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Rational quadratic spline interpolation to monotonic data.

R. Delbourgo and J.A. Gregory

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Abstract

In an earlier paper by Gregory & Delbourgo (1982), a piecewise rational quadratic function is developed which produces a monotonic interpolant to monotonic data. This interpolant gives visually pleasing curves and is of continuity class C^1. In the present paper, the data is restricted to be strictly monotonic and it is shown that it is possible to obtain a monotonic rational quadratic spline interpolant which is of continuity class C^2. An $O(h^4)$ convergence analysis is included.

1. Introduction

A set of data points (x_i, f_i), $i=1,\dots,n$, is given, with $x_1 < x_2 < \dots < x_n$ and such that the values f_i form a strictly monotonic sequence. In the subsequent work it will be assumed that

$$f_1 < f_2 < \dots < f_n,$$

since the case of a strictly decreasing sequence of function values can be treated in a similar manner.

In Gregory and Delbourgo (1982), a piecewise rational quadratic function $s(x) \in C^1[x_1, x_n]$ is constructed which is monotonic on $[x_1, x_n]$ and satisfies

$$s(x_i) = f_i, \; s^{(1)}(x_i) = d_i, \; i = 1, \dots, n,$$

where the derivatives d_i are positive for strictly increasing f_i. The piecewise rational quadratic $s(x)$ is defined as follows: Let

$$\begin{aligned} h_i &= x_{i+1} - x_i, \\ \theta &= (x - x_i) / h_i, \\ \Delta_i &= (f_{i+1} - f_i) / h_i \end{aligned} \tag{1.1}$$

Then for $x \in [x_i, x_{i+1}]$,

$$s(x) = \frac{f_{i+1}\,\theta^2 + \Delta_i^{-1}(f_{i+1}\,d_i + f_i\,d_{i+1})\,\theta(1-\theta) + f_i(1-\theta)^2}{\theta^2 + \Delta_i^{-1}(d_i + d_{i+1})\,\theta(1-\theta) + (1-\theta)^2}. \tag{1.2}$$

The denominator is strictly positive for all $0 \le \theta \le 1$. Also, a differentiation gives the result that for $x \in [x_i, x_{i+1}]$,

$$s^{(1)}(x) = \frac{d_{i+1}\,\theta^2 + 2\Delta_i\,\theta(1-\theta) + d_i(1-\theta)^2}{\{\theta^2 + \Delta_i^{-1}(d_i + d_{i+1})\,\theta(1-0) + (1-\theta)^2\}^2}, \tag{1.3}$$

and hence $s^{(1)}(x) > 0$ throughout any interval $[x_i, x_{i+1}]$.

In the earlier paper by Gregory and Delbourgo (1982), the derivative Values d_i are determined by local approximations which involve the values f_i. These approximations give a $C^1[x_1, x_n]$ interpolant for which an $O(h^3)$ convergence result can be obtained. In the present paper, positive values of the derivatives d_i are determined in an analogous way to cubic polynomial spline interpolation, which make $s(x) \in C^2[x_1, x_n]$. Furthermore, it is shown that an $O(h^4)$ convergence result can be obtained when accurate derivatives d_1 and d_n are available as end conditions.

It should be noted that if the data is monotonic but not strictly monotonic, then there will be intervals $[x_i, x_{i+1}]$ where $\Delta_i = 0$. The requirement that s(x) be monotonic then implies that $s(x) = f_i$, a constant, on $[x_i, x_{i+1}]$. Elsewhere, the data can be divided into strictly monotonic parts and the proposed method of this paper can be applied.

2. The Monotonic Rational Quadratic Spline

If s(x) is a C^2 function then, necessarily, there is no jump discontinuity in the second derivatives of s(x) at the interior knots x_i, $i = 2, \ldots, n-1$, For cubic polynomial splines, such C^2 consistency conditions lead to a set of linear equations each relating three consecutive derivatives d_i. For the piecewise rational quadratic function employed here, corresponding consistency equations arise which will be non-linear. These are derived below and will then be shown to have a unique solution with all $d_i > 0$.

The requirement for C^2 continuity, namely that $s^{(2)}(x_{i+}) - s^{(2)}(x_{i-}) = 0$ at all the interior knots, gives

$$\frac{2}{h_i}[\Delta_i + d_i(1 - \frac{d_i + d_{i+1}}{\Delta_i})] + \frac{2}{h_{i-1}}[\Delta_{i-1} + d_i(1 - \frac{d_{i-1} + d_i}{\Delta_{i-1}})] = 0.$$

This can be written as

$$d_i[-c_i + a_{i-1} d_{i-1} + (a_{i-1} + a_i) d_i + a_i d_{i+1}] = b_i, \quad i=2,\ldots n-1, \tag{2.1}$$

where

$$a_i = 1/(h_i\Delta_i),$$
$$b_i = \Delta_{i-1}/h_{i-1} + \Delta_i/h_i,$$
$$c_i = 1/h_{i-1} + 1/h_i. \tag{2.2}$$

Given d_1 and d_n, (2.1) gives a system of $n-2$ non-linear equations for the unknowns $d_2,\ldots,d_{n-1}$. It should be noted that $c_i > 0$ and, for data which is strictly increasing, $a_i > 0$, $b_i > 0$ for all i in equations (2.1).

The existence and uniqueness of a solution $d_2,\ldots,d_{n-1}$ of the non-linear equations (2.1) with all $d_i > 0$ will first be proved by analysing a Jacobi typeof iteration. It will then be shown that a Gauss-Seidel type of iteration can be used in practice.

Each equation (2.1) is a quadratic in the variable d_i . Solving for the positive root gives

$$d_i = \frac{1}{2(a_{i-1} + a_i)}[c_i - a_{i-1}d_{i-1} - a_1d_{i+1} + \{(C_i - a_{i-1}d_{i-1} - a_id_{i+1})^2 + 4(a_{i-1} + a_i)b_i\}^{\frac{1}{2}}], i = 2,\ldots,n-1, \tag{2.3}$$

A Jacobi iteration may be defined by the equation

$$d_i^{(k+1)} = \frac{1}{2(a_{i-1} + a_i)}[c_i - a_{i-1}d_{i-1}^{(k)} - a_i d_{i+1}^{(k)} + \{(c_i - a_{i-1}d_{i-1}^{(k)} - a_i d_{i+1}^{(k)})^2 + 4(a_{i-1} + a_i)b_i\}^{\frac{1}{2}}], i = 2,\ldots, n-1, \tag{2.4}$$

where $d_1^{(k=1)} = d_1^{(k)} = d_1$ and $d_n^{(k+1)} = d_n^{(k)} = d_n$ are given end condition.

Theorem 2.1. (Existence) For strictly increasing data and given end conditions $d_1 \geq 0$, $d_n \geq 0$, there exits a strictly positive solution $d_2, \ldots, d_{n-1}$ satisfying the non-linear consistency equations.

Proof. A set of functions G_i, $i=1, \ldots, n$, is defined initially on the domain R^n by

$$G_1(\underline{\xi}) = d_1$$

$$G_i(\underline{\xi}) = \frac{1}{2(a_{i-1} + a_i)}[c_i - a_{i-1}\xi_{i-1} - a_i\xi_{i+1} + \{(c_i - a_{i-1}\xi_{i-1} - a_i\xi_{i+1})^2 + 4(a_{i-1} + a_i)b_i)^{\frac{1}{2}}], i = 2, \ldots, n-1$$

$$G_n(\underline{\xi}) = d_n, \tag{2.5}$$

where $\underline{\xi} = (\xi_1, \ldots, \xi_n) \in R^n$. Let $\underline{G} = (G_1, \ldots, G_n)$ and $\underline{d} = (d_1, \ldots, d_n)$. Then the Jacobi iteration (2.4) assumes the form

$$\underline{d}^{(k+1)} = \underline{G}(\underline{d}^{(k)}) .$$

Restricting $\underline{\xi}$ to have positive components, we now show that there exist. constants α_i and β_i such that

$$0 < \alpha_i \leq G_1(\underline{\xi}) \leq \beta < \infty, \; i = 2, \ldots, n-1.$$

Also, for $G_1(\underline{\xi})$ and $G_n(\underline{\xi})$, we may define $\alpha_1 = \beta_1 = d_1$ and $\alpha_n = \beta_n = d_n$. Now, for $i=2, \ldots, n-1$, examination of $G_i(\underline{\xi})$ in the two cases $0 \leq a_{i-1}\xi_{i-1} + a_i\xi_{i+1} \leq c_i$ and $a_{i-1}\xi_{i-1} + a_i\xi_{i+1} > c_i$ gives

$$\beta_i = \frac{1}{2(a_{i-1} + a_i)}[c_i + \{c_i^2 + 4(a_{i-1} + a_i)b_i\}^{\frac{1}{2}}] .$$

Finding a strictly positive value for α_i is slightly more complicated but it can be shown that

$$\alpha_i = \min\left\{\frac{-c_i + (c_i^2 + 4(a_{i-1} + a_i)b_i\}^{\frac{1}{2}}}{2(a_{i-1} + a_i)}, \frac{2b_i}{N_i + \{N_i^2 + 4(a_{i-1} + a_i)b_i\}^{\frac{1}{2}}}\right\},$$

where $N_i = \max\{0, -c_i + (a_{i-1} + a_i)\max_{2 \leq i \leq n-1} \beta_i\}$. Thus if $I_i = [\alpha_i, \beta_i]$,

$i=1,\ldots,n$, then the map $\underline{G}$ can be restricted to the n-dimensional interval $I = I_1 x \ldots x I_n$, where $\underline{G} : I \to I$ and hence maps positive vectors into positive vectors.

Next, $\underline{G}$ is shown to be a contraction mapping on I: Let $\underline{\xi}, \underline{\eta} \in I$ and let

$$X_i = c_i - a_{i-1}\xi_{i-1} - a_i\xi_{i+1}, Y_i = c_i - a_{i-1} \eta_{i-1} - a_i \eta_{i+1}.$$

Then, for $i=2,\ldots,n-1$,

$$G_i(\underline{\xi}) - G_i(\underline{\eta}) = \frac{1}{2(a_{i-1} + a_i)} [X_i - Y_i + \{X_i^2 + 4(a_{i-1} + a_i)b_i\}^{\frac{1}{2}} - \{Y_1^2 + 4(a_{i-I} + a_i)b_i\}^{\frac{1}{2}}\}$$

$$= \frac{X_i - Y_i}{2(a_{i-1} + a_i)} \left[1 + \frac{X_i + Y_i}{\{x_i^2 + 4(a_{i-1} + a_i)b_i\}^{\frac{1}{2}} + \{Y_i^2 + 4(a_i - 1 + a_{i-1} + a_i)b_i\}^{\frac{1}{2}}} \right],$$

and $G_1(\underline{\xi}) - G_1(\underline{\eta}) = 0$, $G_n(\underline{\xi}) - G_n(\underline{\eta}) = 0$. Now

$$|x_i - y_i| / (a_{i-1} + a_i) \leq \|\underline{\xi} - \underline{\eta}\|_\infty, \text{ and}$$

$$\frac{|X_i + Y_i|}{\{X_i^2 + 4(a_{i-1} + a_i)b_i\}^{\frac{1}{2}} + \{Y_i^2 + 4(a_{i-1} + a_i) b_i\}^{\frac{1}{2}}} \leq \frac{|X_i| + |Y_i|}{\{(|X_i| + |Y_i|)^2 + 8(a_{i-1} + a_i)b_i\}^{\frac{1}{2}}}$$

$$= \frac{1}{\{1 + 8(a_{i-1} + a_i) b_i / (|X_i| + |Y_i|)^2\}^{\frac{1}{2}}}$$

$$\leq \frac{1}{\{1 + L\}^{\frac{1}{2}}},$$

where, since each of $|X_i|$ and $|Y_i|$ has an upper bound $c_i + a_{i-1}\beta_{i-1} + a_i\beta_{i+1}$,

$$L = 2 \min_{2\leq i\leq n-1} (a_{i-1} + a_i) b_i / \max_{2\leq i\leq n-1} (c_i + a_{i-1}\beta_{i-1} + a_i\beta_{i+1})^2 > 0.$$

Hence

$$\|\underline{G}(\underline{\xi}) - \underline{G}(\underline{\eta})\|_\infty \leq \frac{1}{2} [1 + 1 / \{1 + L\}^{\frac{1}{2}}] \|\underline{\xi} - \underline{\eta}\|_\infty,$$

from which it follows that $\underline{G}$ is a contraction mapping on I. Thus the

Jacobi iteration converges to a unique fixed point $\underline{d} \in I$, i.e. $\underline{\underline{d}} = \underline{G(d)}$.,

and it follows that $\underline{d}$ is a solution of (2.1), which thus completes the proof.

Equations (2.3) are derived from (2.1) by solving for the positive root. The alternative choice of negative root must lead to a $d_i < 0$, if such a solution exists. Thus uniqueness of a positive solution of (2.1) follows directly from the uniqueness of the solution of $\underline{d} = \underline{G(d)}$, where $\underline{G}$ is a contraction map. Alternatively, uniqueness of a positive solution of (2.1) may be proved directly as follows:

<u>Theorem 2.2.</u> (Uniqueness) The solution of the non-linear consistency equations which satisfies the monotonicity conditions $d_i > 0$ is unique.

<u>Proof.</u> Assume that $d_1, \ldots, d_n$ and $e_1, \ldots, e_n$ are two sets of values each satisfying the consistency equations, where $d_1 = e_1 \geq 0$ and $d_n = e_n \geq 0$ are given and $d_i > 0, e_i > o$, $i = 2, \ldots, n-1$, Then

$$b_i / d_i + c_i - a_{i-1}d_{i-1} - (a_{i-1} + a_i)d_i - a_i d_{i+1} = 0,$$
$$b_i / e_i + c_i - a_{i-1}e_{i-1} - (a_{i-1} + a_i)e_i - a_i e_{i+1} = 0, \quad i = 2, \ldots, n-1.$$

Substraction gives

$$(e_i - d_i)\,[b_i / (d_i e_i) + a_{i-1} + a_i] = a_{i-1}(d_{i-1} - e_{i-1}) + a_i(d_{i+1} - e_{i+1})$$

.Consider the j^{th} equation, where j is chose so that

$$|e_j - d_j| = \max_{2 \leq i \leq n-1} |e_i - d_i|.$$

Then taking moduli gives

$$|e_j - d_j|\{b_j / (d_j e_j) + a_{j-1} + a_j\} \leq (a_{j-1} + a_j)|e_j - d_j|,$$

and thus

$$|e_j - d_j| b_j / (d_j e_j) \leq 0.$$

Hence $d_j = e_j$ and so $d_i = e_i$, $i = 2, \ldots, n = 1$.

In practice a Gauss-Seidel type of iteration can be used to solve (2.3). This iteration is defined by

$$d_i^{(k+1)} = \frac{1}{2(a_{i-1} + a_i)}[c_i - a_{i-1}d_{i=1}^{(k+1)} - a_i d_{i+1}^{(k)} + \{(c_i - a_{i-1}d_{i=1}^{(k+1)} - a_i d_{i+2}^{(k)})^2$$

$$+4(a_{i-1} + a_i)b_i\}^{\frac{1}{2}}],$$
$$i = 2, \ldots, n-1, \qquad (2.6)$$

where $d_1^{(k+1)} = d_1^{(k)} = d_1$ and $d_n^{(k+1)} = d_n^{(k)} = d_n$ are given end conditions.

A convenient starting vector $\underline{d}^{(0)}$ for this iteration is given by

$$d_i^{(0)} = \{b_i / (a_{i-1} + a_i)\}^{\frac{1}{2}}, \quad i = 2, \ldots, n-1.$$

<u>Theorem 2.3.</u> The Gauss-Seidel iteration (2.6) converges to the unique positive solution of the non-linear consistency equations.

<u>Proof.</u> By Theorems 2.1 and 2.2 there exist unique $d_i > 0$ satisfying

$$b_i / d_i + c_i - a_{i-1}d_{i-1} - (a_{i-1} + a_i)d_i - a_i d_{i+1} = 0, \; i = 2, \ldots, n-1.$$

Also, the Gauss-Seidel iterates satisfy

$$b_i / d_i^{(k+1)} + c_i - a_{i-1}d_{i-1}^{(k+1)} - (a_{i-1} + a_i)d_i^{(k+1)} - a_i d_{i+1}^{(k)} = 0, \; i = 2, \ldots, n-1.$$

Subtract and write $d_i^{(k)} = d_i + \varepsilon_i^{(k)}$ Then

$$[b_i / \{d_i(d_i + \varepsilon_i^{(k+1)})\} + a_{i-1} + a_i]\,\varepsilon_i^{(k+1)} = -a_{i-1}\varepsilon_{i-1}^{(k+1)} - a_i\varepsilon_{i+1}^{(k)}$$

Since $d_i + \varepsilon_i^{(k+1)} = d_i^{(k+1)} > 0$, on taking moduli we obtain

$$[b_i / \{d_i(d_i + |\varepsilon_i^{(k+1)}|)\} + a_{i-1} + a_i]|\varepsilon_i^{(k+1)}| \le a_{i-1}|\varepsilon_{i-1}^{(k+1)}| + a_i|\varepsilon_{i+1}^{(k)}|.$$

Consider the j^{th} inequality, where j is chosen so that

$$|\varepsilon_j^{(k+1)}| = \max_{2 \le i \le n-1} |\varepsilon_i^{(k+1)}| = \|\underline{\varepsilon}^{(k+1)}\|_\infty .$$

Then

$$[b_j/\{d_j(d_j + \|\underline{\varepsilon}^{(k+1)}\|_\infty)\} + a_{j-1} + a_j]\|\underline{\varepsilon}^{(k+1)}\|_\infty$$

$$\leq a_{j-1}\|\underline{\varepsilon}^{(k+1)}\|_\infty + a_j\|\underline{\varepsilon}^{(k)}\|_\infty,$$

which reduces to

$$\|\underline{\varepsilon}^{(k+1)}\|_\infty \leq \frac{a_j\|\underline{\varepsilon}^{(k)}\|_\infty}{a_j + b_j/\{d_j(d_j + \|\underline{\varepsilon}^{(k+1)}\|_\infty)\}}$$

It follows that

$$\|\underline{\varepsilon}^{(k+1)}\|_\infty \leq \beta\ \|\underline{\varepsilon}^{(k)}\|_\infty,$$

where

$$\beta = \frac{a_j}{a_j + b_j/\{d_j(d_j + \|\underline{\varepsilon}^{(0)}\|_\infty)\}}$$

And $0 < \beta < 1$. Thus $\|\underline{\varepsilon}^{(k)}\|_\infty \to 0$ as $k \to \infty$ and hence $d_i^{(k+1)} \to d_i$, $i=2,\ldots,n-1$.

3. Convergence Analysis of Rational Quadratic Spline

We begin by quoting a theorem which was given with proof in the earlier paper Gregory and Delbourgo (1982) and which will be required in the subsequent work.

Theorem3.1 Let $f(x) \in C^4[x_1, x_n]$ and $f^{(1)}(x) > 0$ on $[x_1, x_n]$. Let $s(x)$ be the piecewise rational quadratic interpolant such that $s(x_i) = f(x_i)$ and $s^{(1)}(x_i) = d_i \geq o$. then for $x \in [x_i, x_{i+1}]$, $i = 1,\ldots,n-1$

$$|f(x) - s(x)| \leq \frac{h_i}{4c}\|f^{(1)}\| \max\{|f_i^{(1)} - d_i|, |f_{i+1}^{(1)} - d_{i+1}|\}$$

$$+ \frac{h_i^4}{384c}[\|f^{(4)}\|\|f^{(1)}\| + \frac{2}{3}h_i\|f^{(3)}\|^2 + 2\|f^{(2)}\|\|f^{(3)}\|], \quad (3.1)$$

Where $h_i = x_{i+1} - x_i$, c is a constant independent of i whose value is at least

$\frac{1}{2} \min_{x_1, x_n} f^{(1)}(x)$ and $\|\cdot\|$ denotes the uniform norm on $[x_1, \quad x_n]$.

The next theorem establishes an upper bound for $\max_{2 \le i \le n-1} \left| f_i^{(1)} - d_i \right|$ when the d_i are the solutions of the non-linear consistency conditions (2.1).

Theorem 3.2 Let $d_1 = f_n^{(1)}$ and $d_n = f_n^{(1)}$ in the rational quadratic spline interpolant. Then, with the assumptions of Theorem 3.1 and for h sufficiently small,

$$\max_{2 \le i \le n-1} \left| f_i^{(1)} - d_i \right| \le \frac{h^3 K(h) \left\| f^{(1)} \right\|}{2m^3 / \left\| f^{(1)} \right\| - h^3 K(h)}, \tag{3.2}$$

where

$$k(h) - \frac{1}{12} \{7 \left\| f^{(1)} \right\| \quad \left\| f^{(4)} \right\| \quad + \quad \left\| f^{(2)} \right\| \quad \left\| f^{(3)} \right\| \} \quad + o(h), \tag{3.3}$$

and $\quad h = \max h_i, \quad m = \min_{[x_1, x_n]} f^{(1)}(x) > 0$ (3.4)

Thus $\max_{2 \le i \le n-1} \left| f_i^{(1)} - d_i \right| = o(h^3)$.

Proof. Consider the consistency equations

$$b_i / d_i + c_i - a_{i-1} d_{i-1} - (a_{i-1} + a_i) d_i - a_i d_{i+1} = 0$$

and let

$$b_i / f_i^{(1)} + c_i - a_{i-1} f_{i-1}^{(1)} - (a_{i-1} + a_i) f_i^{(1)} - a_i f_{i+1}^{(1)} = E_i, \quad i = 2, \ldots, n-1. \tag{3.5}$$

where, from (3.4), $0 < 1 / f_i^{(1)} < 1 / m$. Subtracting and writing

$$d_i - f_i^{(1)} = \lambda_i \tag{3.6}$$

gives

$$b_i \lambda_i / \{ f_i^{(1)} (f_i^{(1)} + \lambda_i) \} + a_{i-1} \lambda_{i-1} + (a_{i-1} + a_i) \lambda_i + a_i \lambda_{i+1} = E_i, \quad i = 2, \ldots, n-1, \tag{3.7}$$

where we require a bound on $\max_{2\le i\le n-1}|\lambda_i|$. Now, from (3.5) and the definitions (2.2), it follows that

$$E_i\ h_{i-1}\ h_i\ \Delta_{i-1}\ \Delta_i = \{h_i\ \Delta_{i-1}^2\ \Delta_i + h_{i-1}\ \Delta_{i-1}\ \Delta_i^2\} / f_i^{(1)} + (h_i + h_{i-1})\ \Delta_{i-1}\ \Delta_i$$

$$- h_i\ \Delta_i\ (f_{i-1}^{(1)} + f_i^{(1)}) - h_{i-1}\ \Delta_{i-1}\ (f_i^{(1)} + f_{i+1}^{(1)}) .$$

On the right the following Taylor expansions are made:

$$\Delta_{i-1} = f_i^{(1)} - \frac{1}{2} h_{i-1}\ f_i^{(2)} + \frac{1}{6} h_{i-1}^2\ f_i^{(3)} - \frac{1}{24} h_{i-1}^3\ f_{i-\alpha}^{(4)},$$

$$\Delta_i = f_i^{(1)} + \frac{1}{2} h_i\ f_i^{(2)} + \frac{1}{6} h_i^2\ f_i^{(3)} + \frac{1}{24} h_i^3\ f_{i+\beta}^{(4)} ,$$

$$f_{i-1}^{(1)} = f_i^{(1)} - h_{i-1}\ f_i^{(2)} + \frac{1}{2} h_{i-1}^2\ f_i^{(3)} - \frac{1}{6} h_{i-1}^3\ f_{i-\gamma}^{(4)},$$

$$f_{i+1}^{(1)} = f_i^{(1)} - h_i\ f_i^{(2)} + \frac{1}{2} h_i^2\ f_i^{(3)} + \frac{1}{6} h_i^3\ f_{i+\delta}^{(4)} ,$$

where $f_{i-\alpha}^{(4)}$ means $f^{(4)}(x_i - \alpha h_{i-1})$, $0 < \alpha < 1$, etc. After some algebra, the result of these substitutions gives

$$E_i \Delta_{i-1}\ \Delta_i = f_i^{(1)} \{\frac{1}{8}(h_i^2\ f_{i+\beta}^{(4)} - h_{i-1}^2\ f_{i-\alpha}^{(4)}) - \frac{1}{6}(h_i^2\ f_{i+\delta}^{(4)} - h_{i-1}^2\ f_{i-\gamma}^{(4)}\}$$

$$+ \frac{1}{12}(h_i^2 - h_{i-1}^2)\ f_i^{(2)}\ f_i^{(3)} + o(h^3) .$$

Now $\Delta_{i-1}\ \Delta_i \ge m^2$, where m is defined by (3.4). Thus it follows that

$$m^2 |E_i| \le \frac{1}{12} h^2 \{7 \|f^{(1)}\|\ \|f^{(4)}\| + \|f^{(2)}\|\ \|f^{(3)}\|\} + o(h^3) .$$

Hence

$$|E_i| \le m^{-2}\ h^2\ K(h) , \tag{3.8}$$

where $K(h)$ is defined by (3.3). We now consider equation (3.7) with index $i = j$ taken so that $|\lambda_j| = \max_{2 \le i \le n-1} |\lambda_i|$. Then

$$[b_j / \{f_j^{(1)} (f_j^{(1)} + \lambda_j)\} + a_{j-1} + a_j]\ \lambda_j = E_j - a_{j-1}\ \lambda_{j-1} - a_j\ \lambda_{j+1} ,$$

where $|\lambda_j| = \|\underline{\lambda}\|_\infty$, since $\lambda_1 = 0 = \lambda_n$. Taking moduli and noting that

$0 < f_j^{(1)} + \lambda_j \le f_j^{(1)} + \|\underline{\lambda}\|_\infty$ gives

$$[b_j / \{f_j^{(1)} (f_j^{(1)} + \|\underline{\lambda}\|_\infty)\} + a_{j-1} + a_j] \|\underline{\lambda}\|_\infty \le |E_j| + (a_{j-1} + a_j) \|\underline{\lambda}\|_\infty .$$

This inequality reduces to

$$\|\underline{\lambda}\|_\infty \le f_j^{(1)} |E_j| / \{b_j / f_j^{(1)} - |E_j|\}, \tag{3.9}$$

under the assumption that the denominator is positive. Now

$$b_j / f_j^{(1)} = (\Delta_{j-1} / h_{j-1} + \Delta_j / h_j) / f_j^{(1)},$$

$$= (f_{j-\theta}^{(1)} / h_{j-1} + f_{j+\varphi}^{(1)} / h_j) / f_j^{(1)} \text{ for some } 0 < \theta, \phi < 1,$$

$$\ge 2m / \{h \|f^{(1)}\|\} .$$

Thus, from (3.8)

$$b_j / f_j^{(1)} - |E_j| \ge 2m / \{h \|f^{(1)}\|\} - m^{-2} h^2 K(h) \tag{3.10}$$

which is positive for h sufficiently small. Finally, substituting (3.10) and (3.8) in (3.9) gives the desired result.

Remark. When the results of Theorems 3.1 and 3.2 are taken together, it can be seen that $f(x) - s(x) = 0(h^4)$ on the assumption that $d_1 = f_1^{(1)}$ and $d_n = f_n^{(1)}$ are given end conditions.

4. Numerical Results and Discussion

Our first set of results is concerned with the order of convergence of the interpolation scheme. Tables 1 and 2 show the interpolation errors arising from the application of the rational quadratic spline scheme to

$f(x) = \exp(x)$ over [0,1] when the exact choice of end conditions $d_1 = f^{(1)}(0) = 1$ and $d_1 = f^{(1)}(1) = 1 \quad a = \exp(1)$ is made. The knots are taken to be equally spaced with four choices of interval lengths, namely h = 0.2, 0.1, 0.05, 0.025. In one experiment, the errors e_1, e_2, e_3, e_4 corresponding to these four choices of h are evaluated at $\theta = 1/3$, where, for each h, the interval of interpolation is that containing the point x = 0.86. In a second experiment the four intervals containing the point x = 0.86 are selected with $\theta = 2/3$.

error e_1 (h = 0. 2)	error e_1 (h = 0. 1)	error e_3 (h = 0.0 5)	error e_4 (h = 0. 25)	e_1/e_2	e_2/e_3	e_3/e_4
$-.45217\times10^{-5}$	$-.26477\times10^{-6}$	$-.16973\times10^{-7}$	$-.1046\times10^{-8}$	17.08	15.60	16.22

Table 1. Rational quadratic spline interpolation errors at $\theta = 1/3$ in interval containing x = 0.26, f(x) = exp(x).

error e_1 (h = 0. 2)	error e_2 (h = 0. 1)	error e_3 (h = 0. 05)	error e_4 (h = 0. 25)	e_1/e_2	e_2/e_3	e_3/e_4
$-.84774\times10^{-5}$	$-.47378\times10^{-6}$	$-.30788\times10^{-7}$	$-.1902\times10^{-8}$	17.89	15.39	16.19

Table 2. Rational quadratic spline interpolation errors at $\theta = 2/3$ in interval containing x = 0.86, f(x) = exp(x).

The theory of Section 3 shows that a convergence rate of $0(h^4)$ is expected and this is confirmed by both tests which clearly show the tendency of the ratios e_k / e_{k+1} to approach the value 2^4.

Our second set of results is concerned with the application of the rational spline scheme to the monotonic data sets of Tables 3, 4, and 5.

x	7.99	8.09	8.19	8.7	9.2	10	12	15	20
Y	0	2.76429×10^{-5}	4.37498×10^{-2}	0.169183	0.469428	0.943740	0.998636	0.999919	0.999994

Table 3. Monotonic Data Set 1 [Fritsch & Carlson (1980)]

x	22	22.5	22.6	22.7	22.8	22.9	23	23.1	23.2	23.3	23.4	23.5	24
y	523	543	550	557	565	575	590	620	860	915	944	958	986

Table 4. Monotonic Data Set 2 [pruess (1979)]

x	0	2	3	5	6	8	9	11	12	14	15
y	10	10	10	10	10	10	10.5	15	50	60	85

Table 5. Monotonic Data Set 3 [Akima (1970); Fritsch & Carlson (1980)]

Both the Fritsch-Carlson radio-chemical data of Table 3 and the Akima data of Table 5 are used in Gregory & Delbourgo (1982) in connection with the piecewise rational quadratic C^1 scheme proposed there. These data sets are also used by Fritsch & Carlson (1980), where the need for good monotonic interpolants is clearly illustrated by the poor behaviour of other interpolation methods.

In general, to apply the C^2 rational spline scheme of this paper, it is necessary to set the end derivatives d_1 and d_n to suitable non-negative values. Two possible methods are explored below. It should be noted that for the Akima data, s(x) is constant over the interval [0,8] and the rational spline scheme is applied only over [8,15], The condition $d_1=0$ is then imposed at the left hand end point x=8 of this interval, where s (x) will be C^1.

Method 1. This is based on the three point difference approximations

$$d_1 = \Delta_1 + (\Delta_1 - \Delta_2)\, h_1 / (h_1 + h_2) ,$$

if the expression on the right is positive, otherwise d_1 is set to zero;

$$d_n = \Delta_{n-1} + (\Delta_{n-1} - \Delta_{n-2})\; h_{n-1} / (h_{n-2} + h_{n-1}) ,$$

if the expression on the right is positive, otherwise d_1 is set to zero.

Here each of $f_1^{(1)} - d_1$ and $f_n^{(1)} - d_n$ is $0(h^2)$.

Method 2 Non-linear approximations for d_1 and d_n are given by

$$d_1 = \Delta_1 (\Delta_1 / \{ (f_3 - f_1) / (x_3 - x_1) \})^{h_1 / h_2},$$

$$d_n = \Delta_{n-1} (\Delta_{n-1} / \{ f_n - f_{n-2}) / (x_n - x_{n-2}) \})^{h_{n-1} / h_{n-2}} .$$

Here, as in Method 1, each of $f_1^{(1)} - d_1$ and $f_n^{(1)} - d_n$ each $0(h^2)$, as can be shown by a Taylor expansion argument. These approximations are an improvement on the non-linear end conditions quoted in Gregory & Delbourgo (1982) and are identical with these conditions in the case of equal intervals.

Figures 1, 2, and 3 show the results of applying the rational spline scheme to the three given data sets. The scheme is implemented with the end conditions described by Method 2. (End conditions- based on Method 1 gives graphs little different from those shown.) For the purposes of comparison, the C^1 piecewise cubic interpolant using the $\mathcal{L}_2$ monotonicity region recommended by Fritsch & Carlson is shown. Also; the C^1 piecewise rational quadratic interpolant based on the second method of derivative approximation recommended by Gregory & Delbourgo (1982) is shown. For the Data Set 1, the extra degree of continuity of the rational spline scheme is apparent at the knot x = 10 when compared with the C^1 schemes. The Data Set 2 illustrates a behaviour which is to be expected of any spline Scheme. Here, due to the nature of the data, the C^2 constraint has lead to more variation in the curve than that given by the rational quadratic C^1 scheme. However, in general it can be seen that the rational spline scheme produces good curves.

6. Conclusion

A method of constructing a C^2 monotonic interpolant to given monotonic data has been described. This method is based on a rational quadratic spline representation and involves the solution of a non-linear system of consistency equations. The iterative solution of this system means that the method involves more work than existing C^1 methods. However, the method seems to produce visually pleasing curves which have the advantage of being twice continuously differentiable and $0(h^4)$ convergent.

References

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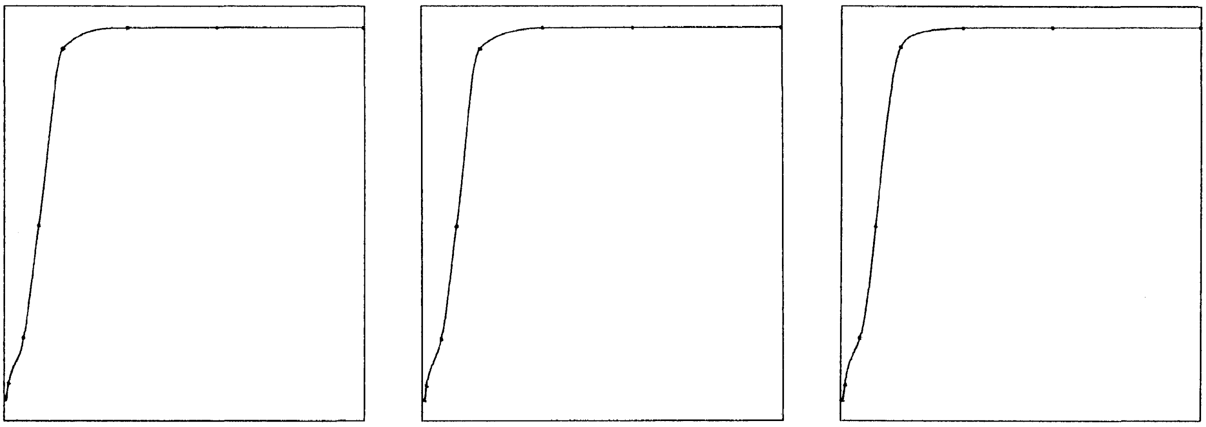

(i) (ii) (iii)

Fig. 1.Results for monotonic data set 1. (i) Fritsch-Carlson; (ii) C^1 piecewise rational quadratic; (iii) C^2 rational quadratic spline.

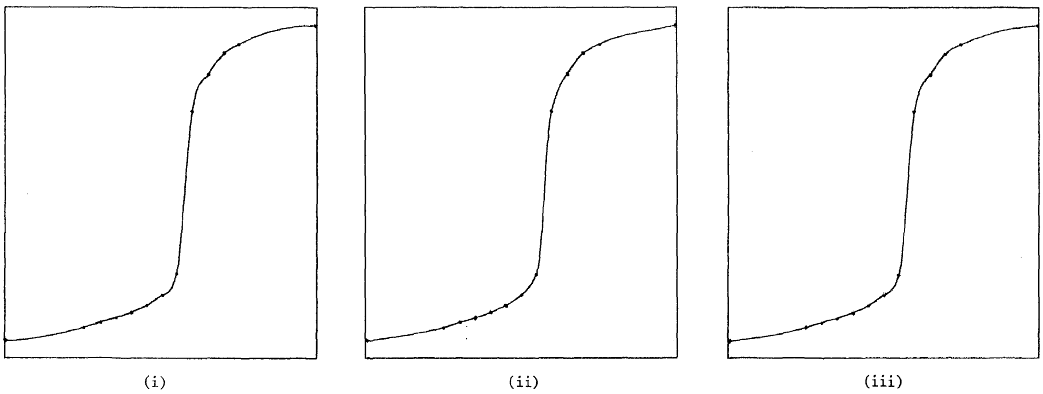

Fig. 2. Results for monotonic data set 2. (i) Fritsch-Carlson; (ii) C piecewise rational quadratic; (iii) C^2 rational quadratic spline.

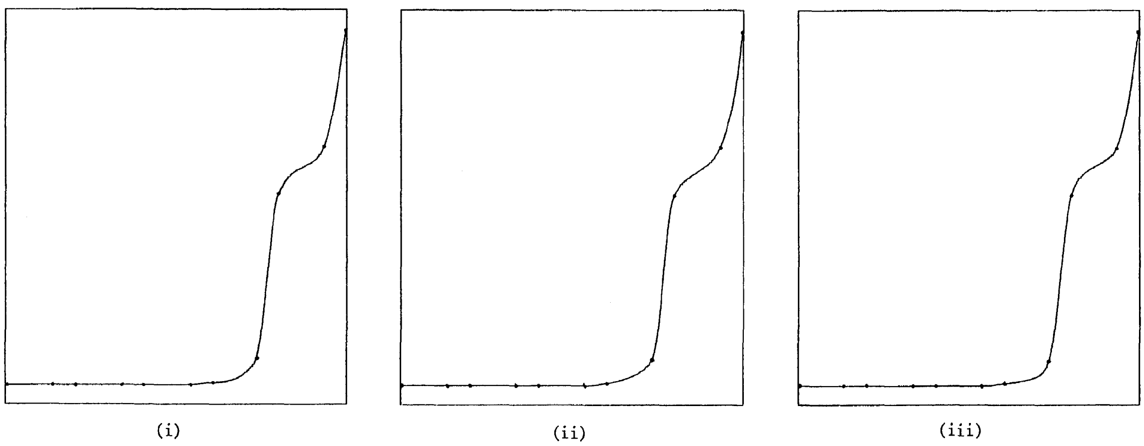

Fig. 3. Results for monotonic data set 3. (i) Fritsch-Carlson; (ii) C^1 piecewise rational quadratic; (iii) C^2 rational quadratic spline.