A NOTE ON THE TWO DIMENSIONAL ELECTROSTATIC FIELD PRODUCED BY A LINE CHARGE NEAR A DIELECTRIC WEDGE.

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Summary

We shall consider the the problem of determining the correct electrostatic field produced when an infinite two dimensional line source is influenced by an adjacent infinite dielectric wedge. This result corrects a number of previous attempts at this problem which are shown to be in error. The method avoids using the Mellin transform which has lead to some of these earlier errors.The method is used to solve a more general problem of the electrostatic field produced by an arbitrary number of line sources located in an arbitrary number of contiguous dielectric wedges.

1. Introduction

In the design and manufacture of of electronic circuit boards, thin film circuit systems, and antennas with novel surface materials the accurate calculation of the fields and currents is important. In the plane of these thin surface electronic systems the fields can be considered to be two dimensional. As such it is necessary that accurate results are required for the field calculations of two dimensional problems. This is especially so for dielectric patch surfaces with corners to ensure the avoidance of spark and resultant material degradation. This will occur if the electrostatic field at the corners result in electrical breakdown. To this end the object of the present work is to point out and correct extant errors for the dielectric wedge problem.

The electric field produced by a line-charge lying between two plane conductors which intersect at an angle $\alpha = \pi/n$,(n an integer), may be solved by the method of images to give a finite sum of the image contributions. When the angle α is not of the form π/n the problem becomes more complicated and has been investigated by several authors. These authors replace the summand of the finite image sum by an integral representation, then interchange the order of summation and integration, sum the integrand and then use analytic continuation to extend the range from rational wedge angles to any wedge angle. If instead of considering the two intersecting planes to be conductors, we assume that two contiguous dielectric wedges occupy the regions defined by, see Figure 1,

$$D_1 : 0 < r < \infty, |z| < \infty, -\beta \leqslant \theta \leqslant \beta, |\beta| > 0,$$

and

$$D_2 : 0 < r < \infty, |z| < \infty, -\alpha \leqslant \phi \leqslant \alpha, |\alpha| > 0,$$

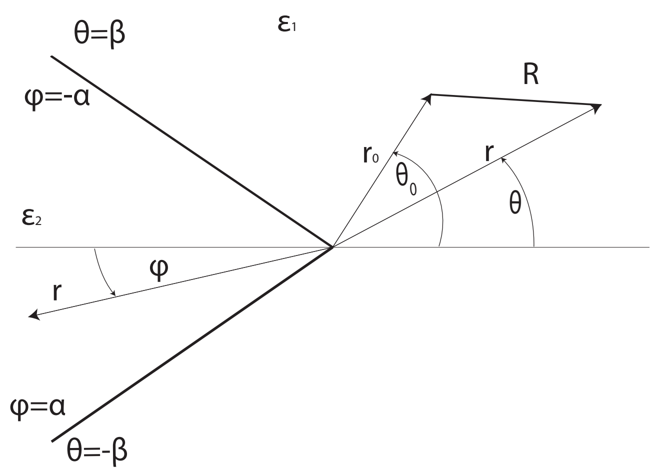

Fig. 1 Geometry of the Dielectric wedge and line source.

with dielectric constants ϵ_1, and ϵ_2 respectively, then the problem is much more complicated.

To the author's knowledge there are six published solutions to the two dimensional problem of the electrostatic field produced by an infinite line charge running parallel to the generators of an infinite dielectric wedge. They are chronologically Rice(**1**), Smythe(**2**), Grinberg(**3**), Lebedev(**4**)(Problem 412 p192), Lewis and McKenna(**5**), and Scharstein(**6**). The solution derived by Grinberg(**3**) is used by Lebedev(**4**)†, and therefore we will concentrate on Grinberg's solution.

Apart from the two authors Rice(**1**), and Smythe(**2**), they all use a Mellin transform approach, which would seem to be the natural method to apply. However the solutions or methods all seem to be in error. The errors are various, but basically come down to the fact that the source representations are inappropriate dimensionally, or if appropriate dimensionally, do not have a Mellin transform. We shall discuss each of the solutions and indicate the flaws in the analysis. The solution that was given by S. O. Rice(**1**) was derived incorrectly from a more generalized problem. In this work Rice derives rigourously, and quite elegantly, the correct solution for the three dimensional problem of a point charge

† Some of these results are obviously dimensionally wrong which can be corrected by replacing $E_r^{(2)}$ by $rE_r^{(2)}$,and $E_\varphi^{(2)}$ by $rE_\varphi^{(2)}$. However the point of interest here is that the first source term of $E_r^{(1)}$ should be replaced by $\frac{r+a\cos(\varphi)}{2R^2}$

near a dielectric wedge. At the end of this paper he then derives the result for an infinite line source by integrating the point source solution over an infinite line. In effect, in carrying out this complicated integration process, to derive his equation (19), he seems to ignore the fact that the integral

$$\lim_{A\to\infty}\int_{-A}^{A}\frac{dz}{\sqrt{z^2+a^2}},$$

is divergent. The end result is the equations (21a) and (21w); which are clearly wrong from a dimensional point of view. We could correct the solution by the use of the dubious argument, often used in electrostatics, that the addition of an arbitrary constant does not effect the result. However, for a given source location there is only one physical solution. Therefore, this lack of uniqueness is not very satisfactory from a mathematical[†] point of view. It is worth noting that from the footnote on page 39 of Rice(**1**) he seems to have had problems dealing directly with the two dimensional problem. The approach of Smythe(**2**) used the logarithmic cosine transform

$$f_c(r) = \int_0^\infty F(\nu)\cos[\nu\log(\frac{r}{a})]d\nu$$

to represent the electrostatic potential due to a line source located parallel to a dielectric wedge for which $a > 0, 0 < r < \infty$. He was unaware that a representation of the above form is not suitable when $0 < r < \infty$ because only the data known for $0 < r < a$ or $a < r < \infty$ is sufficient to uniquely determine $F(\nu)$(i.e. the inverse transform). His results are therefore divergent, see Idemen(**9**).

The solution of this two dimensional problem would seem to be a natural candidate for an application of the Mellin transform. However, care has to be taken in using this method to derive the solution for arbitrary positions of the source and the observation point. A formal application of the Mellin transform approach was used by Scharstein(**6**). Unfortunately there are serious lapses of mathematical rigour in this work that renders the results unreliable. For example, in applying the Mellin transform to equation (13) of (**6**), the author fails to realize that the Mellin transform of $\psi_1(r,\phi)$, that involves the logarithmic line source he considers, does not exist at the limits of integration. For convergence at the lower limit it is required that $\Re(s) > 0$,whereas at the upper limit $\Re(s) < 0$; and therefore the integrated parts are divergent. This renders equation (14) incorrect. This would not be the case for $\psi^o(r,\phi)$ when the transform exists for $-1 < \Re(s) < 0$. Scharstein also makes a number of other omissions and typographical errors which he tries to correct in the publication of a subsequent errata (**7**). However this errata is still not correct. The argument used in section E to derive the incorrect discontinuous constant terms (qua function of ϕ) $w_\pm(r)$ is opaque and suspect.

The errors committed by Grinberg are similar to those of Lewis and McKenna and are of a more subtle nature, related to the choice of the contour of integration of the inverse transform. We shall concentrate on the work of Lewis and McKenna(**5**), this being more readily available; similar lapses of reasoning occur in Grinberg's work. Lewis and McKenna make the problem dimensionless by choosing a line source located at unit distance from the

[†] See K O Friedrichs(**8**) for an interesting discussion of the uniqueness of the two-dimensional electrostatic problems.

edge of the wedge. The application of the Mellin transform is correctly carried out up to the point of choosing the contour of integration of the inverse transform in the expression (15). In (15) they choose $-1 < c < 0$; however this choice excludes the origin which contributes crucially towards the final solution. In effect, in the limit as $c \to 0$ the contour is indented wrongly. This gives rise to the incorrect[‡] solution given by (16) and (17). To obtain the correct solution the contour needs to be indented to the right(left) and the contour closed in the left(right) half plane for $r > 1(r < 1)$. Therefore no single inversion contour can be used for both the large and small r behaviour. It is also worth mentioning that in the limit as the root $-s_0 \to 0$ the contour integral (15) becomes an unbounded principal value integral[§]. A similar mistake is made in the related magnetostatic wedge problem considered by Baker(**10**) when he states how to invert his solution for large and small r in the first paragraph under equation (45).

The boundary value problem will be stated with Gauss's conservation of charge for a two dimensional electrostatic potential field. We shall then derive an appropriate source representation for a two dimensional line source for all locations of the source point and the observation position suitable for wedge shaped regions. This will then be used to derive the correct solution for a line source located near a dielectric wedge. This two wedge domain problem, see figure 1, is then generalized to an n-wedge domain problem, see figure 2.

2. Formulation of the Boundary Value Problem.

The basic equations of electrostatics that we are dealing with for three dimensional dielectric regions are

$$\begin{aligned} \mathbf{E} &= -\nabla u, \qquad \nabla^2 u = -4\pi\rho/\epsilon, \\ Q &= \int_V \rho dv = -\frac{1}{4\pi}\oint_S \epsilon \frac{\partial u}{\partial n} ds, \end{aligned} \tag{2.1}$$

where u is the potential of the electrostatic field $\mathbf{E}$, $\rho(M)$ is the volume density of charge at the point M, ϵ is the dielectric constant of the medium, Q is the total charge inside a volume of space enclosed by a surface S, and ∇ is the Gradient operator. The Figure 1 shows the cross-section of the system at $z = 0$. From the geometry and field equations of the problem all physical quantities will be the same for any other value of z. Thus with $\frac{\partial}{\partial z} = 0$ in equation (2.1) we need only consider the two dimensional Gradient problem, and replace the integral relationship by its two dimensional form

$$Q = \int_A \rho dv = -\frac{1}{2\pi}\oint_C \epsilon \frac{\partial u}{\partial n} ds, \tag{2.2}$$

where Q is the total superficial charge inside an area A enclosed by a contour C, in the plane $z = 0$. The expression (2.2) will be used as a check on the correctness of our solution. For the convenience of comparison purposes we shall adopt the co-ordinate system used

[‡] It can be seen that expression (16) does not reproduce the dominant behaviour of $w_1 \sim r$ as $r \to 0$, and the expression (17) does not give the dominant behaviour $w_1 \sim \log(r)$ as $r \to \infty$.

[§] To give a simple example of this phenomena consider the integral $I(a) = \int_{\imath\infty}^{\imath\infty} \frac{dz}{z(z+a)}, a \neq 0, \imath = \sqrt{-1}$, and the contour indented to the left of the imaginary axis. In the limit as $a \to +0, I(+0) = 0$,whereas in the limit $a \to -0, I(-0) = P\int_{\imath\infty}^{\imath\infty} \frac{dz}{z^2} \to \infty$.

by Rice(**1**) which ensures the solution is single valued in the angular variables. In the two domains S_1 and S_2 two sets of of cylindrical co-ordinates are used, namely, (r, θ, z) and (r, ϕ, z). In accordance with the convention of measuring the angles shown in the figure it follows that $-\beta < \theta < \beta$, $-\alpha < \phi < \alpha$, and $\alpha + \beta = \pi$. Hence working with the potentials $u_1(r, \theta), u_2(r, \phi)$ we have the following boundary value problem to solve:

$$\nabla^2 u_1(r, \theta) = -4\pi q \delta(r - r_0)\delta(\theta - \theta_0)/(r\epsilon_1), \quad (-\beta \leqslant \theta \leqslant \beta), \tag{2.3}$$

$$\nabla^2 u_2(r, \phi) = 0, \quad (-\alpha \leqslant \phi \leqslant \alpha), \tag{2.4}$$

where $u_1(u_2)$ is the potential respectively in region $D_1(D_2)$. The field at infinity where $r \to \infty$ must tend asymptotically to:

$$u_1(r, \theta) \sim -\kappa \log(r), \quad (-\beta \leqslant \theta \leqslant \beta), \tag{2.5}$$

$$u_2(r, \phi) \sim -\kappa \log(r), \quad (-\alpha \leqslant \phi \leqslant \alpha), \tag{2.6}$$

where the constant κ is dependent on the material and charge distribution.

$$[u_1(r, \theta)]_{\theta=\beta} = [u_2(r, \phi)]_{\phi=-\alpha} \quad , \quad \left[\epsilon_1 \frac{\partial u_1(r, \theta)}{\partial \theta}\right]_{\theta=\beta} = \left[\epsilon_2 \frac{\partial u_2(r, \phi)}{\partial \phi}\right]_{\phi=-\alpha}. \tag{2.7}$$

$$[u_1(r, \theta)]_{\theta=-\beta} = [u_2(r, \phi)]_{\phi=\alpha} \quad , \quad \left[\epsilon_1 \frac{\partial u_1(r, \theta)}{\partial \theta}\right]_{\theta=-\beta} = \left[\epsilon_2 \frac{\partial u_2(r, \phi)}{\partial \phi}\right]_{\phi=\alpha}. \tag{2.8}$$

We shall assume without loss of generality that $u_1 = u_0 + u$ where u_0 is a line source of charge q per unit length, located at $r = r_0, \theta = \theta_0$ in D_1 and is given by

$$u_0(r, \theta; r_0, \theta_0) = \frac{2q}{\epsilon_1} \log\left[\frac{\sqrt{r r_0}}{R}\right],$$

where $R = \sqrt{r^2 + r_0^2 - 2 r r_0 \cos(\theta - \theta_0)}$; and that

$$\nabla^2 u_0(r, \theta) = -4\pi q \delta(r - r_0)\delta(\theta - \theta_0)/(r\epsilon_1),$$

and,

$$\nabla^2 u(r, \theta) = 0.$$

3. Representation of the solution

We shall use the integral representation for the line source u_0 valid for all real $r > 0, r_0 > 0, -\beta < \theta < \beta, -\beta < \theta_0 < \beta, 0 < \beta < \pi$,

$$u_0(r, \theta; r_0, \theta_0) = \frac{\imath q}{\epsilon_1} P \int_{-\imath\infty}^{\imath\infty} \frac{\cos \nu(\pi - |\theta - \theta_0|)}{\nu \sin \nu\pi} \left(\frac{r}{r_0}\right)^{\nu} d\nu, \tag{3.1}$$

where $0 < |\theta - \theta_0| < 2\pi$, and P stands for the principal value of the integral. We shall assume a solution of the form

$$u_1(r,\theta;r_0,\theta_0) = \frac{\imath q}{\epsilon_1} P\int_{-\imath\infty}^{\imath\infty} \frac{\cos\nu(\pi-|\theta-\theta_0|)}{\nu\sin\nu\pi}\left(\frac{r}{r_0}\right)^{\nu} d\nu$$
$$+ \quad \frac{\imath q}{\epsilon_1} P\int_{-\imath\infty}^{\imath\infty} (a(\nu)\cos\nu\theta + b(\nu)\sin\nu\theta)\left(\frac{r}{r_0}\right)^{\nu} d\nu, \tag{3.2}$$

$$u_2(r,\phi;r_0,\theta_0) = \frac{\imath q}{\epsilon_1} P\int_{-\imath\infty}^{\imath\infty} (c(\nu)\cos\nu\phi + d(\nu)\sin\nu\phi)\left(\frac{r}{r_0}\right)^{\nu} d\nu. \tag{3.3}$$

Clearly $\nabla^2 u_2 = 0$, $\nabla^2 u_1 = -4\pi q\delta(r-r_0)\delta(\theta-\theta_0)/(\epsilon_1 r)$, and $u_1 \sim -2q\log(R)/(\epsilon_1)$ as $R \to 0$. Substituting the expressions (3.2), (3.3) into the first boundary condition of (2.7) gives,

$$P\int_{-\imath\infty}^{\imath\infty}\left(\frac{\cos\nu(\pi-|\beta-\theta_0|)}{\nu\sin\nu\pi} + a(\nu)\cos\nu\beta + b(\nu)\sin\nu\beta\right)\left(\frac{r}{r_0}\right)^{\nu} d\nu$$
$$= P\int_{-\imath\infty}^{\imath\infty} (c(\nu)\cos\nu\alpha - d(\nu)\sin\nu\alpha)\left(\frac{r}{r_0}\right)^{\nu} d\nu. \tag{3.4}$$

The satisfaction of the first condition of (2.8) is obtained from (3.4) by replacing replacing α by $-\alpha$ and β by $-\beta$ giving

$$P\int_{-\imath\infty}^{\imath\infty}\left(\frac{\cos\nu(\pi-|-\beta-\theta_0|)}{\nu\sin\nu\pi} + a(\nu)\cos\nu\beta - b(\nu)\sin\nu\beta\right)\left(\frac{r}{r_0}\right)^{\nu} d\nu$$
$$= P\int_{-\imath\infty}^{\imath\infty} (c(\nu)\cos\nu\alpha + d(\nu)\sin\nu\alpha)\left(\frac{r}{r_0}\right)^{\nu} d\nu. \tag{3.5}$$

The satisfaction of the equations (3.4) and (3.5) gives

$$a(\nu)\cos\nu\beta + b(\nu)\sin\nu\beta - c(\nu)\cos\nu\alpha + d(\nu)\sin\nu\alpha$$
$$= -\frac{\cos(\pi-(\beta-\theta_0))}{\nu\sin\nu\pi}, \tag{3.6}$$

$$a(\nu)\cos\nu\beta - b(\nu)\sin\nu\beta - c(\nu)\cos\nu\alpha - d(\nu)\sin\nu\alpha$$
$$= -\frac{\cos(\pi-(\beta+\theta_0))}{\nu\sin\nu\pi}. \tag{3.7}$$

Substituting expressions (3.1), (3.2) into the second boundary condition of (2.7) gives,

$$\epsilon_1 P \int_{-\imath\infty}^{\imath\infty} \left(\frac{\sin\nu(\pi - |\beta - \theta_0|)}{\sin\nu\pi} - a(\nu)\nu\sin\nu\beta + b(\nu)\nu\cos\nu\beta \right) \left(\frac{r}{r_0}\right)^\nu d\nu$$
$$= \epsilon_2 P \int_{-\imath\infty}^{\imath\infty} (c(\nu)\sin\nu\alpha + d(\nu)\cos\nu\alpha) \left(\frac{r}{r_0}\right)^\nu d\nu. \tag{3.8}$$

The satisfaction of the second condition of (2.8) is obtained from (3.8) by replacing replacing α by $-\alpha$ and β by $-\beta$ giving

$$\epsilon_1 P \int_{-\imath\infty}^{\imath\infty} \left(\frac{\sin\nu(\pi - |-\beta - \theta_0|)}{\sin\nu\pi} + a(\nu)\nu\sin\nu\beta + b(\nu)\nu\cos\nu\beta \right) \left(\frac{r}{r_0}\right)^\nu d\nu$$
$$= \epsilon_2 P \int_{-\imath\infty}^{\imath\infty} (-c(\nu)\sin\nu\alpha + d(\nu)\cos\nu\alpha) \left(\frac{r}{r_0}\right)^\nu d\nu. \tag{3.9}$$

The satisfaction of the equations (3.8) and (3.9) gives

$$\begin{aligned} -\epsilon_1 a(\nu)\nu\sin\nu\beta + \quad & \epsilon_1 b(\nu)\nu\cos\nu\beta - \epsilon_2 c(\nu)\nu\sin\nu\alpha - \epsilon_2 d(\nu)\nu\cos\nu\alpha \\ & = -\epsilon_1 \frac{\sin(\pi - (\beta - \theta_0))}{\sin\nu\pi}, \end{aligned} \tag{3.10}$$

$$\begin{aligned} \epsilon_1 a(\nu)\nu\sin\nu\beta + \quad & \epsilon_1 b(\nu)\nu\cos\nu\beta + \epsilon_2 c(\nu)\nu\sin\nu\alpha - \epsilon_2 d(\nu)\nu\cos\nu\alpha \\ & = \epsilon_1 \frac{\sin(\pi - (-\beta - \theta_0))}{\sin\nu\pi}. \end{aligned} \tag{3.11}$$

Adding (3.6) and (3.7), and subtracting (3.10) from (3.11) gives

$$a(\nu)\cos\nu\beta - c(\nu)\cos\nu\alpha = -\frac{\cos\nu\alpha\cos\nu\theta_0}{\nu\sin\nu\pi}, \tag{3.12}$$

$$\epsilon_1 a(\nu)\sin\nu\beta + \epsilon_2 c(\nu)\sin\nu\alpha = \epsilon_1 \frac{\sin\nu\alpha\cos\nu\theta_0}{\nu\sin\nu\pi}. \tag{3.13}$$

Subtracting (3.7) from (3.6),and adding (3.10) and (3.11)gives

$$b(\nu)\sin\nu\beta + d(\nu)\sin\nu\alpha = \frac{\sin\nu\alpha\sin\nu\theta_0}{\nu\sin\nu\pi}, \tag{3.14}$$

$$\epsilon_1 b(\nu)\cos\nu\beta - \epsilon_2 d(\nu)\cos\nu\alpha = -\epsilon_1 \frac{\cos\nu\alpha\sin\nu\theta_0}{\nu\sin\nu\pi}. \tag{3.15}$$

Solving equations (3.12) and (3.13) for $a(\nu)$ and $c(\nu)$ (3.14) and (3.15) for $b(\nu)$ and $d(\nu)$ gives

$$a(\nu) = \frac{(\epsilon_1 - \epsilon_2)\cos\nu\theta_0 \sin 2\nu\alpha}{2\nu\sin\nu\pi\Delta_{\alpha\beta}(\nu)}, \tag{3.16}$$

$$c(\nu) = \epsilon_1 \frac{\cos\nu\theta_0}{\nu\Delta_{\alpha\beta}(\nu)}; \tag{3.17}$$

$$b(\nu) = \frac{(\epsilon_2 - \epsilon_1)\sin\nu\theta_0 \sin 2\nu\alpha}{2\nu\sin\nu\pi\Delta_{\beta\alpha}(\nu)}, \tag{3.18}$$

$$d(\nu) = \epsilon_1 \frac{\sin\nu\theta_0}{\nu\Delta_{\beta\alpha}(\nu)}; \tag{3.19}$$

where

$$\begin{aligned} \Delta_{\alpha\beta}(\nu) &= \epsilon_2 \sin\nu\alpha\cos\nu\beta + \epsilon_1\cos\nu\alpha\sin\nu\beta, \\ &= \frac{(\epsilon_1+\epsilon_2)}{2}\sin\nu\pi + \frac{(\epsilon_1-\epsilon_2)}{2}\sin\nu(\beta-\alpha). \end{aligned} \tag{3.20}$$

Then if we let

$$F(\nu,\theta) = a(\nu)\cos\nu\theta + b(\nu)\sin\nu\theta,$$

$$F(\nu,\theta) = \frac{(\epsilon_2-\epsilon_1)\sin 2\nu\alpha}{2\sin\nu\pi}\left[-\frac{\cos\nu\theta\cos\nu\theta_0}{\nu\Delta_{\alpha\beta}(\nu)} + \frac{\sin\nu\theta\sin\nu\theta_0}{\nu\Delta_{\beta\alpha}(\nu)}\right]; \tag{3.21}$$

$$G(\nu,\phi) = c(\nu)\cos\nu\phi + d(\nu)\sin\nu\phi,$$

$$G(\nu,\phi) = \epsilon_1\frac{\cos\nu\phi\cos\nu\theta_0}{\nu\Delta_{\alpha\beta}(\nu)} + \epsilon_1\frac{\sin\nu\phi\sin\nu\theta_0}{\nu\Delta_{\beta\alpha}(\nu)}. \tag{3.22}$$

Hence substituting these results into (3.2) and (3.3) we have

$$u_1(r,\theta;r_0,\theta_0) = \frac{2q}{\epsilon_1}\log\left[\frac{\sqrt{rr_0}}{R}\right] + \frac{\imath q}{\epsilon_1}P\int_{-\imath\infty}^{\imath\infty} F(\nu,\theta)\left(\frac{r}{r_0}\right)^{\nu} d\nu, \tag{3.23}$$

$$u_2(r,\phi;r_0,\theta_0) = \frac{\imath q}{\epsilon_1}P\int_{-\imath\infty}^{\imath\infty} G(\nu,\phi)\left(\frac{r}{r_0}\right)^{\nu} d\nu. \tag{3.24}$$

These results are valid for all $r > 0, r_0 > 0$. We notice that the first term of $F(\nu,\theta)$ and of $G(\nu,\phi)$ has a double pole at the origin. We can replace the principal value integral by an indented closed contour, by adding and subtracting half the residue at the origin. How we indent at the origin and then displace the contour depends on whether $r > r_0$ or $r < r_0$. For $r_0 > r (r_0 < r)$ we can indent, and shift the contour to the right(left); convergence being assured. In carrying out this procedure and using the fact that $F(\nu,\theta)$ and $G(\nu,\phi)$ have no singularities for $0 < |\Re\nu| < \frac{1}{2}$, we get

$$\begin{aligned} u_1(r,\theta;r_0,\theta_0) &= \frac{2q}{\epsilon_1}\log\left[\frac{\sqrt{rr_0}}{R}\right] \mp \frac{q\alpha(\epsilon_2-\epsilon_1)}{\epsilon_1(\epsilon_2\alpha+\epsilon_1\beta)}\log\left(\frac{r}{r_0}\right) \\ &+ \frac{\imath q}{\epsilon_1}\int_{-\imath\infty\pm\frac{1}{2}}^{\imath\infty\pm\frac{1}{2}} F(\nu,\theta)\left(\frac{r}{r_0}\right)^{\nu} d\nu, \end{aligned} \tag{3.25}$$

$$\begin{aligned} u_2(r,\phi;r_0,\theta_0) &= \pm\frac{q\pi}{(\epsilon_2\alpha+\epsilon_1\beta)}\log\left(\frac{r}{r_0}\right) \\ &+ \frac{\imath q}{\epsilon_1}\int_{-\imath\infty\pm\frac{1}{2}}^{\imath\infty\pm\frac{1}{2}} G(\nu,\phi)\left(\frac{r}{r_0}\right)^{\nu} d\nu; \end{aligned} \tag{3.26}$$

where the upper and lower signs correspond to $r_0 \gtrless r$. Infinite series representation can be obtained from the above expressions by summing the residues to the right(left) of $\Re\nu = \frac{1}{2}(\Re\nu = -\frac{1}{2})$. As a check on the results we shall use the result (2.2). Letting $r \to \infty$, with r_0 finite it is not difficult to show that uniformly in the angular variables

$$u_1(r,\theta;r_0,\theta_0) = -\frac{q\pi}{(\epsilon_2\alpha+\epsilon_1\beta)}\log r + o(1); \tag{3.27}$$

$$u_2(r,\phi;r_0,\theta_0) = -\frac{q\pi}{(\epsilon_2\alpha+\epsilon_1\beta)}\log r + o(1). \tag{3.28}$$

Now let the contour C in (2.2) be a large circle of radius r centred at the origin. Then substituting (3.27) and (3.28) into (2.2) gives as $r \to \infty$

$$\begin{aligned} Q &= \frac{1}{2\pi}\int_{-\beta}^{\beta} \frac{q\pi}{(\epsilon_2\alpha+\epsilon_1\beta)}\epsilon_1\frac{\partial \log r}{\partial r} r d\theta \\ &+ \frac{1}{2\pi}\int_{-\alpha}^{\alpha} \frac{q\pi}{(\epsilon_2\alpha+\epsilon_1\beta)}\epsilon_2\frac{\partial \log r}{\partial r} r d\theta \\ &= q. \end{aligned}$$

We remark that the results (3.27) and (3.28) make explicit the constant κ occurring in the far field behavior (2.5) and (2.6).

As a further check we shall derive the near field behavior at the wedge tip as $r \to 0$. In this limit as we shift the contour of integration to the right towards $\Re\nu = 2$ the poles that give rise to residue contributions arise from the roots of

$$\Delta_{\alpha\beta}(\nu) = 0, \quad \Rightarrow \quad \sin\nu\pi = (\frac{\epsilon_2-\epsilon_1}{\epsilon_1+\epsilon_2})\sin\nu(\pi-2\alpha), \tag{3.29}$$

$$\Delta_{\beta\alpha}(\nu) = 0, \quad \Rightarrow \quad \sin\nu\pi = (\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2})\sin\nu(\pi-2\alpha). \tag{3.30}$$

Both of these equations are of the form

$$\sin x = R \sin ax, \tag{3.31}$$

where $x = \nu\pi$, and $a = 1 - \frac{2\alpha}{\pi}$, with $R = \pm\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}$. The transcendental equation (3.31) has been studied in detail by Meixner(**12**). It is shown that for $-1 \le R \le 1$ and $-1 \le a \le 1$ the equation (3.31) has only real simple roots which lie symmetrically on either side of the imaginary ν axis; also that no root lies in the region $0 < \Re\nu < 1/2$, and at least one root lies in the region $1/2 < \Re\nu < 2$. Thus in the range $1/2 < \Re\nu < 2$ the first roots of the equations (3.29) and (3.30) exist. A graphical analysis of the equation (3.31) shows that the smallest root x_1 occurs for R and a of the same sign in the interval $\pi/2 < x_1 < \pi$; whereas when R and a are of the opposite sign the smallest root occurs in the interval $\pi < x_1 < 3\pi/2$. Hence for $\beta - \alpha$ and $\epsilon_2 - \epsilon_1$ of the same(opposite) sign then $\nu_1 < \pi/\alpha(\nu_1 > \pi/\alpha)$, where ν_1 is the smallest root of the equations (3.29) and (3.30). We also note that if $\alpha = \beta$ or $\epsilon_1 = \epsilon_2$ the smallest root for both equations becomes $\nu_1 = \pi/\alpha$. Without loss of generality, we shall assume $\beta > \alpha$(or $\pi > 2\alpha$); in which case the near field as $r \to 0$ becomes for $\epsilon_2 > \epsilon_1$ to

$$\begin{aligned} u_1(r,\theta) \quad &\sim \quad -\frac{2\pi q(\epsilon_2 - \epsilon_1)\sin 2\nu_1\alpha \cos\nu_1\theta \cos\nu_1\theta_0}{\epsilon_1\nu_1 \sin\nu_1\pi(\pi(\epsilon_1+\epsilon_2)\cos\nu_1\pi + (\pi - 2\alpha)(\epsilon_1 - \epsilon_2)\cos\nu_1(\pi - 2\alpha))}\left(\frac{r}{r_0}\right)^{\nu_1} \\ &+ \quad \frac{q\pi}{(\epsilon_2\alpha + \epsilon_1\beta)}\log\left(\frac{r}{r_0}\right), \end{aligned} \tag{3.32}$$

$$\begin{aligned} u_2(r,\phi) \quad &\sim \quad \frac{4\pi q \cos\nu_1\phi \cos\nu_1\theta_0}{\nu_1(\pi(\epsilon_1+\epsilon_2)\cos\nu_1\pi + (\pi - 2\alpha)(\epsilon_1 - \epsilon_2)\cos\nu_1(\pi - 2\alpha))}\left(\frac{r}{r_0}\right)^{\nu_1} \\ &+ \quad \frac{q\pi}{(\epsilon_2\alpha + \epsilon_1\beta)}\log\left(\frac{r}{r_0}\right); \end{aligned} \tag{3.33}$$

whereas for $\epsilon_1 > \epsilon_2$ then

$$\begin{aligned} u_1(r,\theta) \quad &\sim \quad \frac{2\pi q(\epsilon_2 - \epsilon_1)\sin 2\nu_1\alpha \sin\nu_1\theta \sin\nu_1\theta_0}{\epsilon_1\nu_1 \sin\nu_1\pi(\pi(\epsilon_1+\epsilon_2)\cos\nu_1\pi - (\pi - 2\alpha)(\epsilon_1 - \epsilon_2)\cos\nu_1(\pi - 2\alpha))}\left(\frac{r}{r_0}\right)^{\nu_1} \\ &+ \quad \frac{q\pi}{(\epsilon_2\alpha + \epsilon_1\beta)}\log\left(\frac{r}{r_0}\right), \end{aligned} \tag{3.34}$$

$$\begin{aligned} u_2(r,\phi) \quad &\sim \quad \frac{4\pi q \sin\nu_1\phi \sin\nu_1\theta_0}{\nu_1(\pi(\epsilon_1+\epsilon_2)\cos\nu_1\pi - (\pi - 2\alpha)(\epsilon_1 - \epsilon_2)\cos\nu_1(\pi - 2\alpha))}\left(\frac{r}{r_0}\right)^{\nu_1} \\ &+ \quad \frac{q\pi}{(\epsilon_2\alpha + \epsilon_1\beta)}\log\left(\frac{r}{r_0}\right). \end{aligned} \tag{3.35}$$

It is not difficult to show that these expressions satisfy the boundary conditions on the wedge faces. They also exhibit the correct edge field behavior at the apex of the wedge.

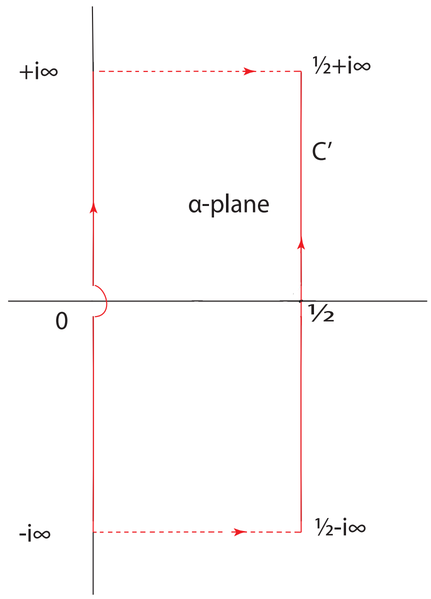

Fig. 2 The closed and indented contour of integration for $r > r_0$.

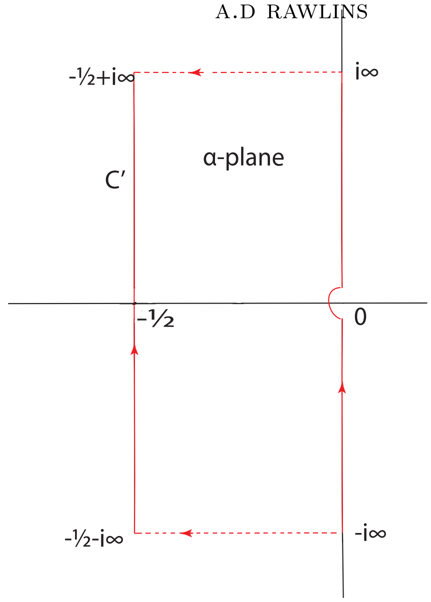

Fig. 3 The closed and indented contour of integration for $r_0 > r$.

4. Generalised problem and solution

We shall now consider a more general problem and indicate how to determine the solution using the results in the body of this paper. We suppose that a number of line sources q, where q is an integer $1 \leq q \leq n$, are located in an arbitrary number n of dielectric sectors, where we assume that n contiguous dielectric wedges occupy the regions defined by, see Figure 3,

$$D_k : 0 < r < \infty, |z| < \infty, \beta_{k-1} \leqslant \theta \leqslant \beta_k,$$

$k = 1, 2, ..., n-1, n; \beta_0 = 0, \beta_n = 2\pi$, with dielectric constants ϵ_k.

We shall assume a solution of the form

$$u_i(r,\theta) = \delta_{i,q} u_q(r,\theta) + \imath P \int_{-\imath\infty}^{\imath\infty} (a_i(\nu)\cos\nu\theta + b_i(\nu)\sin\nu\theta) r^{\nu} d\nu, \qquad i = 1, 2, ..., n, \quad (4.1)$$

where $\delta_{i,q}$ is the Kronecker delta function and

$$u_q(r,\theta) = \frac{2q_q}{\epsilon_q} \log\left(\frac{rr_q}{R_q}\right), \qquad q \in (1, n),$$

with q_q is the charge on the $q-th$ line source; and where $R_q = \sqrt{r^2 + r_q^2 - 2rr_q\cos(\theta - \theta_q)}$.

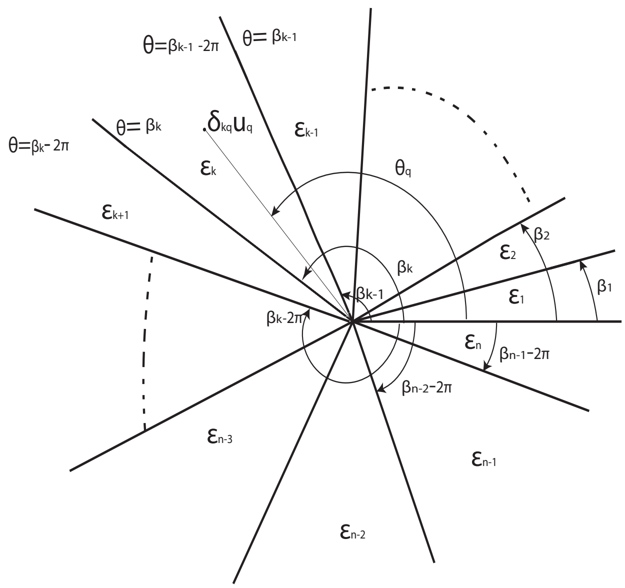

Fig. 4 Geometry of the n contiguous Dielectric wedges and line sources.

Considering an arbitrary k-th sector $(k = 1, 2, ..., n(mod(n))$, see Figure 4, then on the interface with $\theta = \beta_k$ with no interfacial charges we have

$$u_k(r, \beta_k) = u_{k+1}(r, \beta_k - 2\pi) \quad , \quad \epsilon_k \frac{\partial u_k(r, \beta_k)}{\partial \theta} = \epsilon_{k+1} \frac{\partial u_k(r, \beta_k - 2\pi)}{\partial \theta}; \tag{4.2}$$

with $u_{n+1} = u_1$, $\epsilon_{n+1} = \epsilon_1$. By using the integral representation (6.1) for the source term $u_q(r, \theta)$ and substituting the resulting expression given by (4.1) into the boundary conditions (4.2) gives

$$\begin{aligned} &P \int_{-\imath\infty}^{\imath\infty} \left(\delta_{k,q} \frac{\cos \nu(\pi - |\beta_k - \theta_q|)}{r_q^\nu \nu \sin \nu\pi} + a_k(\nu) \cos \nu\beta_k + b_k(\nu) \sin \nu\beta_k \right) r^\nu d\nu \\ &= P \int_{-\imath\infty}^{\imath\infty} \left(\delta_{k+1,q} \frac{\cos \nu(\pi - |\beta_k - 2\pi - \theta_q|)}{r_q^\nu \nu \sin \nu\pi} + a_{k+1}(\nu) \cos \nu\alpha_k - b_{k+1}(\nu) \sin \nu\alpha_k \right) r^\nu d\nu. \end{aligned} \tag{4.3}$$

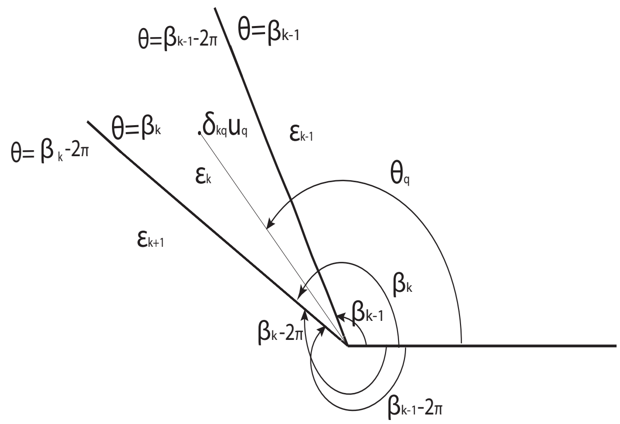

Fig. 5 Geometry of interfaces of the k-th Dielectric wedge and a line source.

$$\epsilon_k P\int_{-\imath\infty}^{\imath\infty}(\delta_{k,q}\frac{\sin\nu(\pi-|\beta_k-\theta_q|)sign(\beta_k-\theta_q)}{r_q^{\nu}\sin\nu\pi}+ \tag{4.4}$$
$$-a_k(\nu)\nu\sin\nu\beta_k+b_k(\nu)\nu\cos\nu\beta_k)r^{\nu}d\nu$$
$$=\epsilon_{k+1}P\int_{-\imath\infty}^{\imath\infty}(\delta_{k+1,q}\frac{\sin\nu(\pi-|\beta_k-2\pi-\theta_q|)sign(\beta_k-2\pi-\theta_q)}{r_q^{\nu}\sin\nu\pi}+$$
$$a_{k+1}(\nu)\nu\sin\nu\alpha_k+b_{k+1}(\nu)\nu\cos\nu\alpha_k)r^{\nu}d\nu, \tag{4.5}$$

where the complimentary angle $\alpha_k = 2\pi-\beta_k$. A sufficient condition for the satisfaction of these equations for $r>0$ is the satisfaction of the equations:

$$a_k(\nu)\cos\nu\beta_k+b_k(\nu)\sin\nu\beta_k-a_{k+1}(\nu)\cos\nu\alpha_k+b_{k+1}(\nu)\sin\nu\alpha_k$$
$$=\delta_{k+1,q}\frac{\cos\nu(\pi-\alpha_k-\theta_q)}{r_q^{\nu}\nu\sin\nu\pi}-\delta_{k,q}\frac{\cos\nu(\pi-\beta_k+\theta_q)}{r_q^{\nu}\nu\sin\nu\pi}, \tag{4.6}$$

$$
\begin{aligned}
&a(\nu)\epsilon_k \sin \nu\beta_k - b_k(\nu)\epsilon_k \cos \nu\beta_k + a_{k+1}(\nu)\epsilon_{k+1} \sin \nu\alpha_k + b_{k+1}(\nu)\epsilon_{k+1} \cos \nu\alpha_k \\
&= \epsilon_k \delta_{k,q} \frac{\sin \nu(\pi - \beta_k + \theta_q)}{r_q^\nu \nu \sin \nu\pi} + \epsilon_{k+1}\delta_{k+1,q} \frac{\sin \nu(\pi - \alpha_k - \theta_q)}{r_q^\nu \nu \sin \nu\pi}.
\end{aligned}
\tag{4.7}
$$

The system of equations may be written in matrix-vector form $A\mathbf{u} = \mathbf{b}$, where,

$$
\mathbf{u} = \begin{bmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \\ \vdots \\ a_k \\ b_k \\ \vdots \\ a_n \\ b_n \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix}
\delta_{1,q} \frac{\cos \nu(\pi - \theta_q)}{r_q^\nu \nu \sin \nu\pi} - \delta_{n,q} \frac{\sin \nu(\pi - \theta_q)}{r_q^\nu \nu \sin \nu\pi} \\
\delta_{2,q} \frac{\cos \nu(\pi - \alpha_1 - \theta_q)}{r_q^\nu \nu \sin \nu\pi} - \delta_{1,q} \frac{\cos \nu(\pi - \beta_1 + \theta_q)}{r_q^\nu \nu \sin \nu\pi} \\
\delta_{2,q}\epsilon_2 \frac{\cos \nu(\pi - \alpha_1 - \theta_q)}{r_q^\nu \nu \sin \nu\pi} + \delta_{1,q}\epsilon_1 \frac{\sin \nu(\pi - \beta_1 + \theta_q)}{r_q^\nu \nu \sin \nu\pi} \\
\delta_{3,q} \frac{\cos \nu(\pi - \alpha_2 - \theta_q)}{r_q^\nu \nu \sin \nu\pi} - \delta_{2,q} \frac{\cos \nu(\pi - \beta_2 + \theta_q)}{r_q^\nu \nu \sin \nu\pi} \\
\delta_{3,q}\epsilon_3 \frac{\cos \nu(\pi - \alpha_2 - \theta_q)}{r_q^\nu \nu \sin \nu\pi} + \delta_{2,q}\epsilon_2 \frac{\sin \nu(\pi - \beta_2 + \theta_q)}{r_q^\nu \nu \sin \nu\pi} \\
\vdots \\
\delta_{k+1,q} \frac{\cos \nu(\pi - \alpha_k - \theta_q)}{r_q^\nu \nu \sin \nu\pi} - \delta_{k,q} \frac{\cos \nu(\pi - \beta_k + \theta_q)}{r_q^\nu \nu \sin \nu\pi} \\
\delta_{k+1,q}\epsilon_{k+1} \frac{\cos \nu(\pi - \alpha_k - \theta_q)}{r_q^\nu \nu \sin \nu\pi} + \delta_{k,q}\epsilon_k \frac{\sin \nu(\pi - \beta_k + \theta_q)}{r_q^\nu \nu \sin \nu\pi} \\
\vdots \\
\delta_{n,q} \frac{\cos \nu(\pi - \alpha_{n-1} - \theta_q)}{r_q^\nu \nu \sin \nu\pi} - \delta_{n-1,q} \frac{\cos \nu(\pi - \beta_{n-1} + \theta_q)}{r_q^\nu \nu \sin \nu\pi} \\
\delta_{n,q}\epsilon_n \frac{\cos \nu(\pi - \alpha_{n-1} - \theta_q)}{r_q^\nu \nu \sin \nu\pi} + \delta_{n-1,q}\epsilon_{n-1} \frac{\sin \nu(\pi - \beta_{n-1} + \theta_q)}{r_q^\nu \nu \sin \nu\pi} \\
\delta_{1,q}\epsilon_1 \frac{\sin \nu(\pi - \theta_q)}{r_q^\nu \nu \sin \nu\pi} + \delta_{n,q}\epsilon_n \frac{\sin \nu(\pi - \theta_q)}{r_q^\nu \nu \sin \nu\pi}
\end{bmatrix},
$$

and

$$
A = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\
c_1^{(\beta)} & s_1^{(\beta)} & -c_1^{(\alpha)} & s_1^{(\alpha)} & 0 & 0 & \cdots & 0 & 0 \\
\epsilon_1 s_1^{(\beta)} & -\epsilon_1 c_1^{(\beta)} & \epsilon_2 s_1^{(\alpha)} & \epsilon_2 c_1^{(\alpha)} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & c_2^{(\beta)} & s_2^{(\beta)} & -c_2^{(\alpha)} & s_2^{(\alpha)} & 0 & \cdots & 0 \\
0 & 0 & \epsilon_2 s_2^{(\beta)} & -\epsilon_2 c_2^{(\beta)} & \epsilon_2 s_2^{(\alpha)} & \epsilon_2 c_2^{(\alpha)} & 0 & \cdots & 0 \\
0 & 0 & \ddots & \ddots & \ddots & \ddots & \cdots & 0 & 0 \\
0 & 0 & c_k^{(\beta)} & s_k^{(\beta)} & -c_k^{(\alpha)} & s_k^{(\alpha)} & 0 & \cdots & 0 \\
0 & 0 & \epsilon_k s_k^{(\beta)} & -\epsilon_k c_k^{(\beta)} & \epsilon_k s_k^{(\alpha)} & \epsilon_k c_k^{(\alpha)} & 0 & \cdots & 0 \\
0 & 0 & \ddots & \ddots & \ddots & \ddots & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c_{n-1}^{(\beta)} & s_{n-1}^{(\beta)} & -c_{n-1}^{(\alpha)} & s_{n-1}^{(\alpha)} \\
0 & 0 & 0 & 0 & 0 & \epsilon_{n-1} s_{n-1}^{(\beta)} & -\epsilon_{n-1} c_{n-1}^{(\beta)} & \epsilon_{n-1} s_{n-1}^{(\alpha)} & \epsilon_{n-1} c_{n-1}^{(\alpha)} \\
0 & \epsilon_1 & 0 & 0 & 0 & 0 & \cdots & 0 & -\epsilon_n
\end{bmatrix},
$$

where

$$
s_k^{(\alpha)} = \sin(\nu\alpha_k), \qquad s_k^{(\beta)} = \sin(\nu\beta_k), \qquad c_k^{(\alpha)} = \cos(\nu\alpha_k), \qquad s_k^{(\alpha)} = \sin(\nu\alpha_k).
$$

In the special case of $n=2, q=1, \alpha_3=\alpha_1, \beta_3=\beta_1$ this system reduces to

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ \cos\nu\beta_1 & \sin\nu\beta_1 & -\cos\nu\alpha_1 & \sin\nu\alpha_1 \\ \epsilon_1\sin\nu\beta_1 & -\epsilon_1\cos\nu\beta_1 & \epsilon_2\sin\nu\alpha_1 & \epsilon_2\cos\nu\alpha_1 \\ 0 & \epsilon_1 & 0 & -\epsilon_2 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} \frac{\cos\nu(\pi-\theta_1)}{r_1^\nu \nu\sin\nu\pi} \\ -\frac{\cos\nu(\pi-\beta_1+\theta_1)}{r_1^\nu \nu\sin\nu\pi} \\ \frac{\epsilon_1\sin\nu(\pi-\beta_1+\theta_1)}{r_1^\nu \nu\sin\nu\pi} \\ \frac{\epsilon_1\sin\nu(\pi-\theta_1)}{r_1^\nu \nu\sin\nu\pi} \end{bmatrix};$$

and when solved gives:

$$a_1(\nu) = a_2(\nu) - \frac{\cos\nu(\pi-\theta_1)}{\nu\sin\nu\pi},$$
$$b_1(\nu) = \frac{\epsilon_2}{\epsilon_1}b_2(\nu) + \frac{\sin\nu(\pi-\theta_1)}{\nu\sin\nu\pi},$$

where

$$a_2(\nu) = \frac{2\epsilon_1\epsilon_2\sin\nu(\beta_1-\theta_1)(\cos\nu\beta_1-\cos\nu\alpha_1)+2\epsilon_1\cos\nu(\beta_1-\theta_1)(\epsilon_2\sin\nu\beta_1+\epsilon_1\sin\nu\alpha_1)}{\nu\Delta(\nu)},$$

$$b_2(\nu) = \frac{-2\epsilon_1^2\cos\nu(\beta_1-\theta_1)(\cos\nu\beta_1-\cos\nu\alpha_1)+2\epsilon_1\sin\nu(\beta_1-\theta_1)(\epsilon_1\sin\nu\beta_1+\epsilon_2\sin\nu\alpha_1)}{\nu\Delta(\nu)},$$

$$\begin{aligned}\Delta(\nu) &= 2\epsilon_1\epsilon_2(1-\cos\nu\alpha_1\cos\nu\beta_1)+(\epsilon_1^2+\epsilon_2^2)\sin\nu\alpha_1\sin\nu\beta_1, \\ &= (\epsilon_1+\epsilon_2)^2\sin^2\nu\pi-(\epsilon_1-\epsilon_2)^2\sin^2\nu(\pi-\alpha_1), \\ &= 4(\epsilon_1\sin\frac{\nu\alpha_1}{2}\cos\frac{\nu\beta_1}{2}+\epsilon_2\sin\frac{\nu\beta_1}{2}\cos\frac{\nu\alpha_1}{2})(\epsilon_1\sin\frac{\nu\beta_1}{2}\cos\frac{\nu\alpha_1}{2}+\epsilon_2\sin\frac{\nu\alpha_1}{2}\cos\frac{\nu\beta_1}{2}).\end{aligned}$$

As a check, after some trigonometrical algebra, involving a change of coordinates to the line of symmetry of the wedge problem in Figure 1 these results agree with the previous results derived for the more symmetrical two media problem.

5. Conclusions

We have brought to light the several errors in the solutions already given by a number of authors to the electrostatic problem of the field produced by a line charge near an infinite dielectric wedge. We rectify these errors by using an approach that uses an appropriate source representation for a line of charge for any location of the line of charge and the observation point. As a check on our result we have derived the explicit edge field behavior near the apex of the wedge. This method avoids the use of the Mellin transform. The Mellin transform can be applied, but with great care, especially on using the inversion formula. There is no strip of convergence in applying the inversion formula, only a broken line. Aspects of this property in the application of Mellin transforms are also covered in a recent publication by Martin (**13**) and Paris and Kaminski(**11**) and references given therein. We also remark that to some extent an analogous situation occurs in the application of the Kontorovich-Lebedev transform to electromagnetic problems; which is overcome by either using a technique that introduces unknown constants, see Jones(**14**), or by an appropriate integral representation of the source,see Osipov(**15**). In the limit as the wave-number tends to zero in these dynamical problems the dynamic line source does not tend uniformly to the electrostatic source result given in the appendix of this work.

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6. Appendix

In this appendix we shall prove the result:

$$2\log\left(\frac{\sqrt{rr_0}}{R}\right) = \imath P\int_{-\imath\infty}^{\imath\infty}\frac{\cos\nu(\pi-|\psi|)}{\nu\sin\nu\pi}\left(\frac{r}{r_0}\right)^{\nu}d\nu, \tag{6.1}$$

where $R = \sqrt{r^2+r_0^2-2rr_0\cos\psi}, -2\pi < \psi < 2\pi, r > 0, r_0 > 0$, and P stands for the Cauchy principal value integral.

Let us denote the integral by I and initially assume $r_0 > r$;

$$I = \imath P\int_{-\imath\infty}^{\imath\infty}\frac{\cos\nu(\pi-|\psi|)}{\nu\sin\nu\pi}\left(\frac{r}{r_0}\right)^{\nu}d\nu, \qquad r_0 > r.$$

It is not difficult to show that for $r_0 > r$ the integrand is exponentially bounded in the region

$\Re\nu > 0$. Then by adding and subtracting the residue of the double pole singularity at the origin we can replace the principal value integral by a closed contour C_+. This contour C_+ is indented to the right at the origin and closed by an infinite semicircle in the right half plane $\Re\nu > 0$. Then

$$I = \imath \oint_{C_+} \frac{\cos\nu(\pi - |\psi|)}{\nu \sin\nu\pi} \left(\frac{r}{r_0}\right)^{\nu} d\nu - \imath\pi Res\left[\imath \frac{\cos\nu(\pi - |\psi|)}{\nu \sin\nu\pi}\right]_{\nu=0},$$

$$I = \imath \int_{\frac{1}{2}-\imath\infty}^{\frac{1}{2}+\imath\infty} \frac{\cos\nu(\pi - |\psi|)}{\nu \sin\nu\pi} \left(\frac{r}{r_0}\right)^{\nu} d\nu + \log\left(\frac{r}{r_0}\right),$$

$$I = 2\sum_{n=1}^{\infty} \frac{\cos n\psi}{n} \left(\frac{r}{r_0}\right)^{n} + \log\left(\frac{r}{r_0}\right).$$

Now substituting the well known identity:

$$\log R = \log r_0 - \sum_{n=1}^{\infty} \frac{\cos n\psi}{n} \left(\frac{r}{r_0}\right)^{n} \qquad r_0 > r,$$

gives

$$I = 2(\log r_0 - \log R) + \log r/r_0 = 2\log\left(\frac{\sqrt{r r_0}}{R}\right).$$

In a similar method it is not difficult to show that for $r > r_0$ the integrand is exponentially bounded in the region $\Re\nu < 0$. Then by adding and subtracting the residue of the double pole singularity at the origin we can replace the principal value integral by a closed contour C_-. This contour C_- is indented to the left at the origin and closed by an infinite semicircle in the left half plane $\Re\nu < 0$. Then

$$I = \imath \oint_{C_-} \frac{\cos\nu(\pi - |\psi|)}{\nu \sin\nu\pi} \left(\frac{r}{r_0}\right)^{\nu} d\nu + \imath\pi Res\left[\imath \frac{\cos\nu(\pi - |\psi|)}{\nu \sin\nu\pi}\right]_{\nu=0},$$

$$I = \imath \int_{\frac{1}{2}-\imath\infty}^{\frac{1}{2}+\imath\infty} \frac{\cos\nu(\pi - |\psi|)}{\nu \sin\nu\pi} \left(\frac{r}{r_0}\right)^{\nu} d\nu - \log\left(\frac{r}{r_0}\right),$$

$$I = 2\sum_{n=1}^{\infty} \frac{\cos n\psi}{n} \left(\frac{r}{r_0}\right)^{-n} + \log\left(\frac{r_0}{r}\right).$$

Now substituting the well known identity:

$$\log R = \log r - \sum_{n=1}^{\infty} \frac{\cos n\psi}{n} \left(\frac{r_0}{r}\right)^{n}, \quad r > r_0,$$

gives

$$I = 2(\log r - \log R) + \log r/r_0 = 2\log\left(\frac{\sqrt{r r_0}}{R}\right);$$

which verifies the result (4.1).