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John A. Gregory and Jianwei Zhou

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John A. Gregory
Jianwei Zhou
Department of Mathematics and Statistics, Brunel University,
Uxbridge, Middlesex UB8 3PH, England.

Abstract

The local support basis representation of the 'weighted v-spline' is derived in terms of double knot cubic B-splines, so providing a convenient form for computing and analysing the representation.

Keywords

Splines, B-splines, tension.

1 Introduction

In this note, the local support basis representation of the weighted v-spline, [Foley'87,88], is considered via a transformation to double knot cubic B-spline form. The weighted v-spline is a C^1 piecewise cubic curve which has specific second derivative jump discontinuity constraints across the knots of the spline. The constraints involve interval and point tension weights, which can be used to influence the shape of the curve, and the transformation to double knot B-spline form provides a convenient computational tool for handling and analysing the local support basis representation.

The weighted v-spline combines the v-spline, [Nielson'74], with the weighted spline, [Salkauskas'84], and thus includes these splines as special cases. Since the v-spline is intimately connected with the β-spline, [Barsky'81], [Goodman'85], there is an immediate relationship between the local support basis representation presented here with that of the β-spline. The development of the local support basis representation builds on the work of [Foley'86], for the weighted spline, and with the work of [Boehm'85] and [Lasser'88], for 'geometric splines', where Bernstein-Bézier representations are discussed, see also [Dierckx and Tytgat'89]. Thus, for completeness, we briefly review the transformation to Bernstein-Bézier form in section 3. Here we will see that the control points of the Bernstein-Bezier representation include those of the double knot B-spline representation developed in section 2, a result that should not surprise the thoughtful reader. In the final section 4, the derivation of the local support basis representation which satisfies interpolation constraints at the knots is considered.

2 The double knot B-spline representation

Let $\{t_i \in I\!R : i \in \mathbb{Z}\}$ define a sequence of distinct knots with interval spacing $h_i := t_{i+1} - t_i > 0, i \in \mathbb{Z}$. Then a *weighted* ν-spline is a C^1 parametric Function $p : I\!R \to I\!R^m$ such that

$$P\big|(t_i t_{i+1}) \in \prod_3 \ , i \in \mathbb{Z} \ , \ \textit{(piecewise cubic curve)} \tag{2.1}$$

and

$$w_i p^{(2)}(t_i^+) - w_{i-1} P^{(2)}(t_i^-) = v_i P^{(1)}(t_i) \ , \ i \in \mathbb{Z} \, . \tag{2.2}$$

The case $w_i = w_{i-1} = 1$ is that of the unweighted ν-spline of [Nielson'74], when (2.2) defines a *geometric* GC^2 constraint across the knot t_i, that is the curve is C^2 under a reparameterization. The case $V_i = 0$ is the weighted spline of [Salkauskas'84], see also [Foley'86]. The general case (2.2) is that described in [Foley'87,88]. For the analysis we take $w_i \neq 0, i \in \mathbb{Z}$, and replace (2.2) by the equivalent form

$$P^{(2)}(t_i^+) - \widehat{w}_i p^{(2)}(t_i^-) = \widehat{v}_i p^{(1)}(t_i), i \in \mathbb{Z}, \tag{2.3}$$

where

$$\widehat{w}_i = w_{i-1} / w_i \ \widehat{v}_i = v_i / w_i \, . \tag{2.4}$$

Assume that there exists a normalized local support basis $\{N_i(t)\}_{i \in Z}$ for the linear space of weighted ν-splines, defined such that

$$N_i(t) = 0, t \notin (t_i, t_{i+4}), \ \textit{(local support)} \tag{2.5}$$

and

$$\sum_{i \in Z} N_i(t) = 1 \, , \ \textit{(partition of unity normalization).} \tag{2.6}$$

Then given the *control points* $\{c_j \in \mathbb{R} : j \in \mathbb{Z}\}$, we wish to represent the Curve

$$\begin{aligned} p(t) &= \sum_{j \in Z} c_j N_j(t) \\ &= \sum_{j=i-3}^{i} c_j N_j(t), t \in [t_i, t_{i+1}], \end{aligned} \tag{2.7}$$

in double knot cubic B-spline form. Indeed, the representation of a weighted v-spline as a double knot B-spline will demonstrate the existence of the local support basis representation.

Thus, let

$$B_j^4(t) := (\hat{t}_{j+4} - \hat{t}_j)\left[\hat{t}_j, \hat{t}_{j+1}, \hat{t}_{j+2}, \hat{t}_{j+3}, \hat{t}_{j+14},\right](.-t)_+^3 \quad , j \in \mathbb{Z}, \tag{2.8}$$

define the normalized cubic B-splines, see [de Boor'78], on the double knot partition

$$\left\{\hat{t}_j \in \mathbb{R} \quad : \hat{t}_{2j} = \hat{t}_{2j+1} = t_j, \in \mathbb{Z}\right\}, \tag{2.9}$$

In particular,

$$B_{2j}^4(t) := (t_{j+2} - t_i)\left[t_j, t_j, t_{j+1}, t_{j+1}, t_{j+2},\right](.-t)_+^3$$

and

$$B_{2j+1}^4(t) := (t_{j+2} - t_j)\left[t_j, t_{j+1}, t_{j+1}, t_{j+2}, t_{j+2},\right](.-t)_+^3$$

are the double knot B-splines with the local support of (t_j, t_{j+2}).

The space of double knot cubic B-splines consists of C^1 piecewise cubic curves which, in general, have second derivative discontinuities across the knots. We thus seek the representation of a weighted f-spline in the double knot cubic B-spline form

$$\begin{aligned} P(t) &:= \sum_{j \in z} b_j B_j^4(t) \\ &:= \sum_{j=2i-2}^{2i+1} b_j B_j^4(t), t \in [t_i, t_{i+1}], \end{aligned} \tag{2.10}$$

with control points $\{b_j \in \mathbb{R}^m : j \in \mathbf{Z}\}$. Following [de Boor'78, pl39] we have

$$p^{(1)}(t) = \sum_{j \in \mathbb{Z}} b_j^{(1)} B_j^3(t), b_j^{(1)} = 3(b_j - b_{j-1}) / (\hat{t}_{j+3} - \hat{t}_j) \tag{2.11}$$

$$p^{(2)}(t) = \sum_{j \in \mathbb{Z}} b_j^{(2)} B_j^2(t),\ b_j^{(2)} = 2(b_j^{(1)} - b_{j-1}^{(1)}) / (\hat{t}_{j+2} - \hat{t}_j), \tag{2.12}$$

where

$$\left.\begin{aligned} B_j^3(t) &:= (\hat{t}_{j+3} - \hat{t})\left[\hat{t}_j, \hat{t}_{j+1}, \hat{t}_{j+2}, \hat{t}_{j+3}\right](.-t)^2+, \\ B_j^2(t) &:= (\hat{t}_{j+2} - \hat{t})\left[\hat{t}_j, \hat{t}_{j+1}, \hat{t}_{j+2},\right](.-t)^1+, \end{aligned}\right\} \tag{2.13}$$

define the normalized quadratic and linear double knot B-splines. It is now a relatively simple exercise to show that the weighted v-spline constraint (2.3), applied to the double knot B-spline representation (2.10), gives

$$(1 - \tau_{i-2}) b_{2i-2} + \tau_{i-2} b_{2i-3} = (1 - \sigma_{i-2}) b_{2i-1} + \sigma_{i-2} b_{2i}, \tag{2.14}$$

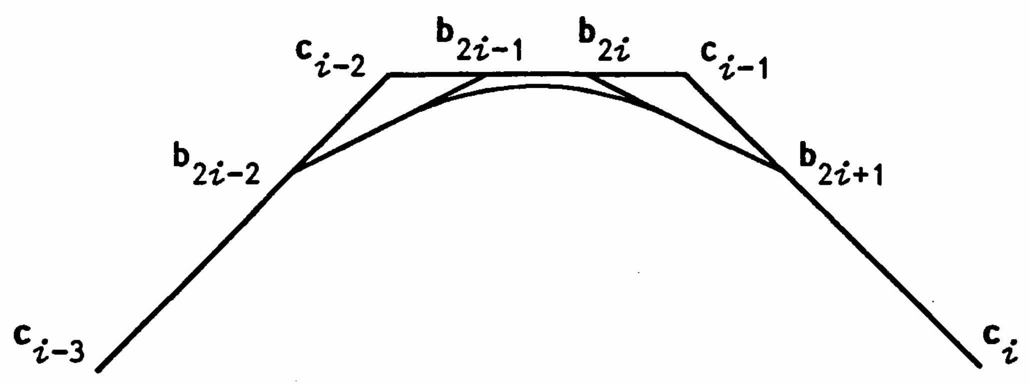

Figure 1: Transformation to double knot B-spline control points

where

$$\tau_{i-2} := 2\hat{w}_i(h_{i-1} + h_i)/(h_{i-1}^2 pi), \sigma_{i-2} := -2(h_{i-1} + h_i)/(h_i^2 pi) \tag{2.15}$$

with

$$\mathrm{pi} := 2/\mathrm{h}_i + 2\hat{\mathrm{w}}_i/\mathrm{h}_{i-1} + \hat{\mathrm{v}}_i. \tag{2.16}$$

Equation (2.14) can be interpreted as the requirement that two lines intersect at a point and it is the simple observation of Proposition 2.1 below that this intersection is the control point c_{i-2} of the local support basis representation (2.7), see figure 1. Thus, given $\{c_i\}_{i\in\mathbb{Z}}$, the control points $\{b_i\}_{i\in\mathbb{Z}}$, are given by solving

$$\left.\begin{array}{l}(1-\sigma_{i-2})b_{2i-1} + \sigma_{i-2}b_{2i} = c_{i-2}, \\ \tau_{i-1}b_{2i-1} + (1-\tau_{i-1})b_{2i} = c_{i-1},\end{array}\right\} i \in \mathbb{Z}, \tag{2.17}$$

that is

$$\left.\begin{array}{l}\mathrm{b}_{2i-1} = [(1-\mathrm{T}_{i-1})/\Delta_i]\mathrm{c}_{i-2} - [\sigma_{i-2}/\Delta_i]\mathrm{c}_{i-1}, \\ \mathrm{b}_{2i} = -[(\mathrm{T}_{i-1}/\Delta_i)]\mathrm{c}_{i-2} + [1-\sigma_{i-2}/\Delta_i]\mathrm{c}_{i-1},\end{array}\right\} i \in \mathrm{B}\mathbb{Z} \tag{2.18}$$

where

$$\Delta_i := 1 - \mathrm{T}_{i-1} - \sigma_{i-2}. \tag{2.19}$$

We now have:

Proposition 2.1 *Suppose* $\mathrm{p}_i \neq 0$ *and* $\Delta_i \neq 0$, $i \in \mathbb{Z}$. *Then*

(i) There exists a normalized local support basis $\{N_i(t)\} i \in \mathbf{Z}$ *for the space of weighted ν-splines.*

(ii) There exists the transformation (2.18) from the control points of the local support basis representation to those of a double knot B-spline represent tation.

Furthermore, if

$$\hat{w}_i > 0 \text{ and } \rho_i := 2/h_i + 2\hat{w}_i/h_{i-1} + \hat{v}_i > 0, \ i \in \mathbb{Z} , \tag{2.20}$$

then

(iii) The normalized local support basis is non-negative and the local support basis representation $P(t), t \in [t_i, t_{i+1}]$, lies in the convex hull of $\{c_i\}_{j=1-3}^{i}$.

(iv) The local support basis representation is variation diminishing, in particular, any (hyper) plane of dimension m — 1 crosses $P(t), t \in [t_i, t_{i+1}]$ no more times than it crosses the control polygon joining the points $\{c_j\}_{j=i-3}^{i}$.

Proof. The proof is straightforward and for brevity we just outline its essential features. For a given $j \in \mathbb{Z}$ let scalar values $b_{2i-1,j}, b_{2i,j}$ be defined by (2.18), with right hand sides having scalar values $c_{i-2,j}, c_{i-1,j}$ defined by

$$c_{i,j} = \delta_{i,j}, \ i \in \mathbb{Z} \tag{2.21}$$

Then $b_{i,j}$, $i \in \mathbb{Z}$ has the support of the integer set {2j + 1,..., 2j + 4} and

$$N_j(t) := \sum_{i \in \mathbb{Z}} b_{i,j} B_i(t) = \sum_{i=2j+1}^{2j+4} b_{i,j} B_i(t) \tag{2.22}$$

defines a weighted v-spline function with the local support of (t_j, t_{j+4}). Moreover, our previous work then demonstrates the existence of the transformation (2.18). In particular, with scalar values $\{c_i = 1 : i \in \mathbb{Z}\}$ we obtain $\{b_i = 1 : i \in \mathbb{Z}\}$ from (2.18) and the partition of unity normalization (2.6) follows from the partition of unity property of the double knot B-splines. The conditions (2.20) guarantee that $\tau_{i-2} < 0, \sigma_{i-2} < 0$, and then $\Delta_i = 1 - \tau_{i-1} - \sigma_{i-2} > 1$, $i \in \mathbb{Z}$. Hence the transformation (2.18) involves convex combinations and it follows that the $b_{i,j}, i \in \mathbb{Z}$ in (2.22) are non-negative. Hence $N_j(t)$ is non-negative and the convex hull property follows from the partition of unity normalization. Also, (2.18) defines a corner cutting process, see figure 1, and hence the variation diminishing property follows from that of the double knot B-spline representation. □

Remark. In terms of the notation of the standard weighted v-spline constraint (2.2), condition (2.20) can be written as

$$w_i > 0, v_i > -2(w_i / h_i + w_{i-1} / h_{i-1}), i \in \mathbb{Z} , \tag{2.23}$$

and it should be noted that negative values of the parameter v_i are allowable whilst still maintaining properties (iii) and (iv) of the Proposition.

Tension properties. The transformation (2.18) can be used to justify the point and interval tension properties of the local support basis representation of the weighted ν-spline with respect to the parameters ν_i and w_i. We first observe that, in terms of the double knot B-spline representation,

$$P(t_i) = \frac{h_i}{h_{i-1}+h_i} b_{2i-2} + \frac{h_{i-1}}{h_{i-1}+h_i} b_{2i-1}. \qquad (2.24)$$

The analysis of the control point transformation (2.18) now leads to the following results, where, for simplicity, it is assumed that the other shape parameters are held constant with respect to each limit process:

(i)Point tension ν_i.

$$\lim_{\nu_i\to\infty} b_{2i-2} = \lim_{\nu_i\to\infty} b_{2i-1} = c_{i-2}. \qquad (2.25)$$

Thus, as ν_i increases, $P(t_i)$ is pulled towards c_{i-2} in the limit a cusp tangent discontinuity is introduced at c_{i-2}.

(ii)Interval tension w_i.

$$\lim_{w_i\to\infty} b_{2i-2} = c_{i-2,} \qquad (2.26)$$

$$\lim_{w_i\to\infty} b_{2i-1} = \frac{2h_i + h_{i+1}}{h_{i-1}+3h_i+h_{i+1}} c_{i-2} + \frac{h_{i-1}+h_i}{h_{i-1}+3h_i+h_{i+1}} c_{i-1}, \qquad (2.27)$$

$$\lim_{w_i\to\infty} b_{2i} = \frac{h_i + h_{i+1}}{h_{i-1}+3h_i+h_{i+1}} c_{i-2} + \frac{h_{i-1}+2h_i}{h_{i-1}+3h_i+h_{i+1}} c_{i-1}, \qquad (2.28)$$

$$\lim_{w_i\to\infty} b_{2i+1} = c_{i-1}. \qquad (2.29)$$

It follows that, as w_i increases, P(t), $t\in[t_i, t_{i+1}]$ is pulled onto a straight line segment between c_{i-2} and c_{i-1}. Furthermore, the end points of this segmentare given by:

$$\lim_{w_i\to\infty} P(t_i) = \frac{3h_i + h_{i+1}}{h_{i-1}+3h_i+h_{i+1}} c_{i-2} + \frac{h_{i-1}}{h_{i-1}+3h_i+h_{i+1}} c_{i-1}, \qquad (2.30)$$

$$\lim_{w_i\to\infty} p(t_{i+1}) = \frac{h_{i+1}}{h_{i-1}+3h_i+h_{i+1}} c_{i-2} + \frac{h_{i-1}+3h_i}{h_{i-1}+3h_i+h_{i+1}} c_{i-1}, \qquad (2.31)$$

Here, the analysis confirms the results of [Foley'86].

Some simple examples of the point and interval tension behaviour for the local support basis are given in figure 2.

3 The Bernstein-Bézier representation

For completeness, we briefly review the transformation of the weighted ν- spline to Bernstein-Bézier (triple knot B-spline) form. This transformation

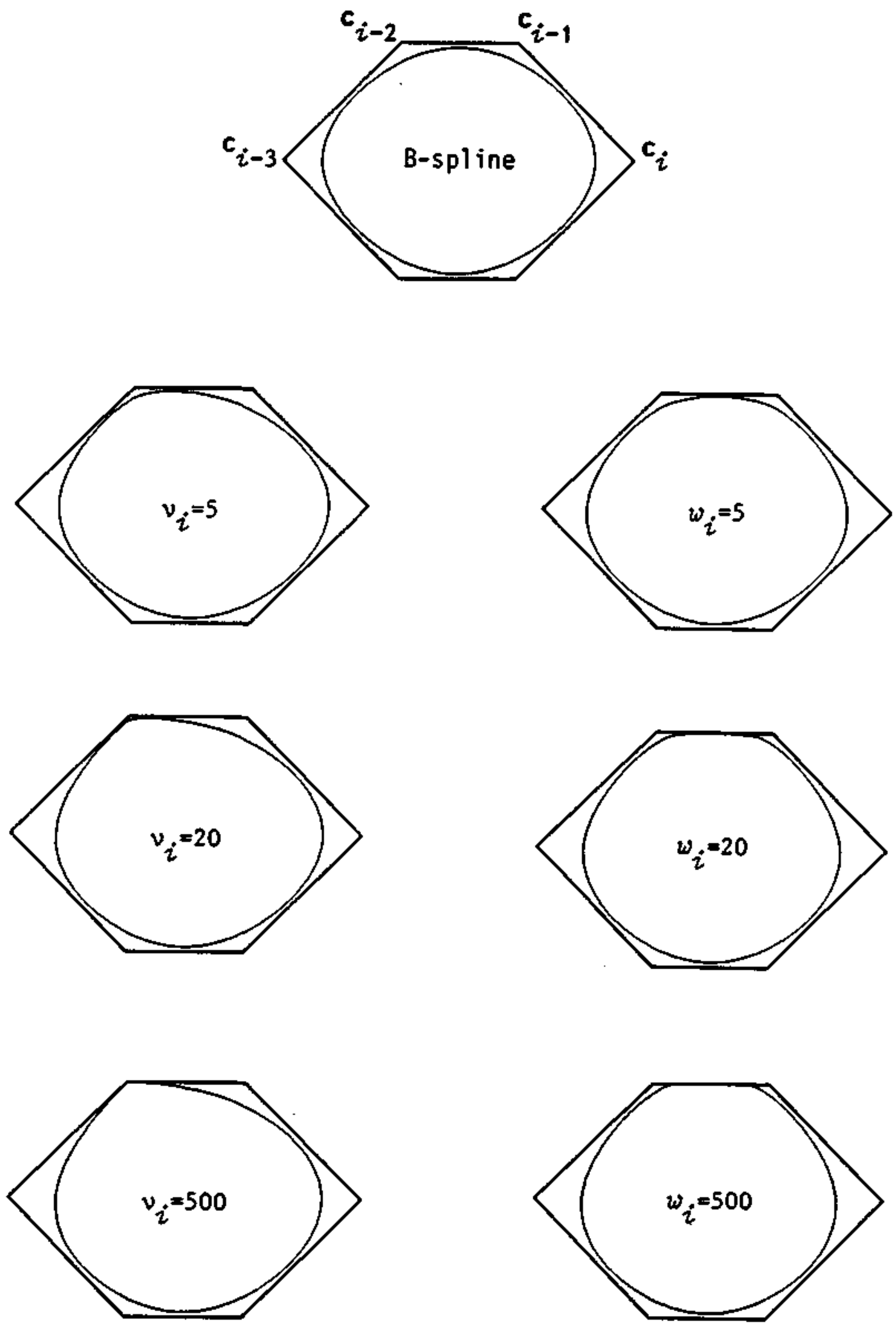

Figure 2: Examples of tension control behaviour

could be easily derived from the double knot B-spline representation but instead we generalize the discussion to the space of C° piecewise cubic *gen-realized* β-spline curves $p: \mathbb{R} \to \mathbb{R}^n$ defined such that

$$\left.\begin{aligned} P^{(1)}\left(t_i^+\right) &= \beta_{1,i} P^{(1)}\left(t_i^-\right), \\ P^{(2)}\left(t_i^+\right) &= \beta_{2,i} P^{(1)}\left(t_i^-\right) + \beta_{3,i} P^{(2)}\left(t_i^-\right), \end{aligned}\right\} i \in \mathbb{Z} \ . \tag{3.1}$$

The derivation of the Bernstein-Bézier form of the local support basis repre-entation then follows from the work of [Boehm'85] and [Lasser'88].

It is well known that there is no loss of generality in taking integer knots in (3.1), since a piecewise affine transformation onto integer knots gives con straint equations with parameters

$$\hat{\beta}_{1,i} := \frac{\beta_{1,i} h_i}{h_{i-1}}, \ \hat{\beta}_{2,i} := \frac{\beta_{2,i} h_i^2}{h_{i-1}}, \ \hat{\beta}_{3,i} := \frac{\beta_{2,i} h_i^2}{h_{i-1}}, \tag{3.2}$$

On the general knot partition, the weighted ν-spline constraint (2.3) is re- covered as the special case

$$\beta_{1,i} = 1, \ \beta_{2,i} = \hat{v}_i, \beta_{3,i} = \hat{w}_i. \tag{3.3}$$

The case $\beta_{3,i} = \beta^2{}_{1,i}$ gives the standard β-spline of [Barsky '81], see [Goodman '85], when (3.1) defines a *geometric* GC^2 constraint across the knot t_i. In fact, in the non-periodic case, the standard β-spline and unweighted ν-spline are equivalent under a piecewise denned affine transformation, although we will not pursue this equivalence here.

We now seek the representation of a generalized β-spline in the C° piece- wise denned Bernstein-Bézier form

$$P\Big|_{(t_i, t_{i+1})}(t) := \sum_{k=0}^{3} b_{3i+k} B_k(\theta), \theta := (t - t_i)/h_i, i \in \mathbb{Z} \ , \tag{3.4}$$

where

$$B_k(\theta) := \binom{3}{k} \theta^k (1-\theta)^{3-k}. \tag{3.5}$$

Imposing the constraints (3.1) on this form gives, after some elimination,

$$b_{3i} = \mu_i b_{3i-1} + (1-\mu_i) b_{3i-1} \tag{3.6}$$

and

$$(1-\tau_{i-2}) b_{3i-1} + \tau_{i-2} b_{3i-1} = (1-\sigma_{i-2}) b_{3i+1} + \sigma_{i-2} b_{3i+2}, \tag{3.7}$$

where

$$\mu_i := \frac{\hat{\beta}_{1,i}}{\left(1+\hat{\beta}_{1,i}\right)}, \ \tau_{i-2} := \frac{-2\left(1+\hat{\beta}_{1,i}\right)\hat{\beta}_{3,i}}{k_i}, \ \sigma_{i-2} := \frac{-2\left(1+\hat{\beta}_{1,i}\right)}{k_i}, \tag{3.8}$$

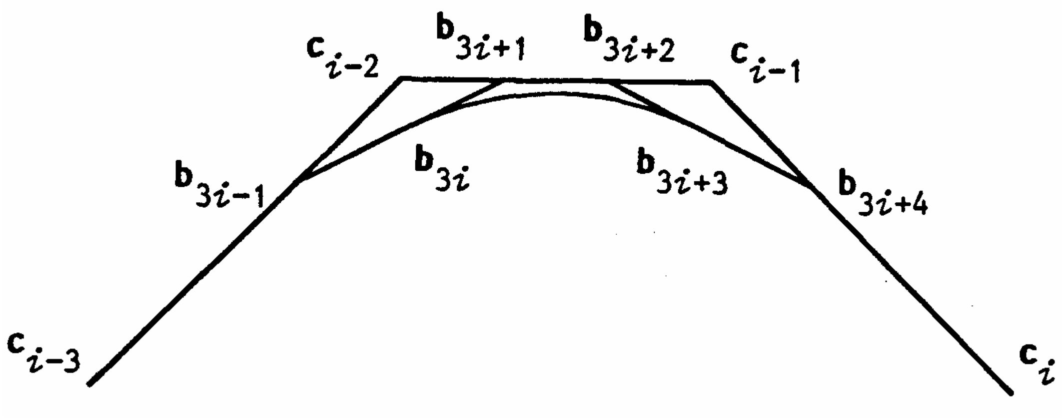

Figure 3: Transformation to Bernstein-Bézier form

with

$$k_i = 2\hat{\beta}_{1,i} + \hat{\beta}_{2,i} + \hat{\beta}_{3,i}. \tag{3.9}$$

Equation (3.7) represents the intersection of two lines at the control point c_{i-2} of the local support representation, see figure 3. We then have

$$\left.\begin{aligned} b_{3i+1} &= [(1-\tau_{i-1})/\Delta_i]c_{i-2} - [\sigma_{i-2}/\Delta_i]c_{i-1}, \\ b_{3i+2} &= -[\tau_{i-1}/\Delta_i]c_{i-2} + [(1-\sigma_{i-2})/\Delta_i]c_{i-1}, \end{aligned}\right\} i \in \mathbb{Z}\ , \tag{3.10}$$

where

$$\Delta_i := 1 - \tau_{i-1} - \sigma_{i-2}. \tag{3.11}$$

Equations (3.10) and (3.6) thus define the transformation from the control points Cj, $i \in \mathbb{Z}$ to those of the Bernstein-Bézier representation.

For the case of the weighted ν-spline, where the β-coefficients are defined by (3.2), we obtain

$$\mu_i = h_i / (h_{i-1} + h_i) \tag{3.12}$$

from (3.8) and coefficients τ_{i-2} , σ_{i-2} which are identical with those of the double knot B-spline transformation defined by (2.15). Thus the interior control points (3.10) of the Bernstein-Bézier representation are identical with the control points (2.18) of the double knot B-spline representation (compare also (2.24) with (3.6)). This is a property which could be observed directly from a study of the behaviour of the double knot cubic B-splines.

4 Interpolation using the local support basis

We conclude by considering the derivation of the local support basis repre-sentation satisfying interpolation constraints at the knots. For brevity, we consider only clamped type end conditions for the interpolation problem. Thus, given the local support basis representation

$$P(t) := \sum_{j=-3}^{n-1} c_j N_j(t) \tag{4.1}$$

of a weighted ν-spline on $[t_0, t_n]$, we seek the c_j, j = -3,..., n-1 such that

$$P(t_i) = y_i, i = 0,....,n, \tag{4.2}$$

with clamped end conditions

$$P^{(1)}(t_0) = d_0, P^{(1)}(t_n) = d_n. \tag{4.3}$$

The representation (4.1) could be derived from the work of [Foley'87], by onverting the piecewise Hermite representation used there to double knot B-spline form. Here, however, we consider a direct derivation of the control points C_j. Substituting the double knot B-spline representation (2.10) in (4.2) and (4.3), with control points given by (2.18), gives

$$\frac{-h_i\tau_{i-2}}{\Delta_{i-1}}c_{i-3} + \left[\frac{h_i(1-\sigma_{i-3})}{\Delta_{i-1}} + \frac{h_{i-1}(1-\tau_{i-1})}{\Delta_i}\right]c_{i-2} + \frac{-h_{i-1}\sigma_{i-2}}{\Delta_{i-1}}c_{i-1}$$
$$= (h_i + h_{i-1})y_i,\ i = 0,...,n, \tag{4.4}$$

with end conditions

$$\frac{-\tau_{-2}}{\Delta_{-1}}c_{-3} = \left[-\frac{1-\sigma_{-3}}{\Delta_{-1}} + \frac{1-\tau_{-1}}{\Delta_0}\right]c_{-2} + \frac{-\sigma_{-2}}{\Delta_0}c_{-1} - \frac{1}{3}(h_{-1} + h_0)d_0, \tag{4.5}$$

$$\frac{-\sigma_{n-2}}{\Delta_n}c_{n-1} = \frac{-\tau_{n-2}}{\Delta_{n-2}}c_{n-3} + \left[\frac{1-\sigma_{n-3}}{\Delta_{n-1}} - \frac{1-\tau_{n-1}}{\Delta_n}\right]c_{n-2} + \frac{1}{3}(h_{n-1} + h_n)d_n. \tag{4.6}$$

Using (4.5) and (4.6) to eliminate c_{-3} and c_{n-1} then gives a tridiagonal system defined by (4.4) for i = 1,..., n — 1, where for i = 0 and i = n the equations become

$$\frac{1-\tau_{-1}}{\Delta_0}c_{-2} + \frac{-\sigma_{-2}}{\Delta_0}c_{-1} = y_0 + \frac{1}{3}h_0 d_0, \tag{4.7}$$

$$\frac{-\tau_{n-2}}{\Delta_{n-1}}c_{n-3} + \frac{1-\sigma_{n-3}}{\Delta_{n-1}}c_{n-2} = y_n - \frac{1}{3}h_{n-1}d_n. \tag{4.8}$$

Let $$\alpha_{i-2} := \frac{(h_{i-1} + h_i)}{w_i p_i}.$$

Then $\alpha_{i-2} > 0$ for ρ_i, satisfying (2.20) and simple substitution shows that

$$(1-\sigma_{i-3})\alpha_{i-2} = \alpha_{i-2} - \tau_{i-2}\alpha_{i-3} > -\tau_{i-2}\alpha_{i-3} > 0, \tag{4.10}$$

$$\left(1-\tau_{i-1}\right)\alpha_{i-2}=\alpha_{i-2}-\sigma_{i-2}\alpha_{i-1}>-\sigma_{i-2}\alpha_{i-1}>0, \tag{4.11}$$

It follows that under the transformation

$$c_i := \alpha_i \hat{c}_i, \tag{4.12}$$

the tridiagonal system in the unknowns $\hat{c}_i$, i = —2,..., n — 2 is diagonally dominant. We thus have:

Proposition 4.1 *For parameters satisfying (2.20), or equivalently (2.23), for i* = 0,...., n—1, *there exists a unique weighted ν-spline on* $[t_0,t_n]$ *satisfying the interpolation conditions (4.2) and (4.3).*

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