{n-1}{r-1}\int_0^{1} (1-z)^{r-1} z^{n-r-t} dz$$

$$= n\binom{n-1}{r-1}\frac{\Gamma(x)\,\Gamma(n-r-t+1)}{\Gamma(n-t+1)} \tag{2.5}$$

Where $\Gamma(r) = \int_0^\infty x^{r-1}e^{-x}dx$ is the gamma function. Using the recurrence relation $\Gamma(r) = (r-1)\ \Gamma(r-1)$, (2.5) becomes

$$M_{X_{(r)}}(t) = \frac{n!}{(n-r)!}\{(n-t)(n-t-1)...(n-r-t+1)\}^{-1} \qquad (2.6)$$

Since

$$\frac{dM_{X_{(r)}}(t)}{dt} = M_{X_{(r)}}(t)\left(\frac{1}{n-t} + \frac{1}{n-t-1} + ... + \frac{1}{(n-r-t+1)}\right)$$

and $e_{r,n} = \{dM_{X_{(r)}}(t)/dt\}_{t=0}$, we obtain

$$e_{r,n} = \frac{1}{n} + \frac{1}{n-1} + ... + \frac{1}{n-r+1}, \qquad r = 1, ..., n \qquad (2.7)$$

The calculation of the exponential scores using (2.7) is straightforward and table 1 gives values of $e_{r,n}$ for n = 2(1) 10 and r = 1(l)n . For larger values of n the approximation

$$e_{r,n} \approx F^{-1}\left(\frac{r}{n+1}\right) = \log\left(\frac{n+1}{n+1-r}\right) \qquad (2.8)$$

is usually adequate.

<u>TABLE 1</u>

Values of the exponential scores $\{e_{r,n}\}$.

r \ n	2	3	4	5	6	7	8	9	10
1	0.500	0.333	0.250	0.200	0.167	0.143	0.125	0.111	0.100
2	1.500	0.833	0.583	0.450	0.367	0.310	0.268	0.236	0.211
3		1.833	1.083	0.783	0.617	0.510	0.435	0.379	0.336
4			2.083	1.283	0.950	0.760	0.635	0.546	0.479
5				2.283	1.450	1.093	0.885	0.746	0.646
6					2.450	1.593	1.2	0.996	0.846
7						2.593	1.718	1.329	1.096
8							2.718	1.829	1.429
9								2.829	1.926
10									2.929

Treating the set of values $\{e_{r,n}\}$ as a finite population of scores, the low order moments of this population are readily found as follows (Cox (1964)). From (2.7) we may write

$$e_{r,n} = e_{r-1,\, n-1} + \frac{1}{n}, \qquad r=1,\dots,n \qquad (2.9)$$

where $e_{0,n} = 0$ for any n. Thus if we let $s_{1,n} = \sum_{r=1}^{n} e_{r,n}$,

we have

$$s_{1,n} = \sum_{r=1}^{n}\left(e_{r-1,n-1} + \frac{1}{n}\right) = s_{1,n-1} + 1 \qquad (2.10)$$

With $s_{1,0} = 0$. Thus

$$s_{1,\,n} = \sum_{j=1}^{n}\left(s_{1,j} - s_{1,j-1}\right) = n \qquad (2.11)$$

Similarly if we let $s_{2,n} = \sum_{r=1}^{n} e_{r,n}^2$, we have

$$s_{2,n} = \sum_{r=1}^{n}\left(e_{r-1,\,n-1} + \frac{1}{n}\right)^2$$

$$= s_{2,n-1} + \frac{2}{n} s_{1,\,n-1} + \frac{1}{n}$$

$$= s_{2,n-1} + 2 - \frac{1}{n} \qquad (2.12)$$

with $s_{2,0} = 0$. Thus

$$s_{2,n} = \sum_{j=1}^{n}\left(s_{2,j} - s_{2,j-1}\right) = \sum_{j=1}^{n}\left(2 - \frac{1}{j}\right) \qquad (2.13)$$

$$= 2n - e_{n,n}$$

Expressions (2.11) and (2.13) give the sum and sum of squares of the set of exponential scores $\{e_{r,n}\}$ for fixed n. If we let $\mu_n(e)$ and $\sigma_n^2(e)$ denote the mean and variance of the scores $\{e_{r,n}\}$, we have

$$\mu_n(e) = \frac{s_{1,n}}{n} = 1 \qquad (2.14)$$

and

$$\sigma_n^2(e) = \frac{1}{n-1}\left\{s_{2,n} - \frac{s_{1,n}^2}{n}\right\} = \frac{n - e_{n,n}}{n-1} \qquad (2.15)$$

3. ASSESSMENT OF FIT

As a first application of the use of exponential scores, we consider the problem of assessing the fit of a two-parameter exponential distribution with p.d.f.

$$f(x) = \sigma^{-1} \exp\{-(x-\mu)/\sigma\}, \qquad x > \mu \tag{3.1}$$

mean $\mu+\sigma$ and variance σ^2, given a random sample of observations $X_1, X_2, \ldots, X_n$ drawn from a single population.

If the true underlying distribution is exponential, the random variables $z_i = (X_i - \mu)/\sigma$ i = 1,2,...,n are independently and identically distributed having the standard exponential distribution with p.d.f. $f(z) = \exp(-z),\ 0 < z < \infty$. Thus if $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$ denote the ordered observations in the sample, we have $E(X_{(i)} = \mu + \sigma E(Z_{(i)})$, that is

$$E(X_{(i)}) = \mu + \sigma e_{i,n}, \qquad i = 1,2,\ldots,n \tag{3.2}$$

Ignoring the random variations, we have the approximation

$$X_{(i)} \approx \mu + \sigma e_{i,n}, \qquad i = 1,2,\ldots,n. \tag{3.3}$$

It follows that if the assumption of an exponential distribution is correct, a plot of $X_{(i)}$ against $e_{i,n}$ should give an approximate straight line relation. Further, if a straight line is fitted to the points, the intercept and slope of the line provide 'preliminary' estimates of the location parameter μ and the scale parameter σ, respectively.

The above plotting procedure only provides an informal graphical assessment, of fit of the two-parameter exponential distribution. A formal test of fit may be made using the statistic

$$W_1 = \frac{n(\overline{X} - X_{(1)})^2}{(n-1)\sum_{i=1}^{n}(X_i - \overline{X})^2} \tag{3.4}$$

which was proposed by Shapiro and Wilk (1972). This statistic arises as the ratio of two estimates of the variance σ^2, the first being the square of the slope of the line fitted to the points $(X_{(i)}, e_{i,n})$ by generalised least squares, the second being the usual sample variance estimate $(n-1)^{-1}\sum_{i=1}^{n}(X_i-\overline{X})^2$ which provides an unbiased estimate of σ^2 whatever the underlying distribution. It should be noted that although the exponential scores do not enter explicitly into the expression for W_1, the derivation of the statistic does utilise the scores and their properties.

Tables of percentage points of the distribution of W_1 when the population has an exponential distribution are given by Shapiro and Wilk (1972) and their upper and lower 10%, 5%, 1% points are shown in table 2 for n = 5(1) 10(2) 20.

In the case when a single parameter exponential distribution with $\mu = 0$ is fitted, a test of fit statistic utilising exponential scores is

$$W_2 = \sum_{i=1}^{n} e_{i,n}\ X_{(i)} \ / \ \sum_{i=1}^{n} X_i \qquad (3.5)$$

This statistic was proposed by Jackson (1967) who showed that when the exponential distribution holds, the mean and variance are given by

$$E(W_2) = 2 - n^{-1}e_{n,n}, \quad \mathrm{var}(W_2) = \frac{n + e_{n,n} - e_{n,n}^2(1+n^{-1})}{n(n+1)} \qquad (3.6)$$

For $n \geq 10$, a reasonable approximation is obtained by taking

$\{W_2 - E(W_2)\}/\{\mathrm{var}(W_2)\}^{\frac{1}{2}}$ to be distributed as N(0,1).

TABLE 2

Upper arid lower 100α% points of the distribution of the W_e statistic when the population has a two-parameter exponential distribution.

	Lower % Points			Upper % Points		
n	α= 0.01	α=0.05	α =0.10	α =0.010	α =0.05	α =0.01
5	0.091	0.119	0.144	0.555	0.668	0.860
6	0.067	0.096	0.117	0.429	0.509	0.678
7	0.059	0.081	0.099	0.347	0.416	0.571
8	0.051	0.071	0.085	0.293	0.350	0.485
9	0.044	0.063	0.075	0.255	0.301	0.402
10	0.040	0.057	0.068	0.218	0.253	0.339
12	0.036	0.049	0.057	0.172	0.202	0.272
14	0.032	0.043	0.050	0.142	0.165	0.213
16	0.028	0.037	0.044	0.119	0.136	0.177
18	0.025	0.033	0.039	0.102	0.116	0.148
20	0.023	0.030	0.035	0.088	0.100	0.129

Example 1. The following data show the ordered times required in hours for a sample of 20 jobs on drill presses in a certain machine shop. An assessment of the fit of an exponential distribution is required.

Job time (hours):	$X_{(i)}$	0.3	0.5	0.6	0.7	0.8	1.1	1.3	1.5	1.8	2.2
Exponential score :	$e_{i.n}$	0.05	0.10	0.16	0.22	0.28	0.35	0.42	0.49	0.58	0.67

Job time (Hours):	$X_{(i)}$	2.4	2.9	3.3	3.8	4.2	4.7	5.5	6.3	7.7	9.9
Exponential score :	$e_{i.n}$	0.77	0.88	1.00	1.15	1.31	1.51	1.76	2.10	2.60	3.60

A plot of $x_{(i)}$ against $e_{i,n}$ is shown in figure 1 and indicates that the fit of an exponential distribution is satisfactory and that the location parameter may be taken to be zero.

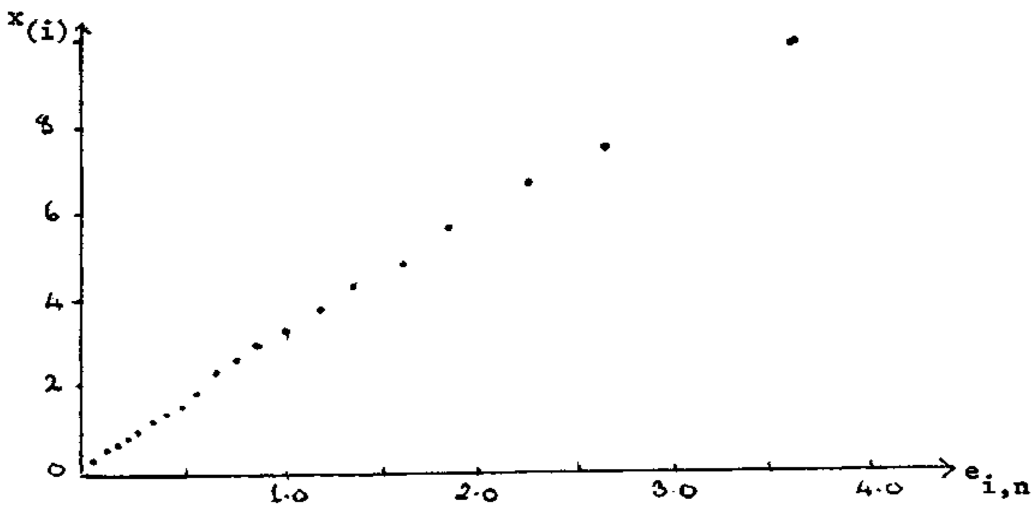


Figure 1. Exponential score plot for assessment of fit

Applying the Shapiro-Wilk test of fit, we have

$$n = 20, \bar{x} = 3.075, x_{(1)} = 0.3, \sum_{i=1}^{20} \left(x - \bar{x}\right)^2 = 133.42 \qquad \text{and}$$

$$W_1 = \frac{20(3.075 - 0.3)^2}{19(133.42)} = 0.061$$

Reference to table 2 shows that the upper and lower 10% critical values of the W_1 statistic when n = 20 are 0.088 and 0.035 respectively, so there is clearly no real evidence against the exponential fit.
For the Jackson statistic, we have W_2 = 1.76, $E(W_2)$ = 1.82 and $var(W_2)$ = 0.0238. Using a single tailed test the observed significance level is approximately

$$\Phi\left\{(1.76 - 1.82) / (1.0278)^{\frac{1}{2}}\right\} = \Phi\left(-0.42\right) = 0.34$$

which confirms the adequacy of fit of the exponential distribution.

4. COMPARISON OF TWO SAMPLES

As a second application of the use of exponential scores, we consider the problem of comparing two independent samples of observations, which we denote by $X_1, X_2,..., X_m$ and $X_{m+1}, X_{m+2},...., X_{m+n}$, respectively. We assume that the samples are drawn from underlying populations with continuous c.d.f.'s $F_1(x)$ and $F_2(x)$ respectively. We wish to test the hypothesis

$$H_0 : F_1(x) = F_2(x) \text{ for all } x. \tag{4.1}$$

For the parametric case when the underlying distributions are assumed to be exponential, we have

$$F_1(x) = 1 - \exp(-\theta_1 x), \qquad F_2(x) = 1 - \exp(-\theta_2 x)$$

and the null hypothesis given by (4.1) is then equivalent to the hypothesis that the population means are equal, or equivalently that $\theta_1 = \theta_2$. In this case we use the test statistic $R = \overline{X_1}/\overline{X2}$, the ratio of sample means. If H_0 is true, R has the F-distribution with (2m, 2n) degrees of freedom. A strong justification for the test arises if the exponential assumption is correct, since the test then provides the uniformly most powerful similar test of H_0 against one-sided alternatives.

For the exponential scores test, we use the statistic first proposed by Savage (1956) ,

$$S_e = \sum_{i=1}^{m} e_{R_i, N} \tag{4.2}$$

where R_i denotes the rank of X_i in the ordered combined samples and $N = m + n$. Thus S_e is simply the sum of exponential scores assigned to the observations in the first sample.

The exact null distribution of S_e is in principle easily found since each of the $\binom{N}{m}$ possible rank orders for the observations in the first sample are equally likely under H_o. Hence

$$P(S_e = s \mid H_o) = \lambda_{m.n}(s) / \binom{N}{m} \tag{4.3}$$

where $\lambda_{m,n}(s)$ in the number of ways of selecting m ranks $(R_1, R_2,..,R_m)$ from the integer set (1, 2,... ,m+n) such that $S_e = s$. The probability distribution of S_e can therefore be constructed by a simple search over the $\binom{N}{m}$ possible rank order vectors for the observations in the first sample, as illustrated in table 3 for the case m = 3, n = 2.

TABLE 3

Calculation of the exact null distributions of S_e for the case m = 3, n = 2.

Ranks for sample 1	Scores for sample 1	S	$P(S_e=S\|H_o)$
1, 2, 3	0.200, 0.450, 0.783	1.433	0.1
1, 2, 4	0.200, 0.450, 1.283	1.933	0.1
1, 2, 5	0.200, 0.450, 2.283	2.933	0.1
1, 3, 4	0.200, 0.783, 1.283	2.266	0.1
1, 3, 5	0.200, 0.783, 2.283	3.266	0.1
1, 4, 5	0.200, 1.283, 2.283	3.766	0.1
2, 3, 4	0.450, 0.783, 1.283	2.516	0.1
2, 3, 5	0.450, 0.783, 2.283	3.516	0.1
2, 4, 5	0.450, 1.283, 2.283	4.016	0.1
3, 4, 5	0.783, 1.283, 2.283	4.349	0.1

In table 4, values of the upper and lower percentage points of the null distribution of S_e are given for nominal significance levels $\alpha = 0.10, 0.05, 0.025$ and sample sizes m = n = 4(1)10(2)20. Because of the discreteness of the distribution of S_e, the actual significance levels are slightly less than the nominal levels for the smaller sample sizes. More extensive tables which cover the case of unequal sample sizes are given by Hajék (1969).

TABLE 4

Upper and lower 100α% points of the null distribution of S_e for the case of equal sample sizes.

	Lower % points			Upper % points		
α	0.025	0.05	0.10	0.10	0.05	0.025
m=n=4	1.46	1.91	2.25	5.75	6.09	6.54
5	2.39	2.66	3.10	6.90	7.34	7.61
6	3.08	3.45	3.91	8.09	8.55	8.92
7	3.78	4.20	4.72	9.28	9.80	10.22
8	4.52	4.98	5.56	10.44	11.02	11.48
9	5.16	5.77	6.40	11.60	12.23	12.74
10	6.03	6.57	7.25	12.75	13.43	13.97
12	7.59	8.21	8.98	15.02	15.79	16.41
14	9.19	9.88	10.72	17.28	18.12	18.81
16	10.81	11.57	12.48	19.52	20.43	21.19
18	12.47	13.28	14.26	21.74	22.72	23.53
20	14.15	15.01	16.06	23.94	24.99	25.85

The calculation of the exact null distribution of S_e is only practical for relatively small sample sizes, so approximations to the percentage points are needed. We first determine the mean and variance of S_e. We have

$$E\left(e_{R_i,N} \mid H_o\right) = \sum_{r=1}^{N} P\left(R_i = r \mid H_o\right) e_{r,N} = \frac{S_{1,N}}{N} \tag{4.4}$$

$$E\left(e^2_{R_i,N} \mid H_o\right) = \sum_{r=1}^{N} P\left(R_i = r \mid H_o\right) e^2_{r,N} = \frac{S_{1,N}}{N} \tag{4.5}$$

$$E\left(e_{R_i,N}\, e_{R_j,N} \mid H_o\right) = \sum_{\substack{r=1 \\ r\neq s}}^{N} \sum_{s=1}^{N} P\left(R_i = r, R_j = s \mid H_o\right) e_{r,N}\, e_{s,N}$$

$$= \frac{S^2_{1,N} - S_{2,N}}{N(N-1)} \tag{4.6}$$

Substituting $S_{1,N} = N$ and $S_{2,N} = 2N - e_{N,N}$, we obtain

$$E(e_{R_i,N} \mid H_o) = 1 \quad , \quad var(e_{R_i,N} \mid H_o) = \frac{N - e_{N,N}}{N}$$
$$cov(e_{R_i,N}, e_{R_j,N} \mid H_o) = -\frac{(N - e_{N,N})}{N(N-1)} \tag{4.7}$$

for $i \neq j = 1,2,\ldots,N$. This leads to

$$E(S_e \mid H_0) = m, \qquad var(S_e \mid H_0) = \frac{mn(N - e_{N,N})}{N(N-1)} \tag{4.8}$$

Taking S_e to be approximately normally distributed with mean and variance given by (4.8), the normal approximation to the 100α th percentile of the null distribution of S_e is

$$S_\alpha \approx m + u_\alpha \left[mn(N - e_{N,N}) \big/ \{N(N-1)\}^{-\frac{1}{2}} \right] \tag{4.9}$$

where u_α is the 100α th percentile of the N(0,1) distribution.
An alternative approximation due to Cox (1964) takes $\bar{e}_x / \bar{e}_y$ to have approximately the F-distribution with (2m, 2n) degrees of freedom, where $\bar{e}_x$ and $\bar{e}_y$ denote the mean scores given to the samples of X's and Y's respectively. Since $S_e = m\bar{e}_X$ and $m\bar{e}_X + n\bar{e}_y = N$, this is equivalent to taking

$$\frac{nS_e}{m(N - S_e)} \overset{approx}{\sim} F_{2m,2n}$$

This leads to an F-approximation for s_α given by

$$s_\alpha \approx \frac{mNF_{2m,2n}(\alpha)}{n + mF_{2m,2n}(\alpha)} \tag{4.10}$$

Numerical studies indicate that (4.9) and (4.10) both give good approximations when the sample sizes are greater than 10.
<u>Example 2</u> The following data show the number of cycles to failure in tests with two types of carbide inserts. The ranks of the observations in the combined samples together with the associated

exponential scores are also given.

Type A carbide			Type B carbide		
X_i	R_i	$e_{R_i,32}$	X_i	R_i	$e_{R_i,32}$
27	1	0.031	32	2	0.064
43	4	0.131	39	3	0.097
58	5	0.167	64	6	0.204
74	7	0.243	81	8	0.283
91	9	0.324	97	10	0.368
109	11	0.413	128	12	0.461
132	13	0.511	149	14	0.563
185	17	0.740	209	18	0.807
221	19	0.878	276	22	1.129
246	20	0.955	341	24	1.340
272	21	1.038	392	26	1.608
329	23	1.229	456	28	1.975
361	25	1.465	536	29	2.225
447	27	1.775	579	31	3.058
550	30	2.558	639	32	4.058

Summing the scores for the type A carbide gives S_e = 13.077. Using a single tailed test, the result is not quite significant at the 10% level so there is only slight evidence that the true mean lives differ. For the parametric test which assumes underlying exponential distributions, the observed value of the test statistic R is $\overline{X}_1/\overline{X}_2$ = (206.00)/(261.81) = 0.787. Referring this value to the F-distribution with (32,32) degrees of freedom, the result is again not quite significant at the 10% level.

It is of course important to know of the loss of power that occurs when using the exponential scores test instead of the parametric R test when the true underlying distributions are exponential. When determining the power of the exponential scores test, we may consider the general alternative

$$1-F_2(x) = \{1-F_1(x)\}^{\theta}$$

where $F_1(x)$ is an arbitrary continuous c.d.f The parameter θ has a simple interpretation since

$$\begin{aligned} P(X_i > X_j) &= \int_{-\infty}^{\infty} f_2(x)\{1 - F_1(x)\}dx \\ &= \int_{-\infty}^{\infty} \{1 - F_1(x)\}^{\theta} f_1(x)\,dx \qquad (4.12) \\ &= \theta/(\theta + 1) \end{aligned}$$

for i = 1,2,...,m, j = m+1, m+2,...,m+n. Any two distributions having c.d.f.'s satisfying (4.11) are said to follow a Lehmann alternative. A special case of interest is when the distributions are Weibull with common power parameter, that is,

$$F_1(x) = 1 - \exp(-\theta_1 X)^{\delta}, \quad F_2(x) = 1 - \exp(-\theta_2 X)^{\delta} \qquad (4.13)$$

for $x > 0$, where $\theta_1,\ \theta_2,\ \delta > 0$. These c.d.f.'s satisfy (4.11) with $\theta = (\theta_2/\theta_1)$, and when δ=1 reduce to exponential forms with θ representing the ratio of the populations means.

The power of the exponential scores test under the Lehmann model was evaluated by Burr and Young (1975a) for one-sided alternatives $\theta > 1$, and a wide range of sample sizes. In the same investigation, the power of the nonparametric Mann-Whitney test based on the statistic

$$U_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} U_{ij} \text{ where } U_{ij} = \begin{cases} 1 \text{ if } X_i > X_j \\ 0 \text{ if } X_i < X_j \end{cases} \qquad (4.14)$$

was also evaluated. Randomised tests were used to keep control over the significance level α. The power of the parametric R-test was also determined for the case when the underlying distributions are exponential using the result that R is then distributed as $\theta\ F_{2m,2n}$.

The broad findings from the power study were

(i) when θ is relatively close to 1, there is only a small loss of power through using the exponential scores test rather than the parametric R tests, but the loss of power becomes quite large for large values of θ,

(ii) the exponential scores test consistently gave a greater power than the Mann-Whitney test.

These findings are illustrated in table 5 for the case m = n = 8.

TABLE 5

Powers of (i) the parametric R-test, (ii) the exponential scores test, (iii) the Mann-Whitney test, for size α, when samples of size m = n =8 are drawn from exponential distributions with a ratio of means equal to θ.

	$\alpha = 0.05$			$\alpha = 0.10$		
θ	(i)	(ii)	(iii)	(i)	(i)	(iii)
1.2	0.097	0.092	0.089	0.176	0.170	0.164
1.4	0.158	0.145	0.136	0.265	0.249	0.235
1.6	0.229	0.202	0.190	0.357	0.331	0.308
1.8	0.305	0.267	0.246	0.446	0.410	0.379
2.0	0.381	0.330	0.302	0.539	0.483	0.445
2.5	0.554	0.477	0.434	0.695	0.638	0.586
3.0	0.689	0.599	0.545	0.507	0.748	0.692
4.0	0.854	0.765	0.705	0.922	0.876	0.825

The results in table 5 show that in small samples, the use of the exponential scores test leads to some loss of efficiency when the true distributions are exponential. Of more interest is an assessment of the gain in efficiency that may result from using the exponential scores test instead of the R test for other distribution forms. When the distributions are Weibull with c.d.f.'s given by (3.13), Burr and Young (1975a) show that the asymptotic relative efficiency of the R-test relative to the S_e test is

$$\mathrm{ARE}(R;S_e) = \frac{\delta^{-2}\{\Gamma(1+\delta^{-1})\}^2}{\Gamma(1+2\delta^{-1}) - \{\Gamma(1+\delta^{-1})\}^2} \qquad (4.15)$$

The values of the A.R.E are 0.100, 0.800, 0.997, 1.000, 0.965 and 0.915 for δ = 0.2, 0.5, 0.9, 1.0, 1.5, 2.0 respectively, showing that the exponential scores test can be appreciably more efficient that the R-test.

Finally, we consider a modified form of the two-sample exponential scores test statistic when a particular type of censoring occurs. Denoting the ordered observations in the combined samples by $X_{(1)} \le X_{(2)} \le \dots \le X_{(N)}$, we assume that because of censoring only $X_{(1)} \le X_{(2)} \dots X_{(c)}$, are observed. This is quite a common situation
in life-testing investigations where an experiment to compare two life distributions may be terminated early because of cost considerations. We let

$$z_i = \begin{cases} 1 \text{ if } X_{(i)} \text{ is an observation from sample 1} \\ 0 \text{ if } X_{(i)} \text{ is an observation from sample 2} \end{cases} \quad (4,16)$$

and let $m_c = \sum_{i=1}^{c} Z_i$ denote the number of uncensored observations which are from the first sample. A distribution free test of the hypothesis that $F_1(x) = F_2(x)$ for all x may be made using the statistic

$$s_e^{(c)} = \sum_{i=1}^{c} Z_i \, e_{i,N} + \frac{m - m_c}{N - c} \sum_{i=c+1}^{N} e_{i,N} \quad (4.17)$$

This statistic differs only by a constant from the statistic originally proposed by Basu (1968) and is the sum of exponential scores for the observations in sample 1 when each censored observation is allocated the average of the largest N-c exponential scores. Burr & Young (l975b) give a theoretical justification for a test based on $s_e^{(c)}$ by showing that its use leads to the locally most powerful rank test of the hypothesis $\theta = 1$ in the Lehmann alternative $F_2(x) = 1-\{1-F_1(x)\}^{\theta}$. They also give tables of percentage points of the null distribution of $s_e^{(c)}$ for sample sizes $3 \le m \le n. \; m+n \le 16$ and r=3(1) m+n-1,and show that simple normal approximations to these points using the moment expressions

$$E(s_e^{(c)} \mid H_o) = m \qquad \operatorname{var}(s_e^{(c)} \mid H_o) = mn(c - e_{c,N})/\{N(N-1)\} \quad (4.18)$$

are adequate even for sample sizes as small as 8.

5. COMPARISON OF k > 2 SAMPLES

The problem of deciding whether differences among the observations in k > 2 samples can be regarded as showing evidence of real differences among the k parent populations or may be attributed to chance is a very common problem in statistics. The formulation of the problem is as follows.

Let $X_{i,j}$,i= 1,...,k, j = 1...,n_i be a set of independent random variables and let $F_i(x)$ be the c.d.f. of X_{ij}. It is assumed that each $F_i(x)$ belongs to the class Ω of absolutely continuous c.d.f.'s. The hypothesis to be tested is

$$H_o : F_i(x) = F(x) \text{ for all } x \tag{5.1}$$

If the common c.d.f. F(x) assumed under H_o is specified, well-known parametric procedures are available. For example, if F(x) is taken to be normal, the analysis of variance statistic

$$R = \frac{\sum_{i=1}^{k} n_i\left(\overline{X}_{i.} - \overline{X}_{..}\right)^2}{k-1} \bigg/ \frac{\sum_{i=1}^{k}\sum_{i=1}^{n_i}\left(X_{ij} - \overline{X}_{i,}\right)^2}{N-k} \tag{5.2}$$

is used, R having the F-distribution with (k-1, N-k) degrees of freedom when H_o is true $(N = \sum_{i=1}^{k} n_i)$.

When F(x) is unspecified, several nonparametric tests have been proposed which include the Kruskal-Wallis rank test (1952), the Mood and Brown median test (1950) and Kiefer's k-sample analogue of the Kolmogorov-Smirnov test (1959). A nonparametric test based on exponential scores was proposed by Downton (1976). If we introduce rank order indicator variables $\{Z_{ij}\}$ where $Z_{ij}=1$ if the jth ordered observation in the combined samples is from the ith population and Z_{ij} = 0 otherwise, Downton's test statistic is

$$T_e = \frac{N-1}{N-e_{N,N}} \sum_{i=1}^{k} n_i(\bar{e}_i - \bar{1})^2 , \tag{5.3}$$

where

$$\bar{e}_i = \frac{1}{n_i} \sum_{J=1}^{N} Z_{ij} e_{j,N} \tag{5.4}$$

Since $\bar{e}_i$. is simply the mean of the exponential scores assigned to the observations in the ith sample, the statistic T_e is the between samples sum of squares of exponential scores scaled by the constant (N-l)/(N-$e_{N,N}$).

To find the exact null distribution of the statistic T_e, we use the result that each of the $N!/\prod_{i=1}^{k} n_i!$ possible exponential score assignments to the observations in the k samples are equally likely when H_o is true. Thus the c.d. f. of T_e is

$$p(T_e \le t / H_o) = \frac{n_1!n_2!....n_k!}{N!} \lambda_{\{n_i\}}(t) \tag{5.5}$$

where $\lambda_{\{n_i\}}(t)$ is the number of exponential score assignments to the k samples for which $T_e \le t$. Values of the upper 100α% points of the null distribution of T_e have been computed by Burr and Young (1977a) for k = 3 and $2 \le n_i \le 5$ and are reproduced in table 6.

TABLE 6

Exact upper 100α % points for T_e when k = 3

Sample sizes	10%	5%	$2\frac{1}{2}$%	Sample sizes	10%	5%	$2\frac{1}{2}$%
2 2 2	4.19	-	-	3 3 3	3.92	4.86	5.81
2 2 3	4.56	4.66	5.07	3 3 4	4.06	4.92	5.79
2 2 4	4.27	5.32	5.48	3 3 5	4.13	5.08	6.02
2 2 5	4.16	5.92	6.09	3 4 4	4.11	4.97	5.99
2 3 3	3.99	5.14	5.34	3 4 5	4.19	5.09	5.91
2 3 4	4.08	5.19	5.93	3 5 5	4.23	5.16	5.98
2 3 5	4.16	5.27	6.54	4 4 4	4.21	5.05	5.84
2 4 4	4.18	5.09	6.26	4 4 5	4.24	5.11	5.93
2 4 5	4.12	5.14	6.25	4 5 5	4.25	5.19	6.03
2 5 5	4.22	5.39	6.18	5 5 5	4.28	5.21	6.08

The calculation of the exact null distribution of T_e is clearly impractical except for small values of k and the $\{n_i\}$, so an approximation is required. Noting that T_e is a special case of a general class of rank statistics considered by Puri (1964), use of his results show that as $n_i \to \infty$, i = 1,...,k, T_e has a limiting X^2-distribution with k - 1 degrees of freedom when H_o is true.

Since

$$E(T_e | H_o) = k - 1, \tag{5.6}$$

the exact and asymptotic null distribution of T_e agree in their first moments. Calculations show that the use of an ordinary chi-square approximation to the upper percentiles of the null distribution of T_e is generally satisfactory even for very small sample sizes, except in the extreme tail of the distribution. Improved approximations can be obtained by taking T_e to be approximately distributed as $a\chi^2_v$ where a and v are chosen to give agreement between the variances as well as the means of the asymptotic and exact null distributions.

The justification for using the distribution-free exponential scores test is that it should have good power properties when the underlying distributions are exponential. Table 7 shows values of the powers of the test obtained by simulation by Burr and Young (1977a) for the Lehmann alternative

$$F_i(x) = 1 - \{1 - F(x)\}^{\theta_i} \quad , i = 1, 2, ..., k \tag{5.7}$$

where F(x) is arbitrary, for the case $k = 3$, $n_i = n$, $\theta_1 = 1$, $\theta_2 = 1 + \delta$, $\theta_3 = 1 + 2\delta$. The corresponding values of the power of the Kruskal-Wallis test are also shown. Randomised tests were used because of the marked discreteness of the distribution of the Kruskal-Wallis statistic. The results in table 7 show that the power of the exponential scores test is consistently higher and that the power differences become quite appreciable for large δ as n increases.

TABLE 7

Values of the powers of size a tests based on (i) the exponential scores statistic T_e , (ii) the Kruskal-Wallis statistic under the alternative $F_i(x) = 1-\{1-F(x)\}^{\theta_i}$ with $\theta_1=1$, $\theta_2=1+\delta$, $\theta_3=1+2\delta$.

		n=4		n=6		n=8	
	δ	(i)	(ii)	(i)	(ii)	(i)	(ii)
α =0.05	0.25	0.065	0.064	0.077	0.074	0.090	0.083
	0.50	0.098	0.092	0.136	0.122	0.177	0.155
	0.75	0.138	0.127	0.210	0.183	0.285	0.242
	1.00	0.181	0.164	0.288	0.245	0.395	0.330
α =0.025	0.25	0.034	0.033	0.042	0.039	0.049	0.045
	0.50	0.055	0.050	0.081	0.071	0.110	0.094
	0.75	0.081	0.073	0.134	0.113	0.193	0.159
	1.00	0.111	0.098	0.194	0.160	0.285	0.229

6. SERIES OF EVENTS AND SOME RELATED SIGNIFICANCE TESTS

We consider a one-dimensional point process in which events occur in a haphazard way in time. For example, the events may refer to accidents, failures of equipment, arrivals of customers at a queue point, etc. We suppose that observations X_1. $X_2,\dots, X_n$ are available on the intervals between the occurrences of (n+1) consecutive events and let $F_i(x)$ denote the c.d. f. of X_i. If the $\{X_i\}$ are independently and identically distributed random variables with $F_i(x) = F(x)$, $i = 1,2,\dots,n$, the series of events forms a renewal process. In the case when the distribution is exponential with $F(x)= 1 - \exp(-\lambda x)$, $x > 0$, the series of events forms a Poisson process with rate λ.

In this section we consider significance tests based on exponential scores which can be used to check the consistency of a renewal process model with an observed series of events, against the alternatives

(i) a trend in the rate of occurrence of events, (ii) serial dependence among the intervals between events.

6.1 Tests For a Renewal Process Against a Trend Alternative

Suppose that as an alternative to a renewal process, it is suspected that there will be a trend with the intervals stochastically increasing or stochastically decreasing in a systematic way. A simple model allowing for trend is

$$F_i(x) = 1 - \{1-F(x)\}^{1+\theta(i-1)}, \quad i = 1,\dots, n \qquad (6.1)$$

We shall refer to (6.1) as a Lehmann trend alternative. If F(x) the c.d.f. of X_1 is fully specified, we obtain exponentially distributed observations by using the transformation $U_i. = - \log \{1-F(X_i)\}$, $i = 1,\dots, n$. The case $\theta = 0$ then corresponds to taking the $\{U_i\}$ to form a Poisson process as the null hypothesis. If F(x) is unspecified, the null hypothesis $\theta = 0$ corresponds to taking the series of events as forming a renewal process. A general class of linear rank statistics which may be used for testing this hypothesis against the trend alternative is

$$\psi_a = \sum_{i=1}^{n} (i - \frac{n+1}{2}) a_{R_i,n} \qquad (6.2)$$

where R_i is the rank of X_i in the ordered set of intervals and $a_{R_i,n}$ is a 'score' used to replace the rank R_i . The scores satisfy

$$a_{1,n} \le a_{2,n} \le \dots \le a_{n,n}$$

so if the intervals between the events tend to increase (decrease) this will be reflected in a relatively large (small) value for ψ_a . If we put $a_{i,n} = e_{i,n}$, we obtain the exponential scores trend statistic

$$\psi_e = \sum_{i=1}^{n} \left(i - \frac{n+1}{2} \right) e_{R_i,n} \qquad (6.3)$$

The use of this statistic is known to provide the locally most powerful rank test of $H_o : \theta = 0$ against one-sided alternatives $\theta > 0$ or $\theta < 0$ in the Lehmann trend alternative given by (6.1), (Burr and Young (1978)). The exact null distribution of ψ_e is easily calculated for small n since each of the n! possible exponential score assignments to the observations are equally likely when H_o is true. Hence we have

$$P(\psi_e \leq \xi \mid H_o) = \lambda_n(\xi)/n! \qquad (6.4)$$

where $\lambda_n(\xi)$ is the number of score assignments for which $\psi_e \leq \xi$.

The mean and variance of ψ_e are also easily found under H_o, using the results given in (4.7) with N replaced by n. We obtain

$$E(\psi_e \mid H_o) = 0 \qquad (6.5)$$

and

$$\operatorname{var}(\psi_e \mid H_o) = \sum_{i=1}^{n}\left(i - \frac{n+1}{2}\right)^2 \operatorname{var}\left(e_{R_i,n} \mid H_o\right)$$

$$+2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(i-\frac{n+1}{2}\right)\left(j-\frac{n+1}{2}\right)\operatorname{cov}\left(e_{R_i,n}, e_{R_j,n} \mid H_0\right)$$

$$= \frac{n - e_{N,N}}{n}\left\{\sum_{i=1}^{n}\left(i-\frac{n+1}{2}\right)^2 - \frac{2}{n-1}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(i-\frac{n+1}{2}\right)\left(j-\frac{n+1}{2}\right)\right\}$$

$$= n(n+1)\,(n-e_{n,n})/12 \qquad (6.6)$$

Values of the upper percentage points of the null distribution of the standardised statistic

$$\psi_e^* = \left\{\frac{12}{n(n+1)(n-e_{n,n})}\right\}^{\frac{1}{2}} \sum_{i=1}^{n} \{i - \frac{1}{2}(n+1)\}\, e_{R_i,n} \qquad (6.7)$$

are given by Burr and Young (1977b) and are reproduced in table 8. It is seen that the distribution of ψ_e^* tends rapidly to the N(0,1) distribution which has upper 10%, 5% and $2\frac{1}{2}$% points equal to 1.28, 1.64, and 1.96 respectively.

TABLE 8

Upper 100α% points of the null distribution of ψ_e^*

n	α=0.10	α =0.05	α =0.025	n	α =0.10	α =0.05	α =0.025
5	1.44	1.60	1.79	16	1.32	1.64	1.91
6	1.36	1.63	1.80	18	1.31	1.65	1.91
7	1.36	1.64	1.83	20	1.31	1.65	1.92
8	1.35	1.64	1.84	22	1.31	1.65	1.92
9	1.34	1.64	1.86	24	1.31	1.65	1.93
10	1.33	1.64	1.88	26	1.31	1.65	1.93
12	1.33	1.64	1.89	28	1.31	1.65	1.93
14	1.32	1.64	1.90	30	1.30	1.65	1.94

The exponential scores test being distribution free provides a valid test for trend whatever the form of F(x). If the distribution is specified and we take $F(x) = 1 - \exp(-\lambda x)$, $x > 0$, the model given by (6.1) is equivalent to X_i having a negative exponential distribution with parameter

$$\lambda_i = \lambda\{1+\theta(i-1)\} \quad , \qquad i = 1,\ldots,n \,. \tag{6.8}$$

When θ=0, the events form a Poisson process. Burr and Young (1978) show that the uniformly most powerful similar test of $H_0 : \theta = 0$ against the one-sided alternative $\theta > 0$ rejects H_0 for sufficiently small values of the statistic

$$R_t = \sum_{i=1}^{n} iX_i \Big/ \sum_{i=1}^{n} X_i \tag{6.9}$$

They also show that the lower 100α% point of the distribution of R_t

When θ=0, say $r_{t,\alpha}$, is given by

$$r_{t,\alpha} = 1 + (n-1)\ u_{n-1,\alpha} \tag{6.10}$$

where $u_{n,\alpha}$ is the lower 100 α % point of the distribution of the mean of n independent random variables each distributed uniformly over (0,1). Exact values of $u_{n,\alpha}$ are given by Stephens (1966) for

n = 3(1)12(2)20. For n > 20, a normal approximation may be used.

Example 3 The following data show the intervals in operating hours between successive failures of a piece of air conditioning equipment. The ranks of the observations and their exponential scores are also shown.

Serial Number(i)	1	2	3	4	5	6	7	8	9	10
Interval(x_i)	257	80	78	87	5	88	59	133	60	9
Rank(r_i)	10	6	5	7	1	8	3	9	4	2
Exponential score ($e_{r_i,10}$)	2.93	085	0.65	1.10	0.10	1.43	0.34	1.93	0.48	0.21

The observed value of the exponential scores trend statistic is

$$\psi_e^* = \frac{12}{(10)(11)(7.07)}(-10.78) = -1.34.$$

Since the null distribution of ψ_e* is symmetrical about zero, use of table 8 shows that the observed value is just significant at the 10% level showing slight evidence of a real downward trend in the intervals. The observed value of the parametric trend statistic is

$$R_t = \sum_{i=1}^{10} ix_i \Big/ \sum_{i=1}^{10} x_i = 4.29$$

From Stephen's tables we have $U_{9,0.05}$ = 0.341, $u_{9,0.10}$ = 0.376 giving $r_{t,0.05}$ = 4.07, $r_{t,0.10}$ = 4,38. The observed result is again significant at the 10% level and shows slightly more evidence of a real downward trend.

The exponential scores test for trend is most likely to be used when the investigator is confident that the underlying distributions have forms close to the exponential. If the exponential form holds exactly, there will be no loss of efficiency in using the nonparametric test based on ψ_e rather than the parametric test based on R_t in very large samples. An assessment of the loss of efficiency in small samples is given by Burr and Young (1978), who report on an investigation to determine the powers of the tests for a Poisson process against the alternative given by (6.8), for various significance levels and sample sizes. Powers were also determined for the nonparametric rank trend test based on the statistic

$$\psi_r = \sum_{i=1}^{n} \left(i - \frac{n+1}{2} \right) R_i \qquad (6.11)$$

for which the scores assigned to the observations are simply the ranks. This statistic is often used for testing for randomness in a sequence of observations against *a* trend in location. They conclude that the exponential scores test has a marked power superiority over the test based on ψ_r in small samples, but that the loss of efficiency through not using the parametric test can be quite large. Their findings are illustrated in table 9.

TABLE 9

Powers of tests of $H_0 : \theta = 0$ against $\theta > 0$ when $F_i(X) = 1-\{1-F(x)\}^{1+(i-1)\theta}$ $(i=1,\ldots,n)$, significance level $\alpha=0.05$, for (i) parametric R_t test with $F(x) = 1-e^{-x}$, (ii) exponential scores trend test, (iii) rank trend test.

θ	(i)	(ii)	(iii)	(i)	(ii)	(iii)
0.2	0.264	0.215	0.192	0.647	0.569	0.483
0.4	0.436	0.342	0.293	0.838	0.759	0.651
0.6	0.552	0.429	0.359	0.906	0.835	0.725
0.8	0.633	0.491	0.407	0.937	0.875	0.767
1.0	0.690	0.536	0.443	0.954	0.898	0.792
1.5	0.779	0.610	0.498	0.973	0.928	0.827
2.0	0.829	0.655	0.532	0.982	0.942	0.845
3.0	0.883	0.706	0.570	0.989	0.955	0.863
4.0	0.912	0.734	0.591	0.992	0.962	0.873

6.2 Tests For Serial Independence

In a renewal process, the intervals between events are independent and identically distributed. Taking this as the null hypothesis, suppose that we are interested in the alternative in which the intervals are serially correlated. Let

$$\rho(X_i, X_{i+k}) = \frac{E[\{X_i - E(X_i)\}\{X_{i+k} - E(X_{i+k})\}]}{\{\mathrm{var}(X_i) \quad \mathrm{var}(X_{i+k})\}^{\frac{1}{2}}}$$

denote the correlation coefficient between X_i and X_{i+k} If we assume that the $\{X_i\}$ have common mean μ and common variance σ^2, and that $\rho(X_i, X_{i+k},) = \rho_k$, for each i, the auto-correlation coefficient of lag k is defined by

$$\rho_k = E\{(X_i - \mu)(X_{i+k} - \mu)\} / \sigma^2. \qquad (6.12)$$

The estimator of ρ_k is the sample serial correlation coefficient of lag k defined by

$$\hat{\rho}_k = \sum^{n-k} (X_i - \bar{X})\,(X_{i+k} - \bar{X}) \,/ \sum_{i=1}^{n} (X_i - \bar{X})^2 \qquad (6.13)$$

$$= \sum_{i=1}^{n-k} Z_i \;\; Z_{i+k} / \sum_{i=1}^{n} Z_i^2 \qquad (6.14)$$

where $Z_i = X_i - \bar{X}$. Under H_0, the random variables Z_i. Z_{i+k}, $i=1,...,n-k$ are identically distributed and hence

$$E\left(\hat{\rho}_k\right) = (n-k)\, E\{(Z_1 \;\; Z_{k+1}) \,/ \sum_{i=1}^{n} Z_i^2\}$$

$$= \frac{(n-k)}{n(n-k)} E\left\{ \sum_{i \neq j}\sum Z_i \;\; Z_j \,/ \sum_{i=1}^{n} Z_i^2 \right\}$$

$$= -\frac{(n-k)}{n(n-1)} \;, \qquad (6.15)$$

since $\sum_{i \neq j}\sum Z_i Z_j = -\sum_{i=1}^{n} Z_i^2$. In particular for the serial correlation coefficient of lag 1, we have

$$E\left(\hat{\rho}_1\right) = -\, 1/n \qquad (6.16)$$

The exact variance of $\hat{\rho}_k$, under H_0 depends on the distribution of the $\{X_i\}$ and Moran (1967) shows that for k = 1 we have

$$\operatorname{var}\left(\hat{\rho}_1 \mid H_0\right) = \frac{n^2 - n + 1}{n^2(n-1)} - \frac{n+1}{n(n+1)} E\left\{ \frac{\sum_{i=1}^{n} Z_i^4}{\left(\sum_{i=1}^{n} Z_i^2\right)^2} \right\} \qquad (6.17)$$

If the $\{X_i\}$ are assumed to be exponentially distributed, we have the approximation

$$\operatorname{var}\left(\hat{p}_1 \middle| H_0\right) \approx \frac{1}{n} - \frac{7}{n^2} + \frac{52}{n^3} - \frac{398}{n^4} + \ldots, \qquad (6.18)$$

this approximation arising by approximating the expectation of the ratio in (6.17) by the ratio of expectations. For large n, we may take

$$\frac{\left(\hat{\rho}_1 + \frac{1}{n}\right)}{\left(\frac{1}{n} - \frac{7}{n^2} + \frac{52}{n^3} - \frac{398}{n^4}\right)^{\frac{1}{2}}} \overset{\text{approx}}{\sim} N(0,1) \qquad (6,19)$$

and so make an approximate test of the hypothesis that the series of events forms a Poisson process against the alternative of first order serial dependence of the intervals. The test must be used with caution in small samples (Lewis (1972)) as the convergence to normality is slow.

The test based on $\hat{\rho}_1$ is only valid when the assumed common distribution of the X_i under the null hypothesis is the negative exponential. When the assumed common distribution is not specified, the null hypothesis is that the series of events forms a renewal process. A general class of rank statistics which may be used to test this null hypothesis against the alternative of first order serial dependence is the product moment score statistic

$$V_a = \sum_{i=1}^{n-1} {}^{a}r_i,n^{a}r_{i+1},n \qquad (6.20)$$

where R_i is rank of X_i. in the ordered set of intervals and $a_{R_i,n}$ is the 'score' given to the observation X_i. If neighbouring intervals tend to be positively correlated, V_a will be relatively large while a negative correlation would result in a relatively small value of V_a.

The moments of the null distribution of V_a can be expressed in terms of sums of powers of the scores $\{a_{r,n}\}$. For the mean, we have

$$E(V_a|H_0) = \sum_{i=1}^{n-1}\left[\sum_{r\neq j} a_{r,n} a_{s,n} \Big/ \{n(n-1)\}\right]$$

$$= \left\{\left(\sum_{r=q}^{n} a_{r,n}\right)^2 - \sum_{r=1}^{n} a_{r,n}^2\right\} \Big/ n \qquad (6,21)$$

Writing

$$V_a^2 = \sum_{i=1}^{n-1} a_{R_i,n}^2\, a_{R_{i+1},n}^2 + 2\sum_{i=1}^{n-2} a_{R_i,n}^2\, a_{R_{i+1},n}^2\, a_{r_{i+2},n}$$

$$+ 2\sum_{i=1}^{n-3}\sum_{j=i+2}^{n-1} a_{R_i,n}\, a_{R_{i+1},n}\, a_{R_j,n}\, a_{R_{j+1},n}$$

we obtain

$$E(V_a^2|H_0) = \frac{1}{n}\sum_{r}\neq\sum_{s} a_{r,n}^2\, a_{s,n}^2 + \frac{2}{n(n-1)}\sum_{r}\neq\sum_{s}\neq\sum_{t} a_{r,n}\, a_{s,n}^2\, a_{t,n}$$

$$+ \frac{2}{n(n-1)}\sum_{r}\neq\sum_{s}\neq\sum_{t}\neq\sum_{u} a_{r,n}\, a_{s,n}^2\, a_{t,n}\, a_{u,n} \qquad (6,22)$$

Two particular classes of scores have been considered in the literature. If the scores are the ranks of the observations, the test statistic is

$$V_r = \sum_{i=1}^{n-1} R_i R_{i+1} \qquad (6.23)$$

with mean and variance given by

$$E(V_r|H_0) = \frac{1}{12}(3n+2)(n^2-1) \qquad (6.24)$$

$$\mathrm{var}(V_r|H_0) = \frac{(n+1)(5n^6+21n^5+501n^4-823n^3+1102n^2-68n-240)}{720(n-2)(n-3)} \qquad (6.25)$$

If the exponential scores are used, the test statistic is

$$V_e = \sum_{i=1}^{n-1} e_{R_i,}\, e_{R_{i+1},n} \qquad (6.26)$$

Use of (4.7) with N replaced by n gives

$$E\left(e_{R_i,n}\ e_{R_{i+1},n} \middle| H_0\right) = \frac{n^2 - 2n + e_{n,n}}{n(n-1)} \qquad (6.27)$$

so the exact mean is

$$E\left(V_e \middle| H_0\right) = n - 2 + n^{-1}\ e_{n,n} \qquad (6.28)$$

Cox and Lewis (1966) give the approximation

$$E\left(V_e \middle| H_0\right) \approx n - 2 + \frac{\log n + \gamma}{n} \qquad (2.29)$$

where γ=0.5772 is Euler's constant, and approximate the variance by

$$\mathrm{var}\left(V_e \middle| H_0\right) \approx \frac{n^3 - 6n^2 + 24n}{(n-2)(n-3)} - 2\log n \qquad (6.30)$$

The test statistics V_r and V_e when standardised have limiting N(0,1) distributions when the series of events forms a renewal process, but caution must again be used with small samples as the convergence to normality is relatively slow.

The theoretical study of the properties of the tests based on $\hat{\rho}_1, V_r$ and V_e requires the specification of a model which creates dependence between neighbouring intervals. A simple approach is to take the conditional distribution of X_i. given X_{i-1}.= x_{i-1} as negative exponential with a parameter depending on the observed value of X_{i-1}. The conditional p.d.f of X_i is then

$$f_i\ (x|X_i = x_{i-1}) = \lambda(x_{i-1})\ \exp\{-x\lambda(-x\lambda\,(x_{i-1})\}. \qquad 0 < x < \infty. \qquad (6.31)$$

The special form

$$\lambda(x_{i-1}) = \lambda_0(1 + \lambda_1 x_{i-1}), \qquad i = 1,2,\ldots\ldots, n \qquad (6.32)$$

with x_o = 0 was considered by Cox (1955).

Example 4. The following data $\{X_i\}$ were obtained by simulation using the model given by (6.31) and (6.32) with n = 30, $\lambda_0 = 5, \lambda_1 = 0.2$.

The ranks $\{R_i\}$ and exponential scores $\{e_{R_i,30}\}$ are also given.

X_i	2.65	2.87	15.49	33.8	49.41	73.15	4.28	4.18	17.24	4.31
R_i	9	10	23	28	29	3	12	11	24	13
$e_{R_i,30}$	0.35	0.40	1.40	2.50	3.00	4.00	0.50	0.45	1.55	0.56

X_i	4.79	12.07	11.65	12.52	8.83	1.75	1.16	17.57	2.29	6.07
R_i	14	21	20	22	17	5	3	25	7	16
$e_{R_i,30}$	0.61	1.17	1.07	1.28	0.81	0.18	0.10	1.71	0.26	0.74

X_i	0.14	9.79	10.24	2.46	1.49	22.96	27.16	5.66	1.92	0.23
R_i	1	18	19	8	4	26	27	15	6	2
$e_{R_i,30}$	0.033	0.89	0.98	0.30	0.14	1.91	2.16	0.68	0.22	0.068

A plot of X_{i+1} against X_i. is shown in figure 2 and shows some evidence of a first order serial dependence. For the parametric test, we have

$$\hat{\rho}_1 = 0.434, \quad E\left(\hat{\rho}_1 | H_0\right) = -0.0333, \quad var\left(\hat{\rho}_1 | H_0\right) = 0.0269$$

Referring the value

$$\{\hat{\rho}_1 - E(\hat{\rho}_1 | H_0)\} \;/\; \{var(\hat{\rho}_1 | H_0)\}^{\frac{1}{2}} = 2.85$$

to the N(0,l) distribution gives an observed significance level approximately equal to 0.0022.

For the exponential scores statistic, we have

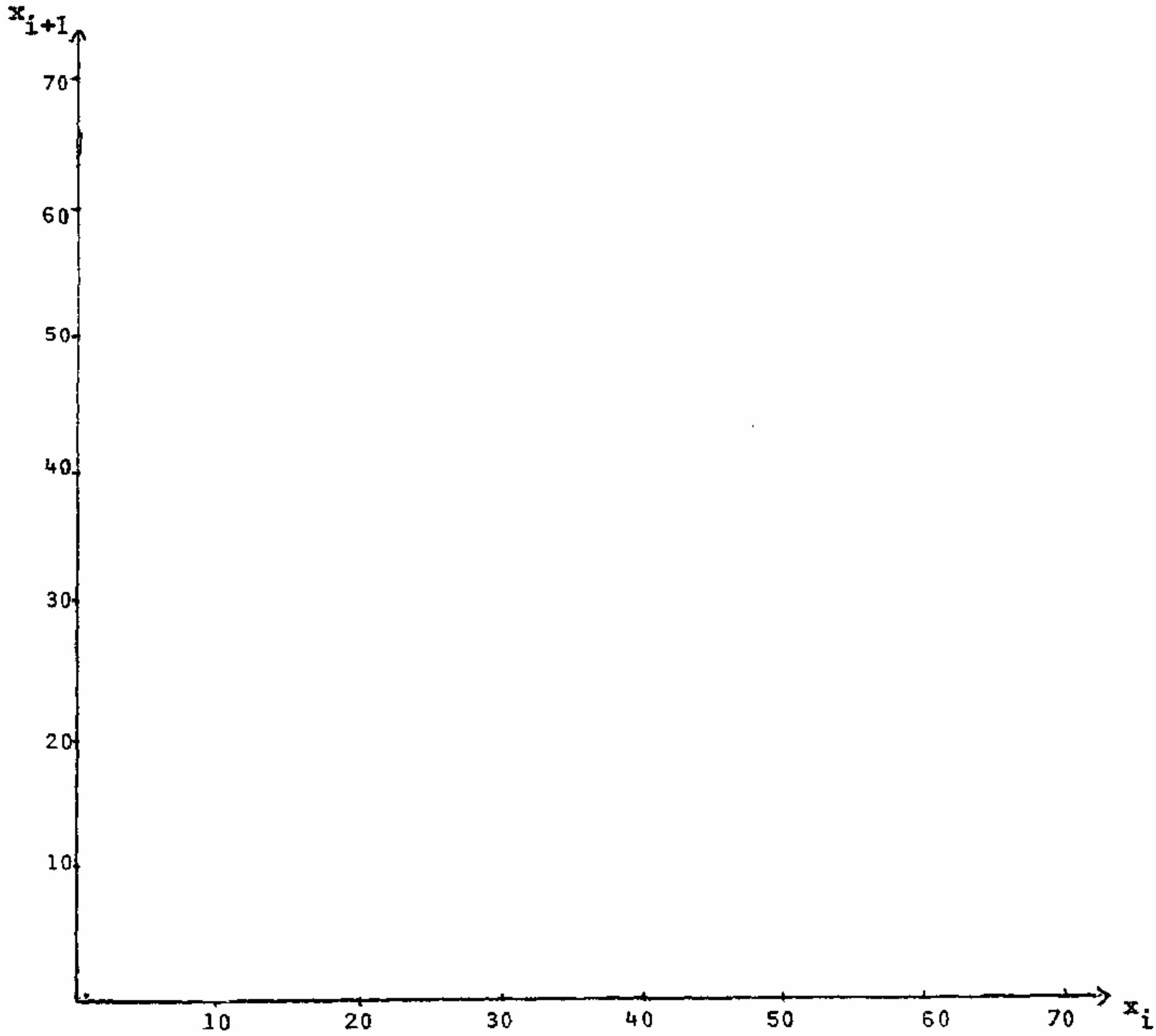
$$V_e = 40.38, \quad E\left(V_e | H_0\right) = 28.13, \quad var\left(V_e | H_0\right) = 22.72,$$

$\{V_e - E\,(V_e | H_0)\} \;/\; \{var(v_e | H_0)\}^{\frac{1}{2}} = 2.57$ and an approximate observed significance level of 0.0051.

Finally, the standardised value of the rank statistic is

$$\{v_r - E(v_r|H_0)\} \;/\; \{var(v_r|H_0)\}^{\frac{1}{2}} = 1.70$$

with an associated observed significance level equal to 0.045. For this data set, the evidence against H_0 shown by the rank test is much less than that given by the exponentical scores test



Figur 2. Scatter diagram plot of x_{i+1} against x_i

REFERENCES

Basu. A.P.(1968). On a generalised Savage statistic with applications to life testing. Ann. Math. Statist., 39, 1591-1604.

Burr, P.C. and Young, D.H. (1975a). Considerations of power of the exponential scores test and other tests under the Lehmann model. Tech. Report STR/4. Brunel University.

Burr, P.C. and Young D.H. (1975b). The exponential scores test for the two-sample problem with censoring. Tech. Report STR/10, Brunel University.

Burr, P.C. and Young, D.H. (1977a). Properties of exponential scores test for the k-sample problem under the Lehmann model. J.R.Statist. Soc. B., 39, 79-85.

Burr, P.C. and Young, D.H. (1977b). Critical values for the exponential scores trend statistic. Biometrica, 64, 640-642.

Burr, P.C. and Young, D.H. (1978).T he power of the exponential scores test for an ordinary renewal process against trend alternatives. Cormmun. Statist., A7, 461-473

Cox, D.R. (1964) Some applications of exponential ordered scores. J.R.Statist. Soc. B., 26, 103-110.

Cox, D.R. and Lewis, P.A.W. (1966). The statistical analysis of series of events, London-Methuen.

Downton, F. (1976) Non-parametric tests for block experiments. Biometrika, 63, 137-142.

Hajek, J. (1969) A course in nonparametric statistics, Holden-Day.

Jackson, O.A.Y. (1967) An analysis of departures from the exponential distribution. J.R.Statist. Soc. B., 29, 540-549.

Kiefer, J. (1959) K-sample analogues of the Kolmogorov-Smirnov and Cramér V. Mises tests. Ann. Math. Statist., 30, 420-444.

Kruskal, W.M. and Wallis, W.A. (1952). Use of ranks in on way criterion analysis of variance. J.A.S.A., 47, 583-621).

Lewis, P.A.W. (1972). Stochastic point processes, Wiley, New York.

Mood, A.M. (1950). Introduction to theory of statistics, McGraw-Hill, New York.

Moran, P.A.P. (1967) Testing for serial correlation with exponentially distributed variates. Biometrika, 54, 395-401.

Puri, M.L. (1964). Asymptotic efficiency of a class of c-sample tests. Ann. Math. Statist., 35, 102-121.

Savage, I.R. (1956). Contributions to the theory of rank order statistics - the two sample case. Ann. Math Statist, 27, 590-615.

Shapiro, S.S. and Wilk, M.B. (1972). An analysis of variance test for the exponential distribution (complete samples). Technometrics, 14, 355-370.

Stephens, M.A. (1966). Statistics connected with the uniform distribution: percentage points and application to testing for randomness of directions. Biometrika, 53, 235-40.