ON THE FAVORITE POINTS OF SYMMETRIC LÉVY PROCESSES

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Abstract. This paper is concerned with asymptotic behavior (at zero and at infinity) of the favorite points of Lévy processes. By exploring Molchan's idea for deriving lower tail probabilities of Gaussian processes with stationary increments, we extend the result of Marcus (2001) on the favorite points to a larger class of symmetric Lévy processes.

1. Introduction

Let $X = \{X_t, t \geq 0, \mathbb{P}^x\}$ be a real-valued Lévy process with characteristic exponent ψ which is given by

$$\mathbb{E}^0[e^{i\lambda X_t}] = e^{-t\psi(\lambda)}.$$

Many sample path properties of X, including the existence and regularity of local times, can be described explicitly in terms of ψ. Building upon the seminal results of Kesten [14], Hawkes [12] showed that the local times of X, denoted by $\{L_t^x, t \geq 0, x \in \mathbb{R}\}$, exist as the Radon-Nikodym derivative of the occupation measure with respect to the Lebesgue measure on $\mathbb{R}$, namely, for all $t \geq 0$ and all Borel measurable functions $f : \mathbb{R} \to \mathbb{R}$,

$$\int_0^t f(X_s)\mathrm{d}s = \int_{\mathbb{R}} f(x) L_t^x \mathrm{d}x,$$

if and only if $\mathrm{Re}\big(1/(1+\psi)\big) \in L^1(\mathbb{R})$. See also Bertoin [6, p.126] for more information. Several authors have studied the a.s. joint continuity of $(t, x) \mapsto L_t^x$ and, finally, Barlow and Hawkes [2], Barlow [1], Marcus and Rosen [17] established sufficient and necessary conditions for the joint continuity of the local times of Lévy processes in terms of their 1-potential kernel densities. They also determined the exact uniform and local moduli of continuity for the local time L_t^x in the space variable x. The approaches in [2, 1] and [17] are different. [2, 1] considered general Lévy processes and their method was based on Dudley's metric entropy method; while [17] considered strongly symmetric Markov processes and made use of an isomorphism theorem of Dynkin, which relates sample path behavior of local times of a symmetric Markov process with those of the associated Gaussian processes. An extension of the results in [17] to local times of non-symmetric Markov processes was established by Eisenbaum and Kaspi [8]. Recently, Marcus and Rosen [19] extended the sufficiency part for the joint continuity of local times in [17] to Markov processes which are not necessarily symmetric. Their arguments are based on an isomorphism theorem of Eisenbaum and Kaspi [9] which relates local times of Markov processes with permanental processes. We refer to the book of Marcus and Rosen [18] for a systematic account on connections between local times of Markov processes and Gaussian processes.

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This paper is concerned with asymptotic properties of the local times of a class of symmetric Lévy processes. Let $X = \{X_t, t \geq 0\}$ be a real-valued symmetric Lévy process. Recall that from [3] the set of favorite points (or most visited sites) of X up to time t is defined by

$$\mathcal{V}_t = \Big\{x \in \mathbb{R} : \, L_t^x = \sup_{y \in \mathbb{R}} L_t^y\Big\}$$

and the favorite points process of X is defined by

$$V_t = \inf \Big\{|x| \in \mathbb{R} : \, L_t^x = \sup_{y \in \mathbb{R}} L_t^y\Big\}.$$

As pointed out by Bass *et al.* [3] (see also Marcus [16]) the choice of V_t is not significant. The results on V_t mentioned in this paper are still valid if it is replaced by any other element of $\mathcal{V}_t$.

Bass and Griffin [4] initiated the study of favorite points and established a quite surprising result for Brownian motion. Indeed one might expect that a real-valued Brownian motion starting from zero admits zero as one of its favorite points in the long run, but Bass and Griffin showed that the favorite points process of a Brownian motion is actually transient, namely, $\lim_{t \to \infty} V_t = \infty$. Their proof relies on the Ray-Knight Theorem. Later, Bass *et al.* [3] proved that when X is a symmetric stable Lévy process in $\mathbb{R}$ of index $\alpha \in (1, 2)$, its favorite points process is still transient. They appealed to a generalized Ray-Knight Theorem [10] and made a clever use of Slepian's Lemma. Marcus [16] adopted the approach of [3] and extended their result to a large class of symmetric Lévy processes. On the other hand, Eisenbaum and Khoshnevisan [11] studied the hitting probability of the set of favorite points $\mathcal{V}_t$. More specifically, they call a compact set $K \subset \mathbb{R}$ *polar* for $\mathcal{V} := \bigcup_{t>0} \mathcal{V}_t$ if

$$\mathbb{P}\big(\exists t > 0 : \mathcal{V}_t \cap K \neq \emptyset\big) = 0.$$

It has been an open problem to establish a criterion for a general compact set K to be polar. However, Eisenbaum and Khoshnevisan [11] proved that all singletons are polar for a large class of symmetric Markov processes, including symmetric stable Lévy processes in $\mathbb{R}$ with $\alpha \in (1, 2)$.

The present paper is mainly motivated by Marcus [16] and a remark of Marcus and Rosen [18, p.527]. In his study of the asymptotic behavior of the favorite points process $\{V_t, t \geq 0\}$ as $t \to \infty$, Marcus [16] assumed that the characteristic exponent $\psi(\lambda)$ of a symmetric Lévy process X in $\mathbb{R}$ satisfies the following two conditions:

(i) $\psi(\lambda)$ is regularly varying at zero with index $1 < \alpha \leq 2$ and $1/(1 + \psi(\lambda)) \in L^1(\mathbb{R}, d\lambda)$;
(ii) $\sigma_0^2(x)$ is increasing on $[0, \infty)$, where

$$\sigma_0^2(x) = \frac{2}{\pi} \int_0^\infty \big(1 - \cos(\lambda x)\big) \frac{\mathrm{d}\lambda}{\psi(\lambda)}, \quad x \in \mathbb{R}. \tag{1.1}$$

We remark first that the regularly varying condition in (i) is somewhat restrictive (e.g., as shown by Choi [7, Proposition 2.4], the characteristic exponent of a general semi-stable Lévy process may not be regularly varying. See Section 5 below for more examples.) Secondly, condition (ii) is not easy to verify even if condition (i) is satisfied. This issue was also noticed by Marcus and Rosen [18, p.527] who suggested to generalize the change of measure argument due to Molchan [20, 21] to obtain a key estimate instead of using Slepian's Lemma which requires the monotonicity condition (ii). In this paper, we will relax the conditions (i) and (ii) significantly and study the asymptotic properties of $\{V_t, t \geq 0\}$ not only as $t \to \infty$, but also as $t \to 0$. Moreover, our results also show that Theorem 1.3 and Proposition 1.4 of Eisenbaum and Khoshnevisan [11] are applicable to Lévy processes that satisfy Condition (C1). Hence all singletons are polar for the set of favorite points of such a Lévy process.

Now we introduce the conditions used in this paper. Recall that for a symmetric Lévy process X in $\mathbb{R}$, its characteristic exponent ψ is nonnegative and has the following Lévy-Khintchine representation:

$$\psi(\lambda) = A^2\lambda^2 + \int_0^\infty \big(1-\cos(x\lambda)\big)\nu(\mathrm{d}x), \tag{1.2}$$

where ν is a Borel measure on $(0,\infty)$ that satisfies $\int_0^\infty (1\wedge x^2)\nu(\mathrm{d}x) < \infty$ and is called the Lévy measure of X. Except in Section 6.1, we tacitly assume that the following holds:

$$\int_1^\infty \frac{d\lambda}{\psi(\lambda)} < \infty \quad \text{and} \quad \int_0^1 \frac{d\lambda}{\psi(\lambda)} = \infty. \tag{1.3}$$

The convergence of the first integral in (1.3) is equivalent to the existence of local times [12], while the second condition in (1.3) is equivalent to the recurrence of X (see Port and Stone [22, Thm 16.2]).

As in Bass *et al.* [3] and Marcus [16], we will focus on symmetric, pure-jump Lévy processes, i.e., we assume that $A = 0$. Some results for symmetric Lévy processes with a Brownian motion part can be derived from those for the pure jump part, see Section 6.

We will further assume that the Lévy measure ν is absolutely continuous $\nu(\mathrm{d}x) = \mathrm{d}x/\theta(x)$. It is possible to consider the case when ν is discrete or singular using the methods in Berman [5] or Luan and Xiao [15], respectively. Since this will increase the size of the paper and most of the deviation is of technical nature, we do not carry it out here.

Our conditions are on the asymptotic behavior of the Lévy measure ν.

(C1) There exist constants $\underline{\alpha}$ and $\overline{\alpha}$ such that

$$1 < \underline{\alpha} = \liminf_{x\to 0} \frac{x/\theta(x)}{\nu(z:|z|\geq x)} \leq \limsup_{x\to 0} \frac{x/\theta(x)}{\nu(z:|z|\geq x)} = \overline{\alpha} < 2.$$

(C2) There exist constants $\underline{\beta}$ and $\overline{\beta}$ such that

$$1 < \underline{\beta} = \liminf_{x\to \infty} \frac{x/\theta(x)}{\nu(z:|z|\geq x)} \leq \limsup_{x\to \infty} \frac{x/\theta(x)}{\nu(z:|z|\geq x)} = \overline{\beta} < 2.$$

Remark 1.1. Some comments about Conditions (C1) and (C2) are in order.

- *Berman* [5] *used (C2) type conditions to prove local nondeterminism property for a centered Gaussian process* $Y = \{Y(t), t \in \mathbb{R}\}$ *with stationary increments whose variance of increment* $\mathbb{E}\big[(Y(t+\lambda)-Y(t))^2\big]$ *can be represented (up to a constant) by (1.2) with* $A = 0$. *Xiao* [26] *extended Berman's argument to the random field setting and proved that a similar condition implies strong local nondeterminism for Gaussian random fields. Here we show that Conditions (C1) and (C2) imply asymptotic properties of the characteristic exponent* $\psi(\lambda)$ *of* X *as* $\lambda\to\infty$ *and* $\lambda\to 0$; *see Lemma 2.1 and 2.5 below.*
- *It follows from Lemma 2.3 that, under Condition (C1), the function* $\sigma_0^2(x)$ *decays at a power rate as* $x \to 0$. *Then, by applying the criterion of Barlow and Hawkes* [2] *or Marcus and Rosen* [17], *one can show that* X *has a jointly continuous local time.*
- *It follows from Condition (C2) and Lemma 2.5 that* $\int_0^1 \frac{\mathrm{d}\lambda}{\psi(\lambda)} = \infty$. *Hence* X *is recurrent thanks to the Spitzer-type criterion (see* [22, Thm 16.2] *or* [6, p.33]*). By* [18, p.140], *the* 0*-potential density of* X *satisfies*

$$u(0,0) = \int_0^\infty \frac{\mathrm{d}\lambda}{\psi(\lambda)} = \infty,$$

and $\mathbb{P}^x(T_0 < \infty) = 1$ *for all* $x \in \mathbb{R}$, *where* $T_0 = \inf\{t > 0 : X_t = 0\}$ *is the first hitting time of zero by* X.

Let us explain the novelty of our techniques. The function $\sigma_0^2(x)$ in (1.1) plays an important role in characterizing the joint continuity and other analytic properties of the local times of X. Set the tail function

$$\phi(x) = 2\int_{1/x}^{\infty} \frac{\mathrm{d}\lambda}{\psi(\lambda)}, \quad x \geq 0.$$

It is clear that $\phi(x)$ is nondecreasing in x. It follows from Lemma 2.3 and Theorem 2.5 in [26] that, under Condition (C1), the functions $\phi(x)$ and $\sigma_0^2(x)$ are comparable as $x \to 0$. In Section 2, we establish further connections between the asymptotic behaviors of $\sigma_0^2(x)$ with those of $\psi(\lambda)$. These allow us to remove the monotonicity assumption (ii) on $\sigma_0^2(x)$ in Marcus [16]. Following [3], through the generalized Ray-Knight Theorem we transfer some important estimates for the local times to certain estimates of the associated Gaussian processes. In [3, 16], the Gaussian estimates were obtained by Slepian's Lemma. Here we obtain these estimates by applying the Cameron-Martin change of measure formula, together with a uniform upper bound for the maximum location of Gaussian processes with stationary increments.

The following are the main results of this paper. Part (ii) of Theorem 1.2 is an extension of Theorem 1.1 of Marcus [16] and Theorem 1.4 is an extension of Theorem 5.2 of Eisenbaum and Khoshnevisan [11].

Theorem 1.2. *Let* X *be a symmetric Lévy process with characteristic exponent* ψ *that satisfies* (1.3).

(i) *Assume that (C1) holds. For all* $a > 2(\overline{\alpha} - 1)/(2 - \overline{\alpha})$,

$$\lim_{t\to 0} \frac{V_t}{\phi^{-1}\left(\frac{L_t^0}{(\log L_t^0)^a}\right)} = \infty \quad \mathbb{P}^0\text{-}a.s.$$

(ii) *Assume that (C2) holds. For all* $a > 2(\overline{\beta} - 1)/(2 - \overline{\beta})$,

$$\lim_{t\to \infty} \frac{V_t}{\phi^{-1}\left(\frac{L_t^0}{(\log L_t^0)^a}\right)} = \infty \quad \mathbb{P}^0\text{-}a.s.$$

Remark 1.3. *Our conditions impose certain asymptotic behavior of the Lévy measure, which in turn implies certain asymptotic behavior of the characteristic exponent and a ratio control for the tail function of the Lévy measure, see* (2.1)-(2.2) *and* (2.10)-(2.11) *in Section 2. In fact, from the proofs below we may see that Part (i) of Theorem 1.2 (resp. Part (ii)) is valid as long as* (2.1)-(2.2) *(resp.* (2.10)-(2.11)*) hold. Both sets of conditions are useful since Lévy processes encountered in the literature may be specified explicitly either by the Lévy measure or by the characteristic exponent. From this and Proposition 2.4 of Choi [7], it follows that Theorem 1.2 holds for any symmetric semi-stable Lévy process in* $\mathbb{R}$ *with index* $\alpha \in (1, 2]$. *In Section 5, we also present an example where (C1) fails, but* (2.1)-(2.2) *hold, thus Part (i) of Theorem 1.2 continues to hold.*

Theorem 1.4. *If Conditions* (1.3) *and (C1) hold, then all singletons are polar for the set* $\mathcal{V}$ *of favorite points. Namely, for every* $x \in \mathbb{R}$,

$$\mathbb{P}^x\big(\exists t > 0 \text{ such that } x \in \mathcal{V}_t\big) = 0.$$

The rest of the paper is organized as follows. Some preliminary results are presented in Section 2. We obtain upper and lower tail estimates for the maxima of Gaussian processes in Section 3. Using these estimates, we prove Theorems 1.2 and 1.4 in Section 4. Some examples are given in Section 5. Possible extensions are discussed in Section 6.

We end the Introduction with some notations. We use $\mathbb{P}^x$ and $\mathbb{E}^x$ to denote the law and the expectation of any Lévy process with starting point $x \in \mathbb{R}$. For simplicity, we write $\mathbb{P}$ and $\mathbb{E}$ when $x = 0$. In Sections 3 and 4, the law and the expectation of any auxiliary Gaussian process are denoted by P and E. We use c, C to denote generic constants whose value may change from line to line.

2. **Preliminaries**

2.1. **General facts**

Introduce the tail function of the Lévy measure ν:

$$\pi(x) = 2\int_{1/x}^{\infty} \frac{\mathrm{d}z}{\theta(z)}, \qquad \forall x > 0.$$

The function $\pi(x)$ is important because, under our condition, it is equivalent to $\psi(\lambda)$ around infinity and has the advantage of being a monotone function. More precisely, the following lemma which is a reminiscence of [26, Lem 2.3 and Thm 2.5] gathers useful facts derived from (C1).

Lemma 2.1. *Assume that (C1) holds. For any $\varepsilon \in (0, \min(\underline{\alpha} - 1, 2 - \overline{\alpha}))$, there exists a finite positive constant K_0 such that for all $y > x > K_0$,*

$$\left(\frac{x}{y}\right)^{\overline{\alpha}+\varepsilon} \le \frac{\pi(x)}{\pi(y)} \le \left(\frac{x}{y}\right)^{\underline{\alpha}-\varepsilon}. \tag{2.1}$$

Moreover,

$$0 < \liminf_{\lambda\to\infty} \frac{\psi(\lambda)}{\pi(\lambda)} \le \limsup_{\lambda\to\infty} \frac{\psi(\lambda)}{\pi(\lambda)} < \infty. \tag{2.2}$$

Proof. The proof which led to [26, Lem. 2.3] yields plainly (2.1) by considering x, y around infinity. Using similar arguments that led to [26, Thm. 2.5], but for large λ rather than for small ones, entails (2.2). Let us prove (2.2) for the sake of completeness. By (1.2), we can write $\psi(\lambda)$ as

$$\psi(\lambda) = \int_0^{1/\lambda} (1-\cos(\lambda x))\frac{\mathrm{d}x}{\theta(x)} + \int_{1/\lambda}^{\infty} (1-\cos(\lambda x))\frac{\mathrm{d}x}{\theta(x)} := I_1 + I_2.$$

By (C1), there exists $0 < r_0 < 1/K_0$ such that for any $x < r_0$,

$$\underline{\alpha} - \varepsilon \le \frac{x/\theta(x)}{\pi(1/x)} \le \overline{\alpha} + \varepsilon.$$

So for $\lambda > K_0$, it follows from (2.1) that

$$\begin{aligned}\frac{I_1}{\pi(\lambda)} &\geq (\underline{\alpha} - \varepsilon) \int_0^{1/\lambda} (1 - \cos(\lambda x)) \frac{\pi(1/x)}{\pi(\lambda)} \frac{\mathrm{d}x}{x} \\ &\geq (\underline{\alpha} - \varepsilon) \int_0^{1/\lambda} (1 - \cos(\lambda x)) \left(\frac{1}{x\lambda}\right)^{\underline{\alpha} - \varepsilon} \frac{\mathrm{d}x}{x} \\ &= (\underline{\alpha} - \varepsilon) \int_0^1 (1 - \cos x) \frac{\mathrm{d}x}{x^{\underline{\alpha}+1-\varepsilon}} = c > 0.\end{aligned}$$

This proves the left inequality in (2.2).

To prove the right inequality in (2.2), we control both I_1 and I_2. Firstly, for $\lambda > K_0$, we still use (2.1) and derive

$$\begin{aligned}\frac{I_1}{\pi(\lambda)} &\leq (\overline{\alpha} + \varepsilon) \int_0^{1/\lambda} (1 - \cos(\lambda x)) \frac{\pi(1/x)}{\pi(\lambda)} \frac{\mathrm{d}x}{x} \\ &\leq (\overline{\alpha} + \varepsilon) \int_0^{1/\lambda} (1 - \cos(\lambda x)) \left(\frac{1}{x\lambda}\right)^{\overline{\alpha} + \varepsilon} \frac{\mathrm{d}x}{x} \\ &= (\overline{\alpha} + \varepsilon) \int_0^1 (1 - \cos x) \frac{\mathrm{d}x}{x^{\overline{\alpha}+1+\varepsilon}} = c < \infty,\end{aligned}$$

where we have used the fact $\overline{\alpha} + \varepsilon < 2$. Next, bounding from above the integrand by 1, we obtain $I_2/\pi(\lambda) \leq 1$ for all λ. Combining the two bounds completes the proof. □ □

Next we show that the ratio control (2.1) and the equivalence (2.2) on ψ as $\lambda \to \infty$ imply similar properties of $\phi(x)$ and the maxima of $\sigma_0^2(x)$ as $x \to 0$. The tool is [13, Prop. 1], see also [24].

Lemma 2.2 ([13, Prop. 1])**.** *Let*

$$H(x) = \int 1 \wedge \left(\frac{\lambda}{x}\right)^2 \frac{\mathrm{d}\lambda}{\psi(\lambda)}$$

and $\widehat{\sigma}_0^2(h) = \max_{|x| \leq h} \sigma_0^2(x)$. *There exists a constant* $c > 0$ *such that for all* $x > 0$,

$$\begin{aligned} c \int_0^{1/x} (\lambda x)^2 \frac{\mathrm{d}\lambda}{\psi(\lambda)} \leq \sigma_0^2(x) &\leq 2H(1/x), \\ cH(1/x) \leq \widehat{\sigma}_0^2(x) &\leq 2H(1/x). \end{aligned}$$

Lemma 2.3. *Assume that (C1) holds. There are positive finite constants* $\alpha_1 < \alpha_2$ *such that for any* $0 < \varepsilon < \alpha_1$, *there exists* $r_0 > 0$ *such that for* $0 < x < y < r_0$,

$$\left(\frac{x}{y}\right)^{\alpha_2 + \varepsilon} \leq \frac{\phi(x)}{\phi(y)} \leq \left(\frac{x}{y}\right)^{\alpha_1 - \varepsilon}. \tag{2.3}$$

Further, one has

$$0 < \liminf_{x \to 0} \frac{\widehat{\sigma}_0^2(x)}{\phi(x)} \leq \limsup_{x \to 0} \frac{\widehat{\sigma}_0^2(x)}{\phi(x)} < \infty. \tag{2.4}$$

Proof. A direct application of [26, Lem. 2.3] yields (2.3) as long as we show

$$0 < \alpha_1 = \liminf_{\lambda \to \infty} \frac{\lambda/\psi(\lambda)}{\phi(1/\lambda)} \leq \limsup_{\lambda \to \infty} \frac{\lambda/\psi(\lambda)}{\phi(1/\lambda)} = \alpha_2 < \infty. \tag{2.5}$$

Let us start with the left inequality in (2.5). By Lemma 2.1, there exist constants $0 < c, C < \infty$ such that for all $\lambda > C$,

$$\frac{\lambda/\psi(\lambda)}{\phi(1/\lambda)} = \frac{\lambda}{2\int_\lambda^\infty \frac{\psi(\lambda)}{\psi(z)}\mathrm{d}z} \geq \frac{c\lambda}{\int_\lambda^\infty \frac{\pi(\lambda)}{\pi(z)}\mathrm{d}z} \geq \frac{c\lambda}{\int_\lambda^\infty (\lambda/z)^{\underline{\alpha}-\varepsilon}\mathrm{d}z} := c_0 > 0.$$

Similarly, there exists $c' < \infty$ such that for $\lambda > C$,

$$\frac{\lambda/\psi(\lambda)}{\phi(1/\lambda)} \leq \frac{c'\lambda}{\int_\lambda^\infty \frac{\pi(\lambda)}{\pi(z)}\mathrm{d}z} \leq \frac{c'\lambda}{\int_\lambda^\infty (\lambda/z)^{\overline{\alpha}+\varepsilon}\mathrm{d}z} := c_0' < \infty,$$

which implies the desired right inequality in (2.5).

Next we show (2.4). Thanks to Lemma 2.2, it suffices to show (2.4) with $\widehat{\sigma}_0^2(x)$ replaced by $H(1/x)$. Observe that

$$H\left(\frac{1}{x}\right) = 2\int_0^{1/x} (\lambda x)^2 \frac{\mathrm{d}\lambda}{\psi(\lambda)} + 2\int_{1/x}^\infty \frac{\mathrm{d}\lambda}{\psi(\lambda)}. \tag{2.6}$$

Thus, the left inequality in (2.4) holds trivially. On the other hand, for $x < 1/C$,

$$\int_0^{1/x} (\lambda x)^2 \frac{\mathrm{d}\lambda}{\psi(\lambda)} = x^2 \left(\int_0^C \lambda^2 \frac{\mathrm{d}\lambda}{\psi(\lambda)} + \int_C^{1/x} \lambda^2 \frac{\mathrm{d}\lambda}{\psi(\lambda)} \right).$$

The first integral in the parenthesis is finite and the second integral goes to infinity as $x \to 0$ ([18, Lem.4.2.2]). Therefore,

$$\int_0^{1/x} (\lambda x)^2 \frac{\mathrm{d}\lambda}{\psi(\lambda)} \leq cx^2 \int_C^{1/x} \lambda^2 \frac{\mathrm{d}\lambda}{\psi(\lambda)} = \frac{cx^2}{\psi(1/x)} \int_C^{1/x} \frac{\psi(1/x)}{\psi(\lambda)} \lambda^2 \mathrm{d}\lambda$$

which by Lemma 2.1 is bounded from above by

$$\begin{aligned} \frac{cx^2}{\psi(1/x)} \int_C^{1/x} \frac{\pi(1/x)}{\pi(\lambda)} \lambda^2 \mathrm{d}\lambda &\leq \frac{cx^2}{\psi(1/x)} \int_C^{1/x} \left(\frac{1}{x\lambda}\right)^{\overline{\alpha}+\varepsilon} \lambda^2 \mathrm{d}\lambda \\ &\leq \frac{c}{\psi(1/x)x} \int_0^1 \left(\frac{1}{\lambda}\right)^{\overline{\alpha}+\varepsilon-2} \mathrm{d}\lambda. \end{aligned} \tag{2.7}$$

We can assemble the last integral in (2.7) into the constant c since $\overline{\alpha} > 1$ and ε is small. To finish the proof of the right inequality in (2.4), it remains to show that

$$\frac{1}{x\psi(1/x)} \leq c\int_{1/x}^\infty \frac{\mathrm{d}\lambda}{\psi(\lambda)}. \tag{2.8}$$

for all $x > 0$ small. Indeed, combining (2.6), (2.7) and (2.8), together with Lemma 2.2, shows that $\widehat{\sigma}_0^2(x) \leq c\phi(x)$ for all $x > 0$ sufficiently small, as desired. By Lemma 2.1, for all $0 < x < 1/C$,

$$\int_{1/x}^\infty \frac{\psi(1/x)}{\psi(\lambda)} \mathrm{d}\lambda \geq c\int_{1/x}^\infty \left(\frac{1}{\lambda x}\right)^{\overline{\alpha}+\varepsilon} \mathrm{d}\lambda = \frac{c}{x}\int_1^\infty \frac{\mathrm{d}\lambda}{\lambda^{\overline{\alpha}+\varepsilon}} \tag{2.9}$$

from which (2.8) follows. The proof is now complete. □ □

Remark 2.4. In fact, the two terms in (2.6) are of the same order under Condition (C1). To see this, it remains to prove that there exists $c > 0$ such that

$$x^2 \int_C^{1/x} \lambda^2 \frac{\mathrm{d}\lambda}{\psi(\lambda)} \geq c\int_{1/x}^\infty \frac{\mathrm{d}\lambda}{\psi(\lambda)} = c\phi(x).$$

This is true since one can reverse inequalities in (2.7) and (2.9) if we replace all $\overline{\alpha} + \varepsilon$ by $\underline{\alpha} - \varepsilon$. Consequently, $\sigma_0^2(x) \asymp \phi(x)$ as $|x| \to 0$ by the first inequality in Lemma 2.2.

Interchanging the roles of zero and infinity, one deduces similar facts under (C2).

Lemma 2.5. *Assume that (C2) holds. For any* $\varepsilon \in (0, \min(\underline{\beta} - 1, 2 - \overline{\beta}))$*, there exists a finite positive constant* r_1 *such that for all* $0 < x < y < r_1$,

$$\left(\frac{x}{y}\right)^{\overline{\beta}+\varepsilon} \le \frac{\pi(x)}{\pi(y)} \le \left(\frac{x}{y}\right)^{\underline{\beta}-\varepsilon}. \tag{2.10}$$

Moreover,

$$0 < \liminf_{\lambda \to 0} \frac{\psi(\lambda)}{\pi(\lambda)} \le \limsup_{\lambda \to 0} \frac{\psi(\lambda)}{\pi(\lambda)} < \infty. \tag{2.11}$$

Lemma 2.6. *Assume that (C2) holds. There are positive finite constants* $\beta_1 < \beta_2$ *such that for any* $0 < \varepsilon < \beta_1$*, there exists* $K_1 > 0$ *such that for all* $y > x > K_1$,

$$\left(\frac{x}{y}\right)^{\beta_2+\varepsilon} \le \frac{\phi(x)}{\phi(y)} \le \left(\frac{x}{y}\right)^{\beta_1-\varepsilon}. \tag{2.12}$$

Further, one has

$$0 < \liminf_{x \to \infty} \frac{\widehat{\sigma}_0^2(x)}{\phi(x)} \le \limsup_{x \to \infty} \frac{\widehat{\sigma}_0^2(x)}{\phi(x)} < \infty. \tag{2.13}$$

Proof of Lemmas 2.5 and 2.6 The ratio control (2.10) is exactly [26, Lem. 2.3], and the equivalence (2.11) is [26, Thm. 2.5]. To prove (2.12), it suffices to show that

$$0 < \beta_1 = \liminf_{\lambda \to 0} \frac{\lambda/\psi(\lambda)}{\phi(1/\lambda)} \le \limsup_{\lambda \to 0} \frac{\lambda/\psi(\lambda)}{\phi(1/\lambda)} = \beta_2 < \infty.$$

This can be handled in exactly the same way as (2.5) is proved, with an application of Lemma 2.5 instead of Lemma 2.1. Let us prove (2.13). Since there is no obvious way to control the value of β_2, one cannot apply directly Lemma 2.1 to derive (2.13). But Lemma 2.2 which holds for all $x > 0$ is still applicable. Observe that the left inequality of (2.13) follows from (2.6). On the other hand, for $x > 1/r_1$,

$$\begin{aligned}\int_0^{1/x} (\lambda x)^2 \frac{\mathrm{d}\lambda}{\psi(\lambda)} &= \frac{1}{x\psi(1/x)} \int_0^1 \lambda^2 \frac{\psi(1/x)}{\psi(\lambda/x)} \mathrm{d}\lambda \\ &\le \frac{c}{x\psi(1/x)} \int_0^1 \lambda^2 \left(\frac{1}{\lambda}\right)^{\overline{\beta}+\varepsilon} \mathrm{d}\lambda = \frac{c}{x\psi(1/x)},\end{aligned}$$

where we have used Lemma 2.5 in the second inequality and $\overline{\beta} < 2$. To finish the proof, it remains to show (2.8) for all large x. For $x > 2/r_1$, write

$$\int_{1/x}^{\infty} \frac{\mathrm{d}\lambda}{\psi(\lambda)} = \int_{1/x}^{r_1} \frac{\mathrm{d}\lambda}{\psi(\lambda)} + \int_{r_1}^{\infty} \frac{\mathrm{d}\lambda}{\psi(\lambda)} := J_1 + J_2.$$

Observe that $J_2 < \infty$ by assumption (which is the necessary and sufficient condition for the existence of local times). Using again Lemma 2.5, one has for $x > 2/r_1$

$$\begin{aligned}J_1 &= \frac{1}{\psi(1/x)} \int_{1/x}^{r_1} \frac{\psi(1/x)}{\psi(\lambda)} \mathrm{d}\lambda \\ &\ge \frac{c}{\psi(1/x)} \int_{1/x}^{r_1} \left(\frac{1}{x\lambda}\right)^{\overline{\beta}+\varepsilon} \mathrm{d}\lambda \ge \frac{c}{x\psi(1/x)} \int_1^2 \left(\frac{1}{\lambda}\right)^{\overline{\beta}+\varepsilon} \mathrm{d}\lambda = \frac{c}{x\psi(1/x)},\end{aligned}$$

as desired. This finishes the proof. □

2.2. **An isomorphism theorem**

We recall a generalized second Ray-Knight Theorem due to Eisenbaum *et al.* [10]; see also Marcus and Rosen [18]. Let $X = \{X_t, t \geq 0\}$ be a strongly symmetric Borel right process with values in $\mathbb{R}$ with continuous α-potential densities $u^\alpha(x, y)$. Then the local times $\{L_t^x, t \geq 0, x \in \mathbb{R}\}$ exist and satisfy

$$\mathbb{E}^x \left[\int_0^\infty e^{-\alpha t} \mathrm{d}L_t^y \right] = u^\alpha(x, y). \tag{2.14}$$

Denote $u_{T_0}(x, y) = \mathbb{E}^x[L_{T_0}^y]$ which is the 0-potential density of X killed at the first time it hits zero. Set $\tau(t) = \inf\{s \geq 0 : L_s^0 > t\}$. We state the generalized second Ray-Knight Theorem in the recurrent case from [10, Thm 1.1] or [18, Thm 8.2.2].

Theorem 2.7. *Assume that the 0-potential density of X satisfies $u(0,0) = \infty$ and $\mathbb{P}^x(T_0 < \infty) > 0$ for all x. Let $\eta = \{\eta_x, x \in \mathbb{R}\}$ be a mean-zero Gaussian process on a probability space $(\Omega_\eta, \mathcal{F}_\eta, \mathrm{P}_\eta)$ with covariance function $u_{T_0}(x, y)$. Then for any $t > 0$ and any countable subset $D \subset \mathbb{R}$, under $\mathbb{P} \times \mathrm{P}_\eta$, in law*

$$\left\{ L_{\tau(t)}^x + \frac{\eta_x^2}{2}; x \in D \right\} = \left\{ \frac{1}{2}(\eta_x + \sqrt{2t})^2; x \in D \right\}. \tag{2.15}$$

As we have mentioned earlier, we will write the probability P_η as P and its expectation as E.

It follows from Theorem 6.1 of Eisenbaum *et al.* [10] that, for a recurrent symmetric Lévy process with characteristic exponent $\psi(\lambda)$, the associated Gaussian process η is centered with stationary increments such that its covariance $u_{T_0}(x, y)$ is given by

$$u_{T_0}(x, y) = \frac{1}{2}(\sigma_0^2(x) + \sigma_0^2(y) - \sigma_0^2(x - y)). \tag{2.16}$$

or equivalently it has variogram $\mathrm{E}[(\eta(x) - \eta(y))^2] = \sigma_0^2(x - y)$. Here, $\sigma_0^2(x)$ is defined in (1.1).

The following is the Cameron-Martin (change of measure) formula for Gaussian processes.

Theorem 2.8 ([18, Thm 11.4.1])**.** *Let $\{G_t; t \in S\}$ be a mean zero Gaussian process with continuous covariance $\Gamma(x, y)$ and let $H(\Gamma)$ be the reproducing kernel Hilbert space generated by Γ. Let $f \in H(\Gamma)$. Then for any measurable functional F,*

$$\mathrm{E}[F(G_\cdot + f(\cdot))] = e^{-\|f\|^2/2}\mathrm{E}[F(G_\cdot)e^{G(f)}],$$

where $\|\cdot\|$ is the norm on $H(\Gamma)$ and $G(f)$ is a Gaussian random variable with mean zero and variance $\|f\|^2$.

For the Gaussian process associated to a Lévy process with characteristic exponent $\psi(\lambda)$ in Theorem 2.7, the reproducing kernel Hilbert space is endowed with the norm

$$\|f - f(0)\|^2 = \frac{1}{2\pi} \int_0^\infty |\psi(\lambda)||\hat{f}(\lambda)|^2 \mathrm{d}\lambda,$$

where $\hat{f}$ is the Fourier transform of f, see [18, Sec 11.7].

2.3. **Location of the leftmost maximum**

The next lemma is adapted from Theorem 3.1(b) of Samorodnitsky and Shen [23], see also [25].

Lemma 2.9. *Let η be a continuous process with stationary increments. Define $\tau_{[a,b]}$ to be the leftmost maximum location for η on the interval $[a,b]$, that is*

$$\tau_{[a,b]} = \inf\Big\{x \in [a,b] : \eta(x) = \max_{u\in[a,b]} \eta(u)\Big\}.$$

Then the distribution function of $\tau_{[a,b]}$ satisfies

$$F_{[a,b]}(x) = F_{[0,b-a]}(x-a), \quad \forall\, x \in \mathbb{R}.$$

Further, the law of $\tau_{[0,T]}$ restricted to $(0,T)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ and its density has the following universal upper bound

$$f_T(t) \le \max\Big\{\frac{1}{t}, \frac{1}{T-t}\Big\}, \quad 0 < t < T.$$

3. **Gaussian tail estimates**

In this section, we estimate the upper and lower tail probabilities for the associated Gaussian processes $\{\eta(x), x \in \mathbb{R}\}$ in the second Ray-Knight Theorem. Recall that $\{\eta(x), x \in \mathbb{R}\}$ is a centered Gaussian process with stationary increments satisfying

$$\mathrm{E}[(\eta(x)-\eta(y))^2] = \sigma_0^2(x-y).$$

The first two lemmas give upper tail estimates for the maxima of η.

Lemma 3.1. *Assume that (C1) holds. There exist finite positive constants c_0, c_1, c_2, c_3, r_1 such that for all $0 < h < r_1$ and $u \ge c_3\sigma_0(h)$,*

$$\mathrm{E}\Big(\sup_{|x|\le h} |\eta(x)|\Big) \le c_0\widehat{\sigma}_0(h),$$

$$\mathrm{P}\Big(\sup_{|x|\le h} |\eta(x)| > u\Big) \le c_1 \exp\Big(-\frac{u^2}{c_2\widehat{\sigma}_0^2(h)}\Big).$$

Proof. By Marcus and Rosen [18, Lem 7.2.2], for any increasing $\widetilde{\sigma}(x)$ satisfying $\widetilde{\sigma}(0) = 0$ and $\sigma_0(x) \le \widetilde{\sigma}(x)$,

$$\mathrm{E}\Big(\sup_{|x|\le h} |\eta(x)|\Big) \le C\Big(\widetilde{\sigma}(h) + \int_0^{1/2} \frac{\widetilde{\sigma}(hu)}{u\sqrt{\log 1/u}} \mathrm{d}u\Big). \tag{3.1}$$

Actually, (3.1) holds with $\widetilde{\sigma}(\cdot)$ replaced by $\widehat{\sigma}_0(\cdot) = \max_{|x|\le|\cdot|} \sigma_0(x)$ (see [18, p.301]). Applying Lemma 2.3 yields that

$$\int_0^{1/2} \frac{\widehat{\sigma}_0(hu)}{u\sqrt{\log 1/u}} \mathrm{d}u \le c\widehat{\sigma}_0(h) \int_0^{1/2} \frac{u^{(\alpha_1-\varepsilon)/2}}{u\sqrt{\log 1/u}} \mathrm{d}u \le c\widehat{\sigma}_0(h) \tag{3.2}$$

for all $h > 0$ sufficiently small. Combining (3.1) and (3.2) yields the first moment estimate.

To derive the upper tail probability, consider $u > 3c_0\widehat{\sigma}_0(h)$ (we take $c_0 \ge 1/\sqrt{2\pi}$) and let a be the median of $\sup_{|x|\le h} \eta(x)$ which satisfies

$$\Big|a - \mathrm{E}\sup_{|x|\le h} \eta(x)\Big| \le \widehat{\sigma}_0(h)/\sqrt{2\pi},$$

see [18, Cor. 5.4.5]. Using the concentration inequality (cf. [18, Rem. 5.4.4]) in the third inequality below, one gets

$$\begin{aligned}
\mathrm{P}\Big(\sup_{|x|\le\delta}|\eta(x)|>u\Big) &\le \mathrm{P}\Big(\sup_{|x|\le\delta}|\eta(x)|-a>u-c_0\widehat{\sigma}_0(h)-\widehat{\sigma}_0(h)/\sqrt{2\pi}\Big)\\
&\le \mathrm{P}\Big(\Big|\sup_{|x|\le\delta}|\eta(x)|-a\Big|\ge u/3\Big)\le 3\Big(1-\Phi\big(\frac{u}{3\widehat{\sigma}_0(\delta)}\big)\Big)\\
&\le 3\exp\Big(-\frac{u^2}{18\widehat{\sigma}_0^2(\delta)}\Big),
\end{aligned}$$

where Φ is the distribution function of $N(0,1)$. For smaller u, this inequality holds trivially with 3 replaced by some generic constant. □ □

Lemma 3.2. *Assume that (C2) holds. There exists $K>0$ such that for all $h>K$, the conclusion in Lemma 3.1 holds with possibly different constants c_0,c_1,c_2,c_3.*

Proof. Let K_1 be the constant in Lemma 2.6 and $h>K_1$. As was shown in the proof of Lemma 3.1, the upper tail probability estimate follows once we establish the first moment estimate for the maxima of $\{\eta(x),x\in\mathbb{R}\}$. Following Marcus [16], we use Dudley's entropy bound ([18, Thm. 6.1.2])

$$\mathrm{E}\Big(\sup_{|x|\le h}\eta(x)\Big)\le 16\sqrt{2}\int_0^{\widehat{\sigma}_0(h)}\sqrt{\log N([-h,h],d_\eta,u)}\,\mathrm{d}u,$$

where $d_\eta(x,y)=\sigma_0(x-y)$ is the canonical metric of the Gaussian process $\{\eta(x);\ x\in\mathbb{R}\}$ and $N([-h,h],d_\eta,u)$ is the minimal number of balls with radius at most u in the metric d_η to cover the interval $[-h,h]$. Observe that by the stationarity of the increments, $d_\eta(x,y)=d_\eta(x-y,0)$, thus $N([-h,h],d_\eta,u)\le(h/K_1)N([-K_1,K_1],d_\eta,u])$. For any increasing function $\widetilde{\sigma}(x)$ that satisfies $\sigma_0(x)\le\widetilde{\sigma}(x)$, one has by the subadditivity of the square root function

$$\begin{aligned}
&\int_0^{\widetilde{\sigma}(K_1)}\sqrt{\log N([-h,h],d_\eta,u)}\mathrm{d}u\\
&\qquad\le\widetilde{\sigma}(K_1)\sqrt{\log(h/K_1)}+\int_0^{\widetilde{\sigma}(K_1)}\sqrt{\log N([-1,1],d_\eta,u)}\mathrm{d}u.
\end{aligned}$$

That the last integral is finite follows from Fernique's Theorem [18, Thm. 6.2.2]. Also, a d_η-ball of radius u covers at least an interval of Euclidean length $\widetilde{\sigma}^{-1}(u)$ because $[0,\widetilde{\sigma}^{-1}(u)]=\{x\ge0:\widetilde{\sigma}(x)\le u\}\subset\{x\ge0:\sigma_0(x)\le u\}$. Hence,

$$\begin{aligned}
\int_{\widetilde{\sigma}(K_1)}^{\widetilde{\sigma}(h)}\sqrt{\log N([-h,h],d_\eta,u)}\mathrm{d}u&\le\int_{\widetilde{\sigma}(K_1)}^{\widetilde{\sigma}(h)}\sqrt{\log\frac{2h}{\widetilde{\sigma}^{-1}(u)}}\mathrm{d}u\\
&=\int_{K_1}^{h}\sqrt{\log\frac{2h}{s}}\mathrm{d}\widetilde{\sigma}(s)\le\widetilde{\sigma}(h)(\log 2)^{1/2}+\frac{1}{2}\int_{K_1/(2h)}^{1/2}\frac{\widetilde{\sigma}(2hs)}{s\sqrt{\log(1/s)}}\mathrm{d}s. \quad (3.3)
\end{aligned}$$

Since $\sigma_0(x-y)=d_\eta(x,y)\le d_\eta(x,(x+y)/2)+d_\eta((x+y)/2,y)=2\sigma_0((x-y)/2)\le2\widehat{\sigma}_0((x-y)/2)$, one gets for all x,

$$\widehat{\sigma}_0(2x)\le2\widehat{\sigma}_0(x).$$

Taking limits (through a sequence of finite measures $\mathrm{d}\widetilde{\sigma}_n$ on $[1,h]$ that converges to $\mathrm{d}\widehat{\sigma}_0$), one obtains (3.3) with $\widetilde{\sigma}$ replaced by $\widehat{\sigma}_0$. To summarize, we have proved that

$$\begin{aligned}\mathrm{E}\Big(\sup_{|x|\le h}\eta(x)\Big) &\le C\left(\sqrt{\log h}+\widehat{\sigma}_0(h)+\int_{K_1/(2h)}^{1/2}\frac{\widehat{\sigma}_0(hs)}{s\sqrt{\log 1/s}}\mathrm{d}s\right)\\ &:= C(A_1+A_2+A_3).\end{aligned}$$

It follows from Lemma 2.6 that $A_1 < cA_2$. Similarly,

$$A_3 \le c\widehat{\sigma}_0(h)\int_0^{1/2}\frac{s^{\beta_1-\varepsilon}}{s\sqrt{\log 1/s}}\mathrm{d}s \le c\widehat{\sigma}_0(h).$$

This ends the proof. □ □

Next, we consider the lower tail probability for the maxima of $\{\eta(x), x\in\mathbb{R}\}$. Lemma 3.3 is the key technical estimate of this paper. As suggested by Marcus and Rosen [18, p.527], we explore Molchan's idea [20, 21] without using scaling.

Lemma 3.3. *Assume that (C1) holds. For all $\gamma < 1/(\overline{\alpha}-1)$, there exists a finite constant K_2 such that for any $h<1$ and $\delta<1$,*

$$\mathrm{P}\Big(\eta(x) < \sqrt{h\widehat{\sigma}_0^2(\delta)};\ \textit{for all } |x|\le\delta\Big) \le K_2 h^{\gamma}.$$

Proof. Set

$$\xi_\delta(x) = \frac{\eta(\delta x)}{\widehat{\sigma}_0(\delta)};\quad |x|\le 1.$$

The probability in question becomes

$$\mathrm{P}(\xi_\delta(x) < \sqrt{h};\ \text{for all } |x|\le 1).$$

It is plain that for each $0<\delta<1$, the mean zero Gaussian process $\{\xi_\delta(x), |x|\le 1\}$ has stationary increments. Furthermore, its variogram has the following spectral representation:

$$\mathrm{E}\big[(\xi_\delta(y+x)-\xi_\delta(y))^2\big] = \frac{\sigma_0^2(\delta x)}{\widehat{\sigma}_0^2(\delta)} = \frac{2}{\pi}\int_0^\infty \frac{1-\cos(x\lambda)}{\delta\widehat{\sigma}_0^2(\delta)\psi(\lambda/\delta)}\mathrm{d}\lambda.$$

Thus, by Section 11.7 in Marcus and Rosen [18] the reproducing kernel Hilbert space of $\{\xi_\delta(x), |x|\le 1\}$ is endowed with the norm

$$\|f-f(0)\|^2 = \frac{1}{2\pi}\int |\delta\widehat{\sigma}_0^2(\delta)\psi(\lambda/\delta)|\,|\widehat{f}(\lambda)|^2\mathrm{d}\lambda.$$

Let f be a non negative smooth function with support in $[-1,1]$ and $f(0)=1$. Set $f_h(x) = \sqrt{h}f(x/h^a)$ and $\bar{f}_h(x) := \sqrt{h}-f_h(x) = f_h(0)-f_h(x)$, where $a>1$ is to be chosen later. The Cameron-Martin formula (Theorem 2.8) applied to $\{\xi_\delta(x), |x|\le 1\}$ yields

$$\begin{aligned}&\mathrm{P}\Big(\xi_\delta(x) < \sqrt{h}; |x|\le 1\Big)\\ &= \mathrm{P}\Big(\xi_\delta(x)-\bar{f}_h(x) < f_h(x); |x|\le 1\Big)\\ &= e^{-\|\bar{f}_h\|^2/2}\mathrm{E}\Big[e^{-\xi_\delta(\bar{f}_h)}\mathbf{1}_{\{\xi_\delta(x)<f_h(x);|x|\le 1\}}\Big],\end{aligned}$$

where $\mathbf{1}_F$ is the indicator of the event F. By using Hölder's inequality, one bounds the last term from above by

$$e^{(p-1)\|\bar{f}_h\|^2/2}\mathrm{P}\big(\xi_\delta(x) < f_h(x); |x|\le 1\big)^{1/q}, \tag{3.4}$$

where $1/p+1/q=1$ with $p,q>1$.

Below we show that there exists a finite constant M (depending only on f) such that

$$\|\bar{f}_h\|^2 \le M \tag{3.5}$$

and

$$\mathrm{P}\big(\xi_\delta(x) < f_h(x); |x|\le 1\big) \le M h^a \text{ with } a = 1/(\overline{\alpha}+\varepsilon-1) \tag{3.6}$$

for all $\delta, h < 1$. As q can be chosen arbitrarily close to 1, and ε arbitrarily small, the desired estimate follows.

To show (3.5), observe that $\hat{f}_h(\lambda) = h^{\frac{1}{2}+a}\hat{f}(\lambda h^a)$ and $\|\bar{f}_h\| = \|f_h - f_h(0)\|$, thus

$$\begin{aligned}\|\bar{f}_h\|^2 &= \frac{1}{\pi}\int_0^\infty |\delta\widehat{\sigma}_0^2(\delta)\psi(\lambda/\delta)|\,|\hat{f}_h(\lambda)|^2 \mathrm{d}\lambda \\ &= \frac{1}{\pi}\delta\widehat{\sigma}_0^2(\delta)h^{1+2a}\int_0^\infty \psi(\lambda/\delta)|\hat{f}(\lambda h^a)|^2\mathrm{d}\lambda \\ &= \frac{1}{\pi}\delta\widehat{\sigma}_0^2(\delta)h^{1+a}\int_0^\infty \psi\left(\frac{\lambda}{\delta h^a}\right)|\hat{f}(\lambda)|^2\mathrm{d}\lambda. \end{aligned}\tag{3.7}$$

We split the last integral into two parts. For $\lambda \le K_0\delta h^a$ (recalling that K_0 is the constant in Lemma 2.1), the integrand concerning ψ is bounded due to the continuity of ψ, then

$$\int_0^{K_0\delta h^a}\psi\left(\frac{\lambda}{\delta h^a}\right)|\hat{f}(\lambda)|^2\mathrm{d}\lambda \le c\int_0^\infty |\hat{f}(\lambda)|^2\mathrm{d}\lambda = c\|f\|_{L^2},$$

since Fourier transform is an isometry from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$. For $\lambda > K_0\delta h^a$, we use Lemma 2.1,

$$\int_{K_0\delta h^a}^\infty \psi\left(\frac{\lambda}{\delta h^a}\right)|\hat{f}(\lambda)|^2\mathrm{d}\lambda \le c\psi\left(\frac{1}{\delta h^a}\right)\int_0^\infty |\lambda|^{\overline{\alpha}+\varepsilon}|\hat{f}(\lambda)|^2\mathrm{d}\lambda,$$

where we can assemble the last integral into the constant c because $\widehat{f}$ is a Scharwz function. Therefore, one obtains for $h<1$ and δ sufficiently small,

$$\|\bar{f}_h\|^2 \le c\delta\widehat{\sigma}_0^2(\delta)h^{1+a}\psi\left(\frac{1}{\delta h^a}\right) \le c\delta\widehat{\sigma}_0^2(\delta)\psi(1/\delta)h^{1+a-a(\overline{\alpha}+\varepsilon)},$$

where we used again Lemma 2.1 in the second inequality.

By Lemma 2.3, one has

$$\delta\widehat{\sigma}_0^2(\delta)\psi(1/\delta) \le c\delta\int_{1/\delta}^\infty \frac{\psi(1/\delta)}{\psi(\lambda)}\mathrm{d}\lambda \le c\delta\int_{1/\delta}^\infty \left(\frac{1}{\delta\lambda}\right)^{\underline{\alpha}-\varepsilon}\mathrm{d}\lambda = c.$$

Letting $a = 1/(\overline{\alpha}+\varepsilon-1)$ yields (3.5).

Now we prove (3.6). Denote by τ_δ the leftmost maximum location of the process $\{\xi_\delta(x), |x|\le 1\}$. Since f_h vanishes outside of $[-h^{1/(\beta-1)}, h^{1/(\beta-1)}]$, one has

$$\mathrm{P}\big(\xi_\delta(x) < f_h(x); |x|\le 1\big) \le \mathrm{P}\big(|\tau_\delta| \le h^{1/(\beta-1)}\big).$$

By Lemma 2.9, the density at zero of the law of each τ_δ is bounded from above by 2, uniformly for all $\delta>0$. This implies (3.6) and completes the proof of the lemma. □ □

We end this section with a similar lower tail estimate, but for large δ.

Lemma 3.4. *Assume that (C2) holds. For all $\gamma < 1/(\overline{\beta}-1)$, there exists a finite constant $K_3 > 1$ such that for any $\delta > 1$ and $h < 1$ so that $\delta h^{1/(\overline{\beta}+\varepsilon-1)} > 1/r_1$ (the constant defined in Lemma 2.5),*

$$\mathrm{P}\Big(\eta(x) < \sqrt{h\widehat{\sigma}_0^2(\delta)}; |x| \le \delta\Big) \le K_3 h^{\gamma}.$$

Remark 3.5. This lemma merely says that when $h \to 0$ and $\delta \to \infty$ in such a way that $\delta h^a \to \infty$ for some $a < 1/(\overline{\beta}-1)$, the lower tail probability behaves like a power function of h.

Proof of Lemma 3.4 Since (3.4) holds for all $h, \delta > 0$, we only need to show that there exists a finite constant M such that (3.5) and (3.6) hold with $a = 1/(\overline{\beta}+\varepsilon-1)$, uniformly for all $h < 1$ and all $\delta > 1$ so that $\delta h^{1/(\overline{\beta}+\varepsilon-1)} > 1/r_1$.

Let us start with (3.5). One has as in (3.7) that

$$\|\bar{f}_h\|^2 = \frac{1}{\pi}\delta\widehat{\sigma}_0^2(\delta)h^{1+a}\int_0^\infty \psi\left(\frac{\lambda}{\delta h^a}\right)|\hat{f}(\lambda)|^2 \mathrm{d}\lambda.$$

Write

$$\int_0^\infty \psi\left(\frac{\lambda}{\delta h^a}\right)|\hat{f}(\lambda)|^2 \mathrm{d}\lambda$$
$$= \int_0^1 + \int_1^{r_1\delta h^a} + \int_{r_1\delta h^a}^\infty \psi\left(\frac{\lambda}{\delta h^a}\right)|\hat{f}(\lambda)|^2 \mathrm{d}\lambda =: B_1 + B_2 + B_3.$$

By Lemma 2.5, one gets

$$B_1 \le \psi\left(\frac{1}{\delta h^a}\right)\int_0^1 \lambda^{\underline{\beta}-\varepsilon}|\widehat{f}(\lambda)|^2 \mathrm{d}\lambda \le c\psi\left(\frac{1}{\delta h^a}\right),$$
$$B_2 \le \psi\left(\frac{1}{\delta h^a}\right)\int_1^{r_1\delta h^a} \lambda^{\overline{\beta}+\varepsilon}|\widehat{f}(\lambda)|^2 \mathrm{d}\lambda \le c\psi\left(\frac{1}{\delta h^a}\right).$$

Further, a variable change, $\psi(\lambda) = o(\lambda^2)$ at infinity ([18, Lem. 4.2.2]), and the fact that $\widehat{f}$ is a Schwarz function implies that

$$B_3 = \delta h^a \int_{r_1}^\infty \psi(\lambda)|\widehat{f}(\delta h^a\lambda)|^2 \mathrm{d}\lambda$$
$$\le c\delta h^a \int_{r_1}^\infty \lambda^2|\delta h^a\lambda|^{-k} \mathrm{d}\lambda = c(\delta h^a)^{1-k}$$

for arbitrarily large $k \in \mathbb{N}$. Setting $k = 3$ and using Lemma 2.5, one obtains $(\delta h^a)^{-2} \le c(\delta h^a)^{-\overline{\beta}-\varepsilon} \le \psi(1/(\delta h^a))$, which implies $B_3 \le c\psi\left(1/(\delta h^a)\right)$. Combining these estimates yields that

$$\|\bar{f}_h\|^2 \le c\delta\widehat{\sigma}_0^2(\delta)h^{1+a}\psi\left(\frac{1}{\delta h^a}\right) \le c\delta\widehat{\sigma}_0^2(\delta)\psi(1/\delta)h^{1+a-a(\overline{\beta}+\varepsilon)},$$

where we have used Lemma 2.5 again in the second inequality. Applying Lemma 2.5 as in the proof of Lemma 2.6 entails that $\delta\widehat{\sigma}_0^2(\delta)\psi(1/\delta) \le c$ for all δ large enough. Letting $a = 1/(\overline{\beta}+\varepsilon-1)$ gives (3.5).

Remark that, by Lemma 2.9, the uniform density bound for the maximum location τ_δ holds also for large δ. Repeating the same lines as in the proof of Lemma 3.3 yields (3.6) which completes the proof. □

4. **Proofs of Theorems 1.2 and 1.4**

Following [4, 3, 16], the strategy is to first find an upper function for the local times with relatively small space variable, then to obtain a lower function for the maximal local times, where the order of the upper and lower functions are the same. This would give enough information to derive asymptotic results for the favorite points. Precisely, we prove the following two lemmas.

Lemma 4.1. *Let* $h_a(t) = \phi^{-1}\left(\frac{t}{(\log 1/t)^a}\right)$ *with* $a > 0$.

(i) *Assume (C1). For all* $\gamma < a/2$,

$$\lim_{t\to 0} \sup_{|x|\le h_a(t)} \frac{(\log 1/t)^\gamma (L^x_{\tau(t)} - t)}{t} = 0 \quad a.s.$$

(ii) *Assume (C2). For all* $\gamma < a/2$,

$$\lim_{t\to \infty} \sup_{|x|\le h_a(t)} \frac{(\log 1/t)^\gamma (L^x_{\tau(t)} - t)}{t} = 0 \quad a.s.$$

Lemma 4.2. *Set* $L^*_t = \sup_x L^x_t$.

(i) *Assume (C1). For any* $\gamma > (\overline{\alpha} - 1)/(2 - \overline{\alpha})$,

$$\lim_{t\to 0} \frac{(\log 1/t)^\gamma (L^*_{\tau(t)} - t)}{t} = \infty \quad a.s.$$

(ii) *Assume (C2). For any* $\gamma > (\overline{\beta} - 1)/(2 - \overline{\beta})$,

$$\lim_{t\to \infty} \frac{(\log 1/t)^\gamma (L^*_{\tau(t)} - t)}{t} = \infty \quad a.s.$$

Proof of Lemma 4.1 Let $0 < \delta < 1$. By Theorem 2.7,

$$\begin{aligned}
&\mathbb{P}\Bigg(\sup_{0\le x\le h_a(t)} L^x_{\tau(t)} - t \ge (1+\varepsilon)\lambda \Bigg) \\
&\le \mathrm{P} \times \mathbb{P}\Bigg(\sup_{0\le x\le h_a(t)} \left(L^x_{\tau(t)} + \frac{\eta^2(x)}{2} - t \right) \ge (1+\varepsilon)\lambda \Bigg) \\
&= \mathrm{P}\Bigg(\sup_{0\le x\le h(t)} \left(\frac{\eta^2(x)}{2} + \sqrt{2t}\eta(x) \right) \ge (1+\varepsilon)\lambda \Bigg) \\
&\le \mathrm{P}\Bigg(\sup_{0\le x\le h_a(t)} \eta^2(x) \ge 2\varepsilon\lambda \Bigg) + \mathrm{P}\Bigg(\sup_{0\le x\le h_a(t)} \sqrt{2t}|\eta(x)| \ge \lambda \Bigg) := P_1 + P_2.
\end{aligned}$$

Set

$$\lambda = \sqrt{\frac{2t \cdot c_3 \widehat{\sigma}_0^2(h_a(t))(1+\varepsilon)\log\log 1/t}{\delta}}.$$

Then the upper tail estimate in Lemma 3.1 yields,

$$P_2 \le c_2 \exp\left(-\frac{(1+\varepsilon)\log\log 1/t}{\delta} \right).$$

By Lemma 2.3, $\widehat{\sigma}_0^2(h_a(t)) \asymp (\log 1/t)^{-a}$ around zero, one deduces

$$\begin{aligned}
P_1 &= \mathrm{P}\Big(\sup_{0\le x\le h_a(t)} |\eta(x)| \ge \sqrt{2\varepsilon\lambda}\Big) \\
&\le c_2 \exp\left(-2\varepsilon\sqrt{\frac{2t(1+\varepsilon)\log\log 1/t}{c_3\widehat{\sigma}_0^2(h_a(t))\delta}}\right) \\
&= c_2 \exp\left(-c\varepsilon\sqrt{\frac{(1+\varepsilon)\log\log 1/t}{\delta/(\log 1/t)^a}}\right) \\
&\le c_2 \exp\left(-\frac{(1+\varepsilon)\log\log 1/t}{\delta}\right)
\end{aligned}$$

for t sufficiently small.

Let $t_k = \exp(-k^\delta)$. When $t = t_k$, both P_1 and P_2 are the general term of a convergent series. By the Borel-Cantelli Lemma, a.s. for all k large enough,

$$\sup_{0\le |x|\le h_a(t_k)} \frac{|\log t_k|^{a/2}(L^x_{\tau(t_k)} - t_k)}{t_k\sqrt{\log\log 1/t_k}} \le c/\sqrt{\delta}. \tag{4.1}$$

Thus, one has a.s. for all large k, and $t_{k-1} < t < t_k$

$$\sup_{0\le |x|\le h_a(t_k)} \frac{|\log t_k|^{a/2}(L^x_{\tau(t)} - t)}{t_k\sqrt{\log\log 1/t_k}} \le c/\sqrt{\delta} + \frac{(t_k - t_{k-1})|\log t_k|^{a/2}}{t_k\sqrt{\log|\log t_k|}},$$

where the second term is bounded from above by $\sqrt{\delta}$. Therefore, a.s.

$$\sup_{|x|\le h_a(t)} \frac{|\log t|^{a/2}(L^x_{\tau(t)} - t)}{t\sqrt{\log\log 1/t}} \le \left(\frac{c}{\sqrt{\delta}} + \sqrt{\delta}\right)\frac{t_k\log\log 1/t_k}{t_{k-1}\log\log t_{k-1}} \le 2\left(\frac{c}{\sqrt{\delta}} + \sqrt{\delta}\right)$$

for all t sufficiently small. The first claim follows.

To show the second claim, the proof is identical except that we use Lemma 3.2 instead of Lemma 3.1, and that we choose $t_k = \exp(k^\delta)$. We omit the details. □

Proof of Lemma 4.2 Define the events

$$\begin{aligned}
E_1 &= \left\{L^x_{\tau(t)} - t \le C_1 th; |x| \le \phi^{-1}(ht)\right\}; \\
E_2 &= \left\{\frac{\eta^2(x)}{2} \le C_1 th; |x| \le \phi^{-1}(ht)\right\}; \\
E_3 &= \left\{L^x_{\tau(t)} + \frac{\eta^2(x)}{2} - t \le 2C_1 th; |x| \le \phi^{-1}(ht)\right\}.
\end{aligned}$$

By the Markov inequality and the second moment estimate in Lemma 3.1, we can choose the constant C_1 in the event E_2 large enough such that $\mathrm{P}(E_2) > 1/2$ uniformly in $h, t < 1$. Observe that by the independence and Theorem 2.7, one has

$$\begin{aligned}
\frac{\mathbb{P}(E_1)}{2} \le \mathbb{P}(E_1)\mathrm{P}(E_2) &\le \mathrm{P}\times\mathbb{P}(E_3) \\
&= \mathrm{P}\left(\frac{\eta^2(x)}{2} + \sqrt{2t}\eta(x) \le 2C_1 th; |x| \le \phi^{-1}(ht)\right) \\
&\le \mathrm{P}\left(\eta(x) \le \sqrt{2}C_1\sqrt{th}\sqrt{h}; |x| \le \phi^{-1}(ht)\right).
\end{aligned}$$

Recall that by Lemma 2.3, $\widehat{\sigma}_0^2(\phi^{-1}(\delta)) \asymp \delta$ for δ sufficiently small, in other words, we can choose $C_2 < \infty$ such that

$$\sqrt{2}C_1\delta \leq C_2\widehat{\sigma}_0(\phi^{-1}(\delta^2)).$$

In particular, for all $\gamma < 1/(\overline{\alpha} - 1)$ and t, h sufficiently small, one has by Lemma 3.3 that

$$\mathbb{P}(E_1) \leq 2\mathrm{P}\left(\eta(x) \leq C_2\sqrt{h\widehat{\sigma}_0^2(\phi^{-1}(th))}; |x| \leq \phi^{-1}(ht)\right) \leq Kh^\gamma$$

with some finite constant K. It follows that

$$\mathbb{P}(L^*_{\tau(t)} - t \leq C_1 th) \leq Kh^\gamma.$$

Now let $0 < d < 2 - \overline{\alpha}$, $t = t_k := \exp(-k^d)$ and $h = h_k := k^{-(\overline{\alpha}-1)(1+\varepsilon)}$ with $0 < \varepsilon < (1-d)/(\overline{\alpha}-1) - 1$. The Borel-Cantelli Lemma implies that almost surely for all k sufficiently large,

$$L^*_{\tau(t_k)} - t_k \geq \frac{C_1 t_k}{k^{(\overline{\alpha}-1)(1+\varepsilon)}}.$$

Let $t_k \leq t < t_{k-1}$, then,

$$L^*_{\tau(t)} - t \geq -(t - t_k) + \frac{C_1 t_k}{k^{(\overline{\alpha}-1)(1+\varepsilon)}}$$

and

$$\frac{L^*_{\tau(t)} - t}{t} \geq -\frac{(t_{k-1} - t_k)}{t_k} + \frac{C_1 t_k}{t_{k-1}k^{(\overline{\alpha}-1)(1+\varepsilon)}} \geq -2k^{-(1-d)} + \frac{C_1(1-\delta)}{k^{(\overline{\alpha}-1)(1+\varepsilon)}}$$

for all $\delta > 0$ and large k. By our choice of d and ε, one deduces

$$\frac{L^*_{\tau(t)} - t}{t} \geq \frac{C_1(1-2\delta)}{k^{(\overline{\alpha}-1)(1+\varepsilon)}} \geq \frac{C_1(1-3\delta)}{(\log 1/t)^{(\overline{\alpha}-1)(1+\varepsilon)/d}}.$$

As d can be chosen arbitrarily close to $2 - \overline{\alpha}$, the first claim of Lemma 4.2 follows.

Similarly, applying the second Ray-Knight Theorem and Lemma 3.4 yields

$$\mathbb{P}(E_1) \leq Kh^\gamma$$

for $h < 1$, $\delta > 1$ such that $\delta h^{1/(\overline{\beta}+\varepsilon-1)} > 1/r_1$. Repeating the arguments in the last paragraph for $t_k = \exp(k^d)$ and $h_k = k^{-(\overline{\beta}-1)(1+\varepsilon)}$ (with some tuning in the choice of ε) gives the second claim of Lemma 4.2. □

We are ready to finish the proof of Theorem 1.2.

Proof of Theorem 1.2. We only proof part (i) asymptotic behavior of $\{V_t, t \geq 0\}$ around zero, the proof of (ii) is similar. Combining Lemma 4.1 (i) and Lemma 4.2 (i) entails that for $\gamma > 2(\overline{\alpha}-1)/(2-\overline{\alpha})$,

$$V_{\tau(t)} \geq \phi^{-1}\left(\frac{t}{(\log t)^\gamma}\right) \tag{4.2}$$

for all t sufficiently small. In other words, for all small t,

$$V_t \geq \phi^{-1}\left(\frac{L^0_t}{(\log L^0_t)^\gamma}\right),$$

which implies that

$$\lim_{t\to 0} \frac{V_t}{\phi^{-1}\left(\frac{L^0_t}{(\log L^0_t)^\gamma}\right)} = \infty.$$

This finishes the proof of Theorem 1.2. □

We end this section by proving Theorem 1.4.

Proof of Theorem 1.4. By Theorem 1.3 and Proposition 1.4 of Eisenbaum and Khoshnevisan [11], it is sufficient to verify that there is a sequence $\{x_n\}$ such that $\lim_{n\to\infty} x_n = 0$ and

$$\lim_{k\to\infty} (\ln k)^{1/2} \sup_{\substack{n,\, m\in\mathbb{N},\\ |n-m|\geq k}} \frac{u_{T_0}(x_m, x_n)}{\sqrt{u_{T_0}(x_m, x_m) u_{T_0}(x_n, x_n)}} = 0, \tag{4.3}$$

where $u_{T_0}(x, y)$ is the function given in (2.16). The verification is similar to the proof of Theorem 5.2 in [11] and we give it for the sake of completeness. It follow from (2.16) that for any $x, y \in \mathbb{R}$,

$$\frac{u_{T_0}(x, y)}{\sqrt{u_{T_0}(x, x) u_{T_0}(y, y)}} \leq \frac{1}{2}\frac{\sigma_0(x)}{\sigma_0(y)} + \frac{|\sigma_0^2(y) - \sigma_0^2(x-y)|}{\sigma_0(x)\sigma_0(y)}. \tag{4.4}$$

Under Condition (C1), we apply Remark 2.4 to see that for any $0 < x < y \leq 1$,

$$\frac{\sigma_0(x)}{\sigma_0(y)} \leq \left(\frac{x}{y}\right)^{\alpha-\varepsilon}, \tag{4.5}$$

where $\alpha > 0$ is a constant and $\varepsilon \in (0, \alpha)$. In order to bound the second term on the right hand side of (4.4), we use the fact that the covariance function $u_{T_0}(x, y)$ of the associated Gaussian process is the 0-potential density of X killed the first time it hits zero. By [18, Formula (7.232), p.324],

$$\sigma_0^2(x) \geq |\sigma_0^2(y) - \sigma_0^2(y-x)|.$$

Therefore, the second term is bounded from above by $\frac{\sigma_0(x)}{\sigma_0(y)}$. We have thus proved

$$\frac{u_{T_0}(x, y)}{\sqrt{u_{T_0}(x, x) u_{T_0}(y, y)}} \leq c\left(\frac{x}{y}\right)^{\alpha-\varepsilon}.$$

Once this is in place, the rest of the proof is almost the same as that in the bottom on page 254 of [11] and is omitted. □

5. **Examples**

In this section, we provide more examples of symmetric Lévy processes (besides the semistable Lévy processes in [7] that we have mentioned earlier) that satisfy the conditions of the present paper, but not those in Marcus [16]. To avoid repetition, we only present examples concerning the asymptotic behavior of the favorite points around zero.

Example 5.1. Let $1 < \alpha < 2$ and $0 < c_1 < c_2$ be constants such that $c_2\alpha/(2c_1) < 2$ and $c_1\alpha/(2c_2) > 1$. For any decreasing sequence $\{b_n; n \geq 0\}$ such that $b_0 = 1$ and $b_n \to 0$, define

$$\theta(x) = \begin{cases} c_1|x|^{\alpha+1}, & |x| \in (b_{2k+2}, b_{2k+1}], \\ c_2|x|^{\alpha+1}, & |x| \in (b_{2k+1}, b_{2k}]. \end{cases}$$

Then (C1) holds with $\underline{\alpha} = c_1\alpha/(2c_2) < c_2\alpha/(2c_1) = \overline{\alpha}$.

Example 5.2. Let $1<\alpha_1<\alpha_2<2$ and $\{b_n; n\geq 0\}$ be a real sequence decreasing to zero such that $b_0=1$, $b_1=\frac{1}{2}$, and for any integer $k\geq 1$,

$$b_{2k}\leq b_{2k-1}/(k+1), \tag{5.1}$$

$$\int_{b_{2k+1}}^{b_{2k}}\frac{\mathrm{d}x}{x^{\alpha_2+1}}\leq\int_{b_{2k}}^{b_{2k-1}}\frac{\mathrm{d}x}{x^{\alpha_1+1}}, \tag{5.2}$$

$$\int_{b_{2k+1}}^{b_{2k}}\frac{\mathrm{d}x}{x^{\alpha_2-1}}\leq\int_{0}^{b_{2k+1}}\frac{\mathrm{d}x}{x^{\alpha_1-1}}. \tag{5.3}$$

This can be done inductively. We choose first b_{2k} satisfying (5.1) which ensures the convergence as $k\to\infty$, then chose b_{2k+1} close to b_{2k} so that (5.2) and (5.3) hold. Let

$$\theta(x)=\begin{cases}|x|^{\alpha_1+1}, & |x|\in(b_{2k+2},b_{2k+1}],\\ |x|^{\alpha_2+1}, & |x|\in(b_{2k+1},b_{2k}].\end{cases}$$

We claim that the following properties hold:

(i) $\pi(\lambda)\asymp|\lambda|^{\alpha_1}$ for $|\lambda|$ large enough;
(ii) $\psi(\lambda)\asymp|\lambda|^{\alpha_1}$ for $|\lambda|$ large enough;
(iii) Condition (C1) fails.

In particular, the conclusion of Lemma 2.1 holds with $\overline{\alpha}=\underline{\alpha}=\alpha_1$ even if (C1) fails. Consequently, Part (i) of Theorem 1.2 still holds with $\overline{\alpha}=\alpha_1$.

Let us show the claims (i)-(iii). Observe by (1.2) and the integrability of $1/\theta(x)$ at infinity ($\mathrm{d}x/\theta(x)$ is a Lévy measure) that

$$c|\lambda|^{\alpha_1}\leq\psi(\lambda)\leq C|\lambda|^{\alpha_2} \tag{5.4}$$

for all $|\lambda|$ large enough. Similarly,

$$c|\lambda|^{\alpha_1}\leq\pi(\lambda)\leq C|\lambda|^{\alpha_2}$$

for $|\lambda|$ sufficiently large. To show (i), it suffices to show $\pi(\lambda)\leq C|\lambda|^{\alpha_1}$ for large $|\lambda|$. For any $\lambda>1$, there exists a constant k_0 such that either $1/\lambda\in(b_{2k_0+2},b_{2k_0+1}]$ or $1/\lambda\in(b_{2k_0+1},b_{2k_0}]$. In the first case,

$$\pi(\lambda)/2\leq\frac{1}{\alpha_1}\left(\lambda^{\alpha_1}-b_{2k_0+1}^{-\alpha_1}+b_{2k_0}^{-\alpha_1}-b_{2k_0-1}^{-\alpha_1}+\cdots+b_2^{-\alpha_1}-b_1^{-\alpha_1}\right)\\ +\frac{1}{\alpha_2}\left(b_{2k_0+1}^{-\alpha_2}-b_{2k_0}^{-\alpha_2}+b_{2k_0-1}^{-\alpha_2}-b_{2k_0-2}^{-\alpha_2}+\cdots+b_1^{-\alpha_2}-b_0^{-\alpha_2}\right)+\int_1^\infty\frac{\mathrm{d}x}{\theta(x)}.$$

Plainly, the first sum is bounded from above by $C\lambda^{\alpha_1}$ and the integral is finite. It follows from (5.2) that the second sum is bounded from above by $C\lambda^{\alpha_1}$. In the second case when $1/\lambda\in(b_{2k_0+1},b_{2k_0}]$,

$$\pi(\lambda)/2\leq\frac{1}{\alpha_1}\left(b_{2k_0}^{-\alpha_1}-b_{2k_0-1}^{-\alpha_1}+\cdots+b_2^{-\alpha_1}-b_1^{-\alpha_1}\right)\\ +\frac{1}{\alpha_2}\left(b_{2k_0+1}^{-\alpha_2}-b_{2k_0}^{-\alpha_2}+b_{2k_0-1}^{-\alpha_2}-b_{2k_0-2}^{-\alpha_2}+\cdots+b_1^{-\alpha_2}-b_0^{-\alpha_2}\right)+\int_1^\infty\frac{\mathrm{d}x}{\theta(x)}. \tag{5.5}$$

Again the both sums are bounded from above by $C\lambda^{\alpha_1}$ by (5.2) and the integral is finite. Hence, $\pi(\lambda)\leq C\lambda^{\alpha_1}$ for $\lambda>1$, as desired.

Now we verify (ii). Recalling (5.4) and Lemma 2.2 (applied to the Lévy measure rather than the spectral measure), we only need to show that

$$\lambda^2 \int_0^{1/\lambda} x^2 \frac{\mathrm{d}x}{\theta(x)} \le C\lambda^{\alpha_1} \tag{5.6}$$

for λ large enough. Let $\lambda > 1$ and k_1 be the integer so that either $1/\lambda \in (b_{2k_1+2}, b_{2k_1+1}]$ or $1/\lambda \in (b_{2k_1+1}, b_{2k_1}]$. In the first case, by (5.3)

$$\lambda^2 \int_0^{1/\lambda} x^2 \frac{\mathrm{d}x}{\theta(x)} \le \lambda^2 \left(\int_0^{1/\lambda} \frac{\mathrm{d}x}{x^{\alpha_1 - 1}} + \int_{b_{2k_1+3}}^{b_{2k_1+2}} \frac{\mathrm{d}x}{x^{\alpha_2 - 1}} + \int_{b_{2k_1+5}}^{b_{2k_1+4}} \frac{\mathrm{d}x}{x^{\alpha_2 - 1}} + \cdots \right) \le \frac{1}{2-\alpha_1}\lambda^{\alpha_1} + \frac{1}{2-\alpha_2}\lambda^{\alpha_1} \le C\lambda^{\alpha_1},$$

as desired. We prove similarly in the second case that (5.6) holds.

Finally, it follows from (5.5) that

$$\limsup_{x\to 0} \frac{x/\theta(x)}{\nu(y : |y| \ge x)} = \infty.$$

Thus (C1) is not satisfied.

6. **Extensions**

6.1. **Transient case**

In this subsection we assume that the characteristic exponent ψ of the pure jump symmetric Lévy process X satisfies

$$\int_1^\infty \frac{d\lambda}{\psi(\lambda)} < \infty \quad \text{and} \quad \int_0^1 \frac{d\lambda}{\psi(\lambda)} < \infty. \tag{6.1}$$

Consequently, X is transient, its local times exist, and its 0-potential density $u(x, y)$ is bounded and continuous. Further, the support of potential operators of X is $\mathbb{R}$, it follows from [6, p.55] and the absolute continuity of potential operators that

$$h_x = \mathbb{P}^x(T_0 < \infty) > 0, \quad \forall x \in \mathbb{R}. \tag{6.2}$$

We state the generalized second Ray-Knight Theorem for transient symmetric Markov processes due to Eisenbaum *et al.* [10], see also [18, Thm. 8.2.3]. Set $\tau^-(t) = \inf\{s : L_s^0 \ge t\}$.

Theorem 6.1. *Assume that $h_x = \mathbb{P}^x(T_0 < \infty) > 0$ for all $x \in \mathbb{R}$. Let $\eta = \{\eta_x, x \in \mathbb{R}\}$ be a mean zero Gaussian process with covariance $u_{T_0}(x, y)$. Then for any $t > 0$ and any countable subset $D \subset \mathbb{R}$, under $\mathbb{P}^0 \times \mathrm{P} \times \mathrm{P}_\rho$, in law*

$$\left\{ L^x_{\tau^-(t \wedge L^0_\infty)} + \frac{\eta_x^2}{2}; x \in D \right\} = \left\{ \frac{1}{2}(\eta_x + h_x\sqrt{2(t\wedge\rho)})^2; x \in D \right\}, \tag{6.3}$$

where ρ is an exponential random variable with mean $u(0, 0)$.

We also need the following fact in [6, p.26] which says that the excessive functions of a Markov process with absolutely continuous potential operators are lower semi-continuous.

Lemma 6.2. *h_x is lower semi-continuous.*

Now let us focus on deriving an asymptotic result for V around zero. It suffices to prove analogue of Lemma 4.1 (i) and 4.2 (i) for transient processes. We reduce the proof to the point where we can apply Gaussian tail estimates as in Section 4.

Lemma 6.3. *Let $h_a(t)$ be as in Lemma 4.1. Under Condition (C1), for all $\gamma < a/2$,*

$$\lim_{t\to 0} \sup_{|x|\le h_a(t)} \frac{|\log t|^\gamma (L^x_{\tau^-(t)} - h_x^2 t)}{t} = 0, \ \ a.s.$$

Proof. Define the events

$$E_4 = \left\{L^0_\infty > 1, \rho > 1\right\};$$

$$E_5 = \left\{\sup_{|x|\le h_a(t\wedge L^0_\infty)} \frac{\eta^2(x)}{2} + h_x\sqrt{2(t\wedge\rho)}\eta(x) + h_x^2(t\wedge\rho - t\wedge L^0_\infty) \ge (1+\varepsilon)\lambda\right\};$$

$$E_6 = \left\{\sup_{|x|\le h_a(t)} \frac{\eta^2(x)}{2} + \sqrt{2t}\eta(x) \ge (1+\varepsilon)\lambda\right\}.$$

Note that $E_4 \cap E_5 \subset E_6$ for $0 < t < 1$. By Theorem 6.1,

$$\mathbb{P}\left(\sup_{|x|\le h_a(t\wedge L^0_\infty)} L^x_{\tau^-(t\wedge L^0_\infty)} - h_x^2(t\wedge L^0_\infty) \ge (1+\varepsilon)\lambda\right) \quad \le \quad \mathrm{P} \quad \times \quad \mathrm{P}_\rho \quad \times \quad \mathbb{P}(E_5).$$

By the independence and the fact that L^0_∞, ρ are exponential with mean $0 < u(0,0) < \infty$, we have for $0 < t < 1$,

$$\mathbb{P}\left(\sup_{|x|\le h_a(t\wedge L^0_\infty)} L^x_{\tau^-(t\wedge L^0_\infty)} - h_x^2(t\wedge L^0_\infty) \ge (1+\varepsilon)\lambda\right) \le e^{\frac{2}{u(0,0)}}\mathrm{P}(E_6),$$

where the Gaussian upper tail estimates have been applied. Set $t_k = \exp(-k^\delta)$ with $0 < \delta < 1$. It follows that under Condition (C1), a.s. for all k sufficiently large, (4.1) holds with $L^x_{\tau(t_k)} - t_k$ replaced by $L^x_{\tau^-(t_k\wedge L^0_\infty)} - h_x^2(t_k\wedge L^0_\infty)$ and $h_a(t_k)$ replaced by $h_a(t_k\wedge L^0_\infty)$. Since a.s. $L^0_\infty > 0$, one deduces

$$\sup_{|x|\le h_a(t_k)} \frac{|\log t_k|^{a/2}(L^x_{\tau^-(t_k)} - h_x^2 t_k)}{t_k\sqrt{\log\log 1/t_k}} \le c/\sqrt{\delta}$$

for all k sufficiently large. A routine interpolation ends the proof. □ □

Lemma 6.4. *Recall $L^*_t = \sup_x L^x_t$. Under (C1), for $\gamma > (\overline{\alpha}-1)/(2-\overline{\alpha})$,*

$$\lim_{t\to 0} \frac{|\log t|^\gamma(L^*_{\tau^-(t)} - h_x^2 t)}{t} = \infty.$$

Proof. Define the events

$$E_7 = \left\{L^x_{\tau^-(t\wedge L^0_\infty)} - h_x^2(t\wedge L^0_\infty) \le C_1 th; |x| \le \phi^{-1}(ht)\right\};$$

$$E_8 = \left\{L^x_{\tau^-(t\wedge L^0_\infty)} + \frac{\eta_x^2}{2} - h_x^2(t\wedge L^0_\infty) < 2C_1 th; |x| \le \phi^{-1}(ht)\right\};$$

$$E_9 = \left\{\frac{\eta(x)^2}{2} + h_x\sqrt{2(t\wedge\rho)}\eta_x + h_x^2(t\wedge\rho - t\wedge L^0_\infty) < 2C_1 th; |x| \le \phi^{-1}(ht)\right\};$$

$$E_{10} = \left\{\frac{\eta(x)^2}{2} + h_x\sqrt{2t}\eta(x) \le 2C_1 th; |x| \le \phi^{-1}(ht)\right\}.$$

Observe that $E_7 \cap E_2 \subset E_8$. For C_1 large, one has $\mathbb{P}(E_7) \le 2\mathrm{P} \times \mathbb{P}(E_8) = 2\mathbb{P} \times \mathrm{P} \times \mathrm{P}_\rho(E_9)$ by Theorem 6.1. That $E_4 \cap E_9 \subset E_{10}$ for $0 < t < 1$ and $\mathrm{P}_\rho \times \mathbb{P}(E_4) = e^{-2/u(0,0)}$ yields

$$\mathbb{P}(E_7) \le 2e^{2/u(0,0)}\mathrm{P}(E_{10}), \quad 0 < t < 1.$$

By Lemma 6.2 and the fact that $\mathbb{P}(T_0 = 0) = 1$, there exists $r > 0$ such that $h_x \ge 1/2$ for $|x| < r$. Hence,

$$\mathrm{P}(E_{10}) \le \mathrm{P}(\eta(x) \le 2\sqrt{2}C_1\sqrt{th}\sqrt{h}; |x| \le \phi^{-1}(ht))$$

for all t sufficiently small. Now the Gaussian lower tail estimates apply and the proof goes exactly as that of Lemma 4.2. □ □

It is worthy to point out that the generalized second Ray-Knight Theorem does not work well when we wish to study the asymptotic behavior of favorite points *around infinity* for transient processes. Indeed, in the transient case, the local time process at each fixed state is finite at infinity $L^x_\infty < \infty$ a.s. In particular, $L^0_\infty < \infty$, hence its inverse $\tau(t)$ is (finite) constant for all t sufficiently large, a.s. It would be natural to consider the local times under the true time scale L^x_t rather than the time changed local time $L^x_{\tau(t)}$, but we have not been able to do so.

On the other hand, we can consider another type of favorite sites at infinity of transient processes. Let $\mathcal{U}_r = \left\{ y \in [-r, r] : \sup_{|x|\le r} L^x_\infty = L^y_\infty \right\}$ be the set of favorite points within $[-r, r]$ at infinity and define the favorite points process by $r \mapsto U_r = \inf\{|y| : y \in \mathcal{U}_r\}$. We expect that Eisenbaum's Isomorphism Theorem [18, Thm. 8.1.1] plays the role of the second Ray-Knight Theorem. This is beyond the scope of the present article, and will be studied in a separate paper.

6.2. **Symmetric Lévy processes with a Gaussian component**

In this subsection, we assume that $A \ne 0$ in (1.2). Denote $\psi_d(\lambda) = \psi(\lambda) - A^2\lambda^2$. It follows from [18, Lem. 4.2.2] that

$$\psi(\lambda) \asymp \begin{cases} \lambda^2 & \lambda > 1, \\ \psi_d(\lambda) & 0 < \lambda < 1. \end{cases} \tag{6.4}$$

Therefore, the local times exist and the process X is transient or recurrent according to $\int_0^1 \frac{1}{\psi_d(\lambda)} \mathrm{d}\lambda$ converges or diverges. It may be seen from the proof that the asymptotic behavior of the favorite points relies on the asymptotic properties of $\sigma_0^2(x)$ defined in (1.1), which is presented in the following lemma.

Lemma 6.5. *Let $\sigma_0^2(x)$, $\widehat{\sigma}_0^2(x)$, $\pi(\lambda)$, $\phi(x)$ be defined in Sections 1 and 2.*

(i) $\sigma_0^2(x) \asymp \phi(x) \asymp |x|$ *as* $|x| \to 0$.
(ii) *Under (C2),* $\psi(\lambda) \asymp \pi(\lambda)$ *as* $\lambda \to 0$ *and* $\widehat{\sigma}_0^2(x) \asymp \phi(x)$ *as* $|x| \to \infty$.

Proof. (i) holds by a Tauberian type result [18, Thm 7.3.1], and (ii) follows from (6.4). □ □

Since the large time behavior of V depends only on the asymptotic property of $\sigma_0^2(x)$ at infinity, we have that under Condition (C2) and recurrence, Part (ii) of Theorem 1.2 holds when $A \ne 0$.

The small time behavior of V cannot be derived from the present proof when $A \ne 0$, the problem being that the lower tail estimate obtained through Cameron-Martin formula is not strong enough. In such case $\sigma_0^2(x) \asymp |x|$ around zero (the associated Gaussian process behaves locally like a Brownian motion), one sees that our proof of Lemma 3.3 gives an upper bound ch^γ for any $\gamma < 1$ but not for $\gamma = 1$. The decay is not fast enough for the subsequent computations involving local times. This kind of phenomenon appeared already in the case of stable Lévy processes as recalled in the following remark.

Remark 6.6. *For α-stable Lévy processes, the Brownian case ($\alpha = 2$) and the pure jump case ($1 < \alpha < 2$) may amount to different treatment regarding the lower tail estimate of the associated Gaussian process. Recall* [18, p.498] *that the associated Gaussian process η_α for α-stable process with $1 < \alpha \le 2$ is a fractional Brownian motion with covariance $C(s,t) = \frac{C_\alpha}{2}(|s|^{\alpha-1} + |t|^{\alpha-1} - |s-t|^{\alpha-1})$. Note that η_2 is a two-sided Brownian motion. The decay of the lower tail estimate obtained by Cameron-Martin formula is good enough for our purposes when $1 < \alpha < 2$, but is not sufficient for the subsequent computations when $\alpha = 2$. Indeed,* [18, Lem. 11.5.1] *shows that for all $1 < \alpha \le 2$, $\varepsilon > 0$,*

$$\mathrm{P}\left(\sup_{|x|\le 1} \eta_\alpha(x) < \lambda\right) \le c\lambda^{\frac{2(1-\varepsilon)}{\alpha}}. \tag{6.5}$$

In the critical case $\alpha = 2$, we have the upper bound $c\lambda^\gamma$ for any $\gamma < 1$, a bound that is not sufficiently small for computations in the sequel, as pointed out by Marcus and Rosen [18, p.519]. *However, the reflection principle and the the independence of $\{\eta_2(x), x \ge 0\}$ and $\{\eta_2(x), x \le 0\}$ actually implies*

$$\mathrm{P}\left(\sup_{|x|\le 1} \eta_2(x) < \lambda\right) \le \sqrt{\frac{2}{\pi}}\lambda, \tag{6.6}$$

a better bound that is sufficient to derive results for V, see [18, Sec. 11.2].

In both references [3, 16], the authors viewed the associated Gaussian process as a time-changed Brownian motion and made use of (6.6) through Slepian's lemma. This approach remains valid for the small time behavior of the favorite points of a Lévy process with both a Gaussian part and a pure jump part. Indeed, by replacing $\sigma_0^2(x)$ by $|x|$ back and forth (and by the argument developed in Section 6.1 if the process is transient), we can remove the monotonicity condition of Marcus [16] so that [16, Thm 1.1] holds as $t \to 0$ when $\sigma_0^2(x) \asymp |x|$ around zero. Summarizing the discussion, we have proved the following theorem. Part (i) states that the small time behavior of V is the same as that of the Brownian part of X without any condition on the pure jump part. Part (ii) states that (under recurrence condition) the large time behavior of V is the same as that of the pure jump part of X subject to (C2).

Theorem 6.7. *Let $A \ne 0$ in* (1.2).

(i) *For $a > 8$,*

$$\lim_{t\to 0} \frac{V_t}{\frac{L_t^0}{(\log L_t^0)^a}} = \infty \quad a.s.$$

(ii) *Under Condition (C2) and $\int_0^1 \frac{1}{\psi_d(\lambda)}\mathrm{d}\lambda = \infty$, for all $a > 2(\overline{\beta}-1)/(2-\overline{\beta})$,*

$$\lim_{t\to\infty} \frac{V_t}{\phi^{-1}\left(\frac{L_t^0}{(\log L_t^0)^a}\right)} = \infty \quad a.s.$$

We finish the paper with an open problem: can one describe the small (resp. large) time behavior of V when the Lévy measure satisfies (C1) with $\overline{\alpha} = 2$ (resp. (C2) with $\overline{\beta} = 2$) and X is a pure jump process? One may still obtain lower tail estimates with the approach of [3, 16]. However, (2.1)-(2.2) fail to hold and it is not clear to us how to find a monotone function which is equivalent to $\sigma_0^2(x)$.

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