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COMPLETE f–MOMENT CONVERGENCE FOR NEGATIVELY SUPERADDITIVE DEPENDENT RANDOM VARIABLES

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Abstract. In this paper, by utilizing the Kolmogorov exponential type inequality of negatively superadditive dependent random arrays and truncated method, we study the complete f-moment convergence for arrays of rowwise NSD random variables. Some sufficient conditions to prove the complete f-moment convergence are obtained, which generalize and improve some known ones.

1. Introduction

Firstly, let us recall some definitions of the negative dependence. The first one is the concept of negatively associated (NA) random variables, which was introduced by Alam and Saxena [1] and carefully studied by Joag-Dev and Proschan [2].

DEFINITION 1.1. A finite family of random variables $\{X_i, 1 \leqslant i \leqslant n\}$ is said to be NA if for every pair of disjoint subsets $A, B \subset \{1, 2, \ldots, n\}$

$$\operatorname{Cov}\left(g\left(X_i, i \in A\right), q\left(X_j, j \in B\right)\right) \leqslant 0,$$

whenever g and q are coordinatewise nondecreasing such that this covariance exists. An infinite family of random variables is NA if every finite subfamily is NA. An array $\{X_{ni}, i \geqslant 1, n \geqslant 1\}$ of random variables is said to be rowwise NA if for all $n \geqslant 1$, $\{X_{ni}, i \geqslant 1\}$ is NA.

Let us recall the concepts of superadditive function and negatively superadditive dependence. The concept of superadditive function is introduced by Kemperman [3] as follows.

DEFINITION 1.2. A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is called superadditive if $\phi(x \vee y) + \phi(x \wedge y) \geqslant \phi(x) + \phi(y)$ for all $x, y \in \mathbb{R}$, where $\vee$ is for componentwise maximum and $\wedge$ is for componentwise minimum.

The next dependence notion is negatively superadditive dependence, which is weaker than negative association. The concept of NSD random variables was introduced by Hu [4] as follows.

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DEFINITION 1.3. A random vector $X = (X_1, X_2, \dots, X_n)$ is said to NSD if

$$E\phi\left(X_1, X_2, \dots, X_n\right) \leqslant E\phi\left(X_1^*, X_2^*, \dots, X_n^*\right), \tag{1.1}$$

where $(X_1^*, X_2^*, \dots, X_n^*)$ are independent such that X_i^* and X_i have the same distribution for each i and ϕ is a superadditive function such that the expectations in (1.1) exist.

A sequence $\{X_n, n \geqslant 1\}$ of random variables is said to be NSD if for all $n \geqslant 1$, $(X_1, X_2, \dots, X_n)$ is NSD.

An array $\{X_{ni}, i \geqslant 1, n \geqslant 1\}$ of random variables is said to be rowwise NSD if for all $n \geqslant 1, \{X_{ni}, i \geqslant 1\}$ is NSD.

Hu [4] gave an example illustrating that NSD does not imply NA, and he posed an open problem whether NA implies NSD. In addition, Hu [4] provided some basic properties and three structural theorems of NSD. Christofides and Vaggelatou [5] solved this open problem and showed that NA implies NSD. NSD structure is an extension of negatively associated structure and sometimes more useful than it and can be used to get many important probability inequalities. Eghbal et al. [6] derived two maximal inequalities and strong law of large numbers of quadratic forms of NSD random variables under the assumption that $\{X_i, i \geqslant 1\}$ is a sequence of nonnegative NSD random variables with $EX_i < \infty$ for some $r > 1$ and all $i \geqslant 1$. Shen et al. [7] obtained the almost sure convergence for NSD sequences and the strong stability for weighted sums of NSD random variables, which extend the corresponding results for independent sequences and NA sequences without necessarily adding any extra condition. Shen et al. [8] investigated the complete convergence and complete moment convergence for arrays of rowwise NSD random variables and presented some sufficient conditions to prove the complete convergence and the complete moment convergence. Wang [9] presented the Rosenthal-type maximal inequalities and Kolmogorov-type exponential inequality for NSD random variables, studied the complete convergence for arrays of rowwise NSD random variables and weighted sums of arrays of rowwise NSD random variables. Wang [9] obtained the Baum-Katz-type result for arrays of rowwise NSD random variables the complete consistency for the estimator of nonparametric regression model based on NSD errors. Wang et al. [10] obtained the complete convergence for weighted sums of NSD random variables and its application in the EV regression model. Zhen [11] investigated the complete moment convergence for maximal partial sum of NSD random variables under some more general conditions.

The concept of complete convergence was introduced by Hsu and Robbins [12] as fallow: a sequence $\{X_n, n \geqslant 1\}$ of random variables is said to converge completely to the constant θ if for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\left(|X_n - \theta| > \varepsilon\right) < \infty.$$

By the Borel-Cantelli lemma, this implies that $X_n \to \theta$ almost surely, and so complete convergence is a stronger concept than almost sure convergence. Hsu and Robbins [12] proved that the sequence of arithmetic means of i.i.d. random variables converge completely to the expected value if the variance of the summands is finite. Chen et al. [13] established the following complete convergence result for arrays of rowwise NA random variables.

THEOREM 1.1. *Let $\{X_{ni}, 1 \leqslant i \leqslant k_n, n \geqslant 1\}$ be an array of rowwise NA random variables and $\{c_n, n \geqslant 1\}$ be a sequence of positive constants. Suppose that the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} P(|X_{ni}| > \varepsilon) < \infty$ *for every* $\varepsilon > 0$;

(ii) for some $\delta > 0$, *there exists* $j \geqslant 1$ *sunch that*

$$\sum_{n=1}^{\infty} c_n \left(\sum_{i=1}^{k_n} Var\left(X_{ni} I\left(|X_{ni}| \leqslant \delta\right)\right) \right)^j < \infty.$$

Then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n P\left(\max_{1 \leqslant m \leqslant k_n} \left| \sum_{i=1}^{m} \left(X_{ni} - EX_{ni} I\left(|X_{ni}| \leqslant \delta\right)\right) \right| > \varepsilon \right) < \infty.$$

The concept of complete moment convergence was introduced by Chow [14] as follow: let $\{X_n, n \geqslant 1\}$ be a sequence of random variables, and $\{a_n, n \geqslant 1\}$, $\{b_n, n \geqslant 1\}$, $q > 0$. If for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} a_n E\left\{ b_n^{-1} |X_n| - \varepsilon \right\}_+^q < \infty,$$

where $a_+ = \max\{0, a\}$, then $\{X_n, n \geqslant 1\}$ is said to be complete q-th moment convergent.

Shen et al. [8] obtained the following complete moment convergence result for NSD random variables.

THEOREM 1.2. *Let $q \geqslant 1$, $\{X_{ni}, 1 \leqslant i \leqslant k_n, n \geqslant 1\}$ be an array of rowwise NSD random variables and $\{c_n, n \geqslant 1\}$ be a sequence of positive constants. Suppose that the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} E|X_{nk}|^q I(|X_{nk}| > \varepsilon) < \infty$ *for every* $\varepsilon > 0$;

(ii) for some $\delta > 0$, *there exists* $\eta > q$ *such that*

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} EX_{nk}^2 I\left(|X_{nk}| \leqslant \delta\right) \right)^{\eta} < \infty;$$

(iii) $\sum_{k=1}^{k_n} E|X_{nk}|^q I\left(|X_{nk}| > \frac{\delta}{128\eta}\right) \to 0$, *as* $n \to \infty$.

Then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n E\left\{ \max_{1 \leqslant m \leqslant k_n} \left| \sum_{k=1}^{m} \left(X_{nk} - EX_{nk} I\left(|X_{nk}| \leqslant \delta\right)\right) \right| - \varepsilon \right\}_+^q < \infty.$$

Recently, Wu et al. [15] introduced the concept of complete f-moment convergence which is stronger than complete moment convergence, as follows.

DEFINITION 1.4. Let $\{S_n, n \geqslant 1\}$ be a sequence of random variable, $\{c_n, n \geqslant 1\}$ be a sequence of positive constants and $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing function with $f(0) = 0$. $\{S_n, n \geqslant 1\}$ is said to converges f-moment completely, if for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n E f\left(\{|S_n| - \varepsilon\}_+\right) < \infty,$$

where $a_+ = \max\{0, a\}$.

Wu et al. [15] established some results on complete f-moment convergence for sums of arrays of rowwise END random variables.

THEOREM 1.3. *Let $\{X_{nk}, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$ be an array of rowwise END random variables, and $\{c_n, n \geqslant 1\}$ be a sequence of positive constants, $f : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function with $f(0) = 0$ and $q \geqslant 1$ be a constant. suppose that the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} E f\left(8\eta |X_{nk}| I\left(|X_{nk}| > \varepsilon\right)\right) < \infty$ *for any* $\varepsilon > 0$*;*

(ii) for some $\delta > 0$*, there exists constants* $0 < p \leqslant 2$ *such that*

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} E\left|X_{nk} I\left(|X_{nk}| \leqslant \delta\right) - E X_{nk} I\left(|X_{nk}| \leqslant \delta\right)\right|^p\right)^{\eta} < \infty;$$

(iii) $\sum_{k=1}^{k_n} E|X_{nk}| I\left(|X_{nk}| > \frac{\delta}{16\eta}\right) \to 0$*, as* $n \to \infty$*;*

(iv) Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ *be the inverse function for* $f(t)$*, that is,* $g(f(t)) = t$*,* $t \geqslant 0$ *and* $s(\mathrm{t}) = \max_{\delta \leqslant x \leqslant g(t)} \frac{x}{f(\mathrm{x})}$*. Assume that the constants* η, δ *and the function* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *satisfy the condition*

$$\int_{f(\delta)}^{\infty} g^{-\eta}(t) s(t) dt < \infty.$$

Then the for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} c_n E f\left(\left\{\left|\sum_{k=1}^{k_n} \left(X_{nk} - E X_{nk} I\left(|X_{nk}| \leqslant \delta\right)\right)\right| - \varepsilon\right\}_+\right) < \infty.$$

Afterwards, many authors were devoted to studying the probability limit theories for complete f-moment convergence and obtained many interesting results. For example, Lu et al. [16] obtained the complete f-moment convergence for sums of arrays of rowwise END random variables under sub-linear expectation space. Lu et al. [17] studied complete f-moment convergence for WOD random variables and gave its application in nonparametric models. Wang [18] obtained a result on complete f-moment convergence for Sung's type weighted sums of END random variables under some general conditions, and gave some corollaries and an application in errors-in-variables regression models. So far, the corresponding research results on the complete

f-moments convergence for arrays of rowwise NSD random variables have not been obtained.

In this paper, we discuss the complete f-moment convergence for arrays of rowwise NSD random variables under the weaker condition than previous results. The result obtained in the paper extends and improves the corresponding Theorem 1.2 obtained by Shen et al. [8] and Theorem 1.3 obtained by Wang et al. [15] for $q = 1$.

This paper is organized as follows: some preliminary lemmas and inequalities for NSD random variables are provided in Sect. 2. The main result and its proof are stated in Sect. 3. Some corollaries of the main result are presented in Sect. 4.

Throughout the paper, we will denote by C a positive generic constant, which may be different in various places.

2. Preliminary lemmas

In this section, we give some important lemmas which will be used to prove our main results. The following one was presented by Hu [4].

LEMMA 2.1. *Let $(X_1, X_2, \ldots, X_n)$ be NSD, then we have following.*

(i) $(-X_1, -X_2, \ldots, -X_n)$ is also NSD.

(ii) if $g_1, g_2, \ldots, g_n$ are all nondecreasing functions, then $(g_1(X_1), g_2(X_2), \ldots, g_n(X_n))$ is NSD.

The next one is the Kolmogorov exponential type inequality for NSD random variables, which was established by Wang et al. [9].

LEMMA 2.2. *Let $\{X_n, n \geqslant 1\}$ be a sequence of NSD random variables with zero mean and finite second moments. Denote $S_n = \sum_{i=1}^{n} X_i$ and $B_n = \sum_{i=1}^{n} EX_i^2$ for each $n \geqslant 1$. Then for all $x > 0$, $y > 0$ and $n \geqslant 1$*

$$P\left(\max_{1 \leqslant k \leqslant n} |S_k| \geqslant x\right) \leqslant 2P\left(\max_{1 \leqslant k \leqslant n} |X_k| \geqslant y\right) + 8\left(\frac{2B_n}{3xy}\right)^{x/12y}.$$

With Lemma 2.2 accounted for, we can get the following complete convergence for arrays of rowwise NSD random variables, which is a generalization of Theorem 1.1. The proof is similar to that of Theorem 1.1 or Lemma 3.1 of Shen [19]. So we omit the details.

LEMMA 2.3. *Let $\{X_{ni}, 1 \leqslant i \leqslant k_n, n \geqslant 1\}$ be an array of rowwise NSD random variables and $\{c_n, n \geqslant 1\}$ be a sequence of positive constants. Suppose that the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} P(|X_{ni}| > \varepsilon) < \infty$ for every $\varepsilon > 0$;

(ii) for some $\delta > 0$, there exists $j \geqslant 1$ such that

$$\sum_{n=1}^{\infty} c_n \left(\sum_{i=1}^{k_n} Var(X_{ni} I(|X_{ni}| \leqslant \delta))\right)^j < \infty.$$

Then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n P\left(\max_{1\leqslant m\leqslant k_n}\left|\sum_{i=1}^{m}\left(X_{ni} - EX_{ni}I\left(|X_{ni}|\leqslant\delta\right)\right)\right| > \varepsilon\right) < \infty.$$

3. The main result and proof

Our main result is as follows.

THEOREM 3.1. *Let* $\{X_{nk}, 1\leqslant k\leqslant k_n, n\geqslant 1\}$ *be an array of rowwise NSD random variables, and* $\{c_n, n\geqslant 1\}$ *be a sequence of positive constants,* $f:\mathbb{R}^+\to\mathbb{R}^+$ *be an increasing function with* $f(0)=0$ *and* $q\geqslant 1$ *be a constant. Suppose that the following conditions are satisfied:*

(a) $\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} Ef\left(192\eta\,|X_{nk}|^q I\left(|X_{nk}|>\varepsilon\right)\right) < \infty$ *for every* $\varepsilon > 0$;

(b) for some $\delta > 0$, *there exists* $\eta > q$ *such that*

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} EX_{nk}^2 I\left(|X_{nk}|\leqslant\delta\right)\right)^{\eta} < \infty;$$

(c) $\sum_{k=1}^{k_n} E\,|X_{nk}|^q I\left(|X_{nk}| > \frac{\delta}{128\eta}\right) \to 0$, *as* $n\to\infty$;

(d) Let $g:\mathbb{R}^+\to\mathbb{R}^+$ *be the inverse function for* $f(t)$, *that is,* $g(f(t)) = t$, $t\geqslant 0$ *and* $S(\mathrm{t}) = \max_{\delta\leqslant x\leqslant g^{1/q}(\mathrm{t})} \frac{x}{f(\mathrm{X})}$. *Assume that the constants* $\eta > q\geqslant 1$, δ *and the function* $f:\mathbb{R}^+\to\mathbb{R}^+$ *satisfy the condition*

$$\int_{f(\delta^q)}^{\infty} g^{-\eta/q}(t)s(t)dt < \infty.$$

Then for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} c_n Ef\left(\left\{\max_{1\leqslant m\leqslant k_n}\left|\sum_{k=1}^{m}\left(X_{nk} - EX_{nk}I\left(|X_{nk}|\leqslant\delta\right)\right)\right| - \varepsilon\right\}_+^q\right) < \infty.$$

Proof. Note that function f is increasing and $\eta > q\geqslant 1$, for every $\varepsilon > 0$, we have

$$(a') \quad \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} Ef\left(|X_{nk}|^q I\left(|X_{nk}|>\varepsilon\right)\right) < \infty.$$

Now we state that the conditions (i) and (ii) of Lemma 2.3 hold. For all $\varepsilon > 0$, it follows from condition (a') and Markov's inequality that

$$\sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} P\left(|X_{ni}|>\varepsilon\right) < C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} Ef\left(|X_{nk}|^q I\left(|X_{nk}|>\varepsilon\right)\right) < \infty,$$

which implies that condition (i) of Lemma 2.3 holds. Obviously according to (b) the condition (ii) of Lemma 2.3 holds. Thus, all the conditions of Lemma 2.3 are satisfied.

Denote $S_m=\sum_{k=1}^{m}(X_{nk}-EX_{nk}I(|X_{nk}|\leqslant\delta))$, $n\geqslant 1$. we may assume that $0<\varepsilon<\delta$, it is easily seen that:

$$\begin{aligned}
&\sum_{n=1}^{\infty}c_nEf\left(\left\{\max_{1\leqslant m\leqslant k_n}|S_m|-\varepsilon\right\}_+^q\right)\\
&=\sum_{n=1}^{\infty}c_n\int_0^{\infty}P\left[f\left(\left\{\max_{1\leqslant m\leqslant k_n}|S_m|-\varepsilon\right\}_+^q\right)>t\right]dt\\
&=\sum_{n=1}^{\infty}c_n\int_0^{\infty}P\left[\left\{\max_{1\leqslant m\leqslant k_n}|S_m|-\varepsilon\right\}_+^q>g(t)\right]dt\\
&=\sum_{n=1}^{\infty}c_n\int_0^{f(\delta^q)}P\left(\max_{1\leqslant m\leqslant k_n}|S_m|>\varepsilon+g^{1/q}(t)\right)dt\\
&\quad+\sum_{n=1}^{\infty}c_n\int_{f(\delta^q)}^{\infty}P\left(\max_{1\leqslant m\leqslant k_n}|S_m|>\varepsilon+g^{1/q}(t)\right)dt\\
&\leqslant f(\delta^q)\sum_{n=1}^{\infty}c_nP\left(\max_{1\leqslant m\leqslant k_n}|S_m|>\varepsilon\right)\\
&\quad+\sum_{n=1}^{\infty}c_n\int_{f(\delta^q)}^{\infty}P\left(\max_{1\leqslant m\leqslant k_n}|S_m|>g^{1/q}(t)\right)dt\\
&\triangleq I_1+I_2.
\end{aligned}$$

In order to prove Theorem 3.1. we only need to show that $I_1<\infty$ and $I_2<\infty$. By Lemma 2.3, $I_1<\infty$. In the following, we will show that $I_2<\infty$.

$$\begin{aligned}
I_2&=\sum_{n=1}^{\infty}c_n\int_{f(\delta^q)}^{\infty}P\left(\max_{1\leqslant m\leqslant k_n}|S_m|>g^{1/q}(t),\bigcup_{k=1}^{k_n}\left\{|X_{nk}|>g^{1/q}(t)\right\}\right)dt\\
&\quad+\sum_{n=1}^{\infty}c_n\int_{f(\delta^q)}^{\infty}P\left(\max_{1\leqslant m\leqslant k_n}|S_m|>g^{1/q}(t),\bigcap_{k=1}^{k_n}\left\{|X_{nk}|\leqslant g^{1/q}(t)\right\}\right)dt\\
&\leqslant\sum_{n=1}^{\infty}c_n\int_{f(\delta^q)}^{\infty}P\left(\bigcup_{k=1}^{k_n}\left\{|X_{nk}|>g^{1/q}(t)\right\}\right)dt\\
&\quad+\sum_{n=1}^{\infty}c_n\int_{f(\delta^q)}^{\infty}P\Bigg(\max_{1\leqslant m\leqslant k_n}\left|\sum_{k=1}^{k_n}\left(X_{nk}I\left(|X_{nk}|\leqslant g^{1/q}(t)\right)-EX_{nk}I(|X_{nk}|\leqslant\delta)\right)\right|\\
&\quad>g^{1/q}(t)\Bigg)dt\\
&\triangleq I_3+I_4.
\end{aligned}$$

It follows from condition (a') and Markov's inequality that

$$
\begin{aligned}
I_3 &\leqslant \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \int_{f(\delta^q)}^{\infty} P\left(f\left(|X_{nk}|^q\right) > t\right) dt \\
&\leqslant C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} Ef\left(|X_{nk}|^q I\left(|X_{nk}| > \delta\right)\right) < \infty.
\end{aligned}
$$

In order to estimate $I_4 < \infty$, for fixed $n \geqslant 1$, $1 \leqslant k \leqslant k_n$, and $t \geqslant f(\delta^q)$. Let

$$
Y_{nk} = -g^{1/q}(t) I\left(X_{nk} < -g^{1/q}(t)\right) + X_{nk} I\left(|X_{nk}| \leqslant g^{1/q}(t)\right) + g^{1/q}(t) I\left(X_{nk} > g^{1/q}(t)\right),
$$

$$
Z_{nk} = -g^{1/q}(t) I\left(X_{nk} < -g^{1/q}(t)\right) + g^{1/q}(t) I\left(X_{nk} > g^{1/q}(t)\right).
$$

It is easily seen that

$$
\begin{aligned}
&P\left(\max_{1 \leqslant m \leqslant k_n} \left| \sum_{k=1}^{k_n} \left(X_{nk} I\left(|X_{nk}| \leqslant g^{1/q}(t)\right) - EX_{nk} I\left(|X_{nk}| \leqslant \delta\right)\right)\right| > g^{1/q}(t)\right) \\
&\quad = P\left(\max_{1 \leqslant m \leqslant k_n} \left| \sum_{k=1}^{m} \left(Y_{nk} - EY_{nk} - Z_{nk} + EZ_{nk} + EX_{nk} I\left(\delta < |X_{nk}| \leqslant g^{1/q}(t)\right)\right)\right| \right. \\
&\qquad \left. > g^{1/q}(t)\right) \\
&\quad \leqslant P\left(\max_{1 \leqslant m \leqslant k_n} \left| \sum_{k=1}^{m} \left(Y_{nk} - EY_{nk} - Z_{nk} + EZ_{nk}\right)\right| \right. \\
&\qquad \left. + \max_{1 \leqslant m \leqslant k_n} \sum_{k=1}^{m} E|X_{nk}| I\left(\delta < |X_{nk}| \leqslant g^{1/q}(t)\right) > g^{1/q}(t)\right).
\end{aligned}
\tag{3.1}
$$

Next, by assumption (c), we obtain that

$$
\begin{aligned}
&\max_{t \geqslant f(\delta^q)} \frac{1}{g^{1/q}(t)} \max_{1 \leqslant m \leqslant k_n} \left| \sum_{k=1}^{m} E|X_{nk}| I\left(\delta < |X_{nk}| \leqslant g^{1/q}(t)\right)\right| \\
&\quad \leqslant \max_{t \geqslant f(\delta^q)} \sum_{k=1}^{k_n} E\frac{|X_{nk}|}{\delta} I\left(\delta < |X_{nk}| \leqslant g^{1/q}(t)\right) \\
&\quad \leqslant \delta^{-q} \sum_{k=1}^{k_n} E|X_{nk}|^q I\left(|X_{nk}| > \delta\right) \to 0, \quad \text{as} \quad n \to \infty.
\end{aligned}
$$

Hence, for all n large enough,

$$
\max_{1 \leqslant m \leqslant k_n} \sum_{k=1}^{m} E\left| X_{nk} I\left(\delta < |X_{nk}| \leqslant g^{1/q}(t)\right)\right| < \frac{g^{1/q}(t)}{2}, \quad t > f(\delta^q),
$$

combining with (3.1), it follows that

$$
\begin{aligned}
&P\left(\max_{1\leqslant m\leqslant k_n}\left|\sum_{k=1}^{m}\left(X_{nk}I\left(|X_{nk}|\leqslant g^{1/q}(t)\right)-EX_{nk}I\left(|X_{nk}|\leqslant\delta\right)\right)\right|>g^{1/q}(t)\right)\\
&\quad\leqslant P\left(\max_{1\leqslant m\leqslant k_n}\left|\sum_{k=1}^{m}\left(Y_{nk}-EY_{nk}-Z_{nk}+EZ_{nk}\right)\right|>\frac{g^{1/q}(t)}{2}\right)\\
&\quad\leqslant P\left(\max_{1\leqslant m\leqslant k_n}\left|\sum_{k=1}^{m}\left(Y_{nk}-EY_{nk}\right)\right|>\frac{g^{1/q}(t)}{4}\right)\\
&\qquad+P\left(\max_{1\leqslant m\leqslant k_n}\left|\sum_{k=1}^{m}\left(Z_{nk}-EZ_{nk}\right)\right|>\frac{g^{1/q}(t)}{4}\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I_4&\leqslant C\sum_{n=1}^{\infty}c_n\int_{f(\delta^q)}^{\infty}P\left(\max_{1\leqslant m\leqslant k_n}\left|\sum_{k=1}^{m}\left(Z_{nk}-EZ_{nk}\right)\right|>\frac{g^{1/q}(t)}{4}\right)dt\\
&\quad+C\sum_{n=1}^{\infty}c_n\int_{f(\delta^q)}^{\infty}P\left(\max_{1\leqslant m\leqslant k_n}\left|\sum_{k=1}^{m}\left(Y_{nk}-EY_{nk}\right)\right|>\frac{g^{1/q}(t)}{4}\right)dt\\
&\triangleq I_5+I_6.
\end{aligned}
$$

Noting that $|Z_{nk}|=g^{1/q}(t)I\left(|X_{nk}|>g^{1/q}(t)\right)$, we obtain by Markov's inequality and the condition (a') that

$$
\begin{aligned}
I_5&\leqslant C\sum_{n=1}^{\infty}c_n\sum_{k=1}^{k_n}\int_{f(\delta^q)}^{\infty}\frac{1}{g^{1/q}(t)}E\left|Z_{nk}\right|dt\\
&\leqslant C\sum_{n=1}^{\infty}c_n\sum_{k=1}^{k_n}\int_{f(\delta^q)}^{\infty}P\left(|X_{nk}|>g^{1/q}(t)\right)dt\\
&\leqslant C\sum_{n=1}^{\infty}c_n\sum_{k=1}^{k_n}Ef\left(|X_{nk}|^qI\left(|X_{nk}|>\delta\right)\right)<\infty.
\end{aligned}
$$

Next, we will prove $I_6<\infty$. Denote $B_n=\sum_{k=1}^{k_n}E\left(Y_{nk}-EY_{nk}\right)^2$. Now we use the fact that $n\geqslant 1$, $\{Y_{nk}-EY_{nk},1\leqslant k\leqslant k_n\}$ is a sequence of NSD random variables applying to it the Kolmogorov inequality (Lemma 2.2) with $x=g^{1/q}(t)/4$ and $y=g^{1/q}(t)/(48\eta)$, where η satisfies the assumption (d). We obtain

$$
\begin{aligned}
I_6&\leqslant C\sum_{n=1}^{\infty}c_n\int_{f(\delta^q)}^{\infty}P\left(\max_{1\leqslant k\leqslant k_n}|Y_{nk}-EY_{nk}|>\frac{g^{1/q}(t)}{48\eta}\right)dt+C\sum_{n=1}^{\infty}c_n\int_{f(\delta^q)}^{\infty}8\left(\frac{128B_n}{g^{2/q}(t)}\right)^{\eta}dt\\
&\leqslant C\sum_{n=1}^{\infty}c_n\int_{f(\delta^q)}^{\infty}P\left(\max_{1\leqslant k\leqslant k_n}|Y_{nk}-EY_{nk}|>\frac{g^{1/q}(t)}{48\eta}\right)dt+C\sum_{n=1}^{\infty}c_n\int_{f(\delta^q)}^{\infty}\left(\frac{B_n}{g^{2/q}(t)}\right)^{\eta}dt\\
&\triangleq I_7+I_8.
\end{aligned}
$$

In order to estimate I_7, it follows that from assumption (c) and Markov's inequality, $\sum_{k=1}^{k_n} P\left(|X_{nk}| > \frac{\delta}{128\eta}\right) \to 0$ as $n \to \infty$. Hence, for all n large enough, we can get $\sum_{k=1}^{k_n} P\left(|X_{nk}| > \frac{\delta}{128\eta}\right) \leqslant \frac{1}{256\eta}$, and we have

$$\begin{aligned}
&\max_{t \geqslant f(\delta^q)} \max_{1 \leqslant k \leqslant k_n} \frac{1}{g^{1/q}(t)} |EY_{nk}| \\
&\quad \leqslant \max_{t \geqslant f(\delta^q)} \max_{1 \leqslant k \leqslant k_n} \frac{1}{g^{1/q}(t)} E|Y_{nk}| \\
&\quad \leqslant \max_{t \geqslant f(\delta^q)} \max_{1 \leqslant k \leqslant k_n} \left[\frac{1}{g^{1/q}(t)} E|X_{nk}| I\left(|X_{nk}| \leqslant \frac{\delta}{128\eta}\right)\right. \\
&\qquad \left. + \frac{1}{g^{1/q}(t)} E|X_{nk}| I\left(\frac{\delta}{128\eta} < |X_{nk}| \leqslant g^{1/q}(t)\right) + P\left(|X_{nk}| > g^{1/q}(t)\right)\right] \\
&\quad \leqslant \max_{t \geqslant f(\delta^q)} \max_{1 \leqslant k \leqslant k_n} \left[\frac{1}{g^{1/q}(t)} \cdot \frac{\delta}{128\eta} + P\left(|X_{nk}| > \frac{\delta}{128\eta}\right) + P\left(|X_{nk}| > g^{1/q}(t)\right)\right] \\
&\quad \leqslant \frac{1}{128\eta} + \sum_{k=1}^{k_n} P\left(|X_{nk}| > \frac{\delta}{128\eta}\right) + \sum_{k=1}^{k_n} P\left(|X_{nk}| > \delta\right) \\
&\quad \leqslant \frac{1}{128\eta} + 2\sum_{k=1}^{k_n} P\left(|X_{nk}| > \frac{\delta}{128\eta}\right) \\
&\quad \leqslant \frac{1}{64\eta}.
\end{aligned}$$

Hence, for all n large enough it follows that,

$$\max_{1 \leqslant k \leqslant k_n} |EY_{nk}| < \frac{g^{1/q}(t)}{64\eta}, \quad g(t) > \delta^q, \tag{3.2}$$

noting that $|Y_{nk}| \leqslant |X_{nk}|$, we have by (3.2) and assumption (a) that

$$\begin{aligned}
I_7 &\leqslant C\sum_{n=1}^{\infty} c_n \int_{f(\delta^q)}^{\infty} P\left(\max_{1 \leqslant k \leqslant k_n} |Y_{nk}| \geqslant \frac{g^{1/q}(t)}{192\eta}\right) dt \\
&\leqslant C\sum_{n=1}^{\infty} c_n \int_{f(\delta^q)}^{\infty} P\left(\max_{1 \leqslant k \leqslant k_n} |X_{nk}| \geqslant \frac{g^{1/q}(t)}{192\eta}\right) dt \\
&\leqslant C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \int_{f(\delta^q)}^{\infty} P\left(|X_{nk}| \geqslant \frac{g^{1/q}(t)}{192\eta}\right) dt \\
&\leqslant C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \int_{f(\delta^q)}^{\infty} Ef\left((192\eta)^q |X_{nk}|^q I(|X_{nk}| > \delta)\right) < \infty.
\end{aligned}$$

Since $\eta > q \geqslant 1$, it follows by the C_r-inequality that

$$\begin{aligned}
I_8 \leqslant & C\sum_{n=1}^{\infty} c_n \int_{f(\delta^q)}^{\infty} g^{-2\eta/q}(t)\left(\sum_{k=1}^{k_n} E\left(Y_{nk}-EY_{nk}\right)^2\right)^{\eta} dt \\
\leqslant & C\sum_{n=1}^{\infty} c_n \int_{f(\delta^q)}^{\infty} g^{-2\eta/q}(t)\left(\sum_{k=1}^{k_n} E\left(Y_{nk}\right)^2\right)^{\eta} dt \\
= & C\sum_{n=1}^{\infty} c_n \int_{f(\delta^q)}^{\infty} g^{-2\eta/q}(t)\left(\sum_{k=1}^{k_n} EX_{nk}^2 I\left(|X_{nk}| \leqslant g^{1/q}(t)\right)+\sum_{k=1}^{k_n} P\left(|X_{nk}| > g^{1/q}(t)\right)\right)^{\eta} dt \\
\leqslant & C\sum_{n=1}^{\infty} c_n \int_{f(\delta^q)}^{\infty} g^{-2\eta/q}(t)\left(\sum_{k=1}^{k_n} EX_{nk}^2 I\left(|X_{nk}| \leqslant \delta\right)\right)^{\eta} dt \\
& + C\sum_{n=1}^{\infty} c_n \int_{f(\delta^q)}^{\infty} g^{-2\eta/q}(t)\left(\sum_{k=1}^{k_n} EX_{nk}^2 I\left(\delta < |X_{nk}| \leqslant g^{1/q}(t)\right)\right)^{\eta} dt \\
& + C\sum_{n=1}^{\infty} c_n \int_{f(\delta^q)}^{\infty} \left(\sum_{k=1}^{k_n} P\left(|X_{nk}| > g^{1/q}(t)\right)\right)^{\eta} dt \\
\triangleq & I_9 + I_{10} + I_{11}.
\end{aligned}$$

Note that function $s(t)$ is nondecreasing, for $\eta > q \geqslant 1$, $g^{-2\eta/q}(t) \leqslant g^{-\eta/q}(t)s(t)$. Combining the assumption (b) and (d), we have

$$I_9 = C\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} EX_{nk}^2 I\left(|X_{nk}| \leqslant \delta\right)\right)^{\eta} \times \int_{f(\delta^q)}^{\infty} g^{-2\eta/q}(t)dt < \infty,$$

for I_{10}, we can get that

$$I_{10} \leqslant C\sum_{n=1}^{\infty} c_n \int_{f(\delta^q)}^{\infty} g^{-2\eta/q}(t)\left(g^{1/q}(t)\sum_{k=1}^{k_n} E|X_{nk}| I\left(\delta < |X_{nk}| \leqslant g^{1/q}(t)\right)\right)^{\eta} dt.$$

By condition (c), it follows that

$$\begin{aligned}
\sum_{k=1}^{k_n} E|X_{nk}| I\left(\delta < |X_{nk}| \leqslant g^{1/q}(t)\right) & \leqslant \sum_{k=1}^{k_n} E|X_{nk}| I\left(|X_{nk}| > \delta\right) \\
& \leqslant \delta^{1-q}\sum_{k=1}^{k_n} E|X_{nk}|^q I\left(|X_{nk}| > \delta\right) \\
& \leqslant \delta^{1-q}\sum_{k=1}^{k_n} E|X_{nk}|^q I\left(|X_{nk}| > \frac{\delta}{128\eta}\right) \to 0.
\end{aligned}$$

Hence, for all n large enough,

$$\sum_{k=1}^{k_n} E|X_{nk}| I\left(\delta < |X_{nk}| \leqslant g^{1/q}(t)\right) < 1,$$

for $\eta > q \geqslant 1$ we can yield that

$$\left(\sum_{k=1}^{k_n} E|X_{nk}| I\left(\delta < |X_{nk}| \leqslant g^{1/q}(t)\right)\right)^{\eta} \leqslant \sum_{k=1}^{k_n} E|X_{nk}| I\left(\delta < |X_{nk}| \leqslant g^{1/q}(t)\right).$$

Therefore by conditions (a) and (d)

$$\begin{aligned}
I_{10} &\leqslant C\sum_{n=1}^{\infty} c_n \int_{f(\delta^q)}^{\infty} g^{-\eta/q}(t) \left(\sum_{k=1}^{k_n} \frac{E|X_{nk}| I\left(\delta < |X_{nk}| \leqslant g^{1/q}(t)\right)}{f\left(|X_{nk}| I\left(\delta < |X_{nk}| \leqslant g^{1/q}(t)\right)\right)}\right) \\
&\quad \times f\left(|X_{nk}| I\left(\delta < |X_{nk}| \leqslant g^{1/q}(t)\right)\right) dt \\
&\leqslant C\sum_{n=1}^{\infty} c_n \int_{f(\delta^q)}^{\infty} g^{-\eta/q}(t) s(t) \left(\sum_{k=1}^{k_n} f\left(|X_{nk}| I\left(\delta < |X_{nk}| \leqslant g^{1/q}(t)\right)\right)\right) dt \\
&\leqslant C\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} f\left(|X_{nk}| I\left(\delta < |X_{nk}| \leqslant g^{1/q}(t)\right)\right)\right) \times \int_{f(\delta^q)}^{\infty} g^{-n/q}(t) s(t) dt \\
&< \infty.
\end{aligned}$$

For $t \geqslant f(\delta^q)$, it follows from Markov's inequality and condition (c) that for all n large enough,

$$\begin{aligned}
\sum_{k=1}^{k_n} P\left(|X_{nk}| > g^{1/q}(t)\right) &\leqslant \sum_{k=1}^{k_n} P\left(|X_{nk}| > \delta\right) \leqslant \sum_{k=1}^{k_n} E|X_{nk}| I\left(|X_{nk}| > \delta\right) \\
&\leqslant \delta^{1-q} \sum_{k=1}^{k_n} E|X_{nk}|^q I\left(|X_{nk}| > \frac{\delta}{128\eta}\right) \to 0,
\end{aligned}$$

which implies that for all n large enough (we recall that $\eta > q \geqslant 1$)

$$\sum_{k=1}^{k_n} P\left(|X_{nk}| > g^{1/q}(t)\right) < 1,$$

and hence

$$\left(\sum_{k=1}^{k_n} P\left(|X_{nk}| > g^{1/q}(t)\right)\right)^{\eta} \leqslant \sum_{k=1}^{k_n} P\left(|X_{nk}| > g^{1/q}(t)\right).$$

Thus, we have by condition (a') and that

$$\begin{aligned}
I_{11} &\leqslant C\sum_{n=1}^{\infty} c_n \int_{f(\delta^q)}^{\infty} \sum_{k=1}^{k_n} P\left(|X_{nk}| > g^{1/q}(t)\right) dt \\
&\leqslant C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} Ef\left(|X_{nk}|^q I\left(|X_{nk}| > \delta\right)\right) \\
&< \infty.
\end{aligned}$$

The proof of the theorem is completed. □

4. Corollaries

According to Theorem 3.1, we can get the following important corollaries.

COROLLARY 4.1. *Let $\{X_{nk}, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$ be an array of rowwise NSD random variables, and $\{c_n, n \geqslant 1\}$ be a sequence of positive constants. Under the conditions of Theorem 3.1, for any $\varepsilon > 0$, we have*

$$\sum_{n=1}^{\infty} c_n P\left(\max_{1\leqslant m\leqslant k}\left|\sum_{k=1}^{m}\left(X_{nk}-EX_{nk}I\left(|X_{nk}|\leqslant\delta\right)\right)\right|>\varepsilon\right)<\infty.$$

Proof.

$$\begin{aligned}
&\sum_{n=1}^{\infty} c_n Ef\left(\left\{\max_{1\leqslant m\leqslant k}\left|\sum_{k=1}^{m}\left(X_{nk}-EX_{nk}I\left(|X_{nk}|\leqslant\delta\right)\right)-\varepsilon\right|\right\}_{+}^{q}\right)\\
&=\sum_{n=1}^{\infty} c_n \int_0^{\infty} P\left(\left[f\left(\left\{\max_{1\leqslant m\leqslant k}\left|\sum_{k=1}^{m}\left(X_{nk}-EX_{nk}I\left(|X_{nk}|\leqslant\delta\right)\right)-\varepsilon\right|\right\}_{+}^{q}\right)>t\right]dt\right.\\
&=\sum_{n=1}^{\infty} c_n \int_0^{\infty} P\left(\left\{\max_{1\leqslant m\leqslant k}\left|\sum_{k=1}^{m}\left(X_{nk}-EX_{nk}I\left(|X_{nk}|\leqslant\delta\right)\right)-\varepsilon\right|\right\}_{+}^{q}>g(t)\right)dt\\
&=\sum_{n=1}^{\infty} c_n \int_0^{\infty} P\left[\max_{1\leqslant m\leqslant k}\left|\sum_{k=1}^{m}\left(X_{nk}-EX_{nk}I\left(|X_{nk}|\leqslant\delta\right)\right)\right|>g^{1/q}(t)+\varepsilon\right]dt\\
&\geqslant\sum_{n=1}^{\infty} c_n \int_0^{f(\varepsilon^q)} P\left[\max_{1\leqslant m\leqslant k}\left|\sum_{k=1}^{m}\left(X_{nk}-EX_{nk}I\left(|X_{nk}|\leqslant\delta\right)\right)\right|>\varepsilon+\varepsilon\right]dt\\
&\geqslant f\left(\varepsilon^q\right)\sum_{n=1}^{\infty} c_n P\left(\max_{1\leqslant m\leqslant k}\left|\sum_{k=1}^{m}\left(X_{nk}-EX_{nk}I\left(|X_{nk}|\leqslant\delta\right)\right)\right|>2\varepsilon\right).
\end{aligned}$$

Thus,

$$\sum_{n=1}^{\infty} c_n P\left(\max_{1\leqslant m\leqslant k}\left|\sum_{k=1}^{m}\left(X_{nk}-EX_{nk}I\left(|X_{nk}|\leqslant\delta\right)\right)\right|>\varepsilon\right)<\infty. \quad \square$$

COROLLARY 4.2. *Let $\{X_{nk}, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$ be an array of rowwise NSD random variables, and $\{c_n, n \geqslant 1\}$ be a sequence of positive constants. Suppose that the following conditions satisfied:*

(a) $\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} E\left(192\eta\,|X_{nk}|^q I\left(|X_{nk}|>\varepsilon\right)\right)<\infty$, *for every* $\varepsilon > 0$*;*

(b) there exists $\eta > q, 0 < p \leqslant 2$*, and* $\delta > 0$ *such that*

$$\sum_{n=1}^{\infty} c_n\left(\sum_{k=1}^{k_n} E\,|X_{nk}|^p I\left(|X_{nk}|\leqslant\delta\right)\right)^{\eta}<\infty;$$

(c) $\sum_{k=1}^{k_n} E|X_{nk}|^q I\left(|X_{nk}| > \frac{\delta}{128\eta}\right) \to 0$, *as* $n \to \infty$;

Then the fall $\varepsilon > 0$

$$\sum_{n=1}^{\infty} c_n E\left\{\max_{1\leqslant m\leqslant k_n}\left|\sum_{k=1}^{m}(X_{nk} - EX_{nk}I(|X_{nk}| \leqslant \delta))\right| - \varepsilon\right\}_+^q < \infty.$$

Proof. Let $f(t) = t$, $t \geqslant 0$, and note that $|X_{nk}|I(|X_{nk}| \leqslant \delta) \leqslant 2\delta$, by assumptions (b), we have

$$\begin{aligned} &C\sum_{n=1}^{\infty} c_n\left(\sum_{k=1}^{k_n} EX_{nk}^2 I(|X_{nk}| \leqslant \delta)\right)^{\eta} \\ &\quad \leqslant C(2\delta)^{(2-p)\eta}\sum_{n=1}^{\infty} c_n\left(\sum_{k=1}^{k_n} E|X_{nk}|^p I(|X_{nk}| \leqslant \delta)\right)^{\eta} < \infty. \end{aligned}$$

Then, combined with the proof of Theorem 3.1, this completes the proof of the corollary. □

REMARK 4.1. We point out that Theorem 1.2 in [8] is a special case of Corollary 4.2 with $p = 2$. Actually, conditions (i) and (iii) of Theorem 1.2 are equivalent to conditions (a) and (c) of Corollary 4.2.

COROLLARY 4.3. *Let* $q \geqslant 1$, $\{X_{nk}, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$ *be an array of rowwise NSD random variables,* $EX_{nk} = 0$, *and* $\{c_n, n \geqslant 1\}$ *be a sequence of positive constants. Suppose that the following conditions are satisfied:*

(a) $\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} E(192\eta|X_{nk}|^q I(|X_{nk}| > \varepsilon)) < \infty$ *for every* $\varepsilon > 0$;

(b) for some $\delta > 0$, *there exists* $\eta > q$ *such that*

$$\sum_{n=1}^{\infty} c_n\left(\sum_{k=1}^{k_n} EX_{nk}^2 I(|X_{nk}| \leqslant \delta)\right)^{\eta} < \infty;$$

(c) $\sum_{k=1}^{k_n} E|X_{nk}|^q I\left(|X_{nk}| > \frac{\delta}{128\eta}\right) \to 0$, *as* $n \to \infty$.

Then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n E\left\{\max_{1\leqslant m\leqslant k_n}\left|\sum_{k=1}^{m} X_{nk}\right| - \varepsilon\right\}_+^q < \infty,$$

and

$$\sum_{n=1}^{\infty} c_n P\left(\max_{1\leqslant m\leqslant k_n}\left|\sum_{k=1}^{m} X_{nk}\right| > \varepsilon\right) < \infty.$$

REMARK 4.2. Corollary 4.3 is similar to Corollary 3.1 in Shen [8]. We omit the proof.

Taking $k_n = n$, $c_n = 1$, $n \geqslant 1$, and replacing X_{nk} by X_{nk}/a_n in Corollary 4.3, $\{a_n, n \geqslant 1\}$ is a sequence of positive real numbers, we get the following corollary:

COROLLARY 4.4. *Let $q \geqslant 1$, $\{X_{nk}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ be an array of rowwise NSD random variables, $EX_{nk} = 0$. Suppose that the following conditions are satisfied:*

(a) $\sum_{n=1}^{\infty} a_n^{-q} \sum_{k=1}^{n} E\left(192\eta\,|X_{nk}|^q I(|X_{nk}| > \varepsilon a_n)\right) < \infty$ *for every $\varepsilon > 0$;*

(b) for some $\delta > 0$, there exists $\eta > q$ such that

$$\sum_{n=1}^{\infty} a_n^{-2\eta} \left(\sum_{k=1}^{n} EX_{nk}^2 I(|X_{nk}| \leqslant \delta a_n) \right)^{\eta} < \infty;$$

(c) $a_n^{-q} \sum_{k=1}^{n} E|X_{nk}|^q I\left(|X_{nk}| > \frac{\delta a_n}{128\eta}\right) \to 0$, *as $n \to \infty$.*

Then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} a_n^{-q} E\left\{ \max_{1 \leqslant m \leqslant n} \left| \sum_{k=1}^{m} X_{nk} \right| - \varepsilon a_n \right\}_+^q < \infty,$$

and

$$\sum_{n=1}^{\infty} P\left(\max_{1 \leqslant m \leqslant n} \left| \sum_{k=1}^{m} X_{nk} \right| > \varepsilon a_n \right)_+ < \infty.$$

Taking $k_n = n$, $c_n = n^{\alpha r - 2}$, $n \geqslant 1$, and replacing X_{nk} by X_{nk}/n^{α} in Corollary 4.3, we get the following corollary.

COROLLARY 4.5. *Let $q \geqslant 1$, $\alpha > 0$, $\alpha r > 0$, $\{X_{nk}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ be an array of rowwise NSD random variables, $EX_{nk} = 0$. Suppose that the following conditions are satisfied:*

(a) $\sum_{n=1}^{\infty} n^{\alpha r - 2 - \alpha q} \sum_{k=1}^{n} E\left(192\eta\,|X_{nk}|^q I(|X_{nk}| > \varepsilon n^{\alpha})\right) < \infty$ *for every $\varepsilon > 0$;*

(b) for some $\delta > 0$, there exists $\eta > q$ such that

$$\sum_{n=1}^{\infty} n^{\alpha r - 2 - 2\alpha\eta} \left(\sum_{k=1}^{n} EX_{nk}^2 I(|X_{nk}| \leqslant \delta n^{\alpha}) \right)^{\eta} < \infty;$$

(c) $n^{-\alpha q} \sum_{k=1}^{n} E|X_{nk}|^q I\left(|X_{nk}| > \frac{\delta n^{\alpha}}{128\eta}\right) \to 0$, *as $n \to \infty$.*

Then for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} n^{\alpha r-2-\alpha q} E\left\{\max_{1\leqslant m\leqslant n}\left|\sum_{k=1}^{m} X_{nk}\right| - \varepsilon n^{\alpha}\right\}_{+}^{q} < \infty,$$

and

$$\sum_{n=1}^{\infty} n^{\alpha r-2} P\left(\max_{1\leqslant m\leqslant n}\left|\sum_{k=1}^{m} X_{nk}\right| > \varepsilon n^{\alpha}\right)_{+} < \infty.$$

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