Research Article

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Localized boundary-domain singular integral equations of the Robin type problem for self-adjoint second-order strongly elliptic PDE systems

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Abstract: The paper deals with the three-dimensional Robin type boundary-value problem (BVP) for a second-order strongly elliptic system of partial differential equations in the divergence form with variable coefficients. The problem is studied by the localized parametrix based potential method. By using Green's representation formula and properties of the localized layer and volume potentials, the BVP under consideration is reduced to the a system of localized boundary-domain singular integral equations (LBDSIE). The equivalence between the original boundary value problem and the corresponding LBDSIE system is established. The matrix operator generated by the LBDSIE system belongs to the Boutet de Monvel algebra. With the help of the Vishik–Eskin theory based on the Wiener–Hopf factorization method, the Fredholm properties of the corresponding localized boundary-domain singular integral operator are investigated and its invertibility in appropriate function spaces is proved.

Keywords: System of partial differential equations with variable coefficients, boundary value problem, localized parametrix, localized boundary-domain singular integral equations, pseudodifferential operators

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1 Introduction

We consider a three-dimensional Robin type boundary-value problem (BVP) for a second-order strongly elliptic system of partial differential equations (PDE) in the divergence form with variable coefficients. The problem is investigated with the help of generalized layer and volume potentials whose kernel functions are constructed by a *localized parametrix*.

In [9, 10], the localized boundary-domain singular integral equations (LBDSIE) approach is developed for the Dirichlet, Neumann, Robin and mixed boundary value problems for a scalar elliptic second-order PDE with variable coefficients (see also [8]). In [11], the LBDSIE system approach was used to investigate the Dirichlet problem for a self-adjoint second-order strongly elliptic system of PDEs with variable coefficients.

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The Robin type BVP treated in the present paper is well investigated by the variational method and also by the usual classical potential method when the corresponding fundamental solution matrix is available in explicit form, e.g., in the case of constant coefficients (see, e.g., [15–17]).

Our goal here is to develop the *localized potentials method* to the Robin type BVP for a second-order strongly elliptic system.

The paper is organized as follows. First, using Green's representation formula and properties of the localized layer and volume potentials, we reduce the Robin type BVP to the *localized boundary-domain singular integral equations system*. Further, we establish that the original boundary value problem is equivalent to the corresponding LBDSIE system, which proved to be a quite nontrivial problem and plays a crucial role in our analysis. Note that the corresponding *localized boundary-domain singular integral operator* (LBDSIO) belongs to the Boutet de Monvel algebra. With the help of the Vishik–Eskin theory, based on the Wiener–Hopf factorization method, we investigate Fredholm properties of the LBDSIO and prove its invertibility in appropriate function spaces. The invertibility of the LBDSIO is very important from the point of view of numerical analysis since the LBDSIE approach leads to very convenient numerical schemes in applications (for details see [18, 22, 23, 25–27]).

For the reader's convenience we present some auxiliary material concerning classes of localizing cut-off functions and properties of localized potentials in two brief appendices.

2 Matrix harmonic parametrix-based operators

2.1 Matrix co-normal operator and Green's identities

Consider a uniformly strongly elliptic second-order formally self-adjoint matrix partial differential operator

$$A(x,\partial_x) = [A_{pq}(x,\partial_x)]_{p,q=1}^3 = \Big[\frac{\partial}{\partial x_k}\Big(a_{kj}^{pq}(x)\frac{\partial}{\partial x_j}\Big)\Big]_{p,q=1}^3,$$

where the coefficients are assumed to be infinitely smooth functions $a_{kj}^{pq} \in C^\infty(\mathbb{R}^3)$ satisfying the following symmetry conditions:

$$a_{kj}^{pq} = a_{jk}^{qp} = a_{pj}^{kq}, \quad j,k,p,q = 1,2,3.$$

Here and in what follows, the summation over repeated indices from 1 to 3 is assumed if not stated otherwise.

Motivated by applications in mathematical physics and, in particular, the theory of elasticity of anisotropic inhomogeneous solids, we assume that the coefficients a_{kj}^{pq} are real and that the quadratic form $a_{kj}^{pq}(x)\eta_{kp}\eta_{qj}$ is uniformly positive-definite with respect to symmetric variables $\eta_{kp} = \eta_{pk} \in \mathbb{R}$, which implies that the principal homogeneous symbol matrix of the operator $A(x,\partial_x)$ with opposite sign, i.e.,

$$A(x,\xi) := [a_{kj}^{pq}(x)\xi_k\xi_j]_{3\times 3}$$

is uniformly positive-definite and there are positive constants c_1 and c_2 such that

$$c_1|\xi|^2|\zeta|^2 \le A(x,\xi)\zeta\cdot\overline{\zeta} \equiv a_{kj}^{pq}(x)\xi_k\xi_j\overline{\zeta}_p\zeta_q \le c_2|\xi|^2|\zeta|^2 \quad \text{for all } x\in\mathbb{R}^3,\text{ all } \xi\in\mathbb{R}^3 \text{ and all } \zeta\in\mathbb{C}^3. \tag{2.1}$$

Here $a\cdot b := \sum_{j=1}^3 a_j b_j$ is the bilinear product of two column-vectors $a, b \in \mathbb{C}^3$.

Further, let $\Omega = \Omega^+$ be a bounded domain in $\mathbb{R}^3$ with a simply connected boundary $\partial\Omega = S \in C^\infty$, $\overline{\Omega} = \Omega \cup S$, and let $n = (n_1, n_2, n_3)$ denote the unit normal vector to S directed outward the domain Ω. Set $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega}$.

We denote by $[H^r(\Omega)]^m = [H_2^r(\Omega)]^m$ and $[H^r(S)]^m = [H_2^r(S)]^m$, $r \in \mathbb{R}$, the Bessel potential spaces for m-dimensional vector functions $u = (u_1,\dots,u_m)^\top$ on a domain Ω and on a closed manifold S without boundary, while $[\mathcal{D}(\mathbb{R}^3)]^m$ and $[\mathcal{D}(\Omega)]^m$ stand for infinitely smooth vector functions with compact support in $\mathbb{R}^3$ and in Ω, respectively, and $[\mathcal{S}(\mathbb{R}^3)]^m$ denotes the Schwartz space of rapidly decreasing vector functions in $\mathbb{R}^3$. Recall that $[H^0(\Omega)]^m = [L_2(\Omega)]^m$ is a space of square integrable vector functions in Ω.

Let us denote by $\gamma^+ u = (\gamma^+ u_1, \dots, \gamma^+ u_m)^\top$ and $\gamma^- u = (\gamma^- u_1, \dots, \gamma^- u_m)^\top$ the traces of a vector function $u = (u_1, \dots, u_m)^\top$ on S from the interior and the exterior of Ω, respectively.

We also need the following subspaces of $[H^1(\Omega)]^3$ and $[H^r(\mathbb{R}^3)]^3$, respectively:

$$[H^{1,0}(\Omega; A)]^3 := \{u \in [H^1(\Omega)]^3 : Au \in [H^0(\Omega)]^3\}, \quad [\widetilde{H}^r(\Omega)]^3 := \{u \in [H^r(\mathbb{R}^3)]^3 : \operatorname{supp} u \in \overline{\Omega}\}.$$

The co-normal derivative $T^+ u$ for a vector function $u \in [H^2(\Omega)]^3$ is well defined in the usual traces sense on the boundary $\partial\Omega$:

$$[T^\pm(x, \partial_x) u(x)]_p = a_{kj}^{pq}(x) n_k(x) \gamma^\pm \partial_{x_j} u_q(x), \quad x \in \partial\Omega, \ p = 1, 2, 3. \tag{2.2}$$

The co-normal derivative operator defined in (2.2) can be extended by continuity to the space $[H^{1,0}(\Omega; A)]^3$ and the following Green's first identity holds (for details see, e.g., [17, Chapter 4] and [19]):

$$\langle T^+ u, \gamma^+ v \rangle_{\partial\Omega} = \int_\Omega [Au \cdot v + E(u, v)]\, dx \quad \text{for all } v \in [H^1(\Omega)]^3, \tag{2.3}$$

where $\langle \cdot, \cdot \rangle_{\partial\Omega}$ denotes the duality between the adjoint spaces $[H^{-1/2}(\partial\Omega)]^3$ and $[H^{1/2}(\partial\Omega)]^3$, which extends the usual bilinear $[L_2(\partial\Omega)]^3$ inner product, while

$$E(u, v) = a_{kj}^{pq}(x) \partial_{x_j} u_q(x) \partial_{x_k} v_p(x).$$

The co-normal derivative $T^- u$ is defined similarly.

Our goal here is to develop the LBDSIE method for the Robin type problem, which reads as follows.

Problem. Find a vector function $u = (u_1, u_2, u_3)^\top \in [H^{1,0}(\Omega, A)]^3$ satisfying the differential equation

$$A(x, \partial_x) u = f \quad \text{in } \Omega \tag{2.4}$$

and the Robin type boundary condition

$$T^+ u + \varkappa \gamma^+ u = \psi_0 \quad \text{on } S = \partial\Omega, \tag{2.5}$$

where $\psi_0 = (\psi_{01}, \psi_{02}, \psi_{03})^\top \in [H^{-1/2}(S)]^3$, $f = (f_1, f_2, f_3)^\top \in [H^0(\Omega)]^3$ and $\varkappa = [\varkappa_{jk}]_{3\times 3}$ is a positive definite smooth matrix function.

Remark 2.1. Condition (2.1) implies that the quadratic form

$$E(u, u) = a_{kj}^{pq}(x) \varepsilon_{qj}(x) \varepsilon_{pk}(x) \quad \text{with} \quad \varepsilon_{qj}(x) = 2^{-1}(\partial_j u_q(x) + \partial_q u_j(x))$$

is positive definite in the symmetric variables ε_{qj}. Therefore, Green's first formula (2.3) and Korn's inequality along with the Lax–Milgram lemma imply that the Robin type BVP possesses a unique weak solution in the space $[H^{1,0}(\Omega; A)]^3$ (cf., e.g., [3, 16, 17, 20]).

2.2 Parametrix-based vector-valued integral operators and Green's third identity

As has been mentioned, our goal here is to develop the LBDSIE method for the Robin type BVP (2.4)–(2.5).

Let $P_\Delta(x) := -\frac{1}{4\pi|x|}$ denote the scalar fundamental solution of the Laplace operator, $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$. Let us define a *localized matrix parametrix* $\Phi = [\Phi_{rq}]_{3\times 3}$ for the matrix Laplace operator $I_3 \Delta$ as

$$\Phi(x) = [\Phi_{rq}(x)]_{3\times 3} := P_{\chi_\Delta}(x) I_3 = \chi(x) P_\Delta(x) I_3 = -\frac{\chi(x)}{4\pi|x|} I_3,$$

where $P_{\chi_\Delta}(x) := \chi(x) P_\Delta(x)$ is a scalar localized harmonic parametrix, I_3 is the unit 3×3 matrix, and χ is a localizing function (see Section A)

$$\chi \in X_+^k, \quad k \geq 5, \quad \text{with } \chi(0) = 1. \tag{2.6}$$

In what follows, we assume that condition (2.6) is satisfied if not stated otherwise.

In [11], the following Green's third identity is derived:

$$\mathbf{b}(y)u(y) + \mathcal{N}u(y) - \mathbf{V}(T^+u)(y) + \mathbf{W}(\gamma^+u)(y) = \mathcal{P}(Au)(y), \quad y \in \Omega, \tag{2.7}$$

where $\mathbf{b}$ is a positive definite matrix due to (2.1),

$$\mathbf{b}(x) = [b_{pq}(x)]_{p,q=1}^3, \quad b_{pq}(x) = \frac{1}{3}[a_{11}^{pq}(x) + a_{22}^{pq}(x) + a_{33}^{pq}(x)], \tag{2.8}$$

$\mathcal{N}$ is a *localized singular integral operator*

$$\mathcal{N}u(y) := \text{v.p.}\int_{\Omega}[A(x,\partial_x)\Phi(x-y)]u(x)\,dx = \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}[A(x,\partial_x)\Phi(x-y)]u(x)\,dx, \tag{2.9}$$

and $\mathbf{V}$, $\mathbf{W}$ and $\mathcal{P}$ are the *localized vector single layer, double layer* and *Newtonian volume potentials*

$$\mathbf{V}g(y) := -\int_S \Phi(x-y)g(x)\,dS_x, \quad \mathbf{W}g(y) := -\int_S [T(x,\partial_x)\Phi(x-y)]^\top g(x)\,dS_x, \tag{2.10}$$

$$\mathcal{P}h(y) := \int_\Omega \Phi(x-y)h(x)\,dx. \tag{2.11}$$

A singular integral operator similar to (2.9) with the integration domain $\mathbb{R}^3$ is denoted by

$$\mathbf{N}u(y) := \text{v.p.}\int_{\mathbb{R}^3}[A(x,\partial_x)\Phi(x-y)]u(x)\,dx. \tag{2.12}$$

Note that

$$\begin{aligned}\text{v.p.}[A(x,\partial_x)\Phi(x-y)]_{pq} &= \text{v.p.}\Big[-\frac{a_{kj}^{pq}(x)}{4\pi}\frac{\partial^2}{\partial x_k\partial x_j}\frac{1}{|x-y|}\Big] + R_{pq}(x,y)\\ &= \text{v.p.}\Big[-\frac{a_{kj}^{pq}(y)}{4\pi}\frac{\partial^2}{\partial x_k\partial x_j}\frac{1}{|x-y|}\Big] + R_{pq}^{(1)}(x,y),\end{aligned} \tag{2.13}$$

where $R_{pq}(x,y)$ and $R_{pq}^{(1)}(x,y)$ possess weak singularities of type $\mathcal{O}(|x-y|^{-2})$ as $x \to y$. From (2.9) and (2.12) it is evident that

$$\mathcal{N}u(y) = (\mathbf{N}\mathring{E}u)(y) \quad \text{for } y \in \Omega,\ u \in [H^r(\Omega)]^3,\ r \geq 0,$$

where $\mathring{E}$ stands for the extension operator by zero from Ω onto Ω^-. From decomposition (2.13) it follows that (see, e.g., [4], [14, Lemma 23.10] and [15, Theorem 8.6.1]) if $\chi \in X^{k+2}$ with an integer $k \geq 3$, then the operator

$$r_\Omega\mathcal{N} = r_\Omega\mathbf{N}\mathring{E} : [H^r(\Omega)]^3 \to [H^r(\Omega)]^3, \quad 0 \leq r < k - \frac{1}{2}, \tag{2.14}$$

is bounded since the principal homogeneous symbol of $\mathbf{N}$ is rational (see (3.1)) and the operators with the kernel functions $R(x,y)$ and $R^{(1)}(x,y)$ map $[H^r(\Omega)]^3$ into $[H^{r+1}(\Omega)]^3$.

In our further analysis, we need the mapping property (2.14) with $r > \frac{3}{2}$, which is guaranteed by the inclusion $\chi \in X^{k+2}$, $k \geq 3$.

In (2.10)–(2.11), the densities g and h are three-dimensional vector functions. Introducing the localized scalar single layer and Newtonian volume potentials

$$V_\Delta g_0(y) = -\int_S P_{\chi_\Delta}(x-y)g_0(x)\,dS_x,$$

$$\mathcal{P}_\Delta h_0(y) = \int_\Omega P_{\chi_\Delta}(x-y)h_0(x)\,dx,$$

with g_0 and h_0 being scalar density functions, we evidently have

$$[\mathbf{V}g(y)]_p = V_\Delta g_p(y), \quad [\mathcal{P}h(y)]_p = \mathcal{P}_\Delta h_p(y), \qquad p = 1,2,3,$$

for arbitrary vector functions $g = (g_1, g_2, g_3)^\top$ and $h = (h_1, h_2, h_3)^\top$.

We will also need the localized Newtonian volume potentials with the integration domain $\mathbb{R}^3$:

$$\mathbf{P}h(y) := \int_{\mathbb{R}^3} \Phi(x-y)h(x)\,dx, \quad \mathbf{P}_\Delta h_0(y) := \int_{\mathbb{R}^3} P_{\chi_\Delta}(x-y)h_0(x)\,dx. \tag{2.15}$$

Further, let us introduce the boundary operators generated by the localized layer potentials:

$$\mathcal{V}g(y) := -\int_S \Phi(x-y)g(x)\,dS_x, \qquad \mathcal{W}g(y) := -\int_S [T(x,\partial_x)\Phi(x-y)]^\top g(x)\,dS_x, \quad y \in S, \tag{2.16}$$

$$\mathcal{W}'g(y) := -\int_S [T(y,\partial_y)\Phi(x-y)]g(x)\,dS_x, \quad \mathcal{L}^\pm g(y) := T^\pm(y,\partial_y)\mathbf{W}g(y), \qquad y \in S. \tag{2.17}$$

The mapping properties of the localized vector-valued potentials are collected in Section B.

Now we prove the following lemma, which is essential in our further analysis.

Lemma 2.2. *Let* $\chi \in X^3_{1+}$, $\varphi \in [H^{3/2}(\partial\Omega)]^3$ *and* $f \in [H^0(\Omega)]^3$. *If*

$$\mathcal{P}f + \mathbf{W}\varphi = 0 \quad \text{in } \Omega, \tag{2.18}$$

then $f = 0$ *in* Ω *and* $\varphi = 0$ *on* $\partial\Omega$.

Proof. Let us introduce the distribution $T^*\psi$ for $\psi \in [H^{s-3/2}(\partial\Omega)]^3$ with $s < \frac{3}{2}$ defined by the relation

$$\langle T^*\psi, h\rangle_{\mathbb{R}^3} := \langle \psi, Th\rangle_{\partial\Omega} \quad \text{for all } h \in [\mathcal{D}(\mathbb{R}^3)]^3, \tag{2.19}$$

where

$$T = T(x,\partial_x) = [T_{pq}(x,\partial_x)]_{3\times 3} := [a^{pq}_{kj}(x)n_k(x)\partial_{x_j}]_{3\times 3}.$$

In view of duality estimates and the trace theorem, there are constants $C_1 > 0$ and $C_2 > 0$ such that

$$\begin{aligned}
|\langle T^*\psi, h\rangle_{\mathbb{R}^3}| &= |\langle \psi, Th\rangle_{\partial\Omega}| \\
&\le C_1 \|\psi\|_{[H^{s-\frac{3}{2}}(\partial\Omega)]^3} \|Th\|_{[H^{\frac{3}{2}-s}(\partial\Omega)]^3} \\
&\le C_2 \|\psi\|_{[H^{s-\frac{3}{2}}(\partial\Omega)]^3} \|h\|_{[H^{3-s}(\mathbb{R}^3)]^3}.
\end{aligned}$$

Therefore, relation (2.19) can be extended by continuity to functions $h \in [H^{3-s}(\mathbb{R}^3)]^3$, implying the continuity of the operator

$$T^* : [H^{s-\frac{3}{2}}(\partial\Omega)]^3 \to [H^{s-3}(\mathbb{R}^3)]^3, \quad s < \frac{3}{2}.$$

Thus, for $\psi \in [H^{s-3/2}(\partial\Omega)]^3$ with $s < \frac{3}{2}$, we have $T^*\psi \in [H^{s-3}(\mathbb{R}^3)]^3$. Moreover, due to the definition (2.19), it is evident that $\operatorname{supp} T^*\psi \subset \partial\Omega$.

Now, let $s = \frac{3}{2} - \varepsilon$ with an arbitrarily small positive number ε. Then

$$T^*\psi \in [\widetilde{H}^{-\frac{3}{2}-\varepsilon}(\Omega^\pm)]^3 \subset [H^{-\frac{3}{2}-\varepsilon}(\mathbb{R}^3)]^3.$$

Now we represent the double layer potential $\mathbf{W}\varphi$ with $\varphi \in [H^{\frac{3}{2}}(\partial\Omega)]^3$ in the form of volume potential, which follows from relation (2.19) and the selfadjointness of the operator $\mathbf{P}$:

$$\begin{aligned}
\langle \mathbf{W}\varphi, h\rangle_{\mathbb{R}^3} &= -\int_{\mathbb{R}^3} \Big(\int_S [T(x,\partial_x)\Phi(x-y)]^\top \varphi(x)\,dS_x \Big) \cdot h(y)\,dy \\
&= -\int_{\partial\Omega} \varphi(x) \cdot \Big(\int_{\mathbb{R}^3} [T(x,\partial_x)\Phi(x-y)]h(y)\,dy \Big)\,dS_x \\
&= -\int_{\partial\Omega} \varphi(x) \cdot T(x,\partial_x)\mathbf{P}h(x)\,dS_x \\
&= -\langle T^*\varphi, \mathbf{P}h\rangle_{\mathbb{R}^3} \\
&= -\langle \mathbf{P}(T^*\varphi), h\rangle_{\mathbb{R}^3}.
\end{aligned}$$

Thus $\mathbf{W}\varphi = -\mathbf{P}(T^*\varphi)$, where $T^*\varphi \in [H^{-\frac{3}{2}-\varepsilon}(\mathbb{R}^3)]^3$ and $\operatorname{supp} T^*\varphi \subset \partial\Omega$.

In view of the mapping property of the operator $\mathbf{P}$ (see Theorem B.1), we have the embedding

$$\mathbf{W}\varphi = -\mathbf{P}(T^*\varphi) \in [H^{\frac{1}{2}-\varepsilon}(\mathbb{R}^3)]^3.$$

Set $\widetilde{f} = \mathring{E}f \in [\widetilde{H}^0(\Omega)]^3$ and define a vector function

$$U = (U_1, U_2, U_3)^\top := \mathbf{P}\widetilde{f} + \mathbf{W}\varphi = \mathbf{P}(\widetilde{f} - T^*\varphi) \quad \text{in } \mathbb{R}^3. \tag{2.20}$$

By (2.18), we have $U = 0$ in Ω, and consequently $U \in [\widetilde{H}^{1/2-\varepsilon}(\Omega^-)]^3 \subset [H^{1/2-\varepsilon}(\mathbb{R}^3)]^3$.

In what follows, we show that $U = 0$ in Ω^- as well.

Since $\mathcal{D}(\Omega^-)$ is dense in $\widetilde{H}^{1/2-\varepsilon}(\Omega^-)$, there exists a sequence of vector functions $U^{(l)} = (U_1^{(l)}, U_2^{(l)}, U_3^{(l)})^\top$, $l = \overline{1,\infty}$, from the space $[\mathcal{D}(\Omega^-)]^3$ such that

$$\lim_{l\to\infty} \|U^{(l)} - U\|_{[H^{\frac{1}{2}-\varepsilon}(\mathbb{R}^3)]^3} = 0. \tag{2.21}$$

Let us show that

$$\lim_{l\to\infty} \int_{\Omega^-} \nabla U_k \cdot \nabla U_k^{(l)}\, dx = \int_{\Omega^-} |\nabla U_k|^2\, dx, \quad k = 1, 2, 3. \tag{2.22}$$

Under the conditions of the lemma, from Theorem B.1 it follows that $U \in [H^2(\Omega^-)]^3$. Applying the duality estimates and relation (2.21), we deduce

$$\begin{aligned}\Big|\int_{\Omega^-} \nabla U_k \cdot (\nabla U_k^{(l)} - \nabla U_k)\, dx\Big| &\le \|\nabla U_k\|_{[H^{\frac{1}{2}+\varepsilon}(\Omega^-)]^3} \|\nabla U_k^{(l)} - \nabla U_k\|_{[\widetilde{H}^{-\frac{1}{2}-\varepsilon}(\Omega^-)]^3} \\ &\le \|\nabla U_k\|_{[H^{\frac{1}{2}+\varepsilon}(\Omega^-)]^3} \|\nabla U_k^{(l)} - \nabla U_k\|_{[H^{-\frac{1}{2}-\varepsilon}(\mathbb{R}^3)]^3} \\ &\le \|\nabla U_k\|_{[H^{\frac{1}{2}+\varepsilon}(\Omega^-)]^3} \|U_k^{(l)} - U_k\|_{H^{\frac{1}{2}-\varepsilon}(\mathbb{R}^3)} \to 0 \quad \text{as } l \to \infty,\ k = 1, 2, 3,\end{aligned}$$

implying (2.22).

In accordance with Theorem B.1, we have $\mathbf{P}^{-1} = \Delta\mathbf{I} - \mathbf{M}_\chi\mathbf{I}$, where the operator $\mathbf{M}_\chi$ is bounded from $H^0(\mathbb{R}^3)$ into $H^0(\mathbb{R}^3)$ due to (B.1). From (2.20) we find $\mathbf{P}^{-1}U = \widetilde{f} - T^*\varphi = 0$ in Ω^-, implying that

$$\mathbf{P}_\Delta^{-1} U_k = \Delta U_k - \mathbf{M}_\chi U_k = 0$$

in Ω^- for $k = 1, 2, 3$. Using Green's identity, we then obtain (no summation over k)

$$\begin{aligned}0 &= \int_{\Omega^-} [\mathbf{P}_\Delta^{-1} U_k] U_k^{(l)}\, dx \\ &= \int_{\Omega^-} \Delta U_k U_k^{(l)}\, dx - \int_{\Omega^-} [\mathbf{M}_\chi U_k] U_k^{(l)}\, dx \\ &= -\int_{\Omega^-} \nabla U_k \cdot \nabla U_k^{(l)}\, dx - \int_{\Omega^-} [\mathbf{M}_\chi U_k] U_k^{(l)}\, dx, \quad k = 1, 2, 3.\end{aligned} \tag{2.23}$$

By passing to the limit in (2.23) and using equality (2.22), we arrive at the relation

$$\int_{\Omega^-} |\nabla U_k|^2\, dx + \int_{\Omega^-} [\mathbf{M}_\chi U_k] U_k dx = 0, \quad k = 1, 2, 3. \tag{2.24}$$

Evidently, $\mathbf{M}_\chi U_k \in H^0(\mathbb{R}^3)$ since $U_k \in H^{1/2-\varepsilon}(\mathbb{R}^3) \subset H^0(\mathbb{R}^3)$. Therefore, taking into account that U_k is supported in $\mathbb{R}^3 \setminus \Omega^+$ and using Plancherel's theorem, we can rewrite (2.24) as follows:

$$\int_{\Omega^-} |\nabla U_k|^2\, dx + \int_{\mathbb{R}^3} \widehat{m}_\chi(\xi) |\mathcal{F}U_k|^2 d\xi = 0 \qquad \text{with} \qquad \mathcal{F}U_k(\xi) = \int_{\mathbb{R}^3} U_k(x) e^{ix\cdot\xi}\, dx, \quad k = 1, 2, 3,$$

where $\mathcal{F}$ stands for the Fourier transform operator. By (B.1) and the inclusion $U_k \in H^2(\Omega^-)$, we get $U_k = 0$ in Ω^-. Thus, $U = 0$ in $\Omega^\pm$ and the jump relation of the localized double layer potential gives (see Theorem B.2)

$$\mathbf{d}\varphi = \gamma^- U - \gamma^+ U = 0 \quad \text{on } \partial\Omega,$$

where $\mathbf{d}$ is given by (B.2) and is positive definite due to (2.1). Therefore, $\varphi = 0$ on $\partial\Omega$.

In view of (2.20), we have $U = \mathbf{P}\widetilde{f} = 0$ in $\mathbb{R}^3$. By the invertibility of the operator $\mathbf{P}$, we finally conclude that $f = 0$ in Ω, which completes the proof. □

3 Properties of the domain matrix operator

In what follows, in our analysis we need an explicit expression of the principal homogeneous symbol matrix $\mathfrak{S}(\mathcal{N}; y, \xi)$ of the singular integral operator $\mathcal{N}$, which due to (2.9), (2.12) and (2.13) reads as

$$[\mathfrak{S}(\mathcal{N}; y, \xi)]_{pq} = [\mathfrak{S}(\mathbf{N}; y, \xi)]_{pq} = \mathcal{F}_{z\to\xi}\Big[-\text{v.p.}\frac{a^{pq}_{kl}(y)}{4\pi}\frac{\partial^2}{\partial z_k \partial z_l}\frac{1}{|z|}\Big] = \frac{A_{pq}(y,\xi)}{|\xi|^2} - b_{pq}(y), \tag{3.1}$$

where $y \in \overline{\Omega}$, $\xi \in \mathbb{R}^3$ and $A_{pq}(y,\xi) = a^{pq}_{kl}(y)\xi_k\xi_l$. Here we have applied that $\mathcal{F}_{z\to\xi}[(4\pi|z|)^{-1}] = |\xi|^{-2}$. The entries of the principal homogeneous symbol matrix $\mathfrak{S}(\mathcal{N}; y, \xi)$ are even rational homogeneous functions of order 0 in ξ. Set

$$\mathbf{B} := \mathbf{b} + \mathbf{N}. \tag{3.2}$$

Relation (3.1) for the principal homogeneous symbol of the operator $\mathbf{B}$ gives

$$\mathfrak{S}(\mathbf{B}; y, \xi) = |\xi|^{-2} A(y,\xi) \quad \text{for all } y \in \overline{\Omega} \text{ and all } \xi \in \mathbb{R}^3 \setminus \{0\}. \tag{3.3}$$

Due to (2.1), the symbol matrix $\mathfrak{S}(\mathbf{B}; y, \xi)$ is positive definite:

$$\mathfrak{S}(\mathbf{B}; y, \xi)\zeta \cdot \bar{\zeta} \geq c_1|\zeta|^2 \quad \text{for all } y \in \overline{\Omega},\ \xi \in \mathbb{R}^3 \setminus \{0\},\ \zeta \in \mathbb{C}^3,$$

where $c_1 > 0$ is the same as in (2.1). Consequently, $\mathbf{B}$ is a strongly elliptic pseudodifferential operator of order 0 with the rational homogeneous symbol and its partial indices of factorization are equal to zero (cf. [5, 7, 21]).

Now let us formulate some auxiliary assertions needed in our further analysis. Let $\tilde{y} \in S = \partial\Omega$ be some fixed point and consider the frozen symbol $\mathfrak{S}(\mathbf{B}; \tilde{y}, \xi) \equiv \mathfrak{S}(\mathbf{B}; \xi)$, where $\mathbf{B}$ denotes the operator $\mathbf{B}$ written in the chosen local co-ordinate system. Further, let $\mathbf{B}_*$ denote the pseudodifferential operator with the symbol $\mathfrak{S}_*(\mathbf{B}; \xi', \xi_3) := \mathfrak{S}(\mathbf{B}; (1 + |\xi'|)\omega, \xi_3)$, where $\omega = \xi'/|\xi'|$, $\xi = (\xi', \xi_3)$, $\xi' = (\xi_1, \xi_2)$. Then the frozen principal homogeneous symbol matrix $\mathfrak{S}(\mathbf{B}; \xi)$ is also the principal homogeneous symbol matrix of the operator $\mathbf{B}_*$. This principal matrix can be factorized with respect to the variable ξ_3 as

$$\mathfrak{S}(\mathbf{B}; \xi) = \mathfrak{S}^{(-)}(\mathbf{B}; \xi)\mathfrak{S}^{(+)}(\mathbf{B}; \xi), \quad \mathfrak{S}^{(\pm)}(\mathbf{B}; \xi) = \frac{1}{\Theta^{(\pm)}(\xi', \xi_3)}\widetilde{A}^{(\pm)}(\xi', \xi_3). \tag{3.4}$$

Here $\Theta^{(\pm)}(\xi', \xi_3) := \xi_3 \pm i|\xi'|$ are the "plus" and "minus" factors of the symbol $\Theta(\xi) := |\xi|^2$, and $\widetilde{A}^{(\pm)}(\xi', \xi_3)$ are the "plus" and "minus" polynomial matrix factors of first order in ξ_3 of the positive definite polynomial symbol matrix $\widetilde{A}(\xi', \xi_3) \equiv \widetilde{A}(\tilde{y}, \xi', \xi_3)$ corresponding to the frozen differential operator $A(\tilde{y}, \partial)$ at the point $\tilde{y} \in S$ (see [12, 13]), i.e.,

$$\widetilde{A}(\xi', \xi_3) = \widetilde{A}^{(-)}(\xi', \xi_3)\widetilde{A}^{(+)}(\xi', \xi_3) \tag{3.5}$$

with $\det \widetilde{A}^{(+)}(\xi', \tau) \neq 0$ for $\operatorname{Im}\tau > 0$ and $\det \widetilde{A}^{(-)}(\xi', \tau) \neq 0$ for $\operatorname{Im}\tau < 0$. Moreover, the entries of the matrices $\widetilde{A}^{(\pm)}(\xi', \xi_3)$ are homogeneous functions of order 1 in $\xi = (\xi', \xi_3)$.

Denote by $a^{(\pm)}(\xi')$ the coefficients at ξ_3^3 in the determinants $\det \widetilde{A}^{(\pm)}(\xi', \xi_3)$. Evidently,

$$a^{(-)}(\xi')a^{(+)}(\xi') = \det \widetilde{A}(0, 0, 1) > 0 \quad \text{for } \xi' \neq 0. \tag{3.6}$$

The factor-matrices $\widetilde{A}^{(\pm)}(\xi', \xi_3)$ have the structure

$$([\widetilde{A}^{(\pm)}(\xi', \xi_3)]^{-1})_{ij} = \frac{1}{\det \widetilde{A}^{(\pm)}(\xi', \xi_3)} p^{(\pm)}_{ij}(\xi', \xi_3), \quad i, j = 1, 2, 3,$$

where $p^{(\pm)}_{ij}(\xi', \xi_3)$ are the co-factors of the matrix $\widetilde{A}^{(\pm)}(\xi', \xi_3)$, which can be written in the form

$$p^{(\pm)}_{ij}(\xi', \xi_3) = c^{(\pm)}_{ij}(\xi')\xi_3^2 + b^{(\pm)}_{ij}(\xi')\xi_3 + d^{(\pm)}_{ij}(\xi'). \tag{3.7}$$

Here $c^{(\pm)}_{ij}$, $b^{(\pm)}_{ij}$ and $d^{(\pm)}_{ij}$, $i, j = 1, 2, 3$, are homogeneous functions of order 0, 1 and 2, respectively, in ξ'.

From the above discussion it follows that the entries of the factor-symbol matrices

$$B^{(\pm)}_{kj}(\omega, r, \xi_3) := \mathfrak{S}^{(\pm)}_{kj}(\mathbf{B}; \xi', \xi_3), \quad k, j = 1, 2, 3,$$

with $\omega = \xi'/|\xi'|$ and $r = |\xi'|$, satisfy the relations

$$\frac{\partial^l B^{(\pm)}_{kj}(\omega, 0, -1)}{\partial r^l} = (-1)^l \frac{\partial^l B^{(\pm)}_{kj}(\omega, 0, +1)}{\partial r^l}, \quad l = 0, 1, 2, \ldots.$$

These relations imply that the entries of the matrices $\mathfrak{S}^{(\pm)}(\mathbf{B};\xi',\xi_3)$ belong to the class of symbols D_0 introduced in [14, Chapter III, Section 10]:

$$\mathfrak{S}^{(\pm)}(\mathbf{B};\xi',\xi_3)\in D_0.$$

Due to (3.4), we have the factorization

$$\mathfrak{S}_*(\mathbf{B};\xi',\xi_3)=\mathfrak{S}_*^{(-)}(\mathbf{B};\xi',\xi_3)\mathfrak{S}_*^{(+)}(\mathbf{B};\xi',\xi_3),$$

where

$$\mathfrak{S}_*^{(\pm)}(\mathbf{B};\xi',\xi_3)=\mathfrak{S}^{(\pm)}(\mathbf{B};(1+|\xi'|)\omega,\xi_3)\quad\text{with } \omega=\frac{\xi'}{|\xi'|}.$$

Denote by Π^+ the Cauchy type integral operator

$$\Pi^+h(\xi):=\frac{i}{2\pi}\lim_{t\to 0+}\int_{-\infty}^{+\infty}\frac{h(\xi',\eta_3)d\eta_3}{\xi_3+it-\eta_3},\qquad \xi\in\mathbb{R}^3, \tag{3.8}$$

which is well defined for a bounded smooth function $h(\xi',\cdot)$ satisfying the decay condition at infinity $h(\xi',\eta_3)=\mathcal{O}(1+|\eta_3|)^{-\varepsilon}$ with some $\varepsilon>0$.

Let $\mathring{E}_+$ be the extension operator by zero from $\mathbb{R}^3_+$ onto the whole space $\mathbb{R}^3$ and let r_+ be the restriction operator to the half-space $\mathbb{R}^3_+$.

The following lemmas are proved in [11].

Lemma 3.1. *The operator $r_+\widetilde{\mathbf{B}}_*\mathring{E}_+:[H^s(\mathbb{R}^3_+)]^3\to[H^s(\mathbb{R}^3_+)]^3$ is invertible for all $s\ge 0$.*

Moreover, for $f\in[H^s(\mathbb{R}^3_+)]^3$, a unique solution $u\in[H^s(\mathbb{R}^3_+)]^3$ of the equation $r_+\mathbf{B}_*\mathring{E}_+u=f$ in $\mathbb{R}^3_+$ can be written in the form $u=r_+u_+$, where

$$u_+:=\mathring{E}_+u=\mathcal{F}^{-1}\{[\mathfrak{S}_*^{(+)}(\widetilde{\mathbf{B}})]^{-1}\Pi^+([\mathfrak{S}_*^{(-)}(\widetilde{\mathbf{B}})]^{-1}\widehat{f_*})\}, \tag{3.9}$$

and $f_*\in[H^s(\mathbb{R}^3)]^3$ is an extension of $f\in[H^s(\mathbb{R}^3_+)]^3$ such that

$$\|f_*\|_{[H^s(\mathbb{R}^3)]^3}\le 2\|f\|_{[H^s(\mathbb{R}^3_+)]^3},$$

while $\widehat{f_*}(\xi)=\mathcal{F}_{x\to\xi}(f_*)$ is the Fourier transform of f_*. Representation (3.9) does not depend on a choice of the extension operator of f to f_*.

Lemma 3.2. *Let $\chi\in X^{k+2}$ with an integer $k\ge 3$. Then the operator $r_\Omega\mathbf{B}\mathring{E}:[H^s(\Omega)]^3\to[H^s(\Omega)]^3$ is Fredholm with zero index for $0\le s<k-\frac{1}{2}$.*

Lemma 3.3. *Let the factor matrix $\widetilde{A}^{(+)}(\xi',\tau)$ be as in (3.5), and let $a^{(+)}$ and $c_{ij}^{(+)}$ be as in (3.6) and (3.7), respectively. Then the following equality holds:*

$$\frac{1}{2\pi i}\int_{\ell^-}[\widetilde{A}^{(+)}(\xi',\tau)]^{-1}\,d\tau=\frac{1}{a^{(+)}(\xi')}C^{(+)}(\xi'),$$

where $C^{(+)}(\xi')=[c_{ij}^{(+)}(\xi')]_{i,j=1}^3$ and $\det[C^{(+)}(\xi')]\ne 0$ for $\xi'\ne 0$. Here ℓ^- is an anticlockwise orientated contour in the lower complex half-plane enclosing all roots of the polynomial $\det\widetilde{A}^{(+)}(\xi',\tau)$ with respect to τ.

Introduce the notation

$$(\mathbf{K}\mathring{E}u)(y):=(\mathbf{b}(y)-I_3)u(y)+(\mathbf{N}\mathring{E}u)(y)\quad\text{for } y\in\Omega, \tag{3.10}$$

where $\mathbf{b}$ is defined in (2.8), $\mathbf{N}$ is defined in (2.12), and $\mathring{E}$ again stands for the extension operator by zero from Ω onto Ω^-. Rewrite Green's third formula (2.7) in a form more convenient for our further purposes:

$$[\mathbf{I}+\mathbf{K}]\mathring{E}u(y)-\mathbf{V}(T^+u)(y)+\mathbf{W}(\gamma^+u)(y)=\mathcal{P}(A(x,\partial_x)u)(y),\qquad y\in\Omega, \tag{3.11}$$

where $\mathbf{I}$ stands for the unit operator.

Relation (3.3) implies that the principal homogeneous symbols of the singular integral operators $\mathbf{K}$ and $\mathbf{I}+\mathbf{K}$ have the form

$$\mathfrak{S}(\mathbf{K};y,\xi)=|\xi|^{-2}A(y,\xi)-I_3,\quad \mathfrak{S}(\mathbf{I}+\mathbf{K};y,\xi)=|\xi|^{-2}A(y,\xi)\qquad\text{for all } y\in\overline{\Omega} \text{ and all } \xi\in\mathbb{R}^3\setminus\{0\},$$

where $A(y,\xi) = [a_{kj}^{pq}(y)\xi_k\xi_j]_{3\times 3}$ is the positive definite matrix due to (2.1), implying the positive definiteness of the symbol matrix $\mathfrak{S}(\mathbf{I}+\mathbf{K}; y, \xi)$.

Note that $\mathbf{K} = \mathbf{B} - \mathbf{I}$, where $\mathbf{B}$ is defined in (3.2). Therefore, from the mapping property (2.14) it follows that if $\chi \in X^k$ with an integer $k \geq 2$, then the operator

$$r_\Omega \mathbf{K}\mathring{E} : [H^r(\Omega)]^3 \to [H^r(\Omega)]^3, \quad 0 \leq r \leq k-2,$$

is bounded.

Assuming that $u \in [H^2(\Omega)]^3$, by Theorem B.1 we see that all summands in (3.11) belong to the space $[H^2(\Omega)]^3$. Applying the co-normal differential operator $T(x,\partial_x)$ to Green's identity (3.11) and using the properties of localized potentials, we arrive at the relation

$$T^+\mathbf{K}\mathring{E}u + (I_3 - \mathbf{d})(T^+u) - \mathcal{W}'(T^+u) + \mathcal{L}(\gamma^+u) = T^+\mathcal{P}(A(x,\partial_x)u) \quad \text{on } S, \tag{3.12}$$

where the localized boundary integral operators $\mathcal{W}'$ and $\mathcal{L} := \mathcal{L}^+$ are generated by the localized single and double layer potentials and are defined in (2.17), and the matrix $\mathbf{d}$ is defined by B.2.

4 LBDSIE formulation of the Robin type problem and the equivalence theorem

Let $u \in [H^2(\Omega)]^3$ be a solution to the Robin type BVP (2.4)–(2.5) with $\psi_0 \in [H^{1/2}(S)]^3$ and $f \in [H^0(\Omega)]^3$. As we have shown above, there hold relations (3.11) and (3.12), which now can be rewritten in the form

$$(\mathbf{I}+\mathbf{K})\mathring{E}u + \mathbf{W}\varphi + \mathbf{V}(\varkappa\varphi) = \mathcal{P}f + \mathbf{V}\psi_0 \quad \text{in } \Omega, \tag{4.1}$$

$$T^+\mathbf{K}\mathring{E}u + \mathcal{L}\varphi + (\mathbf{d} - I_3)\varkappa\varphi + \mathcal{W}'(\varkappa\varphi) = T^+\mathcal{P}(f) + (\mathbf{d} - I_3)\psi_0 + \mathcal{W}'\psi_0 \quad \text{on } S, \tag{4.2}$$

where $\varphi := \gamma^+u \in [H^{3/2}(S)]^3$.

Consider these relations as an LBDSIE system with respect to the unknown vector functions u and φ. The following equivalence theorem holds.

Theorem 4.1. *Let $\chi \in X^4_{1+}$. The Robin type boundary value problem* (2.4)–(2.5) *is equivalent to the LBDSIE system* (4.1)–(4.2) *in the following sense:*

(i) *if a vector function $u \in [H^2(\Omega)]^3$ solves the Robin type BVP* (2.4)–(2.5)*, then it is unique and the pair $(u,\varphi) \in [H^2(\Omega)]^3 \times [H^{3/2}(S)]^3$ with*

$$\varphi = \gamma^+u \tag{4.3}$$

solves the LBDSIE system (4.1)–(4.2) *and, vice versa;*

(ii) *if a pair $(u,\varphi) \in [H^2(\Omega)]^3 \times [H^{3/2}(S)]^3$ solves the LBDSIE system* (4.1)–(4.2)*, then it is unique, the vector function u solves the Robin type BVP* (2.4)–(2.5) *and relation* (4.3) *holds.*

Proof. (i) The first part of the theorem directly follows form relations (3.11), (3.12), (4.3) and Remark 2.1.

(ii) Now, let a pair $(u,\varphi) \in [H^2(\Omega)]^3 \times [H^{3/2}(S)]^3$ solve the LBDSIE system (4.1)–(4.2). Apply the co-normal derivative operator T to equation (4.1), take its trace on S, and compare with (4.2) to obtain

$$T^+u + \varkappa\varphi = \psi_0 \quad \text{on } S. \tag{4.4}$$

Further, since $u \in [H^2(\Omega)]^3$, we can write Green's third formula (3.11), which in view of (4.4) can be rewritten as

$$(\mathbf{I}+\mathbf{K})\mathring{E}u + \mathbf{V}(\varkappa\varphi) - \mathbf{V}\psi_0 + \mathbf{W}(\gamma^+u) = \mathcal{P}(A(x,\partial_x)u) \quad \text{in } \Omega. \tag{4.5}$$

From (4.1) and (4.5) it follows that $\mathbf{W}(\gamma^+u - \varphi) - \mathcal{P}(A(x,\partial_x)u - f) = 0$ in Ω, whence by Lemma 2.2 we get $A(x,\partial_x)u = f$ in Ω, $\gamma^+u = \varphi$ on S, and $T^+u + \varkappa\gamma^+u = \psi_0$ on S due to (4.4). Thus u solves the Robin type BVP (2.4)–(2.5) and, in addition, equation (4.3) holds.

The uniqueness of a solution to the LBDSIE system (4.1)–(4.2) in the class $[H^2(\Omega)]^3 \times [H^{3/2}(S)]^3$ directly follows from the equivalence result proved above and the uniqueness theorem for the Robin type problem (2.4)–(2.5) (see Remark 2.1). □

5 Invertibility of the LBDSIE

From Theorem 4.1 it follows that the LBDSIE system (4.1)–(4.2) with special right-hand side vector functions, is uniquely solvable in the class $H^2(\Omega, A) \times H^{3/2}(S)$. Here we investigate the Fredholm properties of the localized boundary-domain singular integral operator generated by the left-hand side expressions in (4.1)–(4.2) and prove the corresponding invertibility results in appropriate function spaces.

Now, the LBDSIE system (4.1)–(4.2) with arbitrary right-hand side vector functions from the space $[H^2(\Omega)]^3 \times [H^{1/2}(S)]^3$ can be written as

$$(\mathbf{I} + \mathbf{K})\mathring{E}u + \mathbf{W}\varphi + \mathbf{V}_\varkappa \varphi = F_1 \quad \text{in } \Omega, \tag{5.1}$$

$$T^+\mathbf{K}\mathring{E}u + \mathcal{L}\varphi + (\mathbf{d} - I_3)\varkappa\varphi + \mathcal{W}'_\varkappa \varphi = F_2 \quad \text{on } S, \tag{5.2}$$

where $F_1 \in [H^2(\Omega)]^3$ and $F_2 \in [H^{1/2}(S)]^3$, and

$$\mathbf{V}_\varkappa \varphi := \mathbf{V}(\varkappa\varphi), \quad \mathcal{W}'_\varkappa \varphi := \mathcal{W}'(\varkappa\varphi). \tag{5.3}$$

It is well known that for the differential operator $A(x, \partial_x)$ the co-normal derivative operator $T(x, \partial_x)$ generates *a complementing boundary condition* (see, e.g., [1, 6]), so that the following boundary value problem for the system of ordinary differential equations

$$\widetilde{A}\Big(\xi', i\frac{d}{dt}\Big)v(\xi', t) = 0, \quad t \in \mathbb{R}_+ = (0, \infty), \tag{5.4}$$

$$\widetilde{T}\Big(\xi', i\frac{d}{dt}\Big)v(\xi', t)\Big|_{t=0} = 0 \tag{5.5}$$

has only the trivial solution in the Schwartz space $[\mathcal{S}(\mathbb{R}_+)]^3$ of infinitely smooth, rapidly decreasing vector functions at infinity. The differential operators

$$\widetilde{A}\Big(\xi', i\frac{d}{dt}\Big) := \widetilde{A}\Big(\widetilde{y}, \xi', i\frac{d}{dt}\Big) \quad \text{and} \quad \widetilde{T}\Big(\xi', i\frac{d}{dt}\Big) := \widetilde{T}\Big(\widetilde{y}, \xi', i\frac{d}{dt}\Big)$$

correspond to the "frozen" differential and co-normal operators $\widetilde{A}(\widetilde{y}, \partial)$ and $\widetilde{T}(\widetilde{y}, \partial)$, respectively, at the point $\widetilde{y} \in \partial\Omega$ written in the local co-ordinate system.

Note that, in view of decomposition (3.5) of the matrix $\widetilde{A}(\xi', \xi_3)$, we have a similar decomposition of the matrix ordinary differential operator $\widetilde{A}(\xi', i\frac{d}{dt})$:

$$\widetilde{A}\Big(\xi', i\frac{d}{dt}\Big) = \widetilde{A}^{(-)}\Big(\xi', i\frac{d}{dt}\Big)\widetilde{A}^{(+)}\Big(\xi', i\frac{d}{dt}\Big), \tag{5.6}$$

where the entries of the matrix $\widetilde{A}^{(\mp)}(\xi', i\frac{d}{dt})$ are first-order ordinary differential operators.

The above formulated unique solvability condition for problem (5.4)–(5.5) is equivalent to the algebraic *covering condition* for $A(x, \partial)$ and the corresponding conormal derivative operator $T(x, \partial)$ (for details see [1, 16, 24]). The latter condition implies the following assertion.

Lemma 5.1. *Let ℓ^- be the same as in Lemma 3.3. Then the matrix*

$$\int_{\ell^-} \widetilde{T}(\xi', \tau)[\widetilde{A}^{(+)}(\xi', \tau)]^{-1}\, d\tau$$

is non-degenerate for all $\xi' \neq 0$.

Proof. Let us introduce the matrix

$$\mathbf{Q}(t) := \int_{\ell^-} e^{-i\tau t}[\widetilde{A}^{(+)}(\xi', \tau)]^{-1}\, d\tau, \quad 0 < t < \infty, \tag{5.7}$$

and denote its columns by $v^{(1)}(\xi', t)$, $v^{(2)}(\xi', t)$ and $v^{(3)}(\xi', t)$. Clearly, $v^{(k)}(\xi', \cdot) \in [\mathcal{S}(\mathbb{R}_+)]^3$, $k = 1, 2, 3$.

First, let us prove that the vectors $v^{(k)}(\xi', \cdot)$, $k = 1, 2, 3$, are linearly independent solutions of equation (5.4). Indeed, by direct differentiation one can show that the vector functions solve the following system of ordinary differential equations:

$$\widetilde{A}^{(+)}\Big(\xi', i\frac{d}{dt}\Big)v^{(k)}(\xi', t) = 0, \quad 0 < t < \infty.$$

Consequently, due to decomposition (5.6), they are solutions of equation (5.4) as well.

Assume that for some scalar constants α_k, $k = 1, 2, 3$, the following equality holds:

$$\alpha_1 v^{(1)}(\xi', t) + \alpha_2 v^{(2)}(\xi', t) + \alpha_3 v^{(3)}(\xi', t) = 0, \quad t \in \mathbb{R}_+. \tag{5.8}$$

Note that the matrix-function $\mathbf{Q}(t)$ defined by (5.7) is continuous at $t = 0$. Therefore, passing to the limit as $t \to 0$, from (5.8) we obtain the following linear algebraic system of equations with respect to $\alpha = (\alpha_1, \alpha_2, \alpha_3)^\top$: $Q(0)\alpha = 0$. Due to Lemma 3.3,

$$\det \mathbf{Q}(0) = \det\Big(\int_{\ell^-} [\widetilde{A}^{(+)}(\xi', \tau)]^{-1}\, d\tau\Big) \neq 0 \quad \text{for all } \xi' \neq 0,$$

implying $\alpha = (\alpha_1, \alpha_2, \alpha_3)^\top = 0$. Thus the vectors $v^{(k)}(\xi', t)$, $k = 1, 2, 3$, are linearly independent solutions of equation (5.4) and an arbitrary solution of equation (5.4) belonging to the class $[\mathcal{S}(\mathbb{R}_+)]^3$ is representable in the form

$$v(\xi', t) = \sum_{k=1}^{3} a_k v^{(k)}(\xi', t), \tag{5.9}$$

where a_1, a_2, a_3 are some scalar constants. On the other hand, due to the above-mentioned covering condition, the problem (5.4)–(5.5) possesses only the trivial solution.

Substituting (5.9) into (5.5) leads to the following system of linear algebraic equations with respect to the vector $a = (a_1, a_2, a_3)^\top$:

$$\Big(\int_{\ell^-} \widetilde{T}(\xi', \tau)[\widetilde{A}^{(+)}(\xi', \tau)]^{-1}\, d\tau\Big)a = 0.$$

Due to the covering condition, we deduce

$$\det\Big(\int_{\ell^-} \widetilde{T}(\xi', \tau)[\widetilde{A}^{(+)}(\xi', \tau)]^{-1}\, d\tau\Big) \neq 0 \quad \text{for all } \xi' \neq 0,$$

which completes the proof. □

Now, with the above auxiliary results in hand, we can investigate the invertibility of the localized boundary-domain singular integral operator generated by the left-hand side expressions in system (5.1)–(5.2). Denote this operator by

$$\mathfrak{R} := \begin{bmatrix} r_\Omega \mathbf{B}\mathring{E} & r_\Omega \mathbf{W} + r_\Omega \mathbf{V}_\varkappa \\ T^+\mathbf{K}\mathring{E} & \mathcal{L} + (\mathbf{d} - I_3)\varkappa + \mathcal{W}'_\varkappa \end{bmatrix},$$

where $\mathbf{B} = \mathbf{I} + \mathbf{K}$, and the operators $\mathbf{B}$, $\mathbf{K}$, $\mathbf{W}$, $\mathcal{L} := \mathcal{L}^+$, $\mathbf{V}_\varkappa$ and $\mathcal{W}'_\varkappa$ are defined in (3.2), (3.10), (2.10), (2.17), and (5.3), respectively, and the matrix $\mathbf{d}$ is given by (B.2).

Note that the double layer potential can be represented as

$$\begin{aligned}
\mathbf{W}\varphi(y) &= -\int_S [T(x, \partial_x)\Phi(x-y)]^\top \varphi(x)\, dS_x \\
&= \int_S [T(x, \partial_y)\Phi(x-y)]^\top \varphi(x)\, dS_x \\
&= \int_S T^\top(y, \partial_y)\Phi(x-y)\varphi(x)\, dS_x + \int_S [T^\top(x, \partial_y) - T^\top(y, \partial_y)]\Phi(x-y)\varphi(x)\, dS_x \\
&= -T^\top(y, \partial_y)\mathbf{V}\varphi(y) + \int_S k(x, x-y)\varphi(x)\, dS_x,
\end{aligned}$$

where

$$k(x, x-y) = [T^\top(x, \partial_y) - T^\top(y, \partial_y)]\Phi(x-y) = \mathcal{O}(|x-y|^{-1}).$$

This implies that the main part $\mathbf{W}^{(0)}\varphi$ of the potential $\mathbf{W}\varphi$ can be written in the following form (see [10, Appendix B, (B.5)]):

$$\mathbf{W}^{(0)}\varphi(y) := -T^\top(y, \partial_y)\mathbf{V}\varphi(y) = T^\top(y, \partial_y)\mathbf{P}(\gamma^*\varphi)(y), \tag{5.10}$$

where γ^* is the operator adjoint to the trace operator γ on S.

Let us introduce the following boundary operator depending on the parameter $t \in [0, 1]$:

$$T_t \equiv T_t(x, \partial_x) := (1-t)I\partial_n + tT(x, \partial_x). \tag{5.11}$$

Theorem 5.2. *Let a cut-off function $\chi \in X_{1+}^{k+2}$, $k \geq 3$, and the following condition be satisfied*

$$\det \widetilde{T}_t(\xi', -i|\xi'|) \neq 0 \quad \text{for all } \xi' \neq 0 \text{ and all } t \in [0, 1], \tag{5.12}$$

where the matrix $\widetilde{T}_t(\xi', \xi_3)$ is defined by

$$\widetilde{T}_t(\xi', \xi_3) = (1-t)\xi_3 I + t\widetilde{T}(\xi', \xi_3). \tag{5.13}$$

Then the operator

$$\mathfrak{R} : H^{r+1}(\Omega) \times H^{r+\frac{1}{2}}(S) \to H^{r+1}(\Omega) \times H^{r-\frac{1}{2}}(S), \quad \frac{1}{2} < r < k - \frac{3}{2}, \tag{5.14}$$

is invertible.

Proof. We prove the theorem in three steps. First we show that the operator $\mathfrak{R}$ given in (5.14) is Fredholm (step 1), then we establish that $\operatorname{Ind}\mathfrak{R} = 0$ (step 2), and finally we prove that the operator $\mathfrak{R}$ is invertible (step 3).

Step 1. To investigate the Fredholm properties of the operator $\mathfrak{R}$, we again apply the local principal (cf., e.g., [14, Sections 19 and 22]). Due to this principal, we have to check that the *Šapiro–Lopatinskiĭ condition* for the operator $\mathfrak{R}$ holds at an arbitrary "frozen" point $\tilde{y} \in S$. To obtain the explicit form of this condition, we proceed as follows. Let $\widetilde{U}$ be a neighborhood of a fixed point $\tilde{y} \in \overline{\Omega}$ and let $\widetilde{\psi}_0, \widetilde{\varphi}_0 \in \mathcal{D}(\widetilde{U})$ such that

$$\operatorname{supp}\widetilde{\psi}_0 \cap \operatorname{supp}\widetilde{\varphi}_0 \neq \varnothing, \quad \tilde{y} \in \operatorname{supp}\widetilde{\psi}_0 \cap \operatorname{supp}\widetilde{\varphi}_0,$$

and consider the operator $\widetilde{\psi}_0\mathfrak{R}\widetilde{\varphi}_0$. We separate two possible cases: (a) $\tilde{y} \in \Omega$ and (b) $\tilde{y} \in S$.

Case (a). If $\tilde{y} \in \Omega$, then we can choose a neighborhood $\widetilde{U}$ of the point $\tilde{y}$ such that $\overline{\widetilde{U}} \subset \Omega$. Then

$$\widetilde{\psi}_0\mathfrak{R}\widetilde{\varphi}_0 = \widetilde{\psi}_0\mathbf{B}\widetilde{\varphi}_0,$$

where $\mathbf{B}$ is the operator defined by (3.2). As we have already shown in Lemma 3.2, the operator

$$r_\Omega\mathbf{B}\mathring{E} : [H^{r+1}(\Omega)]^3 \to [H^{r+1}(\Omega)]^3$$

is Fredholm with zero index.

Case (b). If $\tilde{y} \in S$, then at the point $\tilde{y}$ we have to "freeze" the operator $\widetilde{\psi}_0\mathfrak{R}\widetilde{\varphi}_0$, which means that we can choose sufficiently small neighborhood $\widetilde{U}$ of this point such that, at the chosen local co-ordinate system with the origin at point $\tilde{y}$ and the third axis coinciding with the normal vector at the point $\tilde{y} \in S$, the following decomposition holds:

$$\widetilde{\psi}_0\mathfrak{R}\widetilde{\varphi}_0 = \widetilde{\psi}_0(\widetilde{\mathfrak{R}}_* + \mathbf{K}_1 + \mathbf{T}_1)\widetilde{\varphi}_0, \tag{5.15}$$

where $\mathbf{K}_1$ is a bounded operator with a small norm

$$\mathbf{K}_1 : H^{r+1}(\mathbb{R}^3_+) \times H^{r+\frac{1}{2}}(\mathbb{R}^2) \to H^{r+1}(\mathbb{R}^3_+) \times H^{r-\frac{1}{2}}(\mathbb{R}^2),$$

$\mathbf{T}_1$ is a bounded smoothing operator

$$\mathbf{T}_1 : H^{r+1}(\mathbb{R}^3_+) \times H^{r+\frac{1}{2}}(\mathbb{R}^2) \to H^{r+2}(\mathbb{R}^3_+) \times H^{r+\frac{1}{2}}(\mathbb{R}^2),$$

and the operator $\widetilde{\mathfrak{R}}_*$ is defined in the upper half-space $\mathbb{R}^3_+$ by

$$\widetilde{\mathfrak{R}}_* := \begin{bmatrix} r_+\mathbf{B}_*\mathring{E}_+ & r_+\widetilde{\mathbf{W}}^{(0)}_* \\ \gamma^+(\widetilde{T}\mathbf{K})_*\mathring{E}_+ & \widetilde{\mathcal{L}}_* \end{bmatrix}$$

and possesses the following mapping property:

$$\widetilde{\mathfrak{R}}_* : [H^{r+1}(\mathbb{R}^3_+)]^3 \times [H^{r+\frac{1}{2}}(\mathbb{R}^2)]^3 \to [H^{r+1}(\mathbb{R}^3_+)]^3 \times [H^{r-\frac{1}{2}}(\mathbb{R}^2)]^3. \tag{5.16}$$

The operators involved in the expression of $\widetilde{\mathfrak{R}}_*$ are defined as follows: for a given operator $\widetilde{M}$ with the symbol $\mathfrak{S}(\widetilde{M};\xi)$, by $\widetilde{M}_*$ we denote the operator in $\mathbb{R}^n$, $n = 2, 3$, constructed by the symbol

$$\mathfrak{S}_*(\widetilde{M};\xi) = \mathfrak{S}(\widetilde{M};(1+|\xi'|)\omega,\xi_3) \quad \text{for } n=3, \qquad \mathfrak{S}_*(\widetilde{M};\xi) = \mathfrak{S}(\widetilde{M};(1+|\xi'|)\omega) \quad \text{for } n=2,$$

where $\omega = \xi'/|\xi'|$, $\xi = (\xi', \xi_3)$, $\xi' = (\xi_1, \xi_2)$.

The generalized Šapiro–Lopatinskiĭ condition is related to the invertibility of operator (5.16). Let us write the system corresponding to the operator $\widetilde{\mathfrak{R}}_*$:

$$r_+\mathbf{B}_*\mathring{E}_+\widetilde{u} + r_+\widetilde{\mathbf{W}}^{(0)}_*\widetilde{\varphi} = \widetilde{F}_1 \quad \text{in } \mathbb{R}^3_+, \tag{5.17}$$

$$\gamma^+(\widetilde{T}\mathbf{K})_*\mathring{E}_+\widetilde{u} + \widetilde{\mathcal{L}}_*\widetilde{\varphi} = \widetilde{F}_2 \quad \text{on } \mathbb{R}^2, \tag{5.18}$$

where $\widetilde{u} \in [H^{r+1}(\mathbb{R}^3_+)]^3$ and $\widetilde{\varphi} \in [H^{r+1/2}(\mathbb{R}^2)]^3$ are the sought vector functions and $\widetilde{F}_1 \in [H^{r+1}(\mathbb{R}^3_+)]^3$ and $\widetilde{F}_2 \in [H^{r-1/2}(\mathbb{R}^2)]^3$ for $\frac{1}{2} < r < k - \frac{3}{2}$ are the known right-hand sides.

Let us introduce the notation

$$\widetilde{f} := \widetilde{F}_1 - r_+\widetilde{\mathbf{W}}^{(0)}_*\widetilde{\varphi} \in [H^{r+1}(\mathbb{R}^3_+)]^3, \quad \widetilde{f}_* := \widetilde{F}_{1*} - \widetilde{\mathbf{W}}^{(0)}_*\widetilde{\varphi} \in [L_2(\mathbb{R}^3)]^3, \tag{5.19}$$

where $\widetilde{F}_{1*} \in [H^{r+1}(\mathbb{R}^3)]^3$ is the extension of the vector function $\widetilde{F}_1$ from $\mathbb{R}^3_+$ to $\mathbb{R}^3$ preserving the function space.

Due to Lemma 3.1, the operator $r_+\mathbf{B}_*\mathring{E}_+ : [H^{r+1}(\mathbb{R}^3_+)]^3 \to [H^{r+1}(\mathbb{R}^3_+)]^3$ is invertible. Therefore, from equation (5.17) in view of the inclusions (5.19), we can define $\mathring{E}_+\widetilde{u}$ and $\widetilde{u}$ (see (3.9)):

$$\begin{aligned} \mathring{E}_+\widetilde{u} &= \mathcal{F}^{-1}\big[[\mathfrak{S}^{(+)}_*(\mathbf{B})]^{-1}\Pi^+([\mathfrak{S}^{(-)}_*(\mathbf{B})]^{-1}\mathcal{F}(\widetilde{f}_*))\big] \in L_2(\mathbb{R}^3), \\ \widetilde{u} &= [r_+\mathbf{B}_*\mathring{E}_+]^{-1}\widetilde{f} \in H^{r+1}(\mathbb{R}^3_+), \end{aligned} \tag{5.20}$$

where the operator Π^+ is defined in (3.8); here $\mathfrak{S}^{\pm}_*(\mathbf{B};\xi)$ denote the so called “plus” and “minus” factors in the factorization of the symbol $\mathfrak{S}_*(\mathbf{B};\xi)$ with respect to the variable ξ_3.

Substituting (5.20) into (5.18) leads to the following pseudodifferential equation for the unknown vector function $\widetilde{\varphi}$:

$$-\gamma^+(\widetilde{T}\mathbf{K})_*\mathcal{F}^{-1}\big[[\mathfrak{S}^{(+)}_*(\mathbf{B})]^{-1}\Pi^+([\mathfrak{S}^{(-)}_*(\mathbf{B})]^{-1}\mathcal{F}(\widetilde{\mathbf{W}}^{(0)}_*\widetilde{\varphi}))\big] + \widetilde{\mathcal{L}}_*\widetilde{\varphi} = \widetilde{F} \quad \text{on } \mathbb{R}^2, \tag{5.21}$$

where

$$\widetilde{F} = \widetilde{F}_2 - \gamma^+(\widetilde{T}\mathbf{K})_*\mathring{E}_+[r_+\mathbf{B}_*\mathring{E}_+]^{-1}\widetilde{F}_1. \tag{5.22}$$

Due to the mapping properties of the operators involved in (5.22) and Lemma 3.1, it follows that

$$\widetilde{F} \in [H^{r-\frac{1}{2}}(\mathbb{R}^2)]^3.$$

Note that

$$\gamma^+(\widetilde{T}\mathbf{K})v(\widetilde{y}') = \big[\mathcal{F}^{-1}_{\xi\to\widetilde{y}}[\widetilde{T}(-i\xi)\mathfrak{S}(\mathbf{K};\xi)\widehat{v}(\xi)]\big]_{\widetilde{y}_3=0+} = \mathcal{F}^{-1}_{\xi'\to\widetilde{y}'}\big[\Pi'[\widetilde{T}(-i\xi)\mathfrak{S}(\mathbf{K};\xi)\widehat{v}(\xi)]\big], \tag{5.23}$$

with the operator Π' defined as follows (for details see [10, Appendix C]):

$$\Pi'(g)(\xi') = \lim_{x_3\to 0+} r_+\mathcal{F}^{-1}_{\xi_3\to x_3}[g(\xi',\xi_3)] = -\frac{1}{2\pi}\int_{\ell^-} g(\xi',\zeta)\,d\zeta,$$

where $g(\xi', \xi_3)$ is rational in ξ_3, does not have real poles for $\xi = (\xi', \xi_3) \in \mathbb{R}^3 \setminus \{0\}$, is homogeneous of order $m \in \mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$ in $\xi = (\xi', \xi_3)$ and infinitely differentiable with respect to real $\xi = (\xi', \xi_3)$ for $\xi' \neq 0$;

here ℓ^- is a contour in the lower complex half-plane orientated counterclockwise and enclosing all poles of the rational function g.

In the case when $S = \mathbb{R}^2$ is the boundary of the half-space, the distribution $\gamma^*\varphi$ represents a direct product $\gamma^*\varphi = \varphi(x_1, x_2) \times \delta(x_3)$. Therefore, in view of (5.23) and (5.10), using the equality

$$\mathcal{F}(\widetilde{\mathbf{W}}_*^{(0)}\widetilde{\varphi}))(\xi) = \widetilde{T}_*^\top(\xi)\mathfrak{S}_*(\mathbf{P};\xi)\widehat{\widetilde{\varphi}}(\xi') = \mathfrak{S}_*(\mathbf{P};\xi)\widetilde{T}_*^\top(\xi)\widehat{\widetilde{\varphi}}(\xi'),$$

we find

$$\begin{aligned}\gamma^+(\widetilde{T}\mathbf{K})_*\mathcal{F}^{-1}_{\xi\to\widetilde{x}}[[\mathfrak{S}_*^{(+)}(\mathbf{B})(\xi)]^{-1}\Pi^+([\mathfrak{S}_*^{(-)}(\mathbf{B})]^{-1}\mathcal{F}(\widetilde{\mathbf{W}}_*^{(0)}\widetilde{\varphi}))(\xi)](\widetilde{y}')\\ = \mathcal{F}^{-1}_{\xi'\to\widetilde{y}'}\{\Pi'[\widetilde{T}_*\mathfrak{S}_*(\mathbf{K})[\mathfrak{S}_*^{(+)}(\mathbf{B})]^{-1}\Pi^+([\mathfrak{S}_*^{(-)}(\mathbf{B})]^{-1}\mathfrak{S}_*(\mathbf{P})\widetilde{T}_*^\top)]\widehat{\widetilde{\varphi}}(\xi')\}.\end{aligned}$$

By the above relation, equation (5.21) can be rewritten in the form

$$\mathcal{F}^{-1}_{\xi'\to\widetilde{y}'}[\mathbf{S}_{\mathfrak{R}*}(\xi')\widehat{\widetilde{\varphi}}(\xi')] = \widetilde{F}(\widetilde{y}') \quad \text{on } \mathbb{R}^2, \tag{5.24}$$

where $\mathbf{S}_{\mathfrak{R}*}(\xi') = \mathbf{S}_{\mathfrak{R}}((1+|\xi'|)\omega)$, $\omega = \xi'/|\xi'|$, with $\mathbf{S}_{\mathfrak{R}}$ being a homogeneous function of order 1 given by the equality

$$\mathbf{S}_{\mathfrak{R}}(\xi') = -\Pi'[\widetilde{T}\mathfrak{S}(\mathbf{K})[\mathfrak{S}^{(+)}(\mathbf{B})]^{-1}\Pi^+([\mathfrak{S}^{(-)}(\mathbf{B})]^{-1}\mathfrak{S}(\mathbf{P})\widetilde{T}^\top)](\xi') + \mathfrak{S}(\widetilde{\mathcal{L}};\xi') \quad \text{for all } \xi' \neq 0. \tag{5.25}$$

If the function $\det \mathbf{S}_{\mathfrak{R}}(\xi')$ is different from zero for all $\xi' \neq 0$, then $\det \mathbf{S}_{\mathfrak{R}*}(\xi') \neq 0$ for all $\xi' \in \mathbb{R}^2$, and the corresponding pseudodifferential operator $\mathbf{R}_*$ generated by the left-hand side expression in (5.24), i.e.,

$$\mathbf{R}_* : H^s(\mathbb{R}^2) \to H^{s-1}(\mathbb{R}^2) \quad \text{for all } s \in \mathbb{R},$$

is invertible, implying that equation (5.24) is uniquely solvable in the space $[H^{r+1/2}(\mathbb{R}^2)]^3$ for $\frac{1}{2} < r < k - \frac{3}{2}$.

In particular, it follows that the system of equations (5.17)–(5.18) is uniquely solvable with respect to $(\widetilde{u}, \widetilde{\varphi})$ in the space

$$[H^{r+1}(\mathbb{R}^3_+)]^3 \times [H^{r+\frac{1}{2}}(\mathbb{R}^2)]^3$$

for arbitrary right-hand sides $\widetilde{F}_1 \in [H^{r+1}(\mathbb{R}^3_+)]^3$ and $\widetilde{F}_2 \in [H^{r-1/2}(\mathbb{R}^2)]^3$ with $\frac{1}{2} < r < k - \frac{3}{2}$.

Consequently, the operator $\widetilde{\mathfrak{R}}_*$ in (5.16) is invertible, which implies that operator (5.15) possesses left and right regularizers. This in turn yields that operator (5.14) possesses left and right regularizers as well. Thus operator (5.14) is Fredholm if the following *Šapiro–Lopatinskiĭ condition* is satisfied:

$$\mathbf{S}_{\mathfrak{R}}(\xi') = -\Pi'\{\widetilde{T}\mathfrak{S}(\mathbf{K})[\mathfrak{S}^{(+)}(\mathbf{B})]^{-1}\Pi^+([\mathfrak{S}^{(-)}(\mathbf{B})]^{-1}\mathfrak{S}(\mathbf{P})\widetilde{T}^\top)\}(\xi') + \mathfrak{S}(\widetilde{\mathcal{L}};\xi') \neq 0 \quad \text{for all } \xi' \neq 0. \tag{5.26}$$

Let us show that condition (5.26) holds true. To this end, let us note that the principal homogeneous symbols $\mathfrak{S}(\mathbf{K})$, $\mathfrak{S}(\mathbf{B})$, $\mathfrak{S}(\mathbf{P})$ and $\mathfrak{S}(\widetilde{\mathcal{L}})$ of the operators $\mathbf{K}$, $\mathbf{B}$, $\mathbf{P}$ and $\widetilde{\mathcal{L}}$ in the chosen local co-ordinate system involved in (5.26) read as

$$\begin{cases} \mathfrak{S}(\mathbf{K};\xi) = |\xi|^{-2}\widetilde{A}(\xi) - I_3, \\ \mathfrak{S}(\mathbf{B};\xi) = |\xi|^{-2}\widetilde{A}(\xi), \\ \mathfrak{S}(\mathbf{P};\xi) = -|\xi|^{-2}I_3, \\ \mathfrak{S}(\widetilde{\mathcal{L}};\xi') = \dfrac{1}{2|\xi'|}\widetilde{T}(\xi', -i|\xi'|)\widetilde{T}^\top(\xi', -i|\xi'|), \end{cases} \qquad \xi = (\xi', \xi_3),\ \xi' = (\xi_1, \xi_2). \tag{5.27}$$

Recall that the matrices $\mathfrak{S}^{(+)}(\mathbf{B})$ and $\mathfrak{S}^{(-)}(\mathbf{B})$ are the so called “plus” and “minus” factors in the factorization of the symbol $\mathfrak{S}(\mathbf{B})$ with respect to the variable ξ_3 and relations (3.4) hold.

Rewrite (5.25) in the form

$$\begin{aligned}\mathbf{S}_{\mathfrak{R}}(\xi') &= -\Pi'[\widetilde{T}(\mathfrak{S}(\mathbf{B}) - I_3)[\mathfrak{S}^{(+)}(\mathbf{B})]^{-1}\Pi^+([\mathfrak{S}^{(-)}(\mathbf{B})]^{-1}\mathfrak{S}(\mathbf{P})\widetilde{T}^\top)](\xi') + \mathfrak{S}(\widetilde{\mathcal{L}};\xi')\\ &= \mathbf{S}^{(1)}_{\mathfrak{R}}(\xi') + \mathbf{S}^{(2)}_{\mathfrak{R}}(\xi') + \mathfrak{S}(\widetilde{\mathcal{L}};\xi'),\end{aligned} \tag{5.28}$$

where

$$\mathbf{S}^{(1)}_{\mathfrak{R}}(\xi') = -\Pi'[\widetilde{T}\mathfrak{S}(\mathbf{B})[\mathfrak{S}^{(+)}(\mathbf{B})]^{-1}\Pi^{+}([\mathfrak{S}^{(-)}(\mathbf{B})]^{-1}\mathfrak{S}(\mathbf{P})\widetilde{T}^{\top})](\xi'), \tag{5.29}$$

$$\mathbf{S}^{(2)}_{\mathfrak{R}}(\xi') = \Pi'[[\mathfrak{S}^{(+)}(\mathbf{B})]^{-1}\Pi^{+}([\mathfrak{S}^{(-)}(\mathbf{B})]^{-1}\mathfrak{S}(\mathbf{P})\widetilde{T}^{\top})](\xi'), \tag{5.30}$$

$$\mathfrak{S}(\widetilde{\mathcal{L}};\xi') = \frac{1}{2|\xi'|}\widetilde{T}(\xi',-i|\xi'|)\widetilde{T}^{\top}(\xi',-i|\xi'|). \tag{5.31}$$

By direct calculations, we find

$$\begin{aligned}
\Pi^{+}([\mathfrak{S}^{(-)}(\mathbf{B})]^{-1}\mathfrak{S}(\mathbf{P})\widetilde{T}^{\top})(\xi') &= \frac{i}{2\pi}\lim_{t\to 0+}\int_{-\infty}^{+\infty}\frac{[\mathfrak{S}^{(-)}(\mathbf{B};\xi',\eta_3)]^{-1}\mathfrak{S}(\mathbf{P};\xi',\eta_3)\widetilde{T}^{\top}(-i\xi',-i\eta_3)d\eta_3}{\xi_3+it-\eta_3}\\
&= -\frac{1}{2\pi}\lim_{t\to 0+}\int_{-\infty}^{+\infty}\frac{[\mathfrak{S}^{(-)}(\mathbf{B};\xi',\eta_3)]^{-1}\widetilde{T}^{\top}(\xi',\eta_3)d\eta_3}{(\xi_3+it-\eta_3)(|\xi'|^2+\eta_3^2)}\\
&= \frac{1}{2\pi}\lim_{t\to 0+}\int_{\ell^-}\frac{[\mathfrak{S}^{(-)}(\mathbf{B};\xi',\tau)]^{-1}\widetilde{T}^{\top}(\xi',\tau)\,d\tau}{(\xi_3+it-\tau)(|\xi'|^2+\tau^2)}\\
&= \frac{1}{2\pi}\lim_{t\to 0+}\frac{2\pi i[\mathfrak{S}^{(-)}(\mathbf{B};\xi',-i|\xi'|)]^{-1}\widetilde{T}^{\top}(\xi',-i|\xi'|)}{(\xi_3+it+i|\xi'|)2(-i|\xi'|)}\\
&= -\frac{[\mathfrak{S}^{(-)}(\mathbf{B};\xi',-i|\xi'|)]^{-1}\widetilde{T}^{\top}(\xi',-i|\xi'|)}{2|\xi'|(\xi_3+i|\xi'|)}.
\end{aligned} \tag{5.32}$$

Now, using (5.32), from (5.29) we derive

$$\begin{aligned}
\mathbf{S}^{(1)}_{\mathfrak{R}}(\xi') &= -\Pi'[\widetilde{T}\mathfrak{S}^{(-)}(\mathbf{B})\mathfrak{S}^{(+)}(\mathbf{B})[\mathfrak{S}^{(+)}(\mathbf{B})]^{-1}\Pi^{+}([\mathfrak{S}^{(-)}(\mathbf{B})]^{-1}\mathfrak{S}(\mathbf{P})\widetilde{T}^{\top})](\xi')\\
&= -\Pi'[\widetilde{T}\mathfrak{S}^{(-)}(\mathbf{B})\Pi^{+}([\mathfrak{S}^{(-)}(\mathbf{B})]^{-1}\mathfrak{S}(\mathbf{P})\widetilde{T}^{\top})](\xi')\\
&= \Pi'\Big[\widetilde{T}(-i\xi',-i\xi_3)\mathfrak{S}^{(-)}(\mathbf{B};\xi',\xi_3)\Big(\frac{[\mathfrak{S}^{(-)}(\mathbf{B};\xi',-i|\xi'|)]^{-1}\widetilde{T}^{\top}(\xi',-i|\xi'|)}{2|\xi'|(\xi_3+i|\xi'|)}\Big)\Big]\\
&= -i\Pi'\Big[\frac{\widetilde{T}(\xi',\xi_3)\mathfrak{S}^{(-)}(\mathbf{B};\xi',\xi_3)}{\xi_3+i|\xi'|}\Big]\Big(\frac{[\mathfrak{S}^{(-)}(\mathbf{B};\xi',-i|\xi'|)]^{-1}\widetilde{T}^{\top}(\xi',-i|\xi'|)}{2|\xi'|}\Big)\\
&= \frac{i}{2\pi}\int_{\ell^-}\frac{\widetilde{T}(\xi',\tau)\mathfrak{S}^{(-)}(\mathbf{B};\xi',\tau)}{\tau+i|\xi'|}\,d\tau\Big(\frac{[\mathfrak{S}^{(-)}(\mathbf{B};\xi',-i|\xi'|)]^{-1}\widetilde{T}^{\top}(\xi',-i|\xi'|)}{2|\xi'|}\Big)\\
&= -\widetilde{T}(\xi',-i|\xi'|)\mathfrak{S}^{(-)}(\mathbf{B};\xi',-i|\xi'|)\frac{[\mathfrak{S}^{(-)}(\mathbf{B};\xi',-i|\xi'|)]^{-1}\widetilde{T}^{\top}(\xi',-i|\xi'|)}{2|\xi'|}\\
&= -\frac{1}{2|\xi'|}\widetilde{T}(\xi',-i|\xi'|)\widetilde{T}^{\top}(\xi',-i|\xi'|).
\end{aligned} \tag{5.33}$$

Similarly, using (5.32) and Lemma 3.3, from (5.30) we get

$$\begin{aligned}
\mathbf{S}^{(2)}_{\mathfrak{R}}(\xi') &= \Pi'[\widetilde{T}[\mathfrak{S}^{(+)}(\mathbf{B})]^{-1}\Pi^{+}([\mathfrak{S}^{(-)}(\mathbf{B})]^{-1}\mathfrak{S}(\mathbf{P})\widetilde{T}^{\top})](\xi')\\
&= -\Pi'\Big\{\widetilde{T}(-i\xi',-i\xi_3)[\mathfrak{S}^{(+)}(\mathbf{B};\xi',\xi_3)]^{-1}\Big(\frac{[\mathfrak{S}^{(-)}(\mathbf{B};\xi',-i|\xi'|)]^{-1}\widetilde{T}^{\top}(\xi',-i|\xi'|)}{2|\xi'|(\xi_3+i|\xi'|)}\Big)\Big\}(\xi')\\
&= i\Pi'[\widetilde{T}(\xi',\xi_3)\frac{[\mathfrak{S}^{(+)}(\mathbf{B};\xi',\xi_3)]^{-1}}{\xi_3+i|\xi'|}](\xi')\Big(\frac{[\mathfrak{S}^{(-)}(\mathbf{B};\xi',-i|\xi'|)]^{-1}\widetilde{T}^{\top}(\xi',-i|\xi'|)}{2|\xi'|}\Big)\\
&= \frac{i}{2|\xi'|}\Big(-\frac{1}{2\pi}\int_{\ell^-}\frac{\widetilde{T}(\xi',\tau)[\mathfrak{S}^{(+)}(\mathbf{B};\xi',\tau)]^{-1}}{\tau+i|\xi'|}\,d\tau\Big)[\mathfrak{S}^{(-)}(\mathbf{B};\xi',-i|\xi'|)]^{-1}\widetilde{T}^{\top}(\xi',-i|\xi'|)\\
&= -\frac{i}{4\pi|\xi'|}\int_{\ell^-}\widetilde{T}(\xi',\tau)[\widetilde{A}^{(+)}(\xi',\tau)]^{-1}\,d\tau(-2i|\xi'|)[\widetilde{A}^{(-)}(\xi',-i|\xi'|)]^{-1}\widetilde{T}^{\top}(\xi',-i|\xi'|)\\
&= -\Big(\frac{1}{2\pi}\int_{\ell^-}\widetilde{T}(\xi',\tau)[\widetilde{A}^{(+)}(\xi',\tau)]^{-1}\,d\tau\Big)[\widetilde{A}^{(-)}(\xi',-i|\xi'|)]^{-1}\widetilde{T}^{\top}(\xi',-i|\xi'|).
\end{aligned} \tag{5.34}$$

Therefore, in view of relations (5.27), (5.28), (5.31), (5.33) and (5.34), we finally obtain

$$\mathbf{S}_{\mathfrak{R}}(\xi') = \mathbf{S}^{(2)}_{\mathfrak{R}}(\xi') = -\Big(\frac{1}{2\pi}\int\limits_{\ell^-} \widetilde{T}(\xi',\tau)[\widetilde{A}^{(+)}(\xi',\tau)]^{-1}\,d\tau\Big)[\widetilde{A}^{(-)}(\xi',-i|\xi'|)]^{-1}\widetilde{T}^{\top}(\xi',-i|\xi'|).$$

Since $\det\widetilde{A}^{(-)}(\xi',-i|\xi'|)\neq 0$ and $\det\widetilde{T}(\xi',-i|\xi'|)\neq 0$ for all $\xi'\neq 0$, by (3.5) and (5.12) and since due to Lemma 5.1 we have

$$\det\Big(\int\limits_{\ell^-}\widetilde{T}(\xi',\tau)[\widetilde{A}^{(+)}(\xi',\tau)]^{-1}\,d\tau\Big)\neq 0 \quad \text{for all } \xi'\neq 0,$$

we deduce that

$$\det\mathbf{S}_{\mathfrak{R}}(\xi')\neq 0 \quad \text{for all } \xi'\neq 0.$$

Thus we have obtained that for the operator $\mathfrak{R}$ the Šapiro–Lopatinskiĭ condition holds. Therefore, the operator

$$\mathfrak{R} : [H^{r+1}(\Omega)]^3\times[H^{r+\frac12}(S)]^3 \to [H^{r+1}(\Omega)]^3\times[H^{r-\frac12}(S)]^3$$

is Fredholm for $\frac12 < r < k-\frac32$.

Step 2. Here we will show that $\operatorname{Ind}\mathfrak{R}=0$. To this end, for $t\in[0,1]$ let us consider the operator

$$\mathfrak{R}_t := \begin{bmatrix} r_\Omega \mathbf{B}_t \mathring{E} & r_\Omega \mathbf{W}_t + r_\Omega \mathbf{V}_\varkappa \\ t\gamma^+(T_t\mathbf{K})\mathring{E} & \mathcal{L}_t + (\mathbf{d}-I_3)\varkappa + \mathcal{W}'_\varkappa \end{bmatrix},$$

where $\mathbf{B}_t = \mathbf{I} + t\mathbf{K}$ and

$$\mathbf{W}_t(g)(y) = -\int\limits_S [T_t(x,\partial_x)P(x-y)]^{\top} g(x)\,dS_x, \quad \mathcal{L}_t(g)(y) = [T_t(y,\partial_y)\mathbf{W}_t(g)(y)]^+, \quad y\in S,\ t\in[0,1],$$

with $T_t(y,\partial_y)$ defined in (5.11). Clearly, $\mathcal{L}=\mathcal{L}_1$.

Now we prove that $\mathfrak{R}_t$ is homotopic to the operator $\mathfrak{R}=\mathfrak{R}_1$. We have to check that for the operator $\mathfrak{R}_t$ the Šapiro–Lopatinskiĭ condition is satisfied for all $t\in[0,1]$. Indeed, in this case the Šapiro–Lopatinskiĭ condition reads as (cf. (5.25))

$$\begin{aligned}\mathbf{S}_{\mathfrak{R}_t}(\xi') &= -\Pi'[\widetilde{T}_t(\mathfrak{S}(\mathbf{B}_t)-I_3)[\mathfrak{S}^{(+)}(\mathbf{B}_t)]^{-1}\Pi^+([\mathfrak{S}^{(-)}(\mathbf{B}_t)]^{-1}\mathfrak{S}(\mathbf{P})\widetilde{T}_t^{\top})](\xi') + \mathfrak{S}(\widetilde{\mathcal{L}}_t;\xi')\\ &= \mathbf{S}^{(1)}_{\mathfrak{R}_t}(\xi') + \mathbf{S}^{(2)}_{\mathfrak{R}_t}(\xi') + \mathfrak{S}(\widetilde{\mathcal{L}}_t;\xi')\neq 0 \quad \text{for all } \xi'\neq 0,\end{aligned} \tag{5.35}$$

where

$$\begin{aligned}\mathbf{S}^{(1)}_{\mathfrak{R}_t}(\xi') &= -\Pi'[\widetilde{T}_t\mathfrak{S}(\mathbf{B}_t)[\mathfrak{S}^{(+)}(\mathbf{B}_t)]^{-1}\Pi^+([\mathfrak{S}^{(-)}(\mathbf{B}_t)]^{-1}\mathfrak{S}(\mathbf{P})\widetilde{T}_t^{\top})](\xi') = -\frac{1}{2|\xi'|}\widetilde{T}_t(\xi',-i|\xi'|)\widetilde{T}_t^{\top}(\xi',-i|\xi'|),\\ \mathbf{S}^{(2)}_{\mathfrak{R}_t}(\xi') &= \Pi'[[\mathfrak{S}^{(+)}(\mathbf{B}_t)]^{-1}\Pi^+([\mathfrak{S}^{(-)}(\mathbf{B}_t)]^{-1}\mathfrak{S}(\mathbf{P}))\}(\xi'),\\ \mathfrak{S}(\widetilde{\mathcal{L}}_t;\xi') &= \frac{1}{2|\xi'|}\widetilde{T}_t(\xi',-i|\xi'|)\widetilde{T}_t^{\top}(\xi',-i|\xi'|).\end{aligned}$$

By direct calculations, we find

$$\begin{aligned}\mathbf{S}^{(2)}_{\mathfrak{R}_t}(\xi') &= \Pi'[\widetilde{T}_t[\mathfrak{S}^{(+)}(\mathbf{B}_t)]^{-1}\Pi^+([\mathfrak{S}^{(-)}(\mathbf{B}_t)]^{-1}\mathfrak{S}(\mathbf{P})\widetilde{T}_t^{\top})](\xi')\\ &= -\Pi'\Big[\widetilde{T}_t(-i\xi',-i\xi_3)[\mathfrak{S}^{(+)}(\mathbf{B}_t;\xi',\xi_3)]^{-1}\Big(\frac{[\mathfrak{S}^{(-)}(\mathbf{B}_t;\xi',-i|\xi'|)]^{-1}\widetilde{T}_t^{\top}(\xi',-i|\xi'|)}{2|\xi'|(\xi_3+i|\xi'|)}\Big)\Big]\\ &= i\Pi'\Big[\frac{\widetilde{T}_t(\xi',\xi_3)[\mathfrak{S}^{(+)}(\mathbf{B}_t;\xi',\xi_3)]^{-1}}{\xi_3+i|\xi'|}\Big]\Big(\frac{[\mathfrak{S}^{(-)}(\mathbf{B}_t;\xi',-i|\xi'|)]^{-1}\widetilde{T}_t^{\top}(\xi',-i|\xi'|)}{2|\xi'|}\Big)\\ &= \frac{i}{2|\xi'|}\Big(-\frac{1}{2\pi}\int\limits_{\ell^-}\frac{\widetilde{T}_t(\xi',\tau)[\mathfrak{S}^{(+)}(\mathbf{B}_t;\xi',\tau)]^{-1}}{\tau+i|\xi'|}\,d\tau\Big)[\mathfrak{S}^{(-)}(\mathbf{B}_t;\xi',-i|\xi'|)]^{-1}\widetilde{T}_t^{\top}(\xi',-i|\xi'|)\\ &= -\frac{i}{4\pi|\xi'|}\int\limits_{\ell^-}\widetilde{T}_t(\xi',\tau)[\widetilde{A}_t^{(+)}(\xi',\tau)]^{-1}\,d\tau(-2i|\xi'|)[\widetilde{A}_t^{(-)}(\xi',-i|\xi'|)]^{-1}\widetilde{T}_t^{\top}(\xi',-i|\xi'|)\\ &= -\Big(\frac{1}{2\pi}\int\limits_{\ell^-}\widetilde{T}_t(\xi',\tau)[\widetilde{A}_t^{(+)}(\xi',\tau)]^{-1}\,d\tau\Big)[\widetilde{A}_t^{(-)}(\xi',-i|\xi'|)]^{-1}\widetilde{T}_t^{\top}(\xi',-i|\xi'|),\end{aligned} \tag{5.36}$$

where

$$A_t(\xi) = (1-t)|\xi|^2 I + tA(\xi) \quad \text{and} \quad A_t(\xi', \xi_3) = A_t^-(\xi', \xi_3) A_t^+(\xi', \xi_3),$$

and $A_t^{\pm}(\xi', \xi_3)$ are the "plus" and "minus" polynomial matrix factors in ξ_3 of the positive definite polynomial symbol matrix $A_t(\xi', \xi_3)$.

By the same approach as in the proof of Lemma 5.1, it can be shown that the first matrix in (5.36) is non-singular for all $\xi' \neq 0$. Therefore, by (5.35), (5.36) and (5.12) we have

$$\mathbf{S}_{\mathfrak{R}_t}(\xi') = \mathbf{S}^{(2)}_{\mathfrak{R}_t}(\xi') \neq 0 \quad \text{for all } \xi' \neq 0 \text{ and for all } t \in [0,1],$$

which implies that for the operator $\mathfrak{R}_t$ the Šapiro–Lopatinskiĭ condition is satisfied. Therefore, the operator

$$\mathfrak{R}_t : [H^{r+1}(\Omega)]^3 \times [H^{r+\frac{1}{2}}(S)]^3 \to [H^{r+1}(\Omega)]^3 \times [H^{r-\frac{1}{2}}(S)]^3$$

is Fredholm for all $\frac{1}{2} < r < k - \frac{3}{2}$ and for all $t \in [0,1]$.

Further, we show that the index of the operator

$$\mathfrak{R}_0 = \begin{bmatrix} r_\Omega I\mathring{E} & r_\Omega \mathbf{W}_0 + r_\Omega \mathbf{V}_\varkappa \\ 0 & \mathcal{L}_0 + (d - I) + \mathcal{W}'_\varkappa \end{bmatrix} : H^{r+1}(\Omega) \times H^{r+\frac{1}{2}}(S) \to H^{r+1}(\Omega) \times H^{r-\frac{1}{2}}(S)$$

is zero. First, we show that the index of the operator $\mathcal{L}_t$ equals zero for all $t \in [0,1]$.

The principal homogeneous symbol matrix of the operator $\mathcal{L}_t$ reads as

$$\mathfrak{S}(\widetilde{\mathcal{L}}_t; \xi') = \frac{1}{2|\xi'|} \widetilde{T}_t(\xi', -i|\xi'|) \widetilde{T}_t^\top(\xi', -i|\xi'|)$$

and is elliptic due to (5.12). Consequently, the operator $\mathcal{L}_t : [H^{r+1/2}(S)]^3 \to [H^{r-1/2}(S)]^3$ is Fredholm for all $t \in [0,1]$. Moreover, it is evident that the principal part of the operator $\mathcal{L}_0 : [H^{1/2}(S)]^3 \to [H^{-1/2}(S)]^3$ is self-adjoint due to the representation $\mathcal{L}_0 g = \mathcal{L}_\Delta g$, where $\mathcal{L}_\Delta$ stands for the trace of the normal derivative of the localized harmonic double layer potential:

$$\mathcal{L}_\Delta g(y) = -\gamma^+ \Big\{ \frac{\partial}{\partial n(y)} \int_S \frac{\partial \Phi(x-y)}{\partial n(x)} g(x)\, dS_x \Big\}.$$

Therefore, $\operatorname{Ind} \mathcal{L} = \operatorname{Ind} \mathcal{L}_t = \operatorname{Ind} \mathcal{L}_0 = 0$ for all $t \in [0,1]$, implying that the index of the operator $\mathfrak{R}_0$ equals zero. Since the operator $\mathfrak{R}_t$ is homotopic to $\mathfrak{R}_0$ for all $t \in [0,1]$, we conclude that

$$\operatorname{Ind} \mathfrak{R} = \operatorname{Ind} \mathfrak{R}_1 = \operatorname{Ind} \mathfrak{R}_t = \operatorname{Ind} \mathfrak{R}_0 = 0$$

for all $t \in [0,1]$ and for all $\frac{1}{2} < r < k - \frac{3}{2}$.

Step 3. From the equivalence theorem (Theorem 4.1) it follows that the null space of the operator $\mathfrak{R}$ in the class $H^2(\Omega) \times H^{3/2}(S)$ is trivial. Therefore, the same is valid also for all $\frac{1}{2} < r < k - \frac{3}{2}$ (see [2, Assertion 10.6]) implying that operator (5.14) is invertible for all $\frac{1}{2} < r < k - \frac{3}{2}$. □

Remark 5.3. Note that condition (5.12) with $t = 1$ is necessary for the operator $\mathfrak{R}$ to be Fredholm (cf. [2, Theorem 12.1]). Further, we consider two particular examples for the elasticity models and show that in the case of isotropic bodies condition (5.12) is satisfied automatically, while in the case of transversally isotropic bodies it is not satisfied in general.

Suppose that the domain Ω is filled with an isotropic inhomogeneous elastic material. The matrix differential operator of elasticity for isotropic inhomogeneous media reads as

$$A(x, \partial_x) = [A_{pq}(x, \partial_x)]_{3\times 3} := \big[\delta_{pq}\{\partial_i(\mu(x)\partial_i)\} + \partial_p((\lambda(x) + \mu(x))\partial_q)\big]_{3\times 3}.$$

The corresponding stress operator has the form $T = T(x, \partial_x) = [T_{pq}(x, \partial_x)]_{3\times 3}$, $p, q = 1, 2, 3$:

$$T_{11} = (\lambda + 2\mu) n_1 \partial_1 + \mu n_2 \partial_2 + \mu n_3 \partial_3, \quad T_{12} = \lambda n_1 \partial_2 + \mu n_2 \partial_1, \quad T_{13} = \lambda n_1 \partial_3 + \mu n_3 \partial_1,$$

$$T_{21} = \mu n_1 \partial_2 + \lambda n_2 \partial_1, \quad T_{22} = \mu n_1 \partial_1 + (\lambda + 2\mu) n_2 \partial_2 + \mu n_3 \partial_3, \quad T_{23} = \lambda n_2 \partial_3 + \mu n_3 \partial_2,$$

and

$$T_{31} = \mu n_1 \partial_3 + \lambda n_3 \partial_1, \quad T_{32} = \mu n_2 \partial_3 + \lambda n_3 \partial_2, \quad T_{33} = \mu n_1 \partial_1 + \mu n_2 \partial_2 + (\lambda + 2\mu) n_3 \partial_3,$$

where $\lambda, \mu \in C^\infty(\mathbb{R}^3)$ are variable Lame's coefficients satisfying the inequalities $\mu > 0$ and $3\lambda + 2\mu > 0$.

In this case, for the matrix $\widetilde{T}(\xi', \xi_3)$ we have

$$\widetilde{T}(\xi', \xi_3) = \begin{bmatrix} \mu\xi_3 & 0 & \mu\xi_1 \\ 0 & \mu\xi_3 & \mu\xi_2 \\ \lambda\xi_1 & \lambda\xi_2 & (\lambda+2\mu)\xi_3 \end{bmatrix},$$

while the matrix $\widetilde{T}_t(\xi', \xi_3)$ for $t \in [0, 1]$ reads as (see (5.13))

$$\widetilde{T}_t(\xi', \xi_3) = \begin{bmatrix} (1-t)\xi_3 + t\mu\xi_3 & 0 & t\mu\xi_1 \\ 0 & (1-t)\xi_3 + t\mu\xi_3 & t\mu\xi_2 \\ t\lambda\xi_1 & t\lambda\xi_2 & (1-t)\xi_3 + t(\lambda+2\mu)\xi_3 \end{bmatrix}.$$

Then it is easy to show that

$$\det \widetilde{T}_t(\xi', -i|\xi'|) = i|\xi'|^3(1 - t + \mu t)[(1-t)^2 + (1-t)t(\lambda + 3\mu) + 2\mu(\lambda+\mu)t^2] \neq 0 \quad \text{for all } \xi' \neq 0 \text{ and all } t \in [0, 1].$$

Therefore, in the isotropic case, condition (5.12) in Theorem 5.2 is satisfied automatically.

Now suppose that the domain Ω is filled with a transversally isotropic inhomogeneous elastic material. The system of differential equations of elasticity for transversally isotropic inhomogeneous media is written in the form

$$\begin{aligned} &[\partial_1(c_{11}\partial_1 u_1) + \partial_2(c_{66}\partial_2 u_1) + \partial_3(c_{44}\partial_3 u_1)] + \partial_1((c_{11} - c_{66})\partial_2 u_2) + \partial_1((c_{13} + c_{44})\partial_3 u_3) = F_1, \\ &\partial_2((c_{11} - c_{66})\partial_1 u_1) + [\partial_1(c_{11}\partial_1 u_2) + \partial_2(c_{66}\partial_2 u_2) + \partial_3(c_{44}\partial_3 u_2)] + \partial_2((c_{13} + c_{44})\partial_3 u_3) = F_2, \\ &\partial_3((c_{13} + c_{44})\partial_1 u_1) + \partial_3((c_{13} + c_{44})\partial_2 u_2) + [\partial_1(c_{44}\partial_1 u_3) + \partial_2(c_{44}\partial_2 u_3) + \partial_3(c_{33}\partial_3 u_3)] = F_3, \end{aligned}$$

and the corresponding stress operator has the form $T = T(x, \partial_x) = [T_{pq}(x, \partial_x)]_{3\times 3}$, $p, q = 1, 2, 3$:

$$\begin{aligned} &T_{11} = c_{11} n_1 \partial_1 + c_{66} n_2 \partial_2 + c_{44} n_3 \partial_3, \quad T_{12} = (c_{11} - 2c_{66}) n_1 \partial_2 + c_{66} n_2 \partial_1, \quad T_{13} = c_{13} n_1 \partial_3 + c_{44} n_3 \partial_1, \\ &T_{21} = c_{66} n_1 \partial_2 + (c_{11} - 2c_{66}) n_2 \partial_1, \quad T_{22} = c_{66} n_1 \partial_1 + c_{11} n_2 \partial_2 + c_{44} n_3 \partial_3, \quad T_{23} = c_{13} n_2 \partial_3 + c_{44} n_3 \partial_2, \end{aligned}$$

and

$$T_{31} = c_{44} n_1 \partial_3 + c_{13} n_3 \partial_1, \quad T_{32} = c_{44} n_2 \partial_3 + c_{13} n_3 \partial_2, \quad T_{33} = c_{44} n_1 \partial_1 + c_{44} n_2 \partial_2 + c_{33} n_3 \partial_3,$$

where $c_{11}, c_{33}, c_{44}, c_{66}, c_{13} \in C^\infty(\mathbb{R}^3)$ are variable elastic coefficients satisfying the inequalities

$$c_{11} > 0, \quad c_{33} > 0, \quad c_{44} > 0, \quad c_{66} > 0, \quad c_{13}^2 < c_{33}(c_{11} - c_{66}).$$

Further, let us assume that a neighborhood of some point $\widetilde{y} \in S$ is a part of the plane which is parallel to the plane of isotropy. In this case, the matrix $\widetilde{T}_t(\xi', \xi_3)$ is written as

$$\widetilde{T}_t(\xi', \xi_3) = (1-t) I \xi_3 + t\widetilde{T}(\xi', \xi_3) \quad \text{for } t \in [0, 1],$$

where

$$\widetilde{T}(\xi', \xi_3) = \begin{bmatrix} c_{44}\xi_3 & 0 & c_{44}\xi_1 \\ 0 & c_{44}\xi_3 & c_{44}\xi_2 \\ c_{13}\xi_1 & c_{13}\xi_2 & c_{33}\xi_3 \end{bmatrix}.$$

Then, for $\xi' \neq 0$ and $t \in [0, 1]$, we have

$$\det \widetilde{T}_t(\xi', -i|\xi'|) = i|\xi'|^3(1 - t + c_{44}t)[(1 - t + c_{44}t)(1 - t + c_{33}t) + c_{13}c_{44}t^2],$$

which implies that condition (5.12) is not satisfied in general.

Indeed, for $t = 1$ and $\xi' \neq 0$ we get

$$\det \widetilde{T}_1(\xi', -i|\xi'|) = \det \widetilde{T}(\xi', -i|\xi'|) = i|\xi'|^3 c_{44}^2(c_{33} + c_{13}),$$

and if $c_{13} = -c_{33}$, then $\det \widetilde{T}_1(\xi', -i|\xi'|) = 0$.

A Classes of cut-off functions

Definition A.1. We say $\chi \in X^k$ for an integer $k \geq 0$ if

$$\begin{aligned} \chi(x) &= \breve{\chi}(|x|), \\ \chi(0) &= 1, \\ \breve{\chi} &\in W_1^k(0,\infty), \\ \varrho\breve{\chi}(\varrho) &\in L_1(0,\infty). \end{aligned}$$

We say $\chi \in X_+^k$ for an integer $k \geq 1$ if $\chi \in X^k$ and $\sigma_\chi(\omega) > 0$ for all $\omega \in \mathbb{R}$, where

$$\sigma_\chi(\omega) := \begin{cases} \dfrac{\hat{\chi}_s(\omega)}{\omega} > 0 & \text{for } \omega \in \mathbb{R} \setminus \{0\}, \\ \displaystyle\int_0^\infty \varrho\breve{\chi}(\varrho)\, d\varrho & \text{for } \omega = 0, \end{cases} \qquad \hat{\chi}_s(\omega) := \int_0^\infty \breve{\chi}(\varrho)\sin(\varrho\omega)\, d\varrho.$$

We say $\chi \in X_{1+}^k$ for an integer $k \geq 1$ if $\chi \in X_+^k$ and $\omega\hat{\chi}_s(\omega) \leq 1$ for all $\omega \in \mathbb{R}$.

Evidently, we have the following imbeddings: $X^{k_1} \subset X^{k_2}$, $X_+^{k_1} \subset X_+^{k_2}$ and $X_{1+}^{k_1} \subset X_{1+}^{k_2}$ for $k_1 > k_2$ (for details see [9]). An example of a cut-off function $\chi_{1k} \in X_{1+}^k$ for $k \geq 2$ is

$$\chi_{1k}(x) = \begin{cases} \left[1 - \dfrac{|x|}{\varepsilon}\right]^k & \text{for } |x| < \varepsilon, \\ 0 & \text{for } |x| \geq \varepsilon. \end{cases}$$

B Properties of localized potentials

Here we collect some theorems describing the mapping properties of the localized potentials (2.10), (2.11) and (2.15)–(2.17). The proofs can be found in [9].

Theorem B.1. *Let $\chi \in X^k$ with $k \geq 3$ have a compact support, and let $t < k - 1$, and $-\frac{1}{2} < s < k - \frac{1}{2}$. Then the following localized operators are continuous:*

$$\mathbf{V} : [H^t(S)]^3 \to [H^{t+\frac{3}{2}}(\Omega^\pm)]^3, \quad \mathbf{W} : [H^t(S)]^3 \to [H^{t+\frac{1}{2}}(\Omega^\pm)]^3, \quad \mathcal{P} : [H^s(\Omega)]^3 \to [H^{s+2}(\Omega)]^3.$$

Moreover, if $\chi \in X_{1+}^3$, then the operator $\mathbf{P} : [H^r(\mathbb{R}^3)]^3 \to [H^{r+2}(\mathbb{R}^3)]^3$ is invertible for all $r \in \mathbb{R}$ and the inverse operator $\mathbf{P}^{-1}$ is representable in the form

$$\mathbf{P}^{-1} = \Delta\mathbf{I} - \mathbf{M}_\chi\mathbf{I},$$

where Δ is the Laplace operator, $\mathbf{I}$ is the unit operator and $\mathbf{M}_\chi$ is a scalar pseudodifferential operator with the symbol $\widehat{m}_\chi(\xi)$ satisfying the inequality

$$0 \leq \widehat{m}_\chi(\xi) \leq C < \infty \quad \text{for all } \xi \in \mathbb{R}^3, \tag{B.1}$$

with some positive constant C.

Theorem B.2. *Let $\chi \in X^3$, $\psi \in H^{-1/2}(S)$, and $\varphi \in H^{1/2}(S)$. Then the following jump relations hold on S:*

$$\gamma^+\mathbf{V}\psi = \gamma^-\mathbf{V}\psi = \mathcal{V}\psi, \quad \gamma^\pm\mathbf{W}\varphi = \mp\mathbf{d}\varphi + \mathcal{W}\varphi, \quad T^\pm\mathbf{V}\psi = \pm\mathbf{d}\psi + \mathcal{W}'\psi,$$

where

$$\mathbf{d}(y) = [\mathbf{d}^{pq}(y)]_{3\times3} := \frac{1}{2}[a_{kj}^{pq}(y)n_k(y)n_j(y)]_{3\times3}, \quad y \in S, \tag{B.2}$$

and $\mathbf{d}(y)$ is positive definite due to (2.1).

Theorem B.3. *Let $\chi \in X^k$ with $k \geq 3$ and $1 < t < k - 1$. Then the following operators are continuous:*

$$\begin{aligned} \mathcal{V} &: [H^t(S)]^3 \to [H^{t+1}(S)]^3, \\ \mathcal{W} &: H^t(S) \to H^t(S), \\ \mathcal{W}' &: [H^t(S)]^3 \to [H^t(S)]^3, \\ \mathcal{L}^{\pm} &: [H^t(S)]^3 \to [H^{t-1}(S)]^3. \end{aligned}$$

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