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Stabilization and Optimal Control for Discrete-time Markov Jump Linear System with Multiplicative Noises and Input Delays: A Complete Solution

Chunyan Han, Zidong Wang, *Fellow, IEEE*, Huanshui Zhang, *Senior Member, IEEE*, and Paresh Date

***Abstract*—This paper investigates the stabilization and optimal control problems for Markov jump linear system (MJLS) with multiplicative noises and input delays. By overcoming the substantive difficulty resulting from the invalidity of the separation principle, we provide a complete solution to the addressed problems by means of: 1) necessary and sufficient solvability, and the analytical formula on optimal finite horizon control in line with a generalized coupled difference Riccati equation; and 2) necessary and sufficient stabilizability, and the explicit expression of the optimal controller on infinite horizon according to a delayed generalized coupled algebraic Riccati equation (D-GCARE). It is shown that the MJLS with multiplicative noises and input delays is mean square stabilizable if and only if the D-GCARE has a specific solution. Our main results are attained through the creation of a novel delayed stochastic Markov maximum principle as well as the construction of a novel class of delayed Markov Lyapunov function.**

***Index Terms*—Stabilization, optimal control, Markov jump linear system, input delay, multiplicative noise, maximum principle**

I. Introduction

In many practical applications, time delays are an essential feature that cannot be neglected. For example, due primarily to data collisions, network congestion and transmission errors, the information transmitted through communication networks of limited bandwidth may suffer time delays which, if not appropriately tackled, will deteriorate the control performance of the closed-loop systems [13], [14], [17], [18], [20], [25], [26], [28], [42]. So far, plenty of research effort has been dedicated to the stabilization and optimal control for deterministic delayed systems [1], [31], [39]. For example, in [39], the control problem was studied for a single deterministic input-delayed system and an optimal controller was developed by the Smith predictor. In [1], a reduction technique was developed to convert the stabilization problem (subject to input delays) to an equivalent delay-free one. In [31], the optimal control and its application were investigated for the linear systems with multiple input/state time-lags, while the controller was attained by solving one type of partial differential/difference Riccati equations. In [44], the linear quadratic regulation for the multiple input-delayed system was investigated by using the duality method as the key technique.

Apart from the time delays, stochastic noises serve as another source for complicating the system analysis, and time-delay systems with multiplicative noises have recently been a research focus attracting much attention from the control community [2]–[5], [23], [24], [34], [36], [47], [48]. In [22], [45], [46], a novel maximum principle was established for solving the optimal finite horizon control for the systems possessing multiplicative noise and input delays simultaneously, where the existence condition and explicit solution for the optimal controller were supplied on the basis of a delayed difference Riccati equation. The stabilizability was investigated for the infinite horizon case. In fact, the results developed in [22], [45], [46] can be regarded as an extension of the deterministic delayed systems. It is important to realize that the optimal controller for systems with additive/multiplicative noises is with a unified form which can be expressed as a multiplication of a deterministic gain and a state predictor.

Markov jump linear systems (MJLSs) with multiplicative noises are usually termed as stochastic MJLSs which have found wide applications in practice such as nuclear fission, heater transfer, population dynamics and immunology, etc. The stabilization and optimal control problems for MJLS have received persistent research attention ever since the 1960s [6]–[10], [19], [21], [30], [37], [40]. In general, there are two methodologies available for deriving the optimal controller for MJLSs, namely, the maximum principle and the dynamic programming approach. According to different versions of the concepts of controllability and observability (detectability), different stabilization conditions have been developed for the MJLS, see [6], [7], [21].

In recent years, the optimal control for stochastic MJLSs has attained an increasing research interest, some recent results have been developed on optimal mean-variance control [11] and indefinite linear quadratic control [12], [15], [16], [32]. As for the stabilization control problem for state delayed MJLSs, a good many excellent results have been published, see e.g. [27], [29], [33], [35], [41], [43], [50]. Unfortunately,

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Chunyan Han is with the School of Electrical Engineering, University of Jinan, Jinan 250022, P. R. China. (e-mail: cyhan823@hotmail.com).

Zidong Wang is with the Department of Computer Science, Brunel University London, Uxbridge, Middlesex, UB8 3PH, United Kingdom (e-mail: Zidong.Wang@brunel.ac.uk).

Huanshui Zhang is with the College of Electrical Engineering and Automation, Shandong University of Science and Technology, Qingdao 266510, P. R. China. (e-mail: hszhang@sdu.edu.cn).

Paresh Date is with the Department of Mathematics, Brunel University London, Uxbridge, Middlesex, UB8 3PH, United Kingdom (e-mail: paresh.date@brunel.ac.uk)

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the optimality issue of the designed controller has not been considered in [27], [29], [33], [35], [41], [43], [50] and, as far as we know, no corresponding results have been found on the input-delayed stochastic MJLS. As such, distinct from the existed stabilization works for state delayed MJLS, the optimal control problems will be investigated for stochastic MJLSs with input delay and the stabilizability will also be revealed within the optimal control framework.

To date, the stabilization and optimal control problems for stochastic MJLSs with *input* delays are still open despite the numerous results on delayed MJLSs. The main reason for such a lack of fundamental progress towards the stochastic MJLSs with input delays is twofold. First, by comparison with the prevenient results for delayed systems with additive/multiplicative noises and for delay-free MJLSs, *the separation principle* is no longer applicable for stochastic MJLS with *input* delays, that is, the optimal controller for input-delayed stochastic MJLSs can *no longer* be expressed as the Smith predictor form owing to the dependence property of jumping parameters between the adjacent time points, this is the basic obstacle faced in this study. We also note that, when the state is unknown, the separation principle does not hold in the case of systems with multiplicative noises, as pointed out in [47], [48]. In this case, the estimation error is dependent on the control input. Only a suboptimal controller can be obtained there by applying "enforced separation principle", and the conditional mean estimate is required for implementing the controller. Second, although the state augmentation approach can be employed to handle the input-delays, the introduction of the extended state would result in an immense burden in computation [38], and the second challenge would then be to reduce the computation complexity while preserving the convenience of the controller design.

The aim of this study is to supply a complete solution to the stabilization and optimal control of stochastic MJLS possessing input delays. The primary innovations of this study are indicated as below. 1) A novel class of delayed stochastic Markov maximum principle (D-SMMP) with regard to optimal control is established and its analytic solution is deduced, which offers the theoretical basis for solving the optimal control of stochastic MJLS subject to input delay. 2) Necessary and sufficient condition, which is given in an explicit form, is established for the existence of the optimal finite-horizon control. The optimal controller, which is in the feedback form of current state and history inputs, is designed by means of a delayed generalized coupled difference Riccati equation (D-GCDRE). 3) A novel type of delayed Lyapunov function, expressed as the optimal cost function, is put forward for the stabilization problem, and then the necessary and sufficient stabilization conditions along with the infinite-horizon optimal controller are derived on the basis of a new D-GCARE.

The structure of this paper is listed as below. In Section II, the optimal control over a finite-horizon is studied. A new maximum principle is introduced first. By applying the new technique, the finite-horizon controller is designed and the existence condition is established. In Section III, the stabilization and optimal controller on infinite horizon are supplied. Section IV numerically illustrates the solvability condition of finite-horizon control, and verifies the infinite-horizon stabilization condition. Section V draws some concluding conclusions.

Notations: Denote by R^n the n-dimensional Euclidean space, $R^{m\times n}$ the set of all $m\times n$ matrices. For a matrix $L\in R^{n\times n}$, let L' be the transpose of L, $L>0$ ($L\geq 0$) be its positive definite (positive semi-definite) matrix. Use $\mathrm{P}(\cdot)$ and $\mathrm{E}(\cdot)$ to denote the occurrence probability and expectation operator, respectively.

II. Finite-horizon Optimal Control

A. Problem Statement

The stochastic MJLS with r-step input delay is considered, which runs up to a final time N:

$$\begin{aligned} x(t+1)=&(A_{\theta_t}+\omega_t\bar{A}_{\theta_t})x(t)\\ &+(B_{\theta_t}+\omega_t\bar{B}_{\theta_t})u(t-r), \end{aligned}\tag{1}$$

where $x(t)\in\mathrm{R}^n$ represents the current state and $u(t)\in\mathrm{R}^m$ represents the input with delay $r\geq 0$. θ_t means a homogeneous discrete-time Markov chain with finite-state space $\Theta\triangleq\{1,2,\cdots,L\}$ and transition probability matrix $\Pi\triangleq(\lambda_{ij})_{L\times L}$. Let $\pi_0=[\pi_0^{(1)},\cdots,\pi_0^{(L)}]$ be the initial distribution of θ_t, so that $\pi_t, 0\leq t\leq N$ can be obtained. Here, ω_t stands for the zero-mean multiplicative noise with covariance μ^2. If $\theta_t=i\in\{1,\cdots,L\}$, we set $A_i,B_i,\bar{A}_i,\bar{B}_i$ for the coefficient matrices of system (1). The initial values x_0 and $u(-j), 1\leq j\leq r$ are known.

As for the sequences $\{\theta_t\}_{t\geq 0}$ and $\{\omega_t\}_{t\geq 0}$, we make several assumptions below.

Assumption 1: For each $t\geq 0$, the $\sigma-$algebra $\mathcal{F}_t$ is independent of the $\sigma-$algebra $\mathcal{G}_t$, where $\mathcal{F}_t=\sigma[\omega_0,\cdots,\omega_t]$ and $\mathcal{G}_t=\sigma[\theta_0,\cdots,\theta_t]$.

Assumption 2: $\{\theta_t\}_{t\geq 0}$ and $\{\omega_t\}_{t\geq 0}$ are independent of the initial values x_0 and $u(i)$ $(i=-r,\cdots,-1)$.

In conjunction with the $\sigma-$algebras $\mathcal{F}_t$ and $\mathcal{G}_t$, we put forward an algebra as follows:

$$\mathcal{H}_t=\begin{cases}\sigma\{\theta_l,\omega_s,0\leq l\leq t,0\leq s\leq t-1\}, & \\ & \text{if } t\geq 1;\\ \sigma\{\theta_0\}, & \text{if } t=0.\end{cases}$$

Obviously, we have $\mathrm{E}\{w_t|\mathcal{H}_t\}=0, \mathrm{E}\{\omega_t\omega_s|\mathcal{H}_t\}=\mu^2\delta_{t,s}$.

For system (1), the state $x(t)$ and jumping parameter θ_t are acquired to the current time. In this situation, the optimal control $u(t)$ is $\mathcal{H}_t$-measurable.

The quadratic cost function with relation to system (1) is defined as

$$\begin{aligned} J_N=&\mathrm{E}[\sum_{t=0}^{N}x(t)'Q_{\theta_t}x(t)+\sum_{t=r}^{N}u(t-r)'R_{\theta_t}u(t-r)\\ &+x(N+1)'P_{\theta_{N+1}}x(N+1)], \end{aligned}\tag{2}$$

where $N>r$ is the terminal time, $P_{\theta_{N+1}},Q_{\theta_t}$, and R_{θ_t} are positive semi-definite matrices of compatible dimensions.

The admissible control set is defined as follows:

$$\begin{aligned} \mathcal{U}_{ad}\triangleq\{&u(-r),\cdots,u(-1),u(0),\cdots,u(N-r)|u(t)\in R^m,\\ &u(t)\text{ is }\mathcal{H}_t-\text{measurable, and }\sum_{t=-r}^{N-r}\mathrm{E}[u(t)'u(t)]<+\infty\}, \end{aligned}\tag{3}$$

and any $u(t)\in\mathcal{U}_{ad}$ is called an admissible control. Therefore, the finite-horizon optimal control for stochastic MJLS with input delay can be described as:

Problem 1: Find an admissible controller $u(t) \in \mathcal{U}_{ad}$ such that (2) is minimized according to system (1).
Note that, in case of no jumping parameters, system (1) would reduce to the stochastic system [46]

$$x(t+1) = (A + \omega_t \bar{A})x(t) + (B + \omega_t \bar{B})u(t-r),$$

where ω_t is a scalar random white noise, and then the $\mathcal{F}_{t-r-1}$-measurable controller can be described as

$$\begin{aligned} u(t-r) &\triangleq \mathrm{E}\{f(\omega_t, t)x(t)|\mathcal{F}_{t-r-1}\} \\ &= \mathrm{E}\{f(\omega_t, t)|\mathcal{F}_{t-r-1}\}\mathrm{E}\{x(t)|\mathcal{F}_{t-r-1}\}, \end{aligned}$$

where $\mathrm{E}\{f(\omega_t, t)|\mathcal{F}_{t-r-1}\}$ denotes the controller gain [46].

B. Establishment of D-SMMP

To begin with, we develop a new type of stochastic maximum principle for Problem 1 to deal with the correlation of θ_t between adjacent times and the presence of input delay.

Lemma 1: (Maximum Principle) Assume that Problem 1 is solvable. Then, the optimal controller $u(t-r)$ is $\mathcal{H}_{t-r}$-measurable and satisfies

$$0 = \mathrm{E}[\Gamma_{\theta_t}(t)'\eta_t + R_{\theta_t}u(t-r)|\mathcal{H}_{t-r}], t = r, \cdots, N, \tag{4}$$

where η_t represents the costate and satisfies

$$\eta_N = P_{\theta_{N+1}}x(N+1), \tag{5}$$

$$\eta_{t-1} = \mathrm{E}[\Phi_{\theta_t}(t)'\eta_t + Q_{\theta_t}x(t)|\mathcal{H}_t],\ t = 0, \cdots, N \tag{6}$$

with

$$\begin{aligned} \Phi_{\theta_t}(t) &= A_{\theta_t} + \omega_t \bar{A}_{\theta_t}, \\ \Gamma_{\theta_t}(t) &= B_{\theta_t} + \omega_t \bar{B}_{\theta_t}. \end{aligned}$$

Proof: See Appendix A. ■

It is obvious that (1), (4), (5), and (6) make up a delayed forward backward stochastic Markov difference equation (D-FBSMDE):

$$\begin{cases} x(t+1) = \Phi_{\theta_t}(t)x(t) + \Gamma_{\theta_t}(t)u(t-r), \\ 0 = \mathrm{E}[\Gamma_{\theta_t}(t)'\eta_t + R_{\theta_t}u(t-r)|\mathcal{H}_{t-r}] \\ \quad \text{for } \ t = r, \cdots, N, \\ \eta_{t-1} = \mathrm{E}[\Phi_{\theta_t}(t)'\eta_t + Q_{\theta_t}x(t)|\mathcal{H}_t] \\ \quad \text{for } \ t = 0, \cdots, N, \\ \eta_N = P_{\theta_{N+1}}x(N+1). \end{cases} \tag{7}$$

Next, we seek for the unique solution to D-FBMDE (7). To this end, some new notations need to be defined. For any jumping parameter matrices $X_{\theta_{t-r}}, X_{\theta_{t-r+1}}, \cdots, X_{\theta_{t-s}}(-1 \leq s < r)$, denote $\bar{X}_{\theta_{t-s},\theta_{t-r}} = \prod_{i=s}^{r} X_{\theta_{t-i}}$, and define a new set of linear evolution operators $\mathcal{L}_{\theta_{t-s-1}}(.), \mathcal{L}_{\theta_{t-s-2}}(.), \cdots, \mathcal{L}_{\theta_{t-r}}(.)$ as

$$\begin{aligned} \mathcal{L}_{\theta_{t-s-1}}(\bar{X}_{\theta_{t-s},\theta_{t-r}}) &\triangleq \mathrm{E}[\bar{X}_{\theta_{t-s},\theta_{t-r}}|\mathcal{H}_{t-s-1}] \\ &= \sum_{\theta_{t-s}=1}^{L} \lambda_{\theta_{t-s-1},\theta_{t-s}}\bar{X}_{\theta_{t-s},\theta_{t-r}}, \\ \mathcal{L}_{\theta_{t-s-2}}(\bar{X}_{\theta_{t-s},\theta_{t-r}}) &\triangleq \mathrm{E}[\bar{X}_{\theta_{t-s},\theta_{t-r}}|\mathcal{H}_{t-s-2}] \\ &= \mathrm{E}[\mathrm{E}[\bar{X}_{\theta_{t-s},\theta_{t-r}}|\mathcal{H}_{t-s-1}]|\mathcal{H}_{t-s-2}] \\ &= \sum_{\theta_{t-s-1}=1}^{L} \lambda_{\theta_{t-s-2},\theta_{t-s-1}}\mathcal{L}_{\theta_{t-s-1}}(\bar{X}_{\theta_{t-s},\theta_{t-r}}), \\ &\vdots \\ \mathcal{L}_{\theta_{t-r}}(\bar{X}_{\theta_{t-s},\theta_{t-r}}) &\triangleq \mathrm{E}[\bar{X}_{\theta_{t-s},\theta_{t-r}}|\mathcal{H}_{t-r}] \\ &= \mathrm{E}[\mathrm{E}[\bar{X}_{\theta_{t-s},\theta_{t-r}}|\mathcal{H}_{t-r+1}]|\mathcal{H}_{t-d}] \\ &= \sum_{\theta_{t-r+1}=1}^{L} \lambda_{\theta_{t-r},\theta_{t-r+1}}\mathcal{L}_{\theta_{t-r+1}}(\bar{X}_{\theta_{t-s},\theta_{t-r}}), \end{aligned} \tag{8}$$

where $\lambda_{\theta_{t-s-1},\theta_{t-s}}(-1 \leq s < r)$ is the transition probability of θ_{t-s}, which is defined in Subsection A.It needs to be stated that $X_{\theta_{t-s}}(-1 \leq s \leq r)$ defined above can be chosen as the identity matrix or some multiplication of jumping parameter matrices concerning with θ_{t-s}.

For simplicity, we still use $\theta_{t-i}(-1 \leq i \leq r)$ to denote the possible realization of the Markov chain in this section, where $\theta_{t-i} \in \{1, 2, \cdots, L\}$. So, θ_{t-i} is used as the summation index in the definition of (8). In the system (1) and performance index (2), θ_t represents the variable of Markov chain.

Furthermore, for any $\mathcal{H}_t$-measurable function $\tilde{\Phi}_{\theta_t}(t)$, we have

$$\begin{aligned} \mathrm{E}\{\tilde{\Phi}_{\theta_t}(t)x(t)|\mathcal{H}_{t-r}\} =& \mathcal{L}_{\theta_{t-r}}(\tilde{\Phi}_{\theta_t}(t)\tilde{F}_{\theta_{t-1},\theta_{t-r}})x(t-r) \\ &+ \sum_{i=1}^{r} \mathcal{L}_{\theta_{t-r}}(\tilde{\Phi}_{\theta_t}(t)\tilde{F}_{\theta_{t-1},\theta_{t-i+1}} \\ &\times (B_{\theta_{t-i}} + \omega_{\theta_{t-i}}\bar{B}_{\theta_{t-i}}))u(t-i-r), \end{aligned} \tag{9}$$

where the matrix $\tilde{F}_{\theta_{t-s},\theta_{t-i}} = (A_{\theta_{t-s}} + \omega_{t-s}\bar{A}_{\theta_{t-s}}) \cdots (A_{\theta_{t-i}} + \omega_{t-i}\bar{A}_{\theta_{t-i}}), \tilde{F}_{\theta_{t-i-1},\theta_{t-i}} = I$ for $s, i = 0, 1, \cdots, r$. It would be found that formula (9) plays a key role in the derivation of the subsequent main results.

Now, we define a novel backward difference equation as below

$$\begin{aligned} W_{\theta_{t-r}}(t-r) =& \mathcal{L}_{\theta_{t-r}}(B'_{\theta_t}P_{\theta_{t+1}}(t+1)B_{\theta_t} + \mu^2\bar{B}'_{\theta_t}P_{\theta_{t+1}}(t+1) \\ &\times \bar{B}_{\theta_t} + R_{\theta_t}) - \sum_{s=0}^{r-1}\mathcal{L}_{\theta_{t-r}}(T^{s+1,s+1}_{\theta_{t-s}}(t-s)), \end{aligned} \tag{10}$$

$$\begin{aligned} T^0_{\theta_{t-r}}(t-r) =& \mathcal{L}_{\theta_{t-r}}((B'_{\theta_t}P_{\theta_{t+1}}(t+1)A_{\theta_t} + \mu^2\bar{B}'_{\theta_t} \\ &\times P_{\theta_{t+1}}(t+1)\bar{A}_{\theta_t})F_{\theta_{t-1},\theta_{t-r}}) \\ &- \sum_{s=0}^{r-1}\mathcal{L}_{\theta_{t-r}}(T^{s+1,0}_{\theta_{t-s}}(t-s)F_{\theta_{t-s-1},\theta_{t-r}}), \end{aligned} \tag{11}$$

$$\begin{aligned} T^1_{\theta_{t-r}}(t-r) =& \mathcal{L}_{\theta_{t-r}}((B'_{\theta_t}P_{\theta_{t+1}}(t+1)A_{\theta_t} + \mu^2\bar{B}'_{\theta_t} \\ &\times P_{\theta_{t+1}}(t+1)\bar{A}_{\theta_t})F_{\theta_{t-1},\theta_{t-r+1}}B_{\theta_{t-r}}) \\ &- \sum_{s=0}^{r-1}\mathcal{L}_{\theta_{t-r}}(T^{s+1,0}_{\theta_{t-s}}(t-s)F_{\theta_{t-s-1},\theta_{t-r+1}}B_{\theta_{t-r}}), \end{aligned} \tag{12}$$

$$\begin{aligned} T^j_{\theta_{t-r}}(t-r) =& \mathcal{L}_{\theta_{t-r}}((B'_{\theta_t}P_{\theta_{t+1}}(t+1)A_{\theta_t} + \mu^2\bar{B}'_{\theta_t} \\ &\times P_{\theta_{t+1}}(t+1)\bar{A}_{\theta_t})F_{\theta_{t-1},\theta_{t-r+j}}B_{\theta_{t-r+j-1}}) \\ &- \sum_{s=0}^{r-j}\mathcal{L}_{\theta_{t-r}}(T^{s+1,0}_{\theta_{t-s}}(t-s)F_{\theta_{t-s-1},\theta_{t-r+j}}B_{\theta_{t-r+j-1}}) \\ &- \sum_{s=r-j+1}^{r-1}\mathcal{L}_{\theta_{t-r}}(T^{s+1,s-(r-j)}_{\theta_{t-s}}(t-s)), \\ & j = 1, 2, \cdots, r, \end{aligned} \tag{13}$$

$$\begin{aligned} T^{i,j}_{\theta_{t-s}}(t-s) =& (T^i_{\theta_{t-s}}(t-s))'W_{\theta_{t-s}}(t-s)^{-1}T^j_{\theta_{t-s}}(t-s), \\ & s, i, j = 0, 1, \cdots, r, \end{aligned} \tag{14}$$

for $t = N, N-1, \cdots, r$, and $T^j_{\theta_{N-s}}(N-s) = 0$, $j = 0, 1, \cdots, r, s = 0, 1, \cdots, r-1$, and the matrix $F_{\theta_{t-s},\theta_{t-i}} = A_{\theta_{t-s}} \cdots A_{\theta_{t-i}}, F_{\theta_{t-i-1},\theta_{t-i}} = I$ for $i = 0, 1, \cdots, r$. Moreover, $P_{\theta_t}(t)$ satisfies a backward difference equation as below

$$\begin{aligned} P_{\theta_t}(t) =& A'_{\theta_t}\mathcal{L}_{\theta_t}(P_{\theta_{t+1}}(t+1))A_{\theta_t} + \mu^2\bar{A}'_{\theta_t}\mathcal{L}_{\theta_t}(P_{\theta_{t+1}}(t+1)) \\ &\times \bar{A}_{\theta_t} + Q_{\theta_t} - T^{0,0}_{\theta_t}(t), \end{aligned} \tag{15}$$

for $t = N, N-1, \cdots, 0$, and $P_{\theta_{N+1}}(N+1) = P_{\theta_{N+1}}$.

In the sequel, the collection of (10)-(15) will be termed as the D-GCDRE, where the couplings caused by jumping parameters and history inputs have been taken into simultaneous consideration. The D-GCDRE is with the same dimension as that of the original system state, which shows that the new developed maximum principle method reduces the computational complexity of the state augmentation. In (14), it is assumed that the inverse $W_{\theta_{t-s}}(t-s)^{-1}$ exists. If this is not the case, the recursion stops and the solution to the D-GCDRE (10)-(15) does not exist.

In addition, the following notations are introduced

$$\begin{aligned}&(\alpha^{r}_{\theta_t,\theta_t}(t,t))'=(\delta^{r}_{\theta_t}(t))', &(16)\\&(\alpha^{r-j}_{\theta_t,\theta_{t-j}}(t,t-j))'\\=&(\delta^{r-j}_{\theta_t}(t))'-\sum_{s=1}^{j}(\alpha^{r-s+1}_{\theta_t,\theta_{t-s+1}}(t,t-s+1))'\\&\times W_{\theta_{t-s}}(t-s)^{-1}T^{r-j+s}_{\theta_{t-s}}(t-s), &(17)\\&j=1,2,\cdots,r-1,\end{aligned}$$

with

$$\begin{aligned}(\delta^{1}_{\theta_t}(t))'=&A'_{\theta_t}\mathcal{L}_{\theta_t}(P_{\theta_{t+1}}(t+1))B_{\theta_t}+\mu^2\bar{A}'_{\theta_t}\\&\times\mathcal{L}_{\theta_t}(P_{\theta_{t+1}}(t+1))\bar{B}_{\theta_t}-T^{0,1}_{\theta_t}(t), &(18)\\(\delta^{j}_{\theta_t}(t))'=&A'_{\theta_t}\mathcal{L}_{\theta_t}(\delta^{j-1}_{\theta_{t+1}}(t+1)')-T^{0,j}_{\theta_t}(t),\\&j=2,3,\cdots,r, &(19)\end{aligned}$$

for $t=N,N-1,\cdots,0$ and $\delta^{j}_{\theta_{N+1}}(N+1)=0, j=1,2,\cdots,N$. Then, the relationship between $\alpha^{r-j}_{\theta_t,\theta_{t-j}}(t,t-j)$ and $T^{j}_{\theta_{t-j}}(t-j)$ is revealed as follows.

Lemma 2: Consider $\alpha^{r-j}_{\theta_t,\theta_{t-j}}(t,t-j)$ and $T^{j}_{\theta_{t-j}}(t-j)$ as in (11)-(13), (16), and (17), then the following relations are achieved

$$\begin{aligned}&\mathrm{E}\{\Phi_{\theta_t}(t)'(\alpha^{r}_{\theta_{t+1},\theta_{t+1}}(t+1,t+1))'|\mathcal{H}_t\}=(T^{0}_{\theta_t}(t))', &(20)\\&\mathrm{E}\{\Phi_{\theta_t}(t)'(\alpha^{r-j}_{\theta_{t+1},\theta_{t-j+1}}(t+1,t-j+1))'|\mathcal{H}_t\}\\=&(\alpha^{r-j+1}_{\theta_t,\theta_{t-j+1}}(t,t-j+1))',\quad j=1,2,\cdots,r-1, &(21)\\&\mathrm{E}\{\Gamma_{\theta_t}(t)'(\alpha^{r-j}_{\theta_{t+1},\theta_{t-j+1}}(t+1,t-j+1))'|\mathcal{H}_{t-j}\}\\=&(T^{j+1}_{\theta_{t-j}}(t-j))',\quad j=0,1,\cdots,r-1. &(22)\end{aligned}$$

Proof: See Appendix B. ■

We now deduce the solution to D-FBSMDE (7).

Theorem 1: Assume that the solution to the D-GCDRE (10)-(15) exists. Then, the unique solution to D-FBSMDE (7) is as below

$$\begin{aligned}\eta_{t-1}=&P_{\theta_t}(t)x(t)-\sum_{s=1}^{r}(\alpha^{r-s+1}_{\theta_t,\theta_{t-s+1}}(t,t-s+1))'\\&\times W_{\theta_{t-s}}(t-s)^{-1}\\&\times\mathrm{E}\{\alpha^{r-s+1}_{\theta_t,\theta_{t-s+1}}(t,t-s+1)x(t)|\mathcal{H}_{t-s}\}. &(23)\end{aligned}$$

Furthermore, (23) can be reformulated as

$$\begin{aligned}\eta_{t-1}=&P_{\theta_t}(t)x(t)-\sum_{s=1}^{r}(\alpha^{r-s+1}_{\theta_t,\theta_{t-s+1}}(t,t-s+1))'\\&\times W_{\theta_{t-s}}(t-s)^{-1}T^{0}_{\theta_{t-s}}(t-s)x(t-s)\\&-\sum_{s=1}^{r}\{\sum_{i=r-s+1}^{r}(\alpha^{r-i+1}_{\theta_t,\theta_{t-i+1}}(t,t-i+1))'\\&\times W_{\theta_{t-i}}(t-i)^{-1}T^{s-(r-i)}_{\theta_{t-i}}(t-i)\}\\&\times u(t-2r+s-1),t=r,r+1,\cdots,N. &(24)\end{aligned}$$

Proof: See Appendix C. ■

Remark 1: Note that the solution to D-FBSMDE (7) is more complicated than that proposed in [46] and lays the foundation for the later deduction of the optimal controller and the associated existence condition.

C. Solution to Problem 1

In line with Theorem 1, we are in a position to propose an analytical solution to Problem 1.

Theorem 2: Problem 1 admits a unique optimal control if and only if

$$W_{\theta_{t-r}}(t-r)>0,\ t=N,N-1,\cdots,r. \tag{25}$$

In this situation, the explicit expression of the optimal controller is supplied by

$$\begin{aligned}u(t-r)=&-W_{\theta_{t-r}}(t-r)^{-1}[T^{0}_{\theta_{t-r}}(t-r)x(t-r)\\&+\sum_{j=1}^{r}T^{j}_{\theta_{t-r}}(t-r)u(t-2r+j-1)],\\&t=r,r+1,\cdots,N, &(26)\end{aligned}$$

and the optimal cost satisfies

$$\begin{aligned}J^*_N=&\mathrm{E}\{\sum_{t=0}^{r-1}x(t)'Q_{\theta_t}x(t)+x(r)'P_{\theta_r}(r)x(r)\\&-x(r)'\sum_{s=1}^{r}(\alpha^{r-s+1}_{\theta_r,\theta_{r-s+1}}(r,r-s+1))'W_{\theta_{r-s}}(r-s)^{-1}\\&\times\mathrm{E}[\alpha^{r-s+1}_{\theta_r,\theta_{r-s+1}}(r,r-s+1)x(r)|\mathcal{H}_{r-s}]\}. &(27)\end{aligned}$$

Proof: See Appendix D. ■

D. The Case of No Jumping Parameters

In the case that there are no jumping parameters, i.e., $A_i=A$, $B_i=B$, $\bar{A}_i=\bar{A}$, $\bar{B}_i=\bar{B}$ for $i=1,2,\cdots,L$, the notations defined in (10)-(19) are simplified to W_{t-r}, T^{i}_{t-r} $(i=0,1,\cdots,r)$, P_t, δ^{j}_{t} $(j=1,\cdots,r)$, $\alpha^{r-j}_{t,t-j}$ $(j=0,1,\cdots,r-1)$, respectively. Set $P^1_t\triangleq P_t, P^j_t\triangleq-(\alpha^{j-1}_{t,t-r+j-1})'W^{-1}_{t-r+j-2}\alpha^{j-1}_{t,t-r+j-1}, j=2,\cdots,r+1$, then the expression for the optimal costate (23) turns into

$$\eta_{t-1}=P^1_tx(t)+\sum_{j=2}^{r+1}P^j_t\hat{x}(t|t-r+i-2),$$

with $\hat{x}(t|t-r+i-2)=\mathrm{E}\{x(t)|\mathcal{H}_{t-r+i-2}\}$, and the optimal controller (26) turns into

$$u(t-r)=-\Upsilon^{-1}_tM_t\hat{x}(t|t-r), \tag{28}$$

with $\hat{x}(t|t-r)=\mathrm{E}\{x(t)|\mathcal{H}_{t-r}\}$, and

$$M_t=B'\sum_{j=1}^{r+1}P^j_{t+1}A+\mu^2\bar{B}'P^1_{t+1}\bar{A},$$

$$\Upsilon_t=B'\sum_{j=1}^{r+1}P^j_{t+1}B+\mu^2\bar{B}'P^1_{t+1}\bar{B}+R.$$

It can be seen that the optimal costate and the optimal controller coincide with the results developed for the stochastic systems with multiplicative noises and input delay simultaneously [46].

E. The Case of No Time Delays

When there is no time delay in the control, the D-GCDRE (10)-(15) is specialized in the following form

$$\begin{aligned}W_{\theta_t}(t)=&B'_{\theta_t}\mathcal{L}_{\theta_t}(P_{\theta_{t+1}}(t+1))B_{\theta_t}\\&+\mu^2\bar{B}'_{\theta_t}\mathcal{L}_{\theta_t}(P_{\theta_{t+1}}(t+1))\bar{B}_{\theta_t}+R_{\theta_t},\\T^0_{\theta_t}(t)=&B'_{\theta_t}\mathcal{L}_{\theta_t}(P_{\theta_{t+1}}(t+1))A_{\theta_t}\\&+\mu^2\bar{B}'_{\theta_t}\mathcal{L}_{\theta_t}(P_{\theta_{t+1}}(t+1))\bar{A}_{\theta_t},\\P_{\theta_t}(t)=&A'_{\theta_t}\mathcal{L}_{\theta_t}(P_{\theta_{t+1}}(t+1))A_{\theta_t}\end{aligned}$$

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$$+\mu^2\bar{A}'_{\theta_t}\mathcal{L}_{\theta_t}(P_{\theta_{t+1}}(t+1))\bar{A}_{\theta_t} + Q_{\theta_t} - T^0_{\theta_t}(t)'W_{\theta_t}(t)^{-1}T^0_{\theta_t}(t).$$

For this situation, the optimal costate (23) is simplified as $\eta_{t-1} = P_{\theta_t}(t)x(t)$, and the optimal controller (26) reduces to $u(t) = -W_{\theta_t}(t)^{-1}T^0_{\theta_t}(t)x(t)$. Furthermore, the optimal performance index becomes $J^*_N = \mathrm{E}\{x'_0P_{\theta_0}(0)x_0\}$. It can be seen that the optimal costate and the optimal controller for the delay-free stochastic MJLS are both linear combinations of the state, which are consistent with the results proposed in [15]. The optimal costate and optimal controller for the stochastic MJLS with input delay are more complicated, which are related to not only the current state, but also the recent r-step history inputs.

Remark 2: In order to specifically reflect the terminal time N, we rewritten $W_{\theta_{t-r}}(t-r), T^i_{\theta_{t-r}}(t-r)$ $(i = 0, 1, \cdots, r)$, $P_{\theta_t}(t)$, $\delta^j_{\theta_t}(t)$ $(j = 1, \cdots, r)$, $\alpha^{r-j}_{\theta_t,\theta_{t-j}}(t, t-j)$ $(j = 0, 1, \cdots, r-1)$, respectively, as $W_{\theta_{t-r}}(t-r, N), T^i_{\theta_{t-r}}(t-r, N)$ $(i = 0, 1, \cdots, r)$, $P_{\theta_t}(t, N)$, $\delta^j_{\theta_t}(t, N)$ $(j = 1, \cdots, r)$, $\alpha^{r-j}_{\theta_t,\theta_{t-j}}(t, t-j, N)$ $(j = 0, 1, \cdots, r-1)$.

III. The Infinite-horizon Optimal Control and Stabilization

A. Problem Statement

In this section, we focus on the infinite-horizon stabilization control of stochastic MJLS with multi-step input delay. For the sake of argument, set $P_{\theta_{N+1}} = 0$ and define the performance index as

$$J(\theta_0, x_0, u) \triangleq \mathrm{E}\left\{\sum_{t=0}^{\infty}x(t)'Qx(t) + \sum_{t=r}^{\infty}u(t-r)'Ru(t-r)\right\} \quad (29)$$

with $Q \geq 0, R \geq 0$.

The admissible control set for the infinite horizon case is given as:

$$\mathcal{U}_\infty \triangleq \{u(-r), \cdots, u(-1), u(0), u(1), \cdots | u(t) \in R^m, \; u(t) \text{ is } \mathcal{H}_t-\text{measurable, and } \sum_{t=-r}^{\infty}\mathrm{E}[u(t)'u(t)] < +\infty\}. \quad (30)$$

Furthermore, some definitions and standard assumptions are made in order to analyze the system stabilization.

Definition 1: The stochastic MJLS (1) is said to be stabilizable in the mean square sense if there exists a $\mathcal{H}_t$-measurable controller

$$u(t) = -K^0_{\theta_t}x(t) - \sum_{i=1}^{r}K^i_{\theta_t}u(t+i-r-1)$$

with constant gain matrices K^i_l for $\theta_t = l$ $(l = 1, 2, \cdots, L, i = 0, 1, \cdots, r)$ satisfying $\lim_{t\to\infty}\mathrm{E}(u(t)'u(t)) = 0$ such that $\lim_{t\to\infty}\mathrm{E}(x(t)'x(t)) = 0$ for any initial values $x_0, u(-r), \cdots, u(0)$.

Definition 2: The following stochastic MJLS

$$x(t+1) = (A_{\theta_t} + \omega_t\bar{A}_{\theta_t})x(t), y(t) = C_{\theta_t}x(t) \quad (31)$$

is called exactly observable if, for any $N > 0$,

$$y(t) \equiv 0, a.s., \forall 0 \leq t \leq N \Rightarrow x_0 = 0.$$

For ease of description, we write (31) as $(\mathcal{A}, \bar{\mathcal{A}}, \mathcal{C})$, where $\mathcal{A} = (A_1, A_2, \cdots, A_L), \bar{\mathcal{A}} = (\bar{A}_1, \bar{A}_2, \cdots, \bar{A}_L)$, and $\mathcal{C} = (C_1, C_2, \cdots, C_L)$ with $C_i = Q^{\frac{1}{2}} (i = 1, \cdots, L)$.

Assumption 3: $R > 0$ and $Q \geq 0$.

Assumption 4: $(\mathcal{A}, \bar{\mathcal{A}}, \mathcal{C})$ is exactly observable.

Now, the stabilization and optimal control on infinite horizon is formulated as below:

Problem 2: Seek for a $\mathcal{H}_t$-measurable controller $u(t) = -K^0_{\theta_t}x(t) - \sum_{i=1}^{r}K^i_{\theta_t}u(t+i-r-1) \in \mathcal{U}_\infty$, which minimizes the cost (29) and stabilizes system (1).

B. Solution to Problem 2

Lemma 3: Under Assumption 3, we obtain for $N \geq r$ that

$$P_{\theta_t}(t, N) \geq 0, \; t = N, \cdots, 1, 0, \quad (32)$$

$$P_{\theta_t}(t, N) - \sum_{s=1}^{r}(\alpha^{r-s+1}_{\theta_t,\theta_{t-s+1}}(t, t-s+1, N))' \times W_{\theta_{t-s}}(t-s, N)^{-1}\alpha^{r-s+1}_{\theta_t,\theta_{t-s+1}}(t, t-s+1, N) \geq 0, \; t = N, \cdots, r. \quad (33)$$

Proof: See Appendix E. ■

Lemma 4: Under Assumptions 3 and 4, there exists a positive integer $N_0 \geq r$, such that

$$P_{\theta_t}(t, N_0) - \sum_{s=1}^{r}(\alpha^{r-s+1}_{\theta_t,\theta_{t-s+1}}(t, t-s+1, N_0))' \times W_{\theta_{t-s}}(t-s, N_0)^{-1}\alpha^{r-s+1}_{\theta_t,\theta_{t-s+1}}(t, t-s+1, N_0) > 0, \; t = N_0, \cdots, r.$$

Proof: See Appendix F. ■

For any $t \geq 0$, in this part, we employ $\{l_r, l_{r-1}, \cdots, l_0, \tilde{l}\}$ to describe the realization of $\{\theta_{t-r}, \theta_{t-r+1}, \cdots, \theta_t, \theta_{t+1}\}$, with $l_r, l_{r-1}, \cdots, l_0, \tilde{l} \in \Theta$. Consider some jumping parameter matrices $X_{\theta_{t+1}}, X_{\theta_t}, \cdots, X_{\theta_{t-r}}$ and let $\bar{X}_{\theta_{t+1},\theta_{t-r}} = \prod_{i=-1}^{r}X_{\theta_{t-i}}$, and then the realization of these matrices can be denoted as $X_{\tilde{l}}, X_{l_0}, \cdots, X_{l_{r-1}}, X_{l_r}, \bar{X}_{\tilde{l},l_r} = X_{\tilde{l}}\prod_{i=0}^{r}X_{l_i}$. In accordance with (8), we define a set of algebraic evolution operators $\mathcal{L}_{l_0}(.), \mathcal{L}_{l_1}(.), \cdots, \mathcal{L}_{l_r}(.)$ as

$$\mathcal{L}_{l_0}(\bar{X}_{\tilde{l},l_r}) = \sum_{\tilde{l}=1}^{L}\lambda_{l_0,\tilde{l}}\bar{X}_{\tilde{l},l_r}, l_0 \in \Theta$$

$$\mathcal{L}_{l_1}(\bar{X}_{\tilde{l},l_r}) = \sum_{l_0=1}^{L}\lambda_{l_1,l_0}\mathcal{L}_{l_0}(\bar{X}_{\tilde{l},l_r}), l_1 \in \Theta,$$

$$\vdots$$

$$\mathcal{L}_{l_r}(\bar{X}_{\tilde{l},l_r}) = \sum_{l_{r-1}=1}^{L}\lambda_{l_r,l_{r-1}}\mathcal{L}_{l_{r-1}}(\bar{X}_{\tilde{l},l_r}), l_r \in \Theta. \quad (34)$$

Also, $X_{\theta_{t-s}} (-1 \leq s \leq r)$ can be chosen as the identity matrix or the product of some matrix multiplication. Now, we define a new D-GCARE as

$$W_{l_r} = \mathcal{L}_{l_r}(B'_{l_0}P_{\tilde{l}}B_{l_0} + \mu^2\bar{B}'_{l_0}P_{\tilde{l}}\bar{B}_{l_0} + R) - \sum_{s=0}^{r-1}\mathcal{L}_{l_r}(T^{s+1,s+1}_{l_s}), \; l_r \in \Theta, \quad (35)$$

$$T^0_{l_r} = \mathcal{L}_{l_r}((B'_{l_0}P_{\tilde{l}}A_{l_0} + \mu^2\bar{B}'_{l_0}P_{\tilde{l}}\bar{A}_{l_0})F_{l_1,l_r}) - \sum_{s=0}^{r-1}\mathcal{L}_{l_r}(T^{s+1,0}_{l_s}F_{l_{s+1},l_r}), \; l_r \in \Theta, \quad (36)$$

$$T^j_{l_r} = \mathcal{L}_{l_r}((B'_{l_0}P_{\tilde{l}}A_{l_0} + \mu^2\bar{B}'_{l_0}P_{\tilde{l}}\bar{A}_{l_0})F_{l_1,l_{r-j}}B_{l_{r-j+1}}) - \sum_{s=0}^{r-j}\mathcal{L}_{l_r}(T^{s+1,0}_{l_s}F_{l_{s+1},l_{r-j}}B_{l_{r-j+1}})$$

$$
\begin{aligned}
&\quad - \sum_{s=r-j+1}^{r-1} \mathcal{L}_{l_r}(T_{l_s}^{s+1,s-(r-j)}),\ l_r \in \Theta, \\
&\quad j = 1,2,\cdots,r, && (37)\\
T_{l_s}^{i,j} &= (T_{l_s}^{i})' W_{l_s}^{-1} T_{l_s}^{j},\ \ s,i,j = 0,1,\cdots,r, && (38)\\
P_{l_0} &= Q + A_{l_0}'\mathcal{L}_{l_0}(P_{\tilde{l}})A_{l_0} + \mu^2 \bar{A}_{l_0}'\mathcal{L}_{l_0}(P_{\tilde{l}})\bar{A}_{l_0} - T_{l_0}^{0,0},\\
&\quad l_0 \in \Theta. && (39)
\end{aligned}
$$

Furthermore, define the algebraic operators $\delta_{l_0}^{j}(j = 1,2,\cdots,r)$, $\alpha_{l_0,l_j}^{r-j}(j=0,1,\cdots,r-1)$ as

$$(\delta_{l_0}^{1})' = A_{l_0}'\mathcal{L}_{l_0}(P_{\tilde{l}})B_{l_0} + \mu^2 \bar{A}_{l_0}'\mathcal{L}_{l_0}(P_{\tilde{l}})\bar{B}_{l_0} - T_{l_0}^{0,1}, l_0 \in \Theta, \tag{40}$$

$$(\delta_{l_0}^{j})' = A_{l_0}'\mathcal{L}_{l_0}((\delta_{\tilde{l}}^{j-1})') - T_{l_0}^{0,j}, j = 2,3,\cdots,r, l_0 \in \Theta, \tag{41}$$

$$(\alpha_{l_0,l_0}^{r})' = (\delta_{l_0}^{r})',\ \ l_0 \in \Theta, \tag{42}$$

$$
\begin{aligned}
(\alpha_{l_0,l_j}^{r-j})' &= (\delta_{l_0}^{r-j})' - \sum_{s=1}^{j}(\alpha_{l_0,l_{s-1}}^{r-s+1})'(W_{l_s})^{-1}T_{l_s}^{r-j+s},\\
&\quad j = 1,2,\cdots,r-1,\ \ l_0, l_j \in \Theta. && (43)
\end{aligned}
$$

Then, the main results of this part can be stated as below.

Theorem 3: Under Assumptions 3 and 4, letting system (1) be mean square stabilizable, we have the following properties.

1) When $N \to \infty$, $P_{l_0}(t,N)$,$W_{l_r}(t-r,N)$, $T_{l_r}^{s}(t-r,N)$ $(s = 0,1,\cdots,r)$ converge, respectively, to P_{l_0}, W_{l_r}, $T_{l_r}^{s}$ $(s = 0,1,\cdots,r)$ for any $t \geq 0$ and $l_0, l_r \in \Theta$. In addition, P_{l_0}, W_{l_r}, $T_{l_r}^{s}$ $(s = 0,1,\cdots,r)$ obey the coupled algebraic Riccati equations (35)-(39).

2) The following inequality is satisfied

$$P_{l_0} - \sum_{s=1}^{r}(\alpha_{l_0,l_{s-1}}^{r-s+1})' W_{l_s}^{-1} \alpha_{l_0,l_{s-1}}^{r-s+1} > 0, l_0, \cdots, l_r \in \Theta. \tag{44}$$

Proof: See Appendix G. ■

Theorem 4: Under Assumptions 3 and 4, system (1) is stabilizable in the mean-square sense if and only if there exists a unique solution to (35)-(39) such that

$$P_{l_0} - \sum_{s=1}^{r}(\alpha_{l_0,l_{s-1}}^{r-s+1})' W_{l_s}^{-1} \alpha_{l_0,l_{s-1}}^{r-s+1} > 0, l_0, \cdots, l_r \in \Theta.$$

On this condition, the analytical solution to the optimal controller can be obtained by

$$u(t-r) = -W_{l_r}^{-1}[T_{l_r}^{0}x(t-r) + \sum_{j=1}^{r} T_{l_r}^{j} u(t-2r+j-1)], \tag{45}$$

which stabilizes (1) and minimizes the performance (29). Furthermore, the minimum value of (29) is with the form

$$
\begin{aligned}
J^* &= \mathrm{E}\{\sum_{t=0}^{r-1} x(t)'Qx(t) + x(r)'P_{\theta_r}x(r) - x(r)'\sum_{s=1}^{r}\\
&\quad \times (\alpha_{\theta_r,\theta_{r-s+1}}^{r-s+1})' W_{\theta_{r-s}}^{-1} \mathrm{E}(\alpha_{\theta_r,\theta_{r-s+1}}^{r-s+1} x(r)|\mathcal{H}_{r-s})\}. && (46)
\end{aligned}
$$

Proof: See Appendix H. ■

Remark 3: So far, we have addressed the stabilization and optimal control problems for one type of MJLS with multiplicative noises and input delay. It has been shown that the stochastic MJLS with input delay is mean square stable under the optimal controller if and only if a certain D-GCARE has a particular positive definite solution. The novelty with respect to the methodology lies mainly in the introduction of a D-FBSMDE as well as the definition of a new type of Lyapunov function.

C. The Case of No Time Delay or Multiplicative Noises

For the case without input delay, the D-GCARE (35)-(39) reduces to

$$
\begin{aligned}
P_{l_0} &= A_{l_0}'(\sum_{\tilde{l}=1}^{L}\lambda_{l_0\tilde{l}}P_{\tilde{l}})A_{l_0} + \mu^2 \bar{A}_{l_0}'(\sum_{\tilde{l}=1}^{L}\lambda_{l_0\tilde{l}}P_{\tilde{l}})\bar{A}_{l_0} + Q\\
&\quad - T_{l_0}'W_{l_0}^{-1}T_{l_0}, l_0 = 1,2,\cdots,L, && (47)\\
W_{l_0} &= B_{l_0}'(\sum_{\tilde{l}=1}^{L}\lambda_{l_0\tilde{l}}P_{\tilde{l}})B_{l_0} + \mu^2 \bar{B}_{l_0}'(\sum_{\tilde{l}=1}^{L}\lambda_{l_0\tilde{l}}P_{\tilde{l}})\bar{B}_{l_0} + R,\\
&\quad l_0 = 1,2,\cdots,L, && (48)\\
T_{l_0} &= B_{l_0}'(\sum_{\tilde{l}=1}^{L}\lambda_{l_0\tilde{l}}P_{\tilde{l}})A_{l_0} + \mu^2 \bar{B}_{l_0}'(\sum_{\tilde{l}=1}^{L}\lambda_{l_0\tilde{l}}P_{\tilde{l}})\bar{A}_{l_0},\\
&\quad l_0 = 1,2,\cdots,L, && (49)
\end{aligned}
$$

and the stabilization condition becomes

$$P_{l_0} > 0, l_0 = 1,2,\cdots,L. \tag{50}$$

Meanwhile, the optimal controller satisfies

$$u(t) = -W_{l_0}^{-1}T_{l_0}x(t), t \geq 0 \tag{51}$$

and the optimal cost is given by

$$J^* = \mathrm{E}\{x_0'P_{\theta(0)}x_0\}. \tag{52}$$

Compared with Theorem 4, it can be found that the stabilization condition as well as the optimal controller for the delayed stochastic MJLS (1) is more complicated because of the existence of input delay r. If there are no input delay and multiplicative noises simultaneously, i.e., $\mu = 0$, the obtained result (50)-(52) is coincided with Proposition 2 in [7].

D. Application to the NCSs with Packet Losses and Input Delay

Consider the following NCSs with packet losses and input delay

$$x(t+1) = A_{\theta_t}x(t) + \gamma_t \bar{B}_{\theta_t}u(t-r), \tag{53}$$

where $\{\gamma_t\}_{t\geq 0}$ is modeled as an i.i.d Bernoulli process. $\gamma_t = 1$ denotes that the data packet has been successfully delivered to the plant, and $\gamma_t = 0$ signifies the dropout. Prob$(\gamma_t = 0) = p$, Prob$(\gamma_t = 1) = 1-p$, where $p \in (0,1)$ is the packet dropout rate. In fact, system (53) can be viewed as a simplified version of system (1). In this case, the D-GCARE (35)-(39) specialized in the following form

$$
\begin{aligned}
W_{l_r} &= (1-p)\mathcal{L}_{l_r}((1-p)\bar{B}_{l_0}'P_{\tilde{l}}\bar{B}_{l_0} + p\bar{B}_{l_0}'P_{\tilde{l}}\bar{B}_{l_0} + R)\\
&\quad - \sum_{s=0}^{r-1}\mathcal{L}_{l_r}((T_{l_s}^{s+1})'W_{l_s}^{-1}T_{l_s}^{s+1}),\ l_r \in \Theta, && (54)\\
T_{l_r}^{0} &= (1-p)\mathcal{L}_{l_r}(\bar{B}_{l_0}'P_{\tilde{l}}A_{l_0}F_{l_1,l_r})\\
&\quad - \sum_{s=0}^{r-1}\mathcal{L}_{l_r}((T_{l_s}^{s+1})'W_{l_s}^{-1}T_{l_s}^{0}F_{l_{s+1},l_r}),\ l_r \in \Theta, && (55)\\
T_{l_r}^{j} &= (1-p)^2\mathcal{L}_{l_r}((\bar{B}_{l_0}'P_{\tilde{l}}A_{l_0}F_{l_1,l_{r-j}}\bar{B}_{l_{r-j+1}})\\
&\quad - (1-p)\sum_{s=0}^{r-j}\mathcal{L}_{l_r}((T_{l_s}^{s+1})'W_{l_s}^{-1}T_{l_s}^{0}F_{l_{s+1},l_{r-j}}\bar{B}_{l_{r-j+1}})\\
&\quad - \sum_{s=r-j+1}^{r-1}\mathcal{L}_{l_r}((T_{l_s}^{s+1})'W_{l_s}^{-1}T_{l_s}^{s-(r-j)}),\\
&\quad l_r \in \Theta,\ j = 1,2,\cdots,r, && (56)\\
P_{l_0} &= Q + A_{l_0}'\mathcal{L}_{l_0}(P_{\tilde{l}})A_{l_0} - (T_{l_0}^{0})'W_{l_0}^{-1}T_{l_0}^{0},\ l_0 \in \Theta, && (57)
\end{aligned}
$$

where the notations $\alpha^{r-j}_{l_0,l_j}(j=0,1,\cdots,r-1)$ and $\delta^{j}_{l_0}(j=2,\cdots,r)$ are kept the same as in (42), (43), and (41), while $\delta^1_{l_0}$ becomes

$$(\delta^1_{l_0})' =(1-p)A'_{l_0}\mathcal{L}_{l_0}(P_{\tilde{l}})\bar{B}_{l_0}-(T^0_{l_0})'W^{-1}_{l_0}T^1_{l_0}, l_0\in\Theta. \quad (58)$$

Then, the stabilization condition can be obtained directly from Theorem 4:

$$P_{l_0}-\sum_{s=1}^{r}(\alpha^{r-s+1}_{l_0,l_{s-1}})'W^{-1}_{l_s}\alpha^{r-s+1}_{l_0,l_{s-1}}>0, l_0,\cdots,l_r\in\Theta.$$

It can be seen that the stabilization condition based on (54)-(57) and (42)-(43) is closely related with the packet dropout rate p and the input delay r. As is well known that the problem of maximum packet loss probability has been well studied for the case of NCSs without delay [51]. When the packet loss, input delay, and jumping parameter exist simultaneously, it is difficult to seek for the maximum value of the allowable packet loss rate, which is worthy of in-depth study in the future.

IV. Numerical Examples

A. The Finite-Horizon Case

To illustrate the theoretical result for the finite-horizon case, a second-order dynamic system is introduced. The system matrices in (1) and the weighting matrices in (2) are given below

$$A_1=\begin{bmatrix}1.2 & 1\\ -0.2 & -0.8\end{bmatrix}, A_2=\begin{bmatrix}0.8 & 0\\ 0 & 0.6\end{bmatrix},$$
$$\bar{A}_1=\begin{bmatrix}0.2 & 0.1\\ -0.15 & -0.05\end{bmatrix}, \bar{A}_2=\begin{bmatrix}0.08 & 0.06\\ 0 & 0.06\end{bmatrix},$$
$$B_1=\begin{bmatrix}1\\ 1\end{bmatrix}, B_2=\begin{bmatrix}2\\ 1\end{bmatrix}, \bar{B}_1=\begin{bmatrix}0.1\\ 0.1\end{bmatrix},$$
$$\bar{B}_2=\begin{bmatrix}0.2\\ 0.1\end{bmatrix}, Q_1=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, Q_2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},$$
$$R_1=1, R_2=2, P_{N+1}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},$$

$\theta(t)\in\{1,\ 2\}$ represents a two-state Markov chain, which is with transition probability $\begin{bmatrix}0.9 & 0.1\\ 0.3 & 0.7\end{bmatrix}$ and initial distribution $(0.5,\ 0.5)$. The input delay $r=2$ and $\mu=0.7$. The initial values $u(-1)=0, u(-2)=0, x(0)=[1\ 1]'$.

Set the time horizon $N=9$. By applying Theorem 2, the calculation result for $W_i(t)$, $T_i^0(t)$, $T_i^1(t)$, $T_i^2(t)$ $(i=1,2)$ are listed in Table I. It is checked that $W_i(t)>0$ for $i=1,2, t=1,\cdots,7$. Therefore, there exists a unique solution to the finite-horizon LQR problem based on Theorem 2 and the optimal value of (2) is $J_N^*=11.1025$.

B. The Infinite-Horizon Case

In this part, we show the validity of the stabilization result. The specifications of system (1), the input delay r, and the initial values of $x_0, u(-1), u(-2)$ remain unchanged. The stochastic property of $\theta(t)$ and the value of μ keep the same as in the previous subsection. However, the weighting matrices become as $Q=\text{diag}\{1,1\}, R=1$. We run 50 Monte Carlo simulations, and select the first trajectory to illustrate the stabilization algorithm. By applying Theorem 4, the computed results are shown as below:

$$P_1=\begin{bmatrix}6.0049 & 3.2575\\ 3.2575 & 4.1545\end{bmatrix}, P_2=\begin{bmatrix}2.8640 & 0.4092\\ 0.4092 & 1.8574\end{bmatrix},$$
$$W_1=10.2104,\ W_2=11.8411,\ T_1^0=\begin{bmatrix}4.5047 & 1.7793\end{bmatrix},$$
$$T_2^0=\begin{bmatrix}2.6296 & 0.9424\end{bmatrix},\ T_1^1=7.3898,\ T_2^1=8.1447,$$
$$T_1^2=4.6907,\ T_2^2=7.1068,\ S_1^2=\begin{bmatrix}4.4359 & 3.3759\end{bmatrix},$$
$$S_2^2=\begin{bmatrix}2.7946 & 0.7970\end{bmatrix},\ S_{11}^1=\begin{bmatrix}3.7527 & 0.3324\end{bmatrix},$$
$$S_{12}^1=\begin{bmatrix}3.1283 & -0.1429\end{bmatrix},\ S_{21}^1=\begin{bmatrix}4.0432 & 2.0710\end{bmatrix},$$
$$S_{22}^1=\begin{bmatrix}3.6497 & 1.9588\end{bmatrix}.$$

It can be shown that the stabilization conditions

$$P_{l_0}-\sum_{s=1}^{2}(\alpha^{r-s+1}_{l_0,l_{s-1}})'W^{-1}_{l_s}\alpha^{r-s+1}_{l_0,l_{s-1}}>0, l_0,l_1,l_2\in\{1,2\}. \quad (59)$$

are all satisfied. If $\theta(t)=1$, the optimal infinite horizon controller satisfies

$$\begin{aligned}u(t)=&-W_1^{-1}[T_1^0x(t)+T_1^1u(t-2)+T_1^2u(t-1)]\\ =&-\begin{bmatrix}0.4412 & 0.1743\end{bmatrix}x(t)-0.7237u(t-2)\\ &-0.4594u(t-1). \end{aligned}\quad (60)$$

If $\theta(t)=2$, the optimal infinite horizon controller obeys

$$\begin{aligned}u(t)=&-W_2^{-1}[T_2^0x(t)+T_2^1u(t-2)+T_2^2u(t-1)]\\ =&-\begin{bmatrix}0.2221 & 0.0796\end{bmatrix}x(t)-0.6878u(t-2)\\ &-0.6002u(t-1). \end{aligned}\quad (61)$$

In view of (46), we can obtain the optimal performance $J_0=11.1071$. The simulation results are supplied in Figs. 1-2. The closed-loop state trajectories subject to one sample path of $\theta(t)\in\{1,2\}$ are plotted in Fig. 2 and the corresponding controller is drawn in Fig. 2. It can be found that the optimal controlled system is stable since the condition is satisfied.

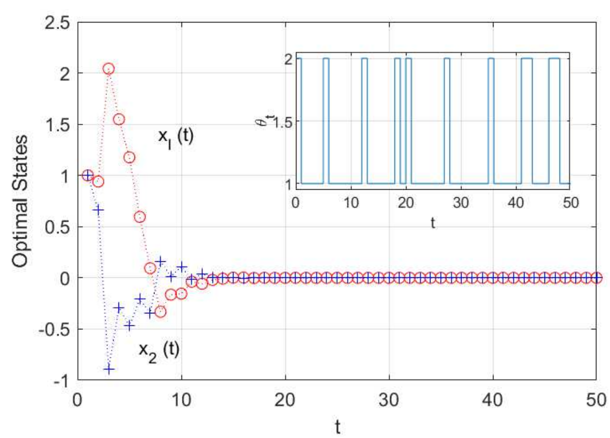

Fig. 1. The optimal state trajectories

C. Comparison with the Delay-Free Case

In this subsection, the differences of the infinite horizon stabilization control between the delay-free case and the input delay case are shown. For this purpose, we still consider the numerical example as in Subsection B, but $r=0$. Based on the GCARE (47)-(49) and the corresponding results (50)-(52), we can obtain the simulation results as follows:

$$P_1=\begin{bmatrix}2.3635 & 1.4999\\ 1.4999 & 2.8990\end{bmatrix}, P_2=\begin{bmatrix}1.2847 & -0.2652\\ -0.2652 & 1.4847\end{bmatrix},$$
$$T_1^0=\begin{bmatrix}3.4837 & 0.3217\end{bmatrix}, T_2^0=\begin{bmatrix}2.7984 & 1.4800\end{bmatrix},$$
$$W_1=8.6976, W_2=10.4456.$$

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TABLE I
CALCULATION RESULTS

t	$W_1(t)$	$W_2(t)$	$T_1^0(t)$	$T_2^0(t)$	$T_1^1(t)$	$T_2^1(t)$	$T_1^2(t)$	$T_2^2(t)$
1	10.1887	11.8120	[4.4942,1.7749]	[2.6229,0.9399]	7.3731	8.1237	4.6787	7.0868
2	10.1612	11.7779	[4.4815,1.7698]	[2.6153,0.9371]	7.3523	8.1001	4.6653	7.0658
3	10.0653	11.6505	[4.4335,1.7492]	[2.5845,0.9257]	7.2767	8.0042	4.6098	6.9741
4	9.9226	11.4614	[4.3766,1.7282]	[2.5490,0.9114]	7.1806	7.8915	4.5522	6.8725
5	9.2490	10.5252	[4.0547,1.5929]	[2.3345,0.8247]	6.6680	7.2106	4.1794	6.2058
6	8.4824	9.6430	[3.7135,1.4680]	[2.1532,0.7648]	6.0906	6.6577	3.8605	5.7781
7	3.6522	5.0974	[1.2999,0.5697]	[0.9334,0.2790]	2.0293	2.7986	1.5094	2.7294

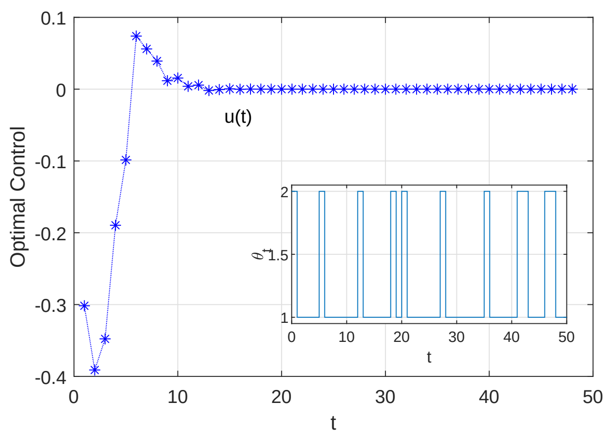

Fig. 2. The optimal control

Obviously, the stabilization condition

$$P_1 > 0, P_2 > 0 \tag{62}$$

are satisfied. The infinite horizon optimal controller

$$u(t) = \begin{cases} -[0.4005 \quad 0.0370]x(t), & \text{if } \theta_t = 1; \\ -[0.2679 \quad 0.1417]x(t), & \text{if } \theta_t = 2, \end{cases} \tag{63}$$

and the optimal performance index $J_0 = 5.2507$. Compared (62), (63) with (59), (60), and (61), respectively, it can be found that the stabilization condition for the delay-free case is much simpler, and the infinite-horizon optimal controller is a linear combination of the current state, which does not involve the delayed input terms.

V. CONCLUSION

This paper has addressed the optimal control and stabilization problems for stochastic MJLS with multi-step input delay. An analytical solution to the finite-horizon case has been given, and a necessary and sufficient condition for the solvability has been proposed for the first time. Later, we have proposed a necessary and sufficient condition for the stabilizability of the stochastic delayed MJLS. To show the existence of such a solution, one just needs to test the satisfaction of a set of algebraic inequalities, which are easily verifiable. To our knowledge, no similar conditions have been developed for the mean square stabilizability subject to delayed stochastic MJLS. It should be noted that our derivations have been mainly based on the subtle usage of the link between the system state/inputs and certain auxiliary variable, thereby avoiding the unnecessarily complicated augmented argument.

APPENDIX

A. Proof of Lemma 1

Proof: For arbitrary $u(t), \Delta u(t) \in \mathcal{U}_{ad}$ and $\varepsilon \in R$, we have $u_t^\varepsilon = u(t) + \varepsilon\Delta u(t) \in \mathcal{U}_{ad}$. Set $x_t^\varepsilon, J_N^\varepsilon$ to be the corresponding state and cost function with u_t^ε, and $x(t)$ and J_N mean the corresponding state and cost function with $u(t)$.

In view of the system (1), it holds

$$\begin{aligned}\Delta x(t+1) &= (A_{\theta_t} + \omega_t \bar{A}_{\theta_t})\Delta x(t) + (B_{\theta_t} + \omega_t \bar{B}_{\theta_t})\varepsilon\Delta u(t-r), \\ \Delta x(0) &= 0, \end{aligned} \tag{64}$$

where $\Delta x(t+1) = x_{t+1}^\varepsilon - x(t+1)$.

Applying the recursive expression, $\Delta x(t+1)$ can be rewritten as

$$\Delta x(t+1) = \sum_{i=0}^{t} \tilde{F}_{\theta_t,\theta_{i+1}} (B_{\theta_i} + \omega_i \bar{B}_{\theta_i})\varepsilon\Delta u(i-r), \tag{65}$$

where $\tilde{F}_{\theta_t,\theta_{i+1}} = (A_{\theta_t} + \omega_t \bar{A}_{\theta_t})(A_{\theta_{t-1}} + \omega_{t-1}\bar{A}_{\theta_{t-1}}) \cdots (A_{\theta_{i+1}} + \omega_{i+1}\bar{A}_{\theta_{i+1}}), i = 0, \cdots, t, \tilde{F}_{\theta_t,\theta_{t+1}} = I$, and $\Delta x(0) = 0$ has been used.

Since $\Delta u(i-r) \in \mathcal{U}_{ad}, i = 0, 1, \cdots, t$, it follows from (3) that $\sum_{i=0}^{t} \mathrm{E}[\Delta u(i-r)'\Delta u(i-r)] < +\infty$. Furthermore, recall that $A_1, A_2, \cdots, A_L, \bar{A}_1, \bar{A}_2, \cdots, \bar{A}_L, B_1, B_2, \cdots, B_L$, and $\bar{B}_1, \bar{B}_2, \cdots, \bar{B}_L$ are constant matrices and $\mathrm{E}(\omega_i^2)$ is finite for $i = 0, 1, \cdots, t$. So, there exists γ satisfying

$$\mathrm{E}\{\Delta x(t+1)'\Delta x(t+1)\} \leq \gamma\varepsilon^2 < +\infty. \tag{66}$$

In what follows, we will deduce the variation of J_N owing to the perturbation of controller $u(t)$. In accordance with (2), J_N^ε can be expressed as

$$\begin{aligned} J_N^\varepsilon = &\mathrm{E}[\sum_{t=0}^{N}(x_t^\varepsilon)'Q_{\theta_t}x_t^\varepsilon + \sum_{t=r}^{N}(u_{t-r}^\varepsilon)'R_{\theta_t}u_{t-r}^\varepsilon \\ &+ (x_{N+1}^\varepsilon)'P_{\theta_{N+1}}x_{N+1}^\varepsilon]. \end{aligned} \tag{67}$$

Recalling that $x_{t+1}^\varepsilon = x(t+1) + \Delta x(t+1)$ and $u_t^\varepsilon = u(t) + \varepsilon\Delta u(t)$, and based on the total derivative of J_N^ε at the point (x, u), we have

$$\begin{aligned} J_N^\varepsilon = &J_N + 2\mathrm{E}[\sum_{t=0}^{N} x(t)'Q_{\theta_t}\Delta x(t) + \sum_{t=r}^{N} u(t-r)'R_{\theta_t}\varepsilon\Delta u(t-r) \\ &+ x(N+1)'P_{\theta_{N+1}}\Delta x(N+1)] + o(\tau), \end{aligned} \tag{68}$$

where

$$\begin{aligned} \tau = &\mathrm{E}\{\mathrm{sqrt}\{\sum_{t=0}^{N}[x_t^\varepsilon - x(t)]'[x_t^\varepsilon - x(t)] + \sum_{t=r}^{N}[u_{t-r}^\varepsilon - u(t-r)]' \\ &\times [u_{t-r}^\varepsilon - u(t-r)] + [x_{N+1}^\varepsilon - x(N+1)]'[x_{N+1}^\varepsilon - x(N+1)]\}\}, \end{aligned}$$

with $sqrt\{\cdot\}$ being the square root and $o(\tau)$ representing the infinitesimal of higher order when $\tau \to 0$.

Moreover, τ^2 satisfies that

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$$\tau^2 \leq \sum_{t=0}^{N} \mathrm{E}[\Delta x(t)'\Delta x(t)] + \varepsilon^2 \sum_{t=r}^{N} \mathrm{E}[\Delta u(t-r)'\Delta u(t-r)] + \mathrm{E}[\Delta x(N+1)'\Delta x(N+1)]. \quad (69)$$

In view of (66) and $\sum_{t=r}^{N} \mathrm{E}[\Delta u(t-r)'\Delta u(t-r)] < +\infty$,we obtain that τ is the same order infinitesimal of ε. Based on this and (65), ΔJ_N can be expressed as

$$\begin{aligned}\Delta J_N &= J_N^{\varepsilon} - J_N\\ &= 2\mathrm{E}\{x(N+1)'P_{\theta_{N+1}} \sum_{i=0}^{N} \tilde{F}_{\theta_N,\theta_{i+1}}(B_{\theta_i}+\omega_i\bar{B}_{\theta_i})\varepsilon\Delta u(i-r)\\ &+\sum_{t=0}^{N} x(t)'Q_{\theta_t} \sum_{i=0}^{t-1} \tilde{F}_{\theta_{t-1},\theta_{i+1}}(B_{\theta_i}+\omega_i\bar{B}_{\theta_i})\varepsilon\Delta u(i-r)\\ &+\sum_{t=r}^{N} u(t-r)'R_{\theta_t}\varepsilon\Delta u(t-r)\}+o(\varepsilon). \quad (70)\end{aligned}$$

Note that $u(-i), i = 1, \cdots, r$ are fixed, we obtain $\Delta u(-i) = 0, i = 1, \cdots, r$. Then, (70) can be rewritten as

$$\begin{aligned}\Delta J_N =& 2\mathrm{E}\{[x(N+1)'P_{\theta_{N+1}}(B_{\theta_N}+\omega_N\bar{B}_{\theta_N})+u(N-r)'R_{\theta_N}]\\ &\times\varepsilon\Delta u(N-r)+\sum_{i=0}^{N-1}[x(N+1)'P_{\theta_{N+1}}\tilde{F}_{\theta_N,\theta_{i+1}}\\ &\times(B_{\theta_i}+\omega_i\bar{B}_{\theta_i})+\sum_{t=i+1}^{N} x(t)'Q_{\theta_t}\tilde{F}_{\theta_{t-1},\theta_{i+1}}(B_{\theta_i}+\omega_i\bar{B}_{\theta_i})\\ &+u(i-r)'R_{\theta_i}]\varepsilon\Delta u(i-r)\}+o(\varepsilon). \quad (71)\end{aligned}$$

Define

$$\eta_i = \mathrm{E}\{\sum_{t=i+1}^{N} \tilde{F}'_{\theta_{t-1},\theta_{i+1}}Q_{\theta_t}x(t)+\tilde{F}_{\theta_N,\theta_{i+1}}P_{\theta_{N+1}}x(N+1)|\mathcal{H}_{i+1}\}, \quad (72)$$

and we have

$$\eta_{i-1} = \mathrm{E}\{Q_{\theta_i}x(i)+(A_{\theta_i}+\omega_i\bar{A}_{\theta_i})'\eta_i|\mathcal{H}_i\}.$$

(5) and (6) are shown.

Substituting (72) into (71) yields

$$\begin{aligned}\Delta J_N =& 2\mathrm{E}\{\sum_{i=0}^{N}[\eta_i'(B_{\theta_i}+\omega_i\bar{B}_{\theta_i})+u(i-r)'R_{\theta_i}]\varepsilon\Delta u(i-r)\}+o(\varepsilon)\\ =& 2\mathrm{E}\{\sum_{i=0}^{N}\mathrm{E}[\eta_i'(B_{\theta_i}+\omega_i\bar{B}_{\theta_i})+u(i-r)'R_{\theta_i}|\mathcal{H}_{i-r}]\varepsilon\Delta u(i-r)\}\\ &+o(\varepsilon).\end{aligned}$$

It is obvious that the necessary condition for the extreme value of performance index is that $\Delta J_N \geq 0$. Since $\Delta u(t)$ is arbitrary for $1 \leq t \leq N$, the necessary condition becomes

$$\mathrm{E}[\eta_i'(B_{\theta_i}+\omega_i\bar{B}_{\theta_i})+u(i-r)'R_{\theta_i}|\mathcal{H}_{i-r}] = 0.$$

The result (4) is evident and the proof is completed. ■

B. Proof of Lemma 2

Proof: Firstly, we prove that equation (20) holds. In view of (16), (18), and (19), we have

$$\begin{aligned}&\mathrm{E}\{\Phi_{\theta_t}(t)'(\alpha^r_{\theta_{t+1},\theta_{t+1}}(t+1,t+1))'|\mathcal{H}_t\}\\ =&\mathcal{L}_{\theta_t}(F'_{\theta_{t+r-1},\theta_t}(\delta^1_{\theta_{t+r}}(t+r))')\\ &-\sum_{s=0}^{r-2}\mathcal{L}_{\theta_t}(F'_{\theta_{t+s},\theta_t}T^{0,r-s}_{\theta_{t+s+1}}(t+s+1))\\ =&\mathcal{L}_{\theta_t}(F'_{\theta_{t+r-1},\theta_t}(A'_{\theta_{t+r}}P_{\theta_{t+r+1}}(t+r+1)B_{\theta_{t+r}}\\ &+\mu^2\bar{A}'_{\theta_{t+r}}P_{\theta_{t+r+1}}(t+r+1)\bar{B}_{\theta_{t+r}}))\\ &-\sum_{s=0}^{r-1}\mathcal{L}_{\theta_t}(F'_{\theta_{t+s},\theta_t}T^{0,r-s}_{\theta_{t+s+1}}(t+s+1)) = (T^0_{\theta_t}(t))',\end{aligned}$$

and (20) is shown.

Next, we will show that (22) is true for $j = 0$. Based on (16), (18), and (19), and recalling the definition of $T^1_{\theta_t}(t)$, we obtain

$$\begin{aligned}&\mathrm{E}\{\Gamma_{\theta_t}(t)'(\alpha^r_{\theta_{t+1},\theta_{t+1}}(t+1,t+1))'|\mathcal{H}_t\}\\ =&\mathcal{L}_{\theta_t}(B'_{\theta_t}F'_{\theta_{t+r-1},\theta_{t+1}}(\delta^1_{\theta_{t+r}}(t+r))')\\ &-\sum_{s=1}^{r-1}\mathcal{L}_{\theta_t}(B'_{\theta_t}F'_{\theta_{t+r-s-1},\theta_{t+1}}T^{0,s+1}_{\theta_{t+r-s}}(t+r-s))\\ =&\mathcal{L}_{\theta_t}(B'_{\theta_t}F'_{\theta_{t+r-1},\theta_{t+1}}(A'_{\theta_{t+r}}P_{\theta_{t+r+1}}(t+r+1)B_{\theta_{t+r}}\\ &+\mu^2\bar{A}'_{\theta_{t+r}}P_{\theta_{t+r+1}}(t+r+1)\bar{B}_{\theta_{t+r}}))\\ &-\sum_{s=0}^{r-1}\mathcal{L}_{\theta_t}(B'_{\theta_t}F'_{\theta_{t+r-s-1},\theta_{t+1}}T^{0,s+1}_{\theta_{t+r-s}}(t+r-s)) = (T^1_{\theta_t}(t))'.\end{aligned}$$

So, (22) is deduced for the case of $j = 0$.

In the sequel, we will show that equation (21) and (22) are true for $j = 1, 2, \cdots, r-1$. The induction method will be employed in this derivation. For this purpose, take any $1 \leq s \leq j-1$, and presume that (21) and (22) are satisfied for all $1 \leq s \leq j-1$. Now, we are in the position to show that (21) and (22) are true for $s = j$. From (17) and the above equations, it follows that

$$\begin{aligned}&\mathrm{E}\{\Phi_{\theta_t}(t)'(\alpha^{r-j}_{\theta_{t+1},\theta_{t-j+1}}(t+1,t-j+1))'|\mathcal{H}_t\}\\ =&A'_{\theta_t}\mathcal{L}_{\theta_t}((\delta^{r-j}_{t+1}(t+1))')-(T^0_{\theta_t}(t))'W_{\theta_t}(t)^{-1}T^{r-j+1}_{\theta_t}(t)\\ &-\sum_{s=1}^{j-1}(\alpha^{r-s+1}_{\theta_t,\theta_{t-s+1}}(t,t-s+1))'W_{\theta_{t-s}}(t-s)^{-1}T^{r-j+s+1}_{\theta_{t-s}}(t-s)\\ =&(\alpha^{r-j+1}_{\theta_t,\theta_{t-j+1}}(t,t-j+1))' \quad (73)\end{aligned}$$

and

$$\begin{aligned}&\mathrm{E}\{\Gamma_{\theta_t}(t)'(\alpha^{r-j}_{\theta_{t+1},\theta_{t-j+1}}(t+1,t-j+1))'|\mathcal{H}_{t-j}\}\\ =&\mathrm{E}\{\Gamma_{\theta_t}(t)'(\delta^{r-j}_{\theta_{t+1}}(t+1))'-\sum_{s=1}^{j}(T^s_{\theta_{t+1-s}}(t+1-s))'\\ &\times W_{\theta_{t+1-s}}(t+1-s)^{-1}T^{r-j+s}_{\theta_{t+1-s}}(t+1-s)]|\mathcal{H}_{t-j}\}\\ =&(T^{j+1}_{\theta_{t-j}}(t-j))'. \quad (74)\end{aligned}$$

From (73) and (74), we know that (21) and (22) are true for $s = j$. By the induction method, we obtain that (21) and (22) are satisfied for all $j = 1, 2, \cdots, r-1$. The proof of Lemma 2 is completed. ■

C. Proof of Theorem 1

Proof: The inductive approach will be used for solving the D-FBMDE. First of all, consider the case of $t = N$. Given (1), (4) and (5), we have

$$\begin{aligned}0 =& \mathrm{E}[\Gamma_{\theta_N}(N)'\eta_N + R_{\theta_N}u(N-r)|\mathcal{H}_{N-r}]\\ =& \mathrm{E}[\Gamma_{\theta_N}(N)'P_{\theta_{N+1}}\Phi_{\theta_N}(N)x(N)|\mathcal{H}_{N-r}]\\ &+W_{\theta_{N-r}}(N-r)u(N-r). \quad (75)\end{aligned}$$

Based on the condition that $W_{\theta_{N-r}}(N-r)$, $T^j_{\theta_{N-r}}(N-r)$ $(j = 0, 1, \cdots, r)$, and $P_{\theta_N}(N)$ are the solutions to (10)-(15), we obtain that the inverse of $W_{\theta_{N-r}}(N-r)$ exists. Therefore, it follows from (75) that

$$\begin{aligned}u(N-r) =& -W_{\theta_{N-r}}(N-r)^{-1}\mathrm{E}[\Gamma_{\theta_N}(N)'P_{\theta_{N+1}}\\ &\times\Phi_{\theta_N}(N)x(N)|\mathcal{H}_{N-r}]. \quad (76)\end{aligned}$$

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Furthermore, applying (1), (5), and (76) to (6), one obtains

$$\begin{aligned}\eta_{N-1} =&\mathrm{E}\{\Phi_{\theta_N}(N)'\eta_N + Q_{\theta_N}x(N)|\mathcal{H}_N\}\\ =&P_{\theta_N}(N)x(N) - \sum_{s=1}^{r}(\alpha^{r-s+1}_{\theta_N,\theta_{N-s+1}}(N,N-s+1))'\\ &\times W_{\theta_{N-s}}(N-s)^{-1}\\ &\times \mathrm{E}\{\alpha^{r-s+1}_{\theta_N,\theta_{N-s+1}}(N,N-s+1)x(N)|\mathcal{H}_{N-s}\},\end{aligned}$$

where the terminal conditions $\alpha^{r-s+1}_{\theta_N,\theta_{N-s+1}}(N,N-s+1) = 0$, $s = 1,2,\cdots,r-1$ have been used. Therefore, (23) is demonstrated for $t = N$.

By the inductive method, we select any k with $1 \leq k \leq N$, and presume that the costates η_{t-1} are given as (23) for all $t \geq k+1$. In the sequel, we need to show that η_{k-1} satisfies (23). To achieve this aim, substituting η_k into (4) and using the transforms (20)-(22), we obtain

$$\begin{aligned}0 =&\mathrm{E}\{\Gamma_{\theta_k}(k)'\eta_k + R_{\theta_k}u(k-r)|\mathcal{H}_{k-r}\}\\ =&\mathrm{E}\{[B'_{\theta_k}\mathcal{L}_{\theta_k}(P_{\theta_{k+1}}(n+1))A_{\theta_k}\\ &+\mu^2\bar{B}'_{\theta_k}\mathcal{L}_{\theta_k}(P_{\theta_{k+1}}(n+1))\bar{A}_{\theta_k}\\ &-(T^1_{\theta_k}(k))'W_{\theta_k}(k)^{-1}T^0_{\theta_k}(k)]x(k)\\ &-\sum_{s=1}^{r-1}(T^{s+1}_{\theta_{k-s}}(k-s))'W_{\theta_{k-s}}(k-s)^{-1}\\ &\times \mathrm{E}[\alpha^{r-s+1}_{\theta_k,\theta_{k-s+1}}(k,k-s+1)x(k)|\mathcal{H}_{k-s}]|\mathcal{H}_{k-r}\}\\ &+W_{\theta_{k-r}}(k-r)u(k-r)\\ =&\mathrm{E}[\alpha^1_{\theta_k,\theta_{k-r+1}}(k,k-r+1)x(k)|\mathcal{H}_{k-r}]\\ &+W_{\theta_{k-r}}(k-r)u(k-r).\end{aligned}\tag{77}$$

Since the solution to (10)-(15) exists, $W_{\theta_{k-r}}(k-r)^{-1}$ exists. Therefore, it follows from (77) that

$$\begin{aligned}u(k-r) =&-W_{\theta_{k-r}}(k-r)^{-1}\\ &\times \mathrm{E}[\alpha^1_{\theta_k,\theta_{k-r+1}}(k,k-r+1)x(k)|\mathcal{H}_{k-r}].\end{aligned}\tag{78}$$

Substituting η_k into (6) and using (1), (20)-(22), (78), one obtains

$$\begin{aligned}\eta_{k-1} =&\mathrm{E}\{Q_{\theta_k}x(k) + \Phi_{\theta_k}(k)'\eta_k|\mathcal{H}_k\}\\ =&[Q_{\theta_k} + A'_{\theta_k}\mathcal{L}_{\theta_k}(P_{\theta_{k+1}}(k+1))A_{\theta_k} + \mu^2\bar{A}'_{\theta_k}\\ &\times \mathcal{L}_{\theta_k}(P_{\theta_{k+1}}(k+1))\bar{A}_{\theta_k} - T^{0,0}_{\theta_k}(k)]x(k)\\ &-\sum_{s=1}^{r-1}(\alpha^{r-s+1}_{\theta_k,\theta_{k-s+1}}(k,k-s+1))'W_{\theta_{k-s}}(k-s)^{-1}\\ &\times \mathrm{E}[\alpha^{r-s+1}_{\theta_k,\theta_{k-s+1}}(k,k-s+1)x(k)|\mathcal{H}_{k-s}]\\ &+[A'_{\theta_k}\mathcal{L}_{\theta_k}(P_{\theta_{k+1}}(k+1))B_{\theta_k} + \mu^2\bar{A}'_{\theta_k}\\ &\times \mathcal{L}_{\theta_k}(P_{\theta_{k+1}}(k+1))\bar{B}_{\theta_k} - T^{0,1}_{\theta_k}(k)\\ &-\sum_{s=1}^{r-1}(\alpha^{r-s+1}_{\theta_k,\theta_{k-s+1}}(k,k-s+1))'\\ &\times W_{\theta_{k-s}}(k-s)^{-1}T^{s+1}_{\theta_{k-s}}(k-s)]u(k-r)\\ =&P_{\theta_k}(k)x(k) - \sum_{s=1}^{r}(\alpha^{r-s+1}_{\theta_k,\theta_{k-s+1}}(k,k-s+1))'\\ &\times W_{\theta_{k-s}}(k-s)^{-1}\\ &\times \mathrm{E}[\alpha^{r-s+1}_{\theta_k,\theta_{k-s+1}}(k,k-s+1)x(k)|\mathcal{H}_{k-s}].\end{aligned}\tag{79}$$

Thus, (23) is proved by the inductive method. This accomplishes the proof of Theorem 1. ■

D. Proof of Theorem 2

Proof: (i) Necessary: Suppose that the solution to Problem 1 is uniquely existed. Using the inductive method, we shall demonstrate that $W_{\theta_{t-r}}(t-r)$ in (10) is reversible and $u(t-r)$ obeys (26). Set

$$\begin{aligned}J(t) \triangleq&\mathrm{E}\{\sum_{i=t}^{N}(x(i)'Q_{\theta_i}x(i) + u(i-r)'R_{\theta_i}u(i-r))\\ &+x(N+1)'P_{\theta_{N+1}}x(N+1)|\mathcal{H}_{t-r}\}\end{aligned}\tag{80}$$

for $t = N,\cdots,r$. For $t = N$, (80) can be written as

$$\begin{aligned}J(N)=&\mathrm{E}\{x(N)'Q_{\theta_N}x(N) + u(N-r)'R_{\theta_N}u(N-r)\\ &+x(N+1)'P_{\theta_{N+1}}x(N+1)|\mathcal{H}_{N-r}\}.\end{aligned}\tag{81}$$

In view of (1), we know that $J(N)$ is with a quadratic form about the state and input terms. Note that system (1) can start at any time with arbitrary initial values. Letting $x(N) = 0$ and substituting (1) into (81), one obtains

$$\begin{aligned}J(N) =&\mathrm{E}\{u(N-r)'(R_{\theta_N} + \Gamma_{\theta_N}(N)'P_{\theta_{N+1}}\\ &\times \Gamma_{\theta_N}(N))u(N-r)|\mathcal{H}_{N-r}\}\\ =&u(N-r)'W_{\theta_{N-r}}(N-r)u(N-r) > 0.\end{aligned}\tag{82}$$

The uniqueness of the optimal controller indicates that $J(N)$ must be positive for arbitrary $u(N-r) \neq 0$. It can be concluded from (82) that $W_{\theta_{N-r}}(N-r) > 0$. Furthermore, based on the necessary condition that Problem 1 is uniquely solvable, we obtain from (76) that

$$\begin{aligned}u(N-r) =&-W_{\theta_{N-r}}(N-r)^{-1}[T^0_{\theta_{N-r}}(N-r)x(N-r)\\ &+\sum_{j=1}^{r}T^j_{\theta_{N-r}}(N-r)u(N-2r+j-1)],\end{aligned}\tag{83}$$

where the following expression

$$\begin{aligned}x(N) =&\Psi(N-1,N-r)x(N-r)\\ &+\sum_{j=0}^{r-1}\Psi(N-1,N-r+j+1)\\ &\times \Gamma_{\theta_{N-r+j}}(N-r+j)u(N-2r+j).\end{aligned}$$

has been employed in the derivation of (83).

Now, select any k with $1 \leq k \leq N$, and presume that $W_{\theta_{t-r}}(t-r)$ is reversible and the explicit solution to $u(t-r)$ is given as (26) for $t \geq k+1$. In what follows, it remains to prove that the existence condition and explicit solution for $u(k-r)$ is also satisfied. Letting $x(k) = 0$, we first derive the quadratic form of $u(k-r)$ in $J(k)$. In light of (1), (4) and (6) for $t \geq k+1$, we obtain

$$\begin{aligned}&\mathrm{E}\{x(t)'\eta_{t-1} - x(t+1)'\eta_t|\mathcal{H}_{k-r+1}\}\\ =&\mathrm{E}\{x(t)'Q_{\theta_t}x(t) + u(t-r)'R_{\theta_t}u(t-r)|\mathcal{H}_{k-r+1}\}.\end{aligned}$$

Adding up the previous equation from $t = k+1$ to $t = N$, we obtain that

$$\begin{aligned}&\mathrm{E}\{x(k+1)'\eta_k - x(N+1)'\eta_N|\mathcal{H}_{k-r+1}\}\\ =&\sum_{t=k+1}^{N}\mathrm{E}\{x(t)'Q_{\theta_t}x(t) + u(t-r)'R_{\theta_t}u(t-r)|\mathcal{H}_{k-r+1}\},\end{aligned}\tag{84}$$

which leads to

$$\begin{aligned}J(k) =&\mathrm{E}\{u(k-r)'R_{\theta_k}u(k-r) + u(k-r)'\Gamma_{\theta_k}(k)'\\ &\times \eta_k|\mathcal{H}_{k-r}\}.\end{aligned}\tag{85}$$

It follows from Theorem 1 that

$$\begin{aligned}\eta_k =&P_{\theta_{k+1}}(k+1)[\Phi_{\theta(k)}(k)x(k) + \Gamma_{\theta(k)}(k)u(k-r)]\\ &-\sum_{s=1}^{r}(\alpha^{r-s+1}_{\theta_{k+1},\theta_{k+2-s}}(k+1,k+2-s))'\end{aligned}$$

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$$\begin{aligned}&\times W_{\theta_{k+1-s}}(k+1-s)^{-1}\\&\times \mathrm{E}\{\alpha^{r-s+1}_{\theta_{k+1},\theta_{k+2-s}}(k+1,k+2-s)\\&\times x(k+1)|\mathcal{H}_{k+1-s}\}.\end{aligned}\tag{86}$$

Substituting (86) into (85) and employing (20)-(22), we arrive at

$$\begin{aligned}J(k)=&u(k-r)'\mathrm{E}\{R_{\theta_k}+\Gamma_{\theta_k}(k)'P_{\theta_{k+1}}(k+1)\Gamma_{\theta_k}(k)\\&-\sum_{s=1}^{r}\Gamma_{\theta_k}(k)'(\alpha^{r-s+1}_{\theta_{k+1},\theta_{k+2-s}}(k+1,k+2-s))'\\&\times W_{\theta_{k+1-s}}(k+1-s)^{-1}\\&\times \mathrm{E}[\alpha^{r-s+1}_{\theta_{k+1},\theta_{k+2-s}}(k+1,k+2-s)\Gamma_{\theta_k}(k)|\mathcal{H}_{k+1-s}]\\&|\mathcal{H}_{k-r}\}u(k-r)\\=&u(k-r)'W_{\theta_{k-r}}(k-r)u(k-r).\end{aligned}\tag{87}$$

Recalling that there exists a unique solution to the optimal controller, we can obtain the positiveness of $J(k)$ directly. Therefore, it follows that $W_{\theta_{k-r}}(k-r)>0$.

To deduce the optimal controller $u(k-r)$, substituting (86) into (4) and using (20)-(22) yield

$$\begin{aligned}u(k-r)=&-W_{\theta_{k-r}}(k-r)^{-1}\mathrm{E}[\alpha^{1}_{\theta_k,\theta_{k-r+1}}(k,k-r+1)\\&\times x(k)|\mathcal{H}_{k-r}]\\=&-W_{\theta_{k-r}}(k-r)^{-1}[T^0_{\theta_{k-r}}(k-r)x(k-r)\\&+\sum_{j=1}^{r}T^j_{\theta_{k-r}}(k-r)u(k-2r+j-1)].\end{aligned}$$

The necessity is shown.

(ii) Sufficiency: Presume that (25) is true, i.e. $W_{\theta_{t-r}}(t-r)>0$ for $t\geq r$. We will show that Problem 1 admits a unique solution. Let

$$\begin{aligned}&V_N(t,x(t))\\=&\mathrm{E}\{x(t)'P_{\theta_t}(t)x(t)-x(t)'\sum_{s=1}^{r}(\alpha^{r-s+1}_{\theta_t,\theta_{t-s+1}}(t,t-s+1))'\\&\times W_{\theta_{t-s}}(t-s)^{-1}\mathrm{E}[\alpha^{r-s+1}_{\theta_t,\theta_{t-s+1}}(t,t-s+1)x(t)|\mathcal{H}_{t-s}]\}.\end{aligned}\tag{88}$$

Applying (1) and (10)-(17), one gets that

$$\begin{aligned}&V_N(t,x(t))-V_N(t+1,x(t+1))\\=&\mathrm{E}\{x(t)'Q_{\theta_t}x(t)+u(t-r)'R_{\theta_t}u(t-r)\\&-[u(t-r)+W_{\theta_{t-r}}(t-r)^{-1}\\&\times \mathrm{E}(\alpha^1_{\theta_t,\theta_{t-r+1}}(t,t-r+1)x(t)|\mathcal{H}_{t-r})]'\\&\times W_{\theta_{t-r}}(t-r)[u(t-r)+W_{\theta_{t-r}}(t-r)^{-1}\\&\times \mathrm{E}(\alpha^1_{\theta_t,\theta_{t-r+1}}(t,t-r+1)x(t)|\mathcal{H}_{t-r})]\}\end{aligned}\tag{89}$$

where the expressions (20)-(22) have played a significant role in the deduction of (89). Adding up (89) from $t=r$ to $t=N$, the index (2) is rewritten as

$$\begin{aligned}J_N=&\mathrm{E}\{\sum_{t=0}^{r-1}x(t)'Q_{\theta_t}x(t)+x(r)'P_{\theta_r}(r)x(r)\\&-x(r)'\sum_{s=1}^{r}(\alpha^{r-s+1}_{\theta_r,\theta_{r-s+1}}(r,r-s+1))'\\&\times W_{\theta_{r-s}}(r-s)^{-1}\mathrm{E}[\alpha^{r-s+1}_{\theta_r,\theta_{r-s+1}}(r,r-s+1)\\&\times x(r)|\mathcal{H}_{r-s}]+\sum_{t=r}^{N}[u(t-r)+W_{\theta_{t-r}}(t-r)^{-1}\\&\times \mathrm{E}(\alpha^1_{\theta_t,\theta_{t-r+1}}(t,t-r+1)x(t)|\mathcal{H}_{t-r})]'\\&\times W_{\theta_{t-r}}(t-r)[u(t-r)+W_{\theta_{t-r}}(t-r)^{-1}\\&\times \mathrm{E}(\alpha^1_{\theta_t,\theta_{t-r+1}}(t,t-r+1)x(t)|\mathcal{H}_{t-r})]\}.\end{aligned}$$

Take notice that for $t\leq r$, $x(t)$ is determined by the initial given value $x_0,u_{-1},\cdots,u_{-r}$. Therefore, the optimum value of J_N depends on the fourth term of the above equation. Since $W_{\theta_{t-r}}(t-r)$ is positive definite for $r\leq t\leq N$, a unique optimal controller subject to Problem 1 is existed and satisfies (26), and the optimal index satisfies (27). The sufficiency is shown. ∎

E. Proof of Lemma 3

Proof: Under the circumstance of $R>0$, following a similar line with the derivation of Lemma 1 and the discussion of Remark 4 in [46], we can show that $W_{\theta_{t-r}}(t-r,N)>0$ with $N\geq r$, $0\leq t\leq N$. Thus, $W_{\theta_{t-r}}(t-r,N)^{-1}$ exists.

Assume that system (1) start at r and set

$$\begin{aligned}S_r=&\sum_{t=r}^{\bar N}\mathrm{E}\{x(t)'Qx(t)+u(t-r)'Ru(t-r)\},\\\bar N=&N-r+t,\end{aligned}\tag{90}$$

where the initial value can be chosen arbitrarily. On the ground of (27), the optimum value of (90) is with the form

$$\begin{aligned}S_r^*=&\mathrm{E}\{x(r)'P_{\theta_r}(r,\bar N)x(r)\\&-x(r)'\sum_{s=1}^{r}(\alpha^{r-s+1}_{\theta_r,\theta_{r-s+1}}(r,r-s+1,\bar N))'\\&\times W_{\theta_{r-s}}(r-s,\bar N)^{-1}\\&\times \mathrm{E}[\alpha^{r-s+1}_{\theta_r,\theta_{r-s+1}}(r,r-s+1,\bar N)x(r)|\mathcal{H}_{r-s}]\}\\=&x(r)'\{P_{\theta_r}(r,\bar N)-\sum_{s=1}^{r}(\alpha^{r-s+1}_{\theta_r,\theta_{r-s+1}}(r,r-s+1,\bar N))'\\&\times W_{\theta_{r-s}}(r-s,\bar N)^{-1}\\&\times \alpha^{r-s+1}_{\theta_r,\theta_{r-s+1}}(r,r-s+1,\bar N)\}x(r)\geq 0.\end{aligned}\tag{91}$$

Note that $x(r)$ can be selected as any value, so (91) implies that

$$\begin{aligned}&P_{\theta_r}(r,\bar N)-\sum_{s=1}^{r}(\alpha^{r-s+1}_{\theta_r,\theta_{r-s+1}}(r,r-s+1,\bar N))'\\&\times W_{\theta_{r-s}}(r-s,\bar N)^{-1}\alpha^{r-s+1}_{\theta_r,\theta_{r-s+1}}(r,r-s+1,\bar N)\geq 0.\end{aligned}\tag{92}$$

Since the notations defined in (10)-(19) keep invariant for N owing to the selection of $P_{\theta_{N+1}}=0$, i.e.,

$$\begin{aligned}W_{\theta_{t-r}}(t-r,N)=&W_{\theta_{t-r-s}}(t-r-s,N-s),\\T^j_{\theta_{t-r}}(t-r,N)=&T^j_{\theta_{t-r-s}}(t-r-s,N-s),\\&j=0,1,\cdots,r,\\P_{\theta_t}(t,N)=&P_{\theta_{t-s}}(t-s,N-s),\end{aligned}$$

$$\begin{aligned}&\alpha^{r-j+1}_{\theta_t,\theta_{t-j+1}}(t,t-j+1,N)\\=&\alpha^{r-j+1}_{\theta_{t-s},\theta_{t-s-j+1}}(t-s,t-s-j+1,N-s),\\&j=1,\cdots,r,\end{aligned}$$

we obtain (33) from (92) directly. Furthermore, (33) implies that

$$\begin{aligned}P_{\theta_t}(t,N)\geq&\sum_{s=1}^{r}(\alpha^{r-s+1}_{\theta_t,\theta_{t-s+1}}(t,t-s+1,N))'\\&\times W_{\theta_{t-s}}(t-s,N)^{-1}\\&\times \alpha^{r-s+1}_{\theta_t,\theta_{t-s+1}}(t,t-s+1,N)\geq 0.\end{aligned}\tag{93}$$

Thus, from (93), one gets $P_{\theta_t}(t,N)\geq 0$. Now, (32) is proven. The proof of Lemma 3 is done. ∎

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F. Proof of Lemma 4

Proof: In order to facilitate the description, we denote

$$
\begin{aligned}
&Y_{\theta_r,\theta_0}(r,0,N)\\
&\triangleq P_{\theta_r}(r,N)-\sum_{s=1}^{r}(\alpha_{\theta_r,\theta_{r-s+1}}^{r-s+1}(r,r-s+1,N))'\\
&\quad\times W_{\theta_{r-s}}(r-s,N)^{-1}\alpha_{\theta_r,\theta_{r-s+1}}^{r-s+1}(r,r-s+1,N).
\end{aligned}
$$

If Assumption 3 is satisfied, it follows from Lemma 3 that $Y_{\theta_r,\theta_0}(r,0,N)\geq 0$ for all $N\geq r$. In the sequel, we just need to show that there exists $N_0\geq r$ such that $Y_{\theta_r,\theta_0}(r,0,N_0)>0$. Assume this is not valid, and we obtain a non-empty set

$$X_N\triangleq\{x\in R^n: x\neq 0, x'Y_{\theta_r,\theta_0}(r,0,N)x=0\}.$$

In light of (90) and (91), we can deduce that $x'Y_{\theta_r,\theta_0}(r,0,N)x \leq x'Y_{\theta_r,\theta_0}(r,0,N+1)x$. Since x is arbitrary, we obtain that $Y_{\theta_r,\theta_0}(r,0,N)\leq Y_{\theta_r,\theta_0}(r,0,N+1)$. Then, if $x'Y_{\theta_r,\theta_0}(r,0,N+1)x=0$, we can deduce that $x'Y_{\theta_r,\theta_0}(r,0,N)x=0$, which implies that $X_{N+1}\subset X_N$. Noting that each X_N is non-empty and with finite-dimension, we can obtain that

$$1\leq\cdots\leq\dim(X_{r+2})\leq\dim(X_{r+1})\leq\dim(X_r)\leq n. \tag{94}$$

It follows from (94) that there must exist an integer N_1, such that for $N\geq N_1$, $\dim(X_N)=\dim(X_{N_1})$ and thus $X_N=X_{N_1}$. It means that $\bigcap_{N\geq r}X_N=X_{N_1}\neq\emptyset$. Therefore, there must exist a nonzero vector $x\in X_{N_1}$ such that $x'Y_{\theta_r,\theta_0}(r,0,N+1)x=0$ for any $N\geq r$.

Set $x(r)=x$ in (91), and we obtain

$$
\begin{aligned}
S_r^* =& x(r)'\{P_{\theta_r}(r,N)-\sum_{s=1}^{r}(\alpha_{\theta_r,\theta_{r-s+1}}^{r-s+1}(r,r-s+1,N))'\\
&\times W_{\theta_{r-s}}(r-s,N)^{-1}\\
&\times\alpha_{\theta_r,\theta_{r-s+1}}^{r-s+1}(r,r-s+1,N)\}x(r)\\
=&0.
\end{aligned} \tag{95}
$$

It follows from the hypothesis $R>0$ and $Q=C'C\geq 0$ that

$$u^*(t-r)=0, Cx^*(t)=0, r\leq t\leq N, N\geq r.$$

Then, system (1) becomes as

$$
\begin{aligned}
x^*(t+1) &= (A_{\theta(t)}+\omega_t\bar{A}_{\theta(t)})x^*(t),\\
Cx^*(t) &= 0, \forall t\geq r.
\end{aligned} \tag{96}
$$

From the observability of (96), we obtain that $x(r)=0$. This contradicts the fact $x\neq 0$. So there exists some $N_0\geq r$ such that $Y_{\theta_r,\theta_0}(r,0,N_0)>0$. This competes the proof of Lemma 4. ∎

G. Proof of Theorem 3

Proof: At the beginning of the derivation, the convergence of the GCDRE (10)-(15) will be shown.

Let us start by proving that $W_{\theta_{t-r}}(t-r,N)$ and $T_{\theta_{t-r}}^{s}(t-r,N)$ $(s=0,1,\cdots,r)$ are convergent. Set

$$
\begin{aligned}
\bar{x}(t) =& \mathrm{col}\{x(t),u(t-1),\cdots,u(t-r)\},\\
\bar{\Phi}_{\theta_t}(t) =& \begin{bmatrix} \Phi_{\theta_t}(t) & 0 & \cdots & 0 & \Gamma_{\theta_t}(t)\\ 0 & 0 & \cdots & 0 & 0\\ 0 & I & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & I & 0 \end{bmatrix},\\
\Delta_{\theta_t} =& \begin{bmatrix} A_{\theta_t} & 0 & \cdots & 0 & B_{\theta_t}\\ 0 & 0 & \cdots & 0 & 0\\ 0 & I & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & I & 0 \end{bmatrix},\\
\bar{\Delta}_{\theta_t} =& \begin{bmatrix} \bar{A}_{\theta_t} & 0 & \cdots & 0 & \bar{B}_{\theta_t}\\ 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},\\
\bar{\Gamma} =& \begin{bmatrix} 0 & I & 0 & \cdots & 0 \end{bmatrix}',\\
\bar{Q} =& \mathrm{diag}\{Q,\overbrace{0,\cdots,0}^{r\ blocks}\},\\
\bar{P}_{\theta_{N+1}} =& \mathrm{diag}\{P_{\theta_{N+1}},\overbrace{0,\cdots,0}^{r\ blocks}\}.
\end{aligned}
$$

Then, system (1) and index (29) become as

$$\bar{x}(t+1)=\bar{\Phi}_{\theta_t}(t)\bar{x}(t)+\bar{\Gamma}u(t), \tag{97}$$

$$\bar{J}=\mathrm{E}\{\sum_{t=0}^{\infty}\bar{x}(t)'\bar{Q}\bar{x}(t)+u(t)'Ru(t)\}. \tag{98}$$

The cost function on finite horizon can be expressed as

$$
\begin{aligned}
\bar{J}_N =& \mathrm{E}\{\sum_{t=0}^{N}\bar{x}(t)'\bar{Q}\bar{x}(t)+u(t)'Ru(t)\\
&+\bar{x}(N+1)'\bar{P}_{\theta_{N+1}}\bar{x}(N+1)\}.
\end{aligned} \tag{99}
$$

The delay-free maximum principle, that is, the necessary optimality condition for the optimal control of system (97) with (99), can be stated as

$$0=\mathrm{E}[\bar{\Gamma}'\bar{\eta}_t+Ru(t)|\mathcal{H}_t], \tag{100}$$

$$\bar{\eta}_{t-1}=\mathrm{E}[\bar{\Phi}_{\theta_t}(t)'\bar{\eta}_t+\bar{Q}x(t)|\mathcal{H}_t], \tag{101}$$

$$\bar{\eta}_N=\bar{P}_{\theta_{N+1}}x(N+1). \tag{102}$$

Applying (100)-(102) and following a similar derivation as that of Theorem 2, one gets

$$u(t)=-\bar{\Upsilon}_{\theta_t}(t,N)^{-1}\bar{M}_{\theta_t}(t,N)\bar{x}(t), \tag{103}$$

$$\bar{\eta}_{t-1}=\bar{P}_{\theta_t}(t,N)\bar{x}(t), \tag{104}$$

where $\bar{\Upsilon}_{\theta_t}(t,N)$, $\bar{M}_{\theta_t}(t,N)$ and $\bar{P}_{\theta_t}(t,N)$ obey the following difference Riccati equations

$$\bar{\Upsilon}_{\theta_t}(t,N)=\bar{\Gamma}'\mathcal{L}_{\theta_t}(\bar{P}_{\theta_{t+1}}(t+1,N))\bar{\Gamma}+R, \tag{105}$$

$$\bar{M}_{\theta_t}(t,N)=\bar{\Gamma}'\mathcal{L}_{\theta_t}(\bar{P}_{\theta_{t+1}}(t+1,N))\Delta_{\theta_t}, \tag{106}$$

$$
\begin{aligned}
\bar{P}_{\theta_t}(t,N) =& \Delta_{\theta_t}'\mathcal{L}_{\theta_t}(\bar{P}_{\theta_{t+1}}(t+1,N))\Delta_{\theta_t}\\
&+\mu^2\bar{\Delta}_{\theta_t}'\mathcal{L}_{\theta_t}(\bar{P}_{\theta_{t+1}}(t+1,N))\bar{\Delta}_{\theta_t}\\
&+\bar{Q}-\bar{M}_{\theta_t}(t,N)'\bar{\Upsilon}_{\theta_t}(t,N)^{-1}\bar{M}_{\theta_t}(t,N).
\end{aligned} \tag{107}
$$

In accordance with the formation of $\bar{x}(t)$, the partitioned forms of $\bar{\eta}_{t-1}$ and $\bar{P}_{\theta_t}(t,N)$ can be expressed as

$$
\begin{aligned}
&\bar{\eta}_{t-1}\\
=&\mathrm{col}\{\bar{\eta}_{t-1}^{(0)},\bar{\eta}_{t-1}^{(1)},\cdots,\bar{\eta}_{t-1}^{(r)}\},\\
&\bar{P}_{\theta_t}(t,N)\\
=&\begin{bmatrix} \bar{P}_{\theta_t}^{(0,0)}(t,N) & \bar{P}_{\theta_t}^{(0,1)}(t,N) & \cdots & \bar{P}_{\theta_t}^{(0,r)}(t,N)\\ \bar{P}_{\theta_t}^{(1,0)}(t,N) & \bar{P}_{\theta_t}^{(1,1)}(t,N) & \cdots & \bar{P}_{\theta_t}^{(1,r)}(t,N)\\ \vdots & \vdots & \ddots & \vdots\\ \bar{P}_{\theta_t}^{(r,0)}(t,N) & \bar{P}_{\theta_t}^{(r,1)}(t,N) & \cdots & \bar{P}_{\theta_t}^{(r,r)}(t,N) \end{bmatrix}.
\end{aligned}
$$

From (104), the second component of $\bar{\eta}_{t-1}$ satisfies

$$
\begin{aligned}
\bar{\eta}_{t-1}^{(1)} =& \bar{P}_{\theta_t}^{(1,0)}(t,N)x(t)+\bar{P}_{\theta_t}^{(1,1)}(t,N)u(t-1)\\
&+\cdots+\bar{P}_{\theta_t}^{(1,r)}(t,N)u(t-r).
\end{aligned} \tag{108}
$$

Meanwhile, (100) can be reduced to

$$0=\mathrm{E}\{\bar{\eta}_t^{(1)}+Ru(t)|\mathcal{H}_t\}. \tag{109}$$

Alternatively, it can be obtained from (4) that

$$0 = \mathrm{E}\{\Gamma_{\theta_{t+r}}(t+r)'\eta_{t+r} + Ru(t)|\mathcal{H}_t\}. \tag{110}$$

Comparing (109) with (110), we obtain

$$\begin{aligned}\mathrm{E}\{\bar{\eta}_t^{(1)}|\mathcal{H}_t\} =&\mathrm{E}\{\Gamma_{\theta_{t+r}}(t+r)'\eta_{t+r}|\mathcal{H}_t\},\\ =&T^0_{\theta_t}(t,N)x(t) + (W_{\theta_t}(t,N) - R)u(t)\\ &+\sum_{j=1}^{r} T^j_{\theta_t}(t,N)u(t-r+j-1). \end{aligned}\tag{111}$$

Comparing (108) with (111), one has

$$T^0_{\theta_t}(t,N) =\mathcal{L}_{\theta_t}(\bar{P}^{(1,0)}_{\theta_{t+1}}(t+1,N))A_{\theta_t}, \tag{112}$$

$$W_{\theta_t}(t,N) - R =\mathcal{L}_{\theta_t}(\bar{P}^{(1,1)}_{\theta_{t+1}}(t+1,N)), \tag{113}$$

$$T^r_{\theta_t}(t,N) =\mathcal{L}_{\theta_t}(\bar{P}^{(1,2)}_{\theta_{t+1}}(t+1,N)), \tag{114}$$

$$\vdots$$

$$T^2_{\theta_t}(t,N) =\mathcal{L}_{\theta_t}(\bar{P}^{(1,r)}_{\theta_{t+1}}(t+1,N)), \tag{115}$$

$$T^1_{\theta_t}(t,N) =\mathcal{L}_{\theta_t}(\bar{P}^{(1,0)}_{\theta_{t+1}}(t+1,N))B_{\theta_t}. \tag{116}$$

Note that, in case system (1) is stabilizable (observable, respectively), then the new extended delay-free system (97) is stabilizable (observable, respectively) too. With the stabilization and observability of system (97), we can obtain that $\bar{P}_{\theta_t}(t,N)$ given in (107) is convergent, i.e., $\lim_{N\to\infty}\bar{P}_{l_0}(t,N) = \bar{P}_{l_0}, l_0 = 1,2,\cdots,L$, where $\bar{P}_{l_0} > 0$ is the unique solution to the following coupled algebraic equations

$$\begin{aligned}\bar{P}_{l_0} =&\Delta'_{l_0}(\sum_{\tilde{l}=1}^{L}\lambda_{l_0\tilde{l}}\bar{P}_{\tilde{l}})\Delta_{l_0} + \mu^2\bar{\Delta}'_{l_0}(\sum_{\tilde{l}=1}^{L}\lambda_{l_0\tilde{l}}\bar{P}_{\tilde{l}})\bar{\Delta}_{l_0} + \bar{Q}\\ &-\bar{M}'_{l_0}\bar{\Upsilon}^{-1}_{l_0}\bar{M}_{l_0}, l_0 = 1,2,\cdots,L,\\ \bar{\Upsilon}_{l_0} =&\bar{\Gamma}'(\sum_{\tilde{l}=1}^{L}\lambda_{l_0\tilde{l}}\bar{P}_{\tilde{l}})\bar{\Gamma} + R, l_0 = 1,2,\cdots,L,\\ \bar{M}_{l_0} =&\bar{\Gamma}'(\sum_{\tilde{l}=1}^{L}\lambda_{l_0\tilde{l}}\bar{P}_{\tilde{l}})\Delta_{l_0}, l_0 = 1,2,\cdots,L.\end{aligned}$$

In view of the partitioned form of $\bar{P}_{\theta_t}(t,N)$, we can obtain that the block matrices $\bar{P}^{(1,0)}_{\theta_{t+1}}(t+1,N)$, $\bar{P}^{(1,1)}_{\theta_{t+1}}(t+1,N)$, $\cdots,\bar{P}^{(1,r)}_{\theta_{t+1}}(t+1,N)$ involved in $\bar{P}_{\theta_{t+1}}(t+1,N)$ are convergent as well. That is $\lim_{N\to\infty}\bar{P}^{(1,0)}_{\tilde{l}}(t+1,N) = \bar{P}^{(1,0)}_{\tilde{l}}, \lim_{N\to\infty}\bar{P}^{(1,1)}_{\tilde{l}}(t+1,N) = \bar{P}^{(1,1)}_{\tilde{l}},\cdots,\lim_{N\to\infty}\bar{P}^{(1,r)}_{\tilde{l}}(t+1,N) = \bar{P}^{(1,r)}_{\tilde{l}}, \tilde{l} = 1,2,\cdots,L$. $\bar{P}_{\tilde{l}} > 0, \tilde{l} = 1,2,\cdots,L$ implies that the diagonal block matrix $\bar{P}^{(1,1)}_{\tilde{l}} \geq 0, \tilde{l} = 1,2,\cdots,L$.

In view of (112)-(116), $T^0_{\theta_t}(t,N),\cdots,T^r_{\theta_t}(t,N)$, $W_{\theta_t}(t,N) - R$ are convergent, i.e.,

$$\begin{aligned}&\lim_{N\to\infty} T^j_{l_0}(t,N) = T^j_{l_0}, l_0 = 1,2,\cdots,L, j = 0,1,\cdots,r,\\ &\lim_{N\to\infty} W_{l_0}(t,N) = W_{l_0}, l_0 = 1,2,\cdots,L,\end{aligned}$$

and $T^j_{l_0}, W_{l_0}, l_0 = 1,2,\cdots,L$ satisfy the D-GCARE (35)-(39). Taking the limit on both sides of (113) subject to N, we obtain that

$$W_{l_0} - R = \sum_{\tilde{l}=1}^{L}\lambda_{l_0,\tilde{l}}\bar{P}^{(1,1)}_{\tilde{l}}.$$

Since $\bar{P}^{(1,1)}_{\tilde{l}} \geq 0$ and $R > 0$, we obtain that $W_{l_0} > 0$ for all $l_0 = 1,2,\cdots,L$. In addition, from the uniqueness of $\bar{P}_{l_0}$, we know that $T^j_{l_0}, W_{l_0}, l_0 = 1,2,\cdots,L$ are the unique solutions to (35)-(39).

The next step is to demonstrate that $P_{\theta_t}(t,N)$ is convergent. Without loss of generality, assume that the initializations $u(-i) = 0, i = 1,2,\cdots,d$, while x_0 is arbitrary. Adding up (89) from $t = 0$ to $t = N$, one gets

$$\begin{aligned}V_N(0,x_0) =& \sum_{t=0}^{N}[V_N(t,x(t)) - V_N(t+1,x(t+1))]\\ =&\sum_{t=0}^{N}\mathrm{E}\{x(t)'Qx(t) + u(t-r)'Ru(t-r)\\ &-[u(t-r) + W_{\theta_{t-r}}(t-r,N)^{-1}\\ &\times\mathrm{E}(\alpha^1_{\theta_t,\theta_{t-r+1}}(t,t-r+1,N)x(t)|\mathcal{H}_{t-r})]'\\ &\times W_{\theta_{t-r}}(t-r,N)\\ &\times[u(t-r) + W_{\theta_{t-r}}(t-r,N)^{-1}\\ &\times\mathrm{E}(\alpha^1_{\theta_t,\theta_{t-r+1}}(t,t-r+1,N)x(t)|\mathcal{H}_{t-r})]\}. \end{aligned}\tag{117}$$

It follows from (117) that

$$\begin{aligned}J_N =& \sum_{t=0}^{N}\mathrm{E}\{x(t)'Qx(t)\} + \sum_{t=r}^{N}\mathrm{E}\{u(t-r)'Ru(t-r)\}\\ =&V_N(0,x_0) - \sum_{t=0}^{r-1}\mathrm{E}\{u(t-r)'Ru(t-r)\}\\ &+\sum_{t=0}^{r-1}\mathrm{E}\{[u(t-r) + W_{\theta_{t-r}}(t-r,N)^{-1}\\ &\times\mathrm{E}(\alpha^1_{\theta_t,\theta_{t-r+1}}(t,t-r+1,N)x(t)|\mathcal{H}_{t-r})]'\\ &\times W_{\theta_{t-r}}(t-r,N)\\ &\times[u(t-r) + W_{\theta_{t-r}}(t-r,N)^{-1}\\ &\times\mathrm{E}(\alpha^1_{\theta_t,\theta_{t-r+1}}(t,t-r+1,N)x(t)|\mathcal{H}_{t-r})]\}\\ =&\mathrm{E}\{x_0'P_{\theta_0}(0,N)x_0\}. \end{aligned}\tag{118}$$

From the arbitrariness of x_0, we have

$$\mathrm{E}\{x_0'P_{\theta_0}(0,N)x_0\} = J^*_N \leq J^*_{N+1} = \mathrm{E}\{x_0'P_{\theta_0}(0,N+1)x_0\},$$

which means that $P_{l_0}(0,N)$ for $\theta_0 = l_0 \in \Theta$ increases according to N. Resemble the deduction of (73)-(77) in [46], it is obtained that there exist constants λ and c such that

$$J = \sum_{t=0}^{\infty}\mathrm{E}\{x(t)'Qx(t)\} + \sum_{t=r}^{\infty}\mathrm{E}\{u(t-r)'Ru(t-r)\} \leq 2\lambda c x_0'x_0.$$

Recall that $0 \leq \mathrm{E}\{x_0'P_{\theta_0}(0,N)x_0\} = J^*_N \leq J = 2\lambda c x_0'x_0$, which means that $0 \leq P_{l_0}(0,N) \leq 2\lambda cI$ for $\theta_0 = l_0 \in \Theta$. This shows the boundedness of $P_{l_0}(0,N)$. Together with the monotonicity of $P_{l_0}(0,N)$, its convergence is obtained. Note that l_0, l_d are the realizations of θ_t and θ_{t-r}, respectively. By letting $N\to\infty$ on both sides of (10)-(15), it can be concluded that P_{l_0}, W_{l_r}, and $T^j_{l_r}(j = 0,1,\cdots,r)$ obey (35)-(39). This completes the first part of the proof of Theorem 3.

The second stage of the deduction is to show the inequality (44). In view of Lemma 4, we know that if system (1) is exact observable, there exists $N_0 \geq r$ such that $Y_{\theta_r,\theta_0}(r,0,N_0) > 0$. Recalling that $Y_{\theta_r,\theta_0}(r,0,N_0)$ is monotonically increasing on N, one yields

$$\begin{aligned}&P_{\theta_r} - \sum_{s=1}^{r}(\alpha^{r-s+1}_{\theta_r,\theta_{r-s+1}})'W^{-1}_{\theta_{r-s}}\alpha^{r-s+1}_{\theta_r,\theta_{r-s+1}}\\ &= \lim_{N\to\infty} Y_{\theta_r,\theta_0}(r,0,N) \geq Y_{\theta_r,\theta_0}(r,0,N_0) > 0.\end{aligned}$$

Therefore, (44) is true. This accomplishes the deduction of Theorem 3. ■

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H. Proof of Theorem 4

Proof: (i) Sufficiency: Presume that $P_{l_0}, W_{l_r}, T^j_{l_r}(j = 0,1,\cdots,r)$, and $\alpha^{r-s+1}_{l_0,l_{s-1}}(s = 1,2,\cdots,r)$ are the solutions to (35)-(43) such that

$$P_{l_0} - \sum_{s=1}^{r}(\alpha^{r-s+1}_{l_0,l_{s-1}})' W^{-1}_{l_s}\alpha^{r-s+1}_{l_0,l_{s-1}} > 0.$$

In the next step, we will demonstrate that system (1) is stabilized by the optimal controller (45). To this end, we define a new type of Lyapunov function with delayed terms and jumping parameters

$$\begin{aligned}V(t,x(t)) =&\mathrm{E}\{x(t)'P_{\theta_t}x(t) - \sum_{s=1}^{r} x(t)'(\alpha^{r-s+1}_{\theta_t,\theta_{t-s+1}})' \\ &\times W^{-1}_{\theta_{t-s}}\mathrm{E}[\alpha^{r-s+1}_{\theta_t,\theta_{t-s+1}}x(t)|\mathcal{H}_{t-s}]\}. \quad (119)\end{aligned}$$

Utilizing (1) and (35)-(43), one has

$$\begin{aligned}&V(t,x(t)) - V(t+1,x(t+1))\\ =&\mathrm{E}\{x(t)'Qx(t) + u(t-r)'Ru(t-r)\\ &- [u(t-r) + W^{-1}_{\theta_{t-r}}\mathrm{E}(\alpha^{1}_{\theta_t,\theta_{t-r+1}}x(t)|\mathcal{H}_{t-r})]'W_{\theta_{t-r}}\\ &\times [u(t-r) + W^{-1}_{\theta_{t-r}}\mathrm{E}(\alpha^{1}_{\theta_t,\theta_{t-r+1}}x(t)|\mathcal{H}_{t-r})]\} \quad (120)\\ =&\mathrm{E}\{x(t)'Qx(t) + u(t-r)'Ru(t-r)\} \geq 0, \quad (121)\end{aligned}$$

which demonstrates that $V(t,x(t))$ monotonically decreasing about t. Furthermore, one yields from (119) that

$$\begin{aligned}&V(t,x(t))\\ =&\mathrm{E}\{x(t)'P_{\theta_t}x(t)\\ &- \sum_{s=1}^{r} x(t)'(\alpha^{r-s+1}_{\theta_t,\theta_{t-s+1}})'W^{-1}_{\theta_{t-s}}\alpha^{r-s+1}_{\theta_t,\theta_{t-s+1}}x(t)\\ &+ \sum_{s=1}^{r}[\alpha^{r-s+1}_{\theta_t,\theta_{t-s+1}}x(t) - \mathrm{E}(\alpha^{r-s+1}_{\theta_t,\theta_{t-s+1}}x(t)|\mathcal{H}_{t-s})]'\\ &\times W^{-1}_{\theta_{t-s}}[\alpha^{r-s+1}_{\theta_t,\theta_{t-s+1}}x(t) - \mathrm{E}(\alpha^{r-s+1}_{\theta_t,\theta_{t-s+1}}x(t)|\mathcal{H}_{t-s})]\}\\ \geq&\mathrm{E}\{x(t)'[P_{\theta_t} - \sum_{s=1}^{r}(\alpha^{r-s+1}_{\theta_t,\theta_{t-s+1}})'W^{-1}_{\theta_{t-s}}\alpha^{r-s+1}_{\theta_t,\theta_{t-s+1}}]x(t)\}\\ \geq&0, \quad (122)\end{aligned}$$

which shows that $V(t,x(t))$ has a lower bound. From (121) and (122), we can obtain the convergence of $V(t,x(t))$.

Next, let m to be any positive integer. Adding up both sides of (121) from $t = m+r$ to $t = m+N$ and mating $m \to +\infty$, one gets

$$\begin{aligned}&\lim_{m\to\infty}\sum_{t=m+r}^{m+N}\mathrm{E}[x(t)'Qx(t) + u(t-r)'Ru(t-r)]\\ =&\lim_{m\to\infty}\{V(m+r,x(m+r))\\ &- V(m+N+1,x(m+N+1))\} = 0. \quad (123)\end{aligned}$$

Note from (91) that

$$\begin{aligned}&\sum_{t=r}^{N}\mathrm{E}[x(t)'Qx(t) + u(t-r)'Ru(t-r)]\\ \geq&\mathrm{E}\{x(r)'[P_{\theta_r}(r,N) - \sum_{s=1}^{r}(\alpha^{r-s+1}_{\theta_r,\theta_{r-s+1}}(r,r-s+1,N))'\\ &\times W_{\theta_{r-s}}(r-s,N)^{-1}\alpha^{r-s+1}_{\theta_r,\theta_{r-s+1}}(r,r-s+1,N)]x(r)\}.\end{aligned}$$

By a time shift of length of m, we have

$$\begin{aligned}&\sum_{t=m+r}^{m+N}\mathrm{E}[x(t)'Qx(t) + u(t-r)'Ru(t-r)]\\ &\geq \mathrm{E}\{x(m+r)'[P_{\theta_r}(r,N)\\ &- \sum_{s=1}^{r}(\alpha^{r-s+1}_{\theta_r,\theta_{r-s+1}}(r,r-s+1,N))'\\ &\times W_{\theta_{r-s}}(r-s,N)^{-1}\\ &\times \alpha^{r-s+1}_{\theta_r,\theta_{r-s+1}}(r,r-s+1,N)]x(m+r)\} \geq 0. \quad (124)\end{aligned}$$

On the ground of (123), one yields

$$\begin{aligned}&\mathrm{E}\{x(m+r)'[P_{\theta_r}(r,N)\\ &- \sum_{s=1}^{r}(\alpha^{r-s+1}_{\theta_r,\theta_{r-s+1}}(r,r-s+1,N))'\\ &\times W_{\theta_{r-s}}(r-s,N)^{-1}\\ &\times \alpha^{r-s+1}_{\theta_r,\theta_{r-s+1}}(r,r-s+1,N)]x(m+r)\} = 0. \quad (125)\end{aligned}$$

Based on the exact observability of system (1), we know from Lemma 4 that there must exist an integer N_0 such that

$$\begin{aligned}&P_{\theta_r}(r,N_0) - \sum_{s=1}^{r}(\alpha^{r-s+1}_{\theta_r,\theta_{r-s+1}}(r,r-s+1,N_0))'\\ &\times W_{\theta_{r-s}}(r-s,N_0)^{-1}\alpha^{r-s+1}_{\theta_r,\theta_{r-s+1}}(r,r-s+1,N_0) > 0,\end{aligned}$$

and therefore (125) indicates that $\lim_{m\to\infty}\mathrm{E}[x(m+r)'x(m+r)] = 0$, which shows that (45) stabilizes (1).

The next step is to demonstrate that the cost function (29) is minimized by (45). Noting from the stabilization of system (1), we have

$$\lim_{t\to\infty}\mathrm{E}\{x(t)'P_{\theta_t}x(t)\} = 0. \quad (126)$$

Alternatively,

$$\begin{aligned}0 \leq&V(t,x(t))\\ =&\mathrm{E}\{x(t)'P_{\theta_t}x(t) - \sum_{s=1}^{r}[x(t)'(\alpha^{r-s+1}_{\theta_t,\theta_{t-s+1}})']W^{-1}_{\theta_{t-s}}\\ &\times \mathrm{E}[\alpha^{r-s+1}_{\theta_t,\theta_{t-s+1}}x(t)|\mathcal{H}_{t-s}]\} \leq \mathrm{E}\{x(t)'P_{\theta_t}x(t)\}. \quad (127)\end{aligned}$$

In light of (126) and (127), we have $\lim_{t\to\infty}V(t,x(t)) = 0$. Adding up (120) from $t = r$ to $t = N$ and letting $N \to \infty$, one gets

$$\begin{aligned}J =&\mathrm{E}\{\sum_{t=0}^{r-1}x(t)'Qx(t) + x(r)'P_{\theta_r}x(r)\\ &- \sum_{s=1}^{r}x(r)'(\alpha^{r-s+1}_{\theta_r,\theta_{r-s+1}})'W^{-1}_{\theta_{r-s}}\mathrm{E}(\alpha^{r-s+1}_{\theta_r,\theta_{r-s+1}}x(r)|\mathcal{H}_{r-s})\\ &+ \sum_{t=r}^{N}[u(t-r) + W^{-1}_{\theta_{t-r}}\mathrm{E}(\alpha^{1}_{\theta_t,\theta_{t-r+1}}x(t)|\mathcal{H}_{t-r})]'\\ &\times W_{\theta_{t-r}}[u(t-r) + W^{-1}_{\theta_{t-r}}\mathrm{E}(\alpha^{1}_{\theta_t,\theta_{t-r+1}}x(t)|\mathcal{H}_{t-r})]\}. \quad (128)\end{aligned}$$

Since $W_{\theta_{t-r}}$ is positive definite, (128) is minimized if and only if

$$\begin{aligned}u(t-r) =& - W^{-1}_{\theta_{t-r}}T^{0}_{\theta_{t-r}}x(t-r)\\ &- \sum_{j=1}^{r}W^{-1}_{\theta_{t-r}}T^{j}_{\theta_{t-r}}u(t-2r+j-1),\end{aligned}$$

and the relevant optimal cost (46) is now attained.

(ii) Necessity: In view of Theorem 3, if system (1) is mean square stabilizable, then the condition

$$P_{\theta_t} - \sum_{s=1}^{r}(\alpha^{r-s+1}_{\theta_t,\theta_{t-s+1}})'W^{-1}_{\theta_{t-s}}\alpha^{r-s+1}_{\theta_t,\theta_{t-s+1}} > 0.$$

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is satisfied. Alternatively, from the derivation process after (116), we can obtain the uniqueness of the solution to (35)-(39). This accomplishes the proof of Theorem 4. ■

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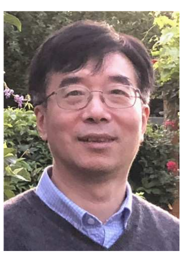
Chunyan Han received the B.Sc. degree and the M.Sc. degree in applied mathematics from Ludong University, Yantai, China, in 2003 and 2006, respectively, and the Ph.D. degree in control theory and control engineering from Shandong University, Jinan, China, in 2010.

Prof. Han has been a visiting scholar with the Department of Computer Science, Brunel University London, Uxbridge, United Kingdom, since Sep. 2019. She is currently also a Professor with the School of Electrical Engineering, University of Jinan, Jinan, China. She has authored or co-authored over 20 papers published in international journals. Her current research interests include optimal control and estimation, stochastic system, time delay system.



Zidong Wang (SM'03-F'14) received the B.Sc. degree in mathematics in 1986 from Suzhou University, Suzhou, China, and the M.Sc. degree in applied mathematics in 1990 and the Ph.D. degree in electrical engineering in 1994, both from Nanjing University of Science and Technology, Nanjing, China.

He is currently a Professor of Dynamical Systems and Computing in the Department of Computer Science, Brunel University London, U.K. From 1990 to 2002, he held teaching and research appointments in universities in China, Germany and the U.K. Prof. Wang's research interests include dynamical systems, signal processing, bioinformatics, control theory and applications. He has published more than 700 papers in international journals. He is a holder of the Alexander von Humboldt Research Fellowship of Germany, the JSPS Research Fellowship of Japan, William Mong Visiting Research Fellowship of Hong Kong.

Prof. Wang serves (or has served) as the Editor-in-Chief for *International Journal of Systems Science*, the Editor-in-Chief for *Neurocomputing*, the Editor-in-Chief for *Systems Science & Control Engineering*, and an Associate Editor for 12 international journals including IEEE Transactions on Automatic Control, IEEE Transactions on Control Systems Technology, IEEE Transactions on Neural Networks, IEEE Transactions on Signal Processing, and IEEE Transactions on Systems, Man, and Cybernetics-Part C. He is a Member of the Academia Europaea, a Member of the European Academy of Sciences and Arts, an Academician of the International Academy for Systems and Cybernetic Sciences, a Fellow of the IEEE, a Fellow of the Royal Statistical Society and a member of program committee for many international conferences.

Huanshui Zhang (SM'06) received the B.S. degree in mathematics from Qufu Normal University, Shandong, China, in 1986, the M.Sc. degree in control theory from Heilongjiang University, Harbin, China, in 1991, and the Ph.D. degree in control theory from Northeastern University, Shenyang, China, in 1997. He was a Postdoctoral Fellow with Nanyang Technological University, Singapore, from 1998 to 2001 and a Research Fellow with Hong Kong Polytechnic University, Hong Kong, from 2001 to 2003.

He currently holds a Professorship with Shandong University of Science and Technology, Shandong, China. He also held visiting appointments as a Research Scientist and Fellow with Nanyang Technological University, Curtin University of Technology, and Hong Kong City University from 2003 to 2006. His research interests include optimal estimation and control, time-delay systems, stochastic systems, signal processing, and wireless sensor networked systems.

Paresh Date received the doctoral degree from the University of Cambridge, Cambridge, U.K., in 2001.

He joined Brunel University London, Uxbridge, U.K., in 2002, where he is currently a Reader and the Director of Research with the Department of Mathematics. His current research interests include state estimation algorithms in nonlinear systems and their applications in financial mathematics.

Dr. Date was a recipient of the Fellowship from the Commonwealth Trust. He is a fellow of the Institute of Mathematics and Its Applications