Chapter 6
Recursive Filtering for Time-Varying Systems with Random Access Protocol

In this chapter, we intend to study the communication-protocol-based recursive filtering problem for time-varying systems. Recursive filtering scheme is a widely employed method which is developed in the sense of minimum filtering error variance. More specifically, such recursive filtering scheme works in a two-step process, namely the *prediction step* and the *correction step*. In the prediction step, the filter computes the one-step prediction of the system state based on the plant dynamics and the current state estimate. In the correction step, the state estimate would be updated by fusing the derived one-step prediction and the new measurement output with the well-designed weighted gain (i.e. the recursive filter gain).

On the other hand, it should be noting that the utilization of networks has also led to certain complicated network-induced effects which could largely jeopardize the control/filtering performance of systems. This chapter is concerned with the recursive filtering problem for a class of networked linear time-varying systems where the communication between the sensors nodes and the remote filter is implemented via a shared network. The transmission order of sensor nodes is orchestrated by the RA protocol scheduling, under which the selected nodes obtaining access to the network could be characterized by a sequence of i.i.d variables. In this chapter, we would like to discuss the effects of RA protocol scheduling on the resultant filtering error covariance and the ultimate boundedness of the filtering error covariance.

The rest of this chapter is organized as follows. In Section 6.1, the networked linear time-varying system with the RA protocol scheduling is introduced and the corresponding recursive filtering problem is formulated. In Section 6.2, the desired filter gain matrix is recursively computed based on two Riccati-like difference equations. Then, the boundedness analysis issue of the filtering error covariance is studied for our developed recursive filtering scheme. Moreover, two numerical simulation examples are given in Section 6.3 to demonstrate the correctness and effectiveness of the main results. Finally, we conclude this chapter in Section 6.4.

6.1 Problem Formulation

6.1.1 The system model and communication model

The plant under consideration is a linear time-varying system described by the following state-space model:

$$\begin{cases} x_{k+1} = A_k x_k + B_k \omega_k \\ y_k = C_k x_k + v_k \end{cases} \tag{6.1}$$

where $x_k \in \mathbb{R}^{n_x}$ and $y_k \in \mathbb{R}^{n_y}$ denote, respectively, the system state and the measurement output before transmitted through the communication network. $\omega_k \in \mathbb{R}^{n_\omega}$ and $v_k \in \mathbb{R}^{n_v}$ represent the process and measurement noises, respectively. The parameters A_k, B_k and C_k are real-valued time-varying matrices of appropriate dimensions.

The initial state x_0, the process noise ω_k and the measurement noise v_k are mutually uncorrelated and have the following statistical properties:

$$\begin{aligned} &\mathbb{E}\{\omega_k\} = \mathbb{E}\{v_k\} = 0, \mathbb{E}\{x_0\} = \bar{x}_0, \mathbb{E}\{\omega_k \omega_k^T\} = Q_k, \\ &\mathbb{E}\{(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)^T\} = P_{0|0}, \mathbb{E}\{v_k v_k^T\} = R_k \end{aligned} \tag{6.2}$$

where $P_{0|0} > 0$, $Q_k > 0$ and $R_k > 0$ are known matrices with appropriate dimensions.

We are now ready to introduce the signal transmission over the communication network. Without loss of generality, it is assumed that sensors of the system are grouped into N sensor nodes according to their spatial distribution. As such, for technical analysis, the measurement output before transmitted can be rewritten as $y_k = \left[y_{1,k}^T \; y_{2,k}^T \; \cdots \; y_{N,k}^T\right]^T$ where $y_{i,k}$ ($i \in \{1, 2, \cdots, N\}$) is the measurement of the i-th sensor node before transmitted.

The communication between the sensors and the remote filter is scheduled by the RA protocol. For the sake of examining the influence of communication protocols, let $\xi(k) \in \{1, 2, \cdots, N\}$ be the selected sensor node obtaining access to the communication network at time instant k. As shown in subsection 4.1, $\xi(k) \in \{1, 2, \cdots, N\}$ could be modeled as a sequence of random variables. The occurrence probability of $\xi(k) = i$ ($\forall i \in \{1, 2, \cdots, N\}$) is given by

$$\text{Prob}\{\xi(k) = i\} = p_i \tag{6.3}$$

where $p_i > 0$ ($i \in \{1, 2, \cdots, M\}$) is the occurrence probability that the node i is selected to transmit data via the communication network and $\sum_{i=1}^{M} p_i = 1$.

Assume that $\xi(k)$ ($k \in \mathbb{N}^+$) is independent of all noise signals. Denote the measurement output after transmitted through the network by

$$\bar{y}_k \triangleq \left[\bar{y}_{1,k}^T \; \bar{y}_{2,k}^T \; \cdots \; \bar{y}_{N,k}^T\right]^T \in \mathbb{R}^{n_y}.$$

In this chapter, we adopt the ZI strategy to compensate the signals received by the filter. Accordingly, the updating rule of $\bar{y}_{i,k}$ ($k \in \mathbb{N}^+$, $i \in \{1,2,\cdots,N\}$) subject to the RA protocol scheduling is set to be

$$\bar{y}_{i,k} = \begin{cases} y_{i,k}, & \text{if } i = \xi(k),\ k \geq 0 \\ 0, & \text{otherwise.} \end{cases} \tag{6.4}$$

According to the updating rule (6.4), it can be seen that

$$\begin{cases} \bar{y}_k = \Phi_{\xi(k)} y_k, & \text{if } k \geq 0 \\ \bar{y}_k = 0, & \text{otherwise} \end{cases} \tag{6.5}$$

where $\Phi_{\xi(k)} = \text{diag}\{\delta(\xi(k)-1)I, \delta(\xi(k)-2)I, \cdots, \delta(\xi(k)-N)I\}$ ($1 \leq i \leq N$) and $\delta(\cdot) \in \{0,1\}$ is the Kronecker delta function.

6.1.2 The recursive filter

In this chapter, we shall adopt the Kalman-type filtering approach to design a recursive filter for the linear time-varying system (6.1) subject to the RA protocol scheduling described by (6.5). The recursive filter is given as follows:

$$\begin{cases} \hat{x}_{k+1|k} = A_k \hat{x}_{k|k} \\ \hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}\left(\bar{y}_{k+1} - \Phi_{\xi(k+1)} C_{k+1} \hat{x}_{k+1|k}\right) \end{cases} \tag{6.6}$$

where $\hat{x}_{k|k}$ is the estimate of x_k at time instant k with $\hat{x}_{0|0} = \bar{x}_0$, $\hat{x}_{k+1|k}$ is the one-step prediction at time instant k, and K_{k+1} is the filter gain to be determined.

The schematic diagram of the filtering system considered in this chapter is shown in Fig. 6.1. The objective of this chapter is to design a recursive filter of the structure (6.6) such that, for *all possible realizations* of the random sequence $\xi(k)$, the filtering error covariance (i.e. $\mathbb{E}\{(x_{k+1} - \hat{x}_{k+1|k+1})(x_{k+1} - \hat{x}_{k+1|k+1})^T\}$) can be derived recursively and subsequently minimized.

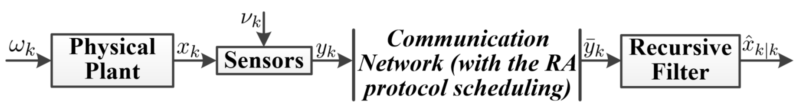

Fig. 6.1 Schematic structure for the plant and the filter over a network (with the RA protocol scheduling)

Remark 6.1. Due to the RA protocol scheduling behavior, the filtering performance is largely affected by the *stochastic parameter matrix* $\Phi_{\xi(k)}$. The main difficulty of this chapter would be the handling of such a stochastic parameter matrix in design of the time-varying filter parameter K_{k+1} and the analysis on the boundedness issue of the filtering error covariance. It can be observed that the filter gain K_{k+1} should be calculated recursively based on the filtering error covariance $P_{k|k}$, and therefore the boundedness of the filtering error covariance is very important to ensure the non-divergence of the filtering algorithm. Moreover, the boundedness of filtering error covariance often serves as an indispensable prerequisite to the boundedness guarantee of the filtering error in mean square. Note that, the problem studied in this chapter is distinct from the filtering issue with the sensor scheduling strategy. The protocol scheduling behavior is determined by the network agreements which are generated according to certain standards (e.g. IEEE 802 standards). Hence, the scheduling behavior could not be designed for the filtering task in this chapter. On the other hand, the sensor scheduling (as shown in [142, 172]) could be regarded as the "design variable". Hence, for the filtering problem with sensor scheduling, the filtering performance is related to the sensor selection strategy. In this chapter, we focus our attention on the recursive filtering problems with RA protocol scheduling, in which the protocol scheduling behavior is modeled by the i.i.d sequence of random variables.

6.2 Main Results

In this section, we aim to establish a unified framework to deal with the addressed recursive filtering problem under the RA protocol scheduling. Before proceeding further, we recall the following lemma which will be used in the subsequent developments.

Lemma 6.1. *(Jensens inequality) [119] Let vector $x_i \in \mathbb{R}^n$ $(i = 0, 1, \cdots, m)$ and scalar constant $a_i > 0$ $(i = 0, 1, \cdots, m)$ be given. The following inequality*

$$f\left(\frac{1}{M}\sum_{i=1}^{m} a_i x_i\right) \leq \frac{1}{M}\sum_{i=1}^{m} a_i f(x_i) \tag{6.7}$$

holds for convex function $f : \mathbb{R}^n \to \mathbb{R}^n$ where $M = \sum_{i=1}^{m} a_i$.

Lemma 6.2. *For matrices M, N, X and P with appropriate dimensions, the following equations hold:*

$$\frac{\partial \mathrm{tr}\{MXN\}}{\partial X} = M^T N^T, \quad \frac{\partial \mathrm{tr}\{MX^T N\}}{\partial X} = NM,$$
$$\frac{\partial \mathrm{tr}\{MXNX^T L\}}{\partial X} = M^T L^T X N^T + LMXN.$$

6.2.1 Design of the filter gain

Let us denote the one-step prediction error as $e_{k+1|k} = x_{k+1} - \hat{x}_{k+1|k}$ and the filtering error as $e_{k+1|k+1} = x_{k+1} - \hat{x}_{k+1|k+1}$. Subtracting (6.6) from (6.1), we have

$$e_{k+1|k} = A_k e_{k|k} + B_k \omega_k. \tag{6.8}$$

Similarly, the filtering error can be written as:

$$e_{k+1|k+1} = \big(I - K_{k+1}\Phi_{\xi(k+1)}C_{k+1}\big)e_{k+1|k} - K_{k+1}\Phi_{\xi(k+1)}\nu_{k+1}. \tag{6.9}$$

In light of (6.8) and (6.9), the covariance for the one-step prediction error and filtering error are calculated in the following theorem.

Theorem 6.1. *Consider the filtering error system given by (6.9). The one-step prediction error covariance $P_{k+1|k} \triangleq \mathbb{E}\{e_{k+1|k}e_{k+1|k}^T\}$ and the filtering error covariance $P_{k+1|k+1} \triangleq \mathbb{E}\{e_{k+1|k+1}e_{k+1|k+1}^T\}$ (with the initial condition $P_{0|0}$) are given by the following difference equations*

$$\begin{aligned} P_{k+1|k} =& A_k P_{k|k} A_k^T + B_k Q_k B_k^T, & (6.10)\\ P_{k+1|k+1} =& \sum_{i=1}^{N} p_i \mathcal{K}_{i,k+1} P_{k+1|k} \mathcal{K}_{i,k+1}^T + \sum_{i=1}^{N} p_i K_{k+1}\Phi_i R_{k+1}\Phi_i K_{k+1}^T. & (6.11)\end{aligned}$$

where $\mathcal{K}_{i,k+1} = I - K_{k+1}\Phi_i C_{k+1}$. Moreover, the trace of the filtering error covariance $P_{k+1|k+1}$ is minimized by the following filter gain:

$$K_{k+1} = P_{k+1|k} C_{k+1}^T \bar{\Phi} \left(\sum_{i=1}^{N} p_i \Phi_i \mathcal{R}_{k+1} \Phi_i \right)^{-1} \tag{6.12}$$

where $\mathcal{R}_{k+1} = C_{k+1}P_{k+1|k}C_{k+1}^T + R_{k+1}$ and $\bar{\Phi} = \mathrm{diag}\{p_1 I, p_2 I, \cdots, p_N I\}$.

Proof. First, it is easy to conclude from (6.8) that the one-step prediction error covariance $P_{k+1|k}$ satisfies

$$P_{k+1|k} = A_k P_{k|k} A_k^T + B_k Q_k B_k^T. \tag{6.13}$$

Next, let us consider the filtering error covariance $P_{k+1|k+1}$. Noting (6.9), we have

$$\begin{aligned} &P_{k+1|k+1} = \mathbb{E}\{e_{k+1|k+1}e_{k+1|k+1}^T\}\\ =&\mathbb{E}\big\{\mathcal{K}_{\xi(k+1),k+1}e_{k+1|k}e_{k+1|k}^T\mathcal{K}_{\xi(k+1),k+1}^T + K_{k+1}\Phi_{\xi(k+1)}\nu_{k+1}\nu_{k+1}^T\Phi_{\xi(k+1)}^T K_{k+1}^T\big\}. \end{aligned} \tag{6.14}$$

On the other hand, it can be seen from the definition of $\Phi_{\xi(k+1)}$ that

$$\Phi_{\xi(k+1)} = \sum_{i=1}^{N} \delta(\xi(k+1)-i)\Phi_i. \tag{6.15}$$

Since

$$\delta(\xi(k+1)-i)\delta(\xi(k+1)-j) = \begin{cases} 0, & i \neq j \\ \delta(\xi(k+1)-i), & i = j \end{cases}$$

and $\mathbb{E}\{\delta(\xi(k+1)-i)\} = \sum_{j=1}^{N} p_j \delta(j-i) = p_i$, we have

$$\begin{aligned}
&\mathbb{E}\big\{\mathcal{K}_{\xi(k+1),k+1} e_{k+1|k} e_{k+1|k}^T \mathcal{K}_{\xi(k+1),k+1}^T\big\} \\
=&\mathbb{E}\Bigg\{\big(I - K_{k+1}\sum_{i=1}^{N} \delta(\xi(k+1)-i)\Phi_i C_{k+1}\big) e_{k+1|k} e_{k+1|k}^T \big(I \\
&- K_{k+1}\sum_{i=1}^{N} \delta(\xi(k+1)-i)\Phi_i C_{k+1}\big)^T\Bigg\} \\
=&P_{k+1|k} - \mathbb{E}\Bigg\{\sum_{i=1}^{N} \delta(\xi(k+1)-i) K_{k+1}\Phi_i C_{k+1} e_{k+1|k} e_{k+1|k}^T\Bigg\} \\
&- \mathbb{E}\Bigg\{\sum_{i=1}^{N} \delta(\xi(k+1)-i) e_{k+1|k} e_{k+1|k}^T C_{k+1}^T \Phi_i K_{k+1}^T\Bigg\} \\
&+ \mathbb{E}\Bigg\{\sum_{i=1}^{N} \delta(\xi(k+1)-i) K_{k+1}\Phi_i C_{k+1} e_{k+1|k} e_{k+1|k}^T C_{k+1}^T \Phi_i K_{k+1}^T\Bigg\} \\
=&\sum_{i=1}^{N} p_i \big(I - K_{k+1}\Phi_i C_{k+1}\big) P_{k+1|k} \big(I - K_{k+1}\Phi_i C_{k+1}\big)^T.
\end{aligned} \tag{6.16}$$

Similarly, we have

$$\mathbb{E}\big\{K_{k+1}\Phi_{\xi(k+1)} v_{k+1} v_{k+1}^T \Phi_{\xi(k+1)}^T K_{k+1}^T\big\} = \mathbb{E}\Bigg\{\sum_{i=1}^{N} p_i K_{k+1}\Phi_i R_{k+1}\Phi_i K_{k+1}^T\Bigg\}. \tag{6.17}$$

Substituting the inequalities (6.16)-(6.17) into (6.14) yields

$$P_{k+1|k+1} = \sum_{i=1}^{N} p_i \mathcal{K}_{i,k+1} P_{k+1|k} \mathcal{K}_{i,k+1}^T + \sum_{i=1}^{N} p_i K_{k+1}\Phi_i R_{k+1}\Phi_i K_{k+1}^T.$$

Next, let us show that the filter gain given by (6.12) is optimal in the sense that it minimizes the trace of the filtering error covariance $P_{k+1|k+1}$. Taking the partial derivative of $\mathrm{tr}\{P_{k+1|k+1}\}$ with respect to K_{k+1} and letting the derivative be zero, we have

$$\frac{\partial \mathrm{tr}\{P_{k+1|k+1}\}}{\partial K_{k+1}} = -2\sum_{i=1}^{N} p_i\Big(\mathcal{K}_{i,k+1}P_{k+1|k}C_{k+1}^T\Phi_i\Big) + 2K_{k+1}\sum_{i=1}^{N} p_i\big(\Phi_i R_{k+1}\Phi_i\big) = 0. \tag{6.18}$$

Based on the above equation, the optimal filter gain K_{k+1} can be determined as

$$K_{k+1} = P_{k+1|k}C_{k+1}^T\bar{\Phi}\left(\sum_{i=1}^{N} p_i\Phi_i\mathcal{R}_{k+1}\Phi_i\right)^{-1}. \tag{6.19}$$

which is identical to (6.12). This completes the proof.

Remark 6.2. So far, we have completed the design issue of the recursive filter for time-varying systems with the RA protocol scheduling. By using Lemma 6.1, it is easy to see that $\sum_{i=1}^{N} p_i\Phi_i R_{k+1}\Phi_i \geq (\sum_{i=1}^{N} p_i\Phi_i)R_{k+1}(\sum_{i=1}^{N} p_i\Phi_i) = \bar{\Phi}R_{k+1}\bar{\Phi} > 0$. As such, it can be concluded that the matrix $\sum_{i=1}^{N} p_i\Phi_i\big(C_{k+1}P_{k+1|k}C_{k+1}^T + R_{k+1}\big)\Phi_i$ is nonsingular. The computation of K_{k+1} is carried out by solving two discrete-time Riccati-like difference equations, which are suitable for online implementation.

Remark 6.3. In our proposed filter design, the information of $\xi(k+1)$ is adopted in the structure of the recursive filter (as shown in (6.6)). The filter gain matrix K_{k+1} is calculated by minimizing the trace of the filtering error covariance $P_{k+1|k+1}$. According to the definition of $P_{k+1|k+1}$ (which is given in Theorem 6.1), the filter gain matrix is derived recursively in the sense of probability. Obviously, it is of both theoretical significance and practical importance for filter design to minimize $\mathrm{tr}\{P_{k+1|k+1}\}$. Another valuable strategy for the recursive filter design is to minimize the *conditional covariance matrix* $\mathbb{E}\{e_{k+1|k+1}e_{k+1|k+1}^T|\xi(k+1)\}$, in which the filter gain K_{k+1} could be computed based on the exact value of $\xi(k+1)$.

6.2.2 Boundedness analysis of the filtering error covariance

In this subsection, let us consider the boundedness of the filtering error covariance $P_{k+1|k+1}$. Throughout the rest of the chapter, the following assumption is made.

Assumption 6.1 *Let $\bar{\Phi} = \mathrm{diag}\{p_1 I, p_2 I, \cdots, p_N I\}$. There exist real numbers $\underline{q}$, $\bar{q}$, $\bar{r}$ and $\underline{r}$, such that the following matrix inequalities are satisfied for every $k \geq 0$ and $i \in \{1,2,\cdots.N\}$:*

$$\underline{q}I \leq B_k Q_k B_k^T \leq \bar{q}I,\ \ \Phi_i R_k \Phi_i \leq \bar{r}I,\ \ \bar{\Phi}R_k\bar{\Phi} \geq \underline{r}I,\ \ C_k^T C_k \leq \bar{c}I,$$
$$C_k C_k^T \leq \hat{c}I,\ \ \underline{a}I \leq A_k A_k^T \leq \bar{a}I.$$

Theorem 6.2. *Under Assumption 6.1, there exists a positive constant $\underline{\varepsilon}$ such that the error covariance $P_{k|k}$ of the recursive filtering for systems (6.1) satisfies*

$$P_{k|k} \geq \underline{\varepsilon}I \tag{6.20}$$

for every $k > 0$ with the lower bound $\underline{\varepsilon}$ given as follows:

$$\underline{\varepsilon} = \left(\underline{q}^{-1} + \underline{r}^{-1}\bar{p}^2\bar{c}\right)^{-1}. \tag{6.21}$$

where $\bar{p} = \max_{i=1,2,\cdots,N}\{p_i\}$.

Proof. Considering (6.11) and (6.12), we have

$$\begin{aligned} P_{k+1|k+1} =& P_{k+1|k} - P_{k+1|k}C_{k+1}^T\bar{\Phi}\left(\sum_{i=1}^{N} p_i\Phi_i\mathcal{R}_{k+1}\Phi_i\right)^{-1}\bar{\Phi}C_{k+1}P_{k+1|k} \\ =& P_{k+1|k} - P_{k+1|k}C_{k+1}^T\bar{\Phi}\left(\Xi_{k+1} + \bar{\Phi}C_{k+1}P_{k+1|k}C_{k+1}^T\bar{\Phi}\right)^{-1}\bar{\Phi}C_{k+1}P_{k+1|k} \\ =& \left(P_{k+1|k}^{-1} + C_{k+1}^T\bar{\Phi}\Xi_{k+1}^{-1}\bar{\Phi}C_{k+1}\right)^{-1} \end{aligned} \tag{6.22}$$

where $\Xi_{k+1} = \sum_{i=1}^{N} p_i\left(\tilde{\Phi}_iC_{k+1}P_{k+1|k}C_{k+1}^T\tilde{\Phi}_i + \Phi_iR_{k+1}\Phi_i\right)$ and $\tilde{\Phi}_i = \Phi_i - \bar{\Phi}$.

Noting that $\Xi_{k+1} > \sum_{i=1}^{N} p_i\Phi_iR_{k+1}\Phi_i \geq \bar{\Phi}R_{k+1}\bar{\Phi}$, it follows from (6.22) that

$$P_{k+1|k+1} \geq \left(P_{k+1|k}^{-1} + C_{k+1}^T\bar{\Phi}\left(\bar{\Phi}R_{k+1}\bar{\Phi}\right)^{-1}\bar{\Phi}C_{k+1}\right)^{-1}. \tag{6.23}$$

On the other hand, it is easy to see that

$$P_{k+1|k} \geq B_kQ_kB_k^T \geq \underline{q}I. \tag{6.24}$$

Subsequently, by considering (6.23) and (6.24), we have

$$\begin{aligned} P_{k+1|k+1}^{-1} \leq& \underline{q}^{-1}I + C_{k+1}^T\bar{\Phi}\left(\bar{\Phi}R_{k+1}\bar{\Phi}\right)^{-1}\bar{\Phi}C_{k+1} \\ \leq& \underline{q}^{-1}I + \underline{r}^{-1}C_{k+1}^T\bar{\Phi}\bar{\Phi}C_{k+1} \leq \left(\underline{q}^{-1} + \underline{r}^{-1}\bar{p}^2\bar{c}\right)I \end{aligned} \tag{6.25}$$

which implies that $P_{k|k} \geq \underline{\varepsilon}I$ for every $k > 0$. The proof is complete.

Theorem 6.2 provides a uniform lower bound of the filtering error covariance for the recursive filtering. Next, we shall study the upper bound of the filtering error covariance.

Theorem 6.3. *Under the Assumption 6.1, there exists an upper bound μ_k such that the filtering error covariance $P_{k|k}$ satisfies*

$$P_{k|k} \leq \mu_kI, \quad k \geq 0 \tag{6.26}$$

where $\mu_k = \mu_0\bar{a}^k + \bar{q}\sum_{i=0}^{k-1}\bar{a}^i$ and $\mu_0 = \lambda_{\max}\{P_{0|0}\}$.

Proof. The proof of this theorem is performed by *mathematical induction*, which is divided into two steps, namely, the initial step and the inductive step.

Initial step. For $k = 0$, it can be immediately known from the definition of μ_0 that

$$P_{0|0} \le \lambda_{\max}\{P_{0|0}\}I = \mu_0 I. \tag{6.27}$$

Inductive step. Now that the assertion of this theorem is true for $t = 0$. Next, given that the assertion is true for $t = k$ (i.e. $P_{t|t} \le \mu_t I$), we aim to show that the same assertion is true for $t = k+1$ (i.e. $P_{t+1|t+1} \le \mu_{t+1} I$). Obviously, it follows from (6.22) that

$$P^{-1}_{k+1|k+1} = P^{-1}_{k+1|k} + C^T_{k+1}\bar{\Phi}\Xi^{-1}_{k+1}\bar{\Phi}C_{k+1} \tag{6.28}$$

where $\Xi_{k+1} = \sum_{i=1}^{N} p_i\big(\tilde{\Phi}_i C_{k+1}P_{k+1|k}C^T_{k+1}\tilde{\Phi}_i + \Phi_i R_{k+1}\Phi_i\big)$ and $\tilde{\Phi}_i = \Phi_i - \bar{\Phi}$. Then, it is easy to see that

$$\begin{aligned} P_{k+1|k+1} &\le P_{k+1|k} = A_k P_{k|k} A_k^T + B_k Q_k B_k^T \\ &\le \mu_k A_k A_k^T + B_k Q_k B_k^T \le \bar{a}\mu_k + \bar{q}I \\ &= \mu_0 \bar{a}^{k+1} + \bar{q}\sum_{i=0}^{k}\bar{a}^i \triangleq \mu_{k+1}. \end{aligned} \tag{6.29}$$

Hence, by the induction, it can be concluded that the assertion of this theorem is true for $k \ge 0$. The proof is complete.

Remark 6.4. By now, we have derived an upper bound function of the error covariance matrix for our developed recursive filtering algorithm. Obviously, the upper bound $\bar{a}$ has an important impact on our derived upper bound function μ_k. It can be found that the derived upper bound function μ_k is divergent if $\bar{a} \ge 1$. When $\bar{a} < 1$, the derived upper bound function would converge to a fixed value.

The following proposition gives a uniform upper bound of μ_k when $\bar{a} < 1$.

Proposition 6.1. *Under Assumption 6.1, if $\bar{a} < 1$, then the upper bound of the error covariance matrix μ_k is exponentially bounded with the uniform upper bound $\mu_0\bar{a} + \frac{\bar{q}}{1-\bar{a}}$ for $k > 0$.*

Proof. The proof is straightforward and is therefore omitted for the conciseness.

In the case of $\bar{a} \ge 1$, the following theorem gives a sufficient condition to guarantee the boundedness of the error covariance matrix.

Theorem 6.4. *Let Assumption 6.1 hold. Suppose that there exist two positive scalars $\sigma > 0$ and $m > 0$ such that the following inequalities*

$$\begin{aligned} \sum_{i=k-m+1}^{k+1} &\Big(\underline{\gamma}_i \phi^T(i,k+1)C_i^T\bar{\Phi}\big(\sigma\bar{a}\hat{c}(1-\underline{p})^2 I + \Theta_i\big)^{-1}\bar{\Phi}C_i\phi(i,k+1)\Big) \\ &\ge \sigma^{-1}I, \qquad k \ge m-1 \qquad (6.30)\\ \mu_k &\le \sigma, \qquad 0 \le k < m-1 \qquad (6.31) \end{aligned}$$

hold for all $k \ge 0$, where

$$\underline{p} = \min_{1\le i\le N}\{p_i\},\ \tilde{\Phi}_j = \Phi_j - \bar{\Phi},\ \phi(k+1,k+1) = I,\ \phi(i,k+1) = A_i^{-1}A_{i+1}^{-1}\cdots A_k^{-1},$$

$$\mu_0 = \lambda_{\max}\{P_{0|0}\},\ \bar{\Phi} = \mathrm{diag}\{p_1 I, p_2 I, \cdots, p_N I\},\ \mu_k = \mu_0\bar{a}^k + \bar{q}\sum_{i=0}^{k-1}\bar{a}^i,$$

$$\Theta_i = \sum_{j=1}^{N} p_j\big(\tilde{\Phi}_j C_i B_{i-1} Q_{i-1} B_{i-1}^T C_i^T \tilde{\Phi}_j + \Phi_j R_i \Phi_j\big),\ \underline{\gamma}_i = (1+\bar{q}\underline{a}^{-1}\underline{\varepsilon}^{-1})^{i-k-1}.$$

Then, the error covariance satisfies $P_{k|k} \le \sigma I$ for $k \ge 0$.

Proof. The proof of this theorem is performed by mathematical induction as follows.

Initial step: Obviously, by employing Theorem 6.3 and (6.31), we have $P_{i|i} \le \mu_i I \le \sigma I$ for $0 \le i < m-1$.

Inductive step: We know that the assertion of this theorem (i.e. $P_{i|i} \le \sigma I$) is true for $i < m-1$. Next, given that the assertion is true for $i = k$, we aim to show that the same assertion is true for $i = k+1$. Since the assertion is true for $i = k$ (i.e. $P_{k|k} \le \sigma I$), it follows from (6.10) that

$$P_{k+1|k} = A_k P_{k|k} A_k^T + B_k Q_k B_k^T \le \sigma\bar{a}I + B_k Q_k B_k^T. \tag{6.32}$$

On the other hand, according to the results proposed in Theorem 6.2, we have

$$\begin{aligned} P_{k+1|k} &= A_k P_{k|k} A_k^T + B_k Q_k B_k^T = A_k\big(P_{k|k} + A_k^{-1} B_k Q_k B_k^T A_k^{-T}\big)A_k^T \\ &\le A_k\big(P_{k|k} + \bar{q}\underline{a}^{-1}\underline{\varepsilon}^{-1} P_{k|k}\big)A_k^T = (1+\bar{q}\underline{a}^{-1}\underline{\varepsilon}^{-1}) A_k P_{k|k} A_k^T. \end{aligned} \tag{6.33}$$

Furthermore, it can be derived from (6.22) that

$$P_{k+1|k+1}^{-1} = P_{k+1|k}^{-1} + C_{k+1}^T\bar{\Phi}\left(\sum_{i=1}^{N} p_i\big(\Phi_i R_{k+1}\Phi_i + \tilde{\Phi}_i C_{k+1} P_{k+1|k} C_{k+1}^T \tilde{\Phi}_i\big)\right)^{-1}\bar{\Phi}C_{k+1}. \tag{6.34}$$

Then, by substituting the inequalities (6.32) and (6.33) into (6.34), we have

$$\begin{aligned} P_{k+1|k+1}^{-1} \ge\ & P_{k+1|k}^{-1} + C_{k+1}^T\bar{\Phi}\big(\sigma\bar{a}\hat{c}(1-\underline{p})^2 I + \Theta_{k+1}\big)^{-1}\bar{\Phi}C_{k+1} \\ \ge\ & (1+\bar{q}\underline{a}^{-1}\underline{\varepsilon}^{-1})^{-1} A_k^{-T} P_{k|k}^{-1} A_k^{-1} + C_{k+1}^T\bar{\Phi}\big(\sigma\bar{a}\hat{c}(1-\underline{p})^2 I \\ & + \Theta_{k+1}\big)^{-1}\bar{\Phi}C_{k+1} \ge \cdots \\ \ge\ & \sum_{i=k-m+1}^{k+1}(1+\bar{q}\underline{a}^{-1}\underline{\varepsilon}^{-1})^{i-k-1}\phi^T(i,k+1)C_i^T\bar{\Phi}\big(\sigma\bar{a}\hat{c}(1-\underline{p})^2 I \\ & + \Theta_i\big)^{-1}\bar{\Phi}C_i\phi(i,k+1) + (1+\bar{q}\underline{a}^{-1}\underline{\varepsilon}^{-1})^{-m-1}\phi^T(k-m,k+1) \\ & \times P_{k-m|k-m}^{-1}\phi(k-m,k+1) \\ \ge\ & \sum_{i=k-m+1}^{k+1}\underline{\gamma}_i\phi^T(i,k+1)C_i^T\bar{\Phi}\big(\sigma\bar{a}\hat{c}(1-\underline{p})^2 I + \Theta_i\big)^{-1}\bar{\Phi}C_i\phi(i,k+1) \end{aligned}$$

$$\geq \sigma^{-1} I, \tag{6.35}$$

which results in

$$P_{k+1|k+1} \leq \sigma I. \tag{6.36}$$

Hence, by induction, it can be concluded that the assertion of this theorem is true for $k \geq 0$. The proof is complete.

Remark 6.5. The derived inequality (6.30) in Theorem 6.4 is of the form similar to the *uniform observability condition* [32]. Considering the time-varying matrices $\underline{A}_k$ and $\underline{C}_k$, the pair $[\underline{C}_k, \underline{A}_k]$ is said to be *uniformly observable* if there exist a positive integer $0 < m < \infty$ and constants $\bar{\varepsilon}$, $\underline{\varepsilon}$ with $0 < \underline{\varepsilon} \leq \bar{\varepsilon} < \infty$ such that the condition $\bar{\varepsilon} I \geq \sum_{i=k-m+1}^{k+1} \left(\overrightarrow{\psi}^T(i,k+1) \underline{C}_i^T \underline{C}_i \overrightarrow{\psi}(i,k+1) \right) \geq \underline{\varepsilon} I$ holds for all $k \geq m-1$, where $\overrightarrow{\psi}(k+1,k+1) = I$ and $\overrightarrow{\psi}(i,k+1) = \underline{A}_i^{-1} \underline{A}_{i+1}^{-1} \cdots \underline{A}_k^{-1}$. Obviously, when the sensor nodes of the plant could simultaneously get access to the network and transmit signals, the inequality (6.30) reduces to the well-known uniform observability condition based on Assumption 6.1. So far, we have investigated the recursive filtering problems for linear time-varying systems subject to the RA protocol scheduling. It can be observed from the filter design that all the important factors contributing to the system complexity are reflected in the main results. These factors include 1) the time-varying system parameters; 2) the noise information (characterized by Q_k and R_k); 3) the initial condition of the system (i.e. $P_{0|0}$); and 4) the information about the RA protocol scheduling (determined by (6.3)).

Remark 6.6. It is worth pointing out that, the recursive filtering technology developed in this chapter is not a global optimal filtering scheme. The main purpose of this chapter is to develop the recursive filtering scheme with a verifiable boundedness condition of the filtering error covariance matrix rather than design the optimal state estimation technology. If we adopt the well-known Kalman filtering to deal with the filtering problem of the linear time-varying system (6.1) subject to the RA protocol scheduling, the observability Gramian matrix would be related to the scheduling matrices in a sliding time-window with fixed finite length. Hence, it is quite difficult to verify the corresponding uniform observability condition. Nevertheless, the observability-like condition proposed in Theorem 6.4 is verifiable. In other words, for the problem considered in this chapter, our developed filtering scheme possesses the advantage that the non-divergent of the filtering algorithm is verifiable compared with the Kalman filtering.

Remark 6.7. The main results of this chapter can be regarded as the analysis and synthesis for the linear time-varying system with a random parameter $\xi(k)$. Such an issue has so far attracted considerable attention due to its clear engineering insight. Typical examples include the research results shown in [8, 75], which concern the design problems of linear-quadratic-Gaussian (LQG) controllers for linear systems with random parameters. Compared with the results in [8, 75], the effects on the system dynamics induced by the random parameter $\xi(k)$ in this chapter is more complex, which in turn increases the difficulty on the filter design. Moreover, we have analyzed the effects on the filtering performance induced by $\xi(k)$ (i.e. the boundedness analysis issue proposed in Section 6.2.2).

6.3 Illustrative Examples

In this section, two numerical examples are presented to demonstrate the effectiveness of the proposed recursive filtering algorithm for system (6.1) subject to the RA protocol scheduling.

Example 1: Consider a linear time-varying system (6.1) with the following system parameters:

$$A_k = \begin{bmatrix} 0.2\sin(0.05\pi k) & 0.5 & 0.5 \\ 0.6 & 0.8 & 0.7 \\ 0.1 & -0.1 & 0.5 \end{bmatrix}, \quad B_k = \begin{bmatrix} 0.1\cos(0.05\pi k) & 0.1 & 0.2 \\ 0.2 & 0.1 & -0.1 \\ 0.1 & -0.1 & 0.3 \end{bmatrix},$$

$$Q_k = 0.16I, \quad C_k = \begin{bmatrix} 0.5 & 0.7 & -0.6 \\ -0.2 & -0.3 & 0.4 \end{bmatrix}, \quad R_k = 0.09I, \quad P_{0|0} = 6.25I.$$

The sensors of this system are grouped into two sensor nodes and the occurrence probability about the RA protocol scheduling is taken to be:

$$\begin{cases} \text{Prob}\{\xi(k) = 1\} = 0.4 \\ \text{Prob}\{\xi(k) = 1\} = 0.6 \end{cases}$$

Then, by constructing the correspond filter and applying Theorem 6.1, the values of $\{K_k\}_{k>0}$ are derived as follows:

Table 6.1 The values of $\{K_k\}_{k>0}$

k	1	2	3	$\cdots$
K_k	$\begin{bmatrix} 0.582 & -1.476 \\ 1.059 & -2.645 \\ 0.216 & 0.545 \end{bmatrix}$	$\begin{bmatrix} 0.561 & -2.229 \\ 1.055 & -4.201 \\ 0.156 & -0.435 \end{bmatrix}$	$\begin{bmatrix} 0.534 & -1.522 \\ 1.025 & -2.967 \\ 0.068 & -0.119 \end{bmatrix}$	$\cdots$

Set the simulation run length to be 300. Based on the derived filter gain matrices, numerical simulation results are given in Figs. 6.2-6.5. Figs. 6.2-6.3 plot the state trajectories and their estimates under the RA protocol scheduling. Fig. 6.4 depicts the trajectory of the filtering error $e_{k|k}$ and Fig. 6.5 shows the selected sensor node obtaining access to the communication network under the RA protocol scheduling. All the simulation results confirm that the filtering performance is well achieved.

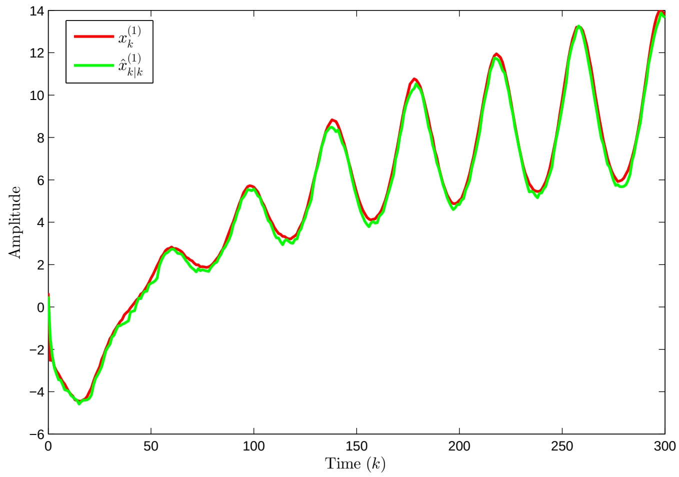

Fig. 6.2 The state trajectories of $x_k^{(1)}$ and $\hat{x}_{k|k}^{(1)}$ under the RA protocol scheduling

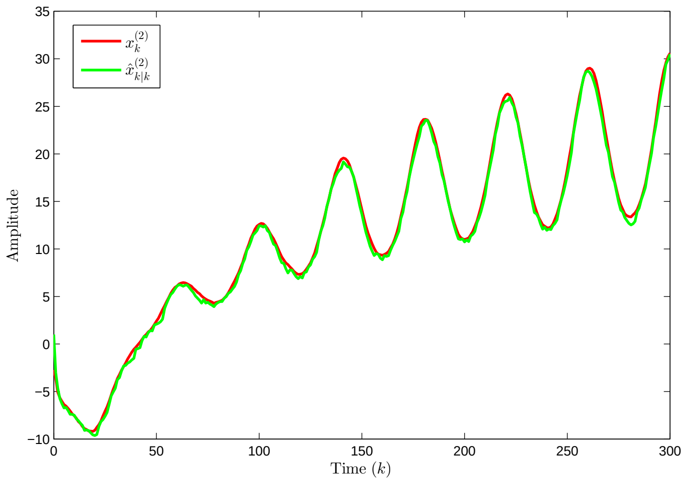

Fig. 6.3 The state trajectories of $x_k^{(2)}$ and $\hat{x}_{k|k}^{(2)}$ under the RA protocol scheduling

Example 2: In this example, we shall verify correctness of the uniform lower bound and uniform upper bound derived in this chapter a *second-order* system:

$$\begin{cases} x_{k+1} = \begin{bmatrix} 1.01 & 0.02 \\ -0.02 & 1.02 \end{bmatrix} x_k + \begin{bmatrix} 0.48 \\ 0.46 \end{bmatrix} \omega_k \\ y_k = \begin{bmatrix} 1.66 & 0 \\ 0 & 1.68 \end{bmatrix} x_k + v_k \end{cases}$$

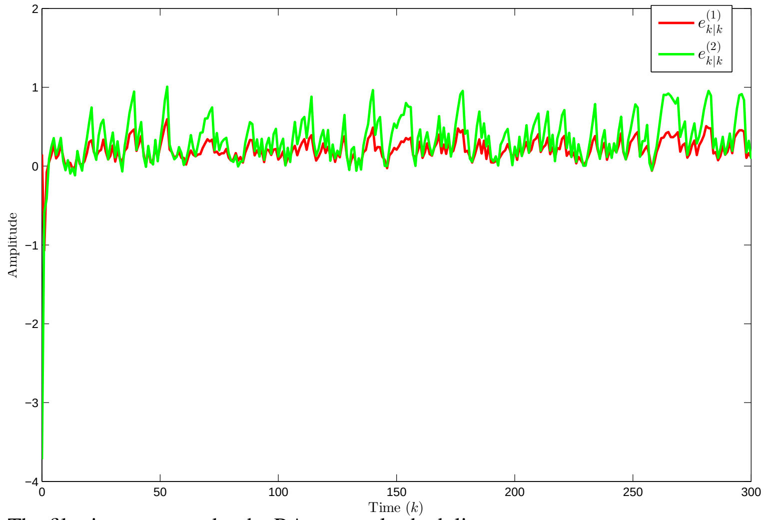

Fig. 6.4 The filtering errors under the RA protocol scheduling

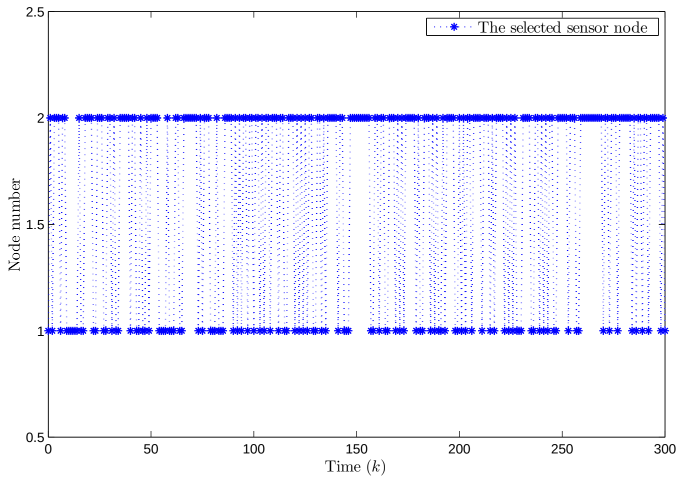

Fig. 6.5 The selected node obtaining access to the communication network under the RA protocol scheduling

with the covariance matrix $P_{0|0} = 10.24I$, $Q_k = 6.25I$ and $R_k = 4I$. The sensors are grouped into two nodes and the corresponding occurrence probability is selected as $p_1 = p_2 = 0.5$. It is easy to check that the this system is an *unstable* system. By employing Theorem 6.2, we obtain the uniform lower bound of the filtering error covariance (i.e. $P_{k|k} \geq 0.6841I$). On the other hand, considering the inequality (6.30) with $\sigma = 20$ and $m = 2$, we have

$$\sum_{i=k-m+1}^{k+1} \underline{\gamma}_i \phi^T(i,k+1) C_i^T \bar{\Phi}\big(\sigma \bar{a}\hat{c}(1-\underline{p})^2 I + \Theta_i\big)^{-1} \bar{\Phi} C_i \times \phi(i,k+1) = \mathrm{diag}\{0.0554, 0.0565\},$$

which implies that $\sum_{i=k-m+1}^{k+1} \underline{\gamma}_i \phi^T(i,k+1) C_i^T \bar{\Phi}\big(\sigma \bar{a}\hat{c}(1-\underline{p})^2 I + \Theta_i\big)^{-1} \bar{\Phi} C_i \phi(i,k+1) \geq \sigma^{-1} I$. Furthermore, it is easy to check that $\mu_0 = 10.24$ and $\mu_1 = 12.0978$, which means that the inequality (6.31) is satisfied. Then, it can be derived that the $P_{k|k} \leq 20I$ in virtue of Theorem 6.4. The results are shown in Figs. 6.6-6.7. The simulation results have confirmed our theoretical analysis on the uniform lower bound and the uniform upper bound of the filtering covariance $P_{k|k}$.

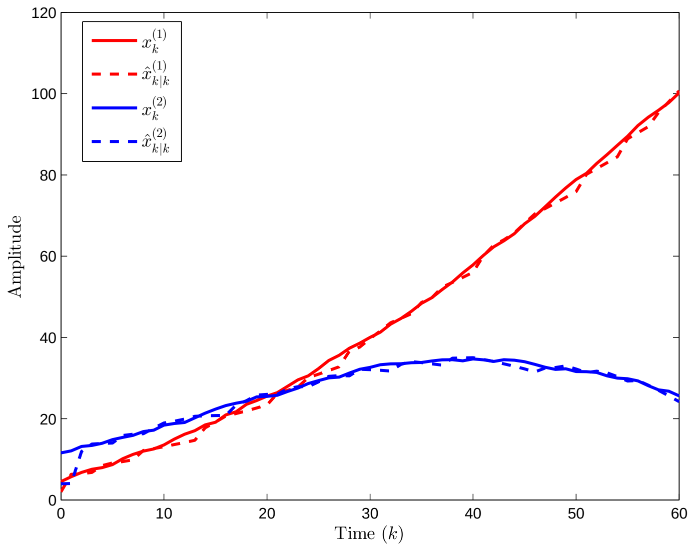

Fig. 6.6 The state trajectories of the plant and the filter under the RA protocol scheduling

6.4 Summary

In this chapter, the recursive filtering problem has been addressed for a class of linear time-varying systems subject to the scheduling of the so-called RA protocol. The scheduling behavior of the RA protocol has been modeled by the i.i.d sequence of random variables with known occurrence probabilities. The corresponding recursive filter has been presented to generate the state estimates. The filter gain has been calculated recursively to guarantee an minimized trace of the filtering error covariance. Furthermore, the boundedness problem of the filtering error covariance has

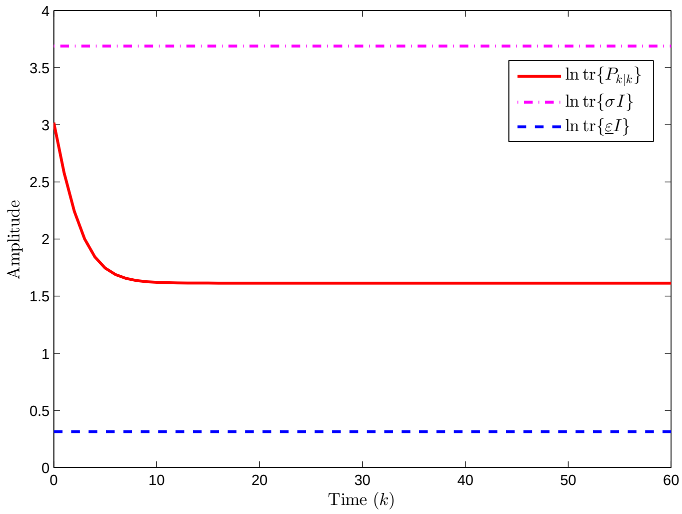

Fig. 6.7 The trace of $P_{k|k}$ and the their corresponding bounds

been studied. The uniform lower and upper bounds have been derived. Finally, two examples have been provided to illustrate the correctness and effectiveness of the filtering algorithm developed in this chapter.