Quadratic Filtering for Discrete Time-Varying Non-Gaussian Systems Under Binary Encoding Schemes ⋆

Shaoying Wang [a,b], Zidong Wang [c], Hongli Dong [b,d,e,∗], Bo Shen [f], Yun Chen [g]

[a] *College of Science, Binzhou University, Binzhou 256603, China*

[b] *Artificial Intelligence Energy Research Institute, Northeast Petroleum University, Daqing 163318, China*

[c] *Department of Computer Science, Brunel University London, Uxbridge, Middlesex, UB8 3PH, United Kingdom*

[d] *Sanya Offshore Oil and Gas Research Institute, Northeast Petroleum University, Sanya 572025, China*

[e] *Heilongjiang Provincial Key Laboratory of Networking and Intelligent Control, Northeast Petroleum University, Daqing 163318, China*

[f] *College of Information Sciences and Technology, Donghua University, Shanghai 201620, China*

[g] *Key Lab for IoT and Information Fusion Technology of Zhejiang, School of Automation, Hangzhou Dianzi University, Hangzhou 310018, China*

Abstract

This paper is concerned with the recursive quadratic filtering problem for a class of linear discrete-time systems subject to non-Gaussian noises. Considering its robustness against channel noises, the binary encoding scheme is utilized in the process of data transmission from sensors to the filter. Under such a scheme, the original signal is first encoded into a bit string, and then transmitted via memoryless binary symmetric channels (with certain crossover probabilities). Subsequently, the received bit string is recovered by a decoder at the receiver end. The primary purpose of this paper is to design a recursive quadratic filter for the underlying non-Gaussian systems with a minimized upper bound on the filtering error covariance. For this purpose, an augmented system is first constructed by aggregating the original vectors and their second-order Kronecker powers. Accordingly, an upper bound on the filtering error covariance is obtained in the form of solutions to certain Riccati-like difference equations, and the obtained bound is then minimized by properly choosing the filter parameter. Moreover, sufficient conditions are established to guarantee the boundedness of filtering error covariance. Finally, a numerical example is provided to demonstrate the effectiveness of the proposed quadratic filtering algorithm.

Key words: Quadratic filtering, non-Gaussian noises, binary encoding schemes, matrix difference equations, boundedness.

1 Introduction

The past several decades have witnessed a surge of research interest devoted to the filtering problem for stochastic systems due to its successful applications in many fields ranging from signal processing, target tracking, fault detection, to navigation, see e.g. [4,26,29,32,34, 38,43,44]. The main objective of the filtering issue is to obtain the accurate/satisfactory state estimates of a target plant based on the possibly corrupted measurements. Up to now, much research enthusiasm has been attracted towards the development of filtering techniques under various performance requirements, and some representative approaches include the Kalman filter, the set-membership filter [19] and the H_∞ filter algorithms [3,11,15,17,31,42]. Among others, the Kalman filter has been well recognized as the optimal recursive minimum-variance estimator for linear systems subject to Gaussian noises with exactly known statistics [10, 30, 33].

In practical applications, it is often the case that the systems undergo heavy-tailed and/or skewed non-Gaussian noises due to various reasons such as severe maneuvering and measurement outliers (incurred by unreliable sensors), see [16, 22, 41, 47, 49] and the references therein. In this case, those existing filtering schemes developed under the assumption of Gaussian noise are no longer applicable and, accordingly, considerable research effort has been made on the non-Gaussian filtering problem with a great number of results reported in [22,24,41,45]. For instance, the quadratic deconvolution filter and fixed-lag smoother have been designed in [48] for linear discrete-time non-Gaussian systems, and the asymptotic stabili-

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∗ Corresponding author.

Email addresses: shaoying2004@163.com (Shaoying Wang), Zidong.Wang@brunel.ac.uk (Zidong Wang), shiningdhl@gmail.com (Hongli Dong), bo.shen@dhu.edu.cn (Bo Shen), yunchen@hdu.edu.cn (Yun Chen).

ty has also been discussed. In [41], the error square constrained filtering issue has been addressed for systems with polytopic uncertainty and non-Gaussian noises. Recently, by resorting to the variational Bayesian inference, a suboptimal filter has been proposed in [47] for the target tracking system with non-Gaussian noises.

Owing to its distinctive advantages in handling non-Gaussian noises (as compared to the existing methods), the polynomial filtering approach has recently received a rapidly growing research interest [1, 2, 5, 8, 14, 23]. As a special kind of the polynomial filters, the so-called quadratic filter is capable of achieving a satisfactory tradeoff between the computational complexity and the filtering accuracy. In this regard, significant research effort has been directed towards the quadratic filtering (QF) problem for non-Gaussian systems from various perspectives [6,7,13,25,27,37,48]. For example, the least-squares QF approach has been put forward in [9] for discrete-time stochastic systems with random parameter matrices. By introducing an output injection term, the feedback QF scheme has been presented in [6] for linear discrete-time systems, in which the filtering performance relies on the filter gain of the output term. In [21], a Kalman-like quadratic filter has been proposed for linear discrete time-varying non-Gaussian systems with multiplicative noises, and an optimized upper bound on the filtering error covariance (FEC) has also been established.

On another research frontier, the binary encoding scheme (BES), which is one of the popular signal transmission strategies, has gained considerable research attention for its distinctive advantages such as easy realization, high reliability, and reduced consumption of network resource [12, 20, 28, 35, 36]. Under the BES, the signals are first encoded into a set of binary bits and subsequently transmitted via binary symmetric channels. Nevertheless, due to unavoidable channel noises or cyber-attacks, the binary bits might undergo random bit errors which, if not properly addressed, would inevitably result in the information distortion and even performance deterioration. As such, some initial research attention has been paid onto the filtering/estimation problems under BESs, see [18, 33, 39, 40, 46] and the references therein. Among others, the moving-horizon estimation issues have been considered in [20] for linear discrete-time dynamic networks with BESs, in which the stochastically ultimate boundedness of the estimation errors has been analyzed as well. Unfortunately, when it comes to the BES-based QF problem, the corresponding results have been very few (if not none), and this constitutes the main motivation of our current investigation.

In light of the aforementioned discussions, in this paper, we endeavor to study the QF problem for non-Gaussian systems under the BES, and this appears to be a rather challenging research issue because of the following three fundamental difficulties: 1) how to develop a quadratic filter in the presence of random bit errors and non-Gaussian noises? 2) how to obtain the statistical properties of the augmented noises which are composed of the original noises and their Kronecker products? and 3) how to analyze the influence from the quantization/flipping errors onto the filtering performance? It is, therefore, the primary purpose of this paper to offer satisfactory answers to these three questions.

The main contributions of this paper can be highlighted as follows: *1) the QF problem is, for the first time, studied for a class of time-varying non-Gaussian systems with BESs; 2) the statistical properties (including the second-, third- and fourth-order moments) of the quantization errors, flipping errors, and augmented noises are obtained; 3) an effective quadratic filter is proposed with a locally minimized upper bound on the FEC, which is suitable for online implementations; and 4) sufficient conditions are provided to ensure the boundedness of the FEC.*

The organization of this paper is as follows. In Section 2, the time-varying non-Gaussian systems under BESs are introduced. The recursive QF problem is formulated in Section 3. In Section 4, the QF algorithm is designed by utilizing the statistical characteristics of involved flipping errors as well as augmented noises, and the boundedness analysis of the FEC is conducted for the presented filtering scheme. A numerical example is given in Section 5 to demonstrate the effectiveness of the designed filter, which is followed by the conclusion in Section 6.

Notation. $\mathbb{R}^m$ is the m-dimensional Euclidean space. $\mathbb{P}\{*\}$ denotes the occurrence probability of the event $*$ and $\mathbb{E}\{*\}$ stands for the expectation of the random variable $*$. $\mathrm{diag}\{A_1, A_2, \cdots, A_m\}$ denotes a block diagonal matrix with diagonal blocks $A_1, A_2, \cdots, A_m$. $\mathrm{col}\{a_1, a_2, \cdots, a_m\}$ means a column vector $[a_1^T, a_2^T, \cdots, a_m^T]^T$. $A \leq B (A < B)$ means that $B - A$ is positive semi-definite (positive definite), where A and B are real symmetric matrices. For a vector y, $y^{[i]}$ represents the ith-order Kronecker power. To be specific, $y^{[0]} = 1$ and $y^{[i]} = y \otimes y^{[i-1]} (i \geq 1)$, where $\otimes$ is the Kronecker product. Moreover, the ith-order moment of $y^{[i]}$ is defined as $\varphi_y^{(i)} \triangleq \mathbb{E}\{y^{[i]}\}$. vec(A) is the vectorization of an $m \times l$ matrix A, which is an $ml \times 1$ column vector formed by stacking the columns of A on top of one another. st(A) denotes the inverse operation transferring vec(A) to A. $K_{m,l}$ is a commutative matrix, i.e., $K_{m,l}(x \otimes y) = y \otimes x$ with $x \in R^m$ and $y \in R^l$. $\mathrm{tr}(A)$ means the trace of matrix A, and $\mathrm{Sym}\{A\}$ denotes $A + A^T$.

2 Problem Statement

Consider the following class of discrete time-varying non-Gaussian systems

$$\begin{cases} x_{z+1} = \Gamma_z x_z + w_z \\ y_z = D_z x_z + v_z \end{cases} \tag{1}$$

where $x_z \in \mathbb{R}^{n_x}$ is the system state to be estimated, $y_z \in \mathbb{R}^{n_y}$ denotes the observation. The process noise $w_z \in \mathbb{R}^{n_x}$ and the measurement noise $v_z \in \mathbb{R}^{n_y}$ are the non-Gaussian sequences. Γ_z and D_z are known time-varying matrices with compatible dimensions. The initial state x_0 and the random sequences w_z and v_z are assumed to be zero-mean, uncorrelated, white and bounded. Moreover, the 2nd, 3rd and 4th moments of all the random variables are known.

In this paper, to cater for the engineering reality, the BES is utilized during the process of signal transmission. Specifically, the measured output y_z is first encoded into a set of binary bits and then forwarded to a remote filter via binary symmetric channels. Let $y_{i,z}$ be the ith element of the measurement y_z with its range being $[-L, L]$, where $L \in \mathbb{R}$ is a known positive scalar determined by the sensor specification. A binary encoder is employed to transform the original signal $y_{i,z}$ into a binary bit string of length S, thereby resulting in 2^S points denoted by $\Upsilon \triangleq \{u_1, u_2, \cdots, u_{2^S}\}$. These 2^S points uniformly divide the interval $[-L, L]$ into $2^S - 1$ small intervals with length

$$h = \frac{2L}{2^S - 1}.$$

To begin with, the original signal $y_{i,z}$ is pretreated by a probabilistic uniform quantizer, i.e., $\mathcal{F}_z : y_{i,z} \longrightarrow q(y_{i,z}, S)$. When $u_i \le y_{i,z} \le u_{i+1}$, the output $q(y_{i,z}, S)$ is generated by the following probabilistic rule

$$\begin{cases} \mathbb{P}\{q(y_{i,z}, S) = u_i\} = 1 - \delta_{i,z} \\ \mathbb{P}\{q(y_{i,z}, S) = u_{i+1}\} = \delta_{i,z} \end{cases} \tag{2}$$

where

$$\delta_{i,z} \triangleq \frac{y_{i,z} - u_i}{h}$$

and $0 \le \delta_{i,z} \le 1$.

Defining the quantization error as $e_{i,z} \triangleq q(y_{i,z}, S) - y_{i,z}$, we have

$$\begin{cases} \mathbb{P}\{e_{i,z} = -\delta_{i,z} h\} = 1 - \delta_{i,z} \\ \mathbb{P}\{e_{i,z} = (1 - \delta_{i,z})h\} = \delta_{i,z}. \end{cases} \tag{3}$$

It is notable that the output signal $q(y_{i,z}, S)$ can be described by a basis of binary bits, i.e.,

$$q(y_{i,z}, S) = -L + \sum_{j=1}^{S} b_{j,z}^{(i)} 2^{j-1} h \tag{4}$$

which indicates that the signal $y_{i,z}$ is encoded into the following binary bit string

$$\mathcal{U}_z \triangleq \{b_{1,z}^{(i)}, b_{2,z}^{(i)}, \cdots, b_{S,z}^{(i)}\}, \quad b_{j,z}^{(i)} \in \{0, 1\}.$$

The converted binary bit string is then transmitted via a memoryless binary symmetric channel, in which every bit might undergo the random bit errors (i.e., flip with a small probability). In this case, the received bit string can be described as

$$\tilde{\mathcal{U}}_z \triangleq \{\tilde{b}_{1,z}^{(i)}, \tilde{b}_{2,z}^{(i)}, \cdots, \tilde{b}_{S,z}^{(i)}\}, \tilde{b}_{j,z}^{(i)} \in \{0, 1\}$$

where

$$\tilde{b}_{j,z}^{(i)} = \alpha_{j,z}(1 - b_{j,z}^{(i)}) + (1 - \alpha_{j,z}) b_{j,z}^{(i)} \tag{5}$$

with $\mathbb{P}\{\alpha_{j,z} = 1\} = \bar{\alpha}$ and $\mathbb{P}\{\alpha_{j,z} = 0\} = 1 - \bar{\alpha}$. The random variables $\alpha_{j,z}$ are assumed to be uncorrelated with x_0, w_z and v_z.

Based on the received bit string $\tilde{\mathcal{U}}_z$, the signal $y_{i,z}$ can be recovered by

$$\tilde{q}(y_{i,z}, S) = -L + \sum_{j=1}^{S} \tilde{b}_{j,z}^{(i)} 2^{j-1} h. \tag{6}$$

From Lemma 2 in [20], we know that

$$\mathbb{E}[\tilde{q}(y_{i,z}, S)] = (1 - 2\bar{\alpha}) q(y_{i,z}, S). \tag{7}$$

For the convenience of subsequent analysis, let $\tilde{y}_{i,z} \triangleq \tilde{q}(y_{i,z}, S)$ and $\lambda_{i,z} \triangleq \tilde{q}(y_{i,z}, S) - \mathbb{E}[\tilde{q}(y_{i,z}, S)]$. Then, the restored signal $\tilde{y}_{i,z}$ can be represented as

$$\begin{aligned} \tilde{y}_{i,z} =& (1 - 2\bar{\alpha}) q(y_{i,z}, S) + \lambda_{i,z} \\ =& (1 - 2\bar{\alpha})(y_{i,z} + e_{i,z}) + \lambda_{i,z} \\ =& (1 - 2\bar{\alpha}) y_{i,z} + (1 - 2\bar{\alpha}) e_{i,z} + \lambda_{i,z}. \end{aligned} \tag{8}$$

As a result, the considered system can be reformulated as

$$\begin{cases} x_{z+1} = \Gamma_z x_z + w_z \\ y_z = D_z x_z + v_z \\ \tilde{y}_z = \Upsilon y_z + \Upsilon e_z + \Lambda_z \end{cases} \tag{9}$$

where

$$\begin{aligned} &\Upsilon \triangleq \mathrm{diag}\{1 - 2\bar{\alpha}, 1 - 2\bar{\alpha}, \cdots, 1 - 2\bar{\alpha}\}, \\ &\Lambda_z \triangleq \mathrm{col}\{\lambda_{1,z}, \lambda_{2,z}, \cdots, \lambda_{n_y,z}\}, \\ &e_z \triangleq \mathrm{col}\{e_{1,z}, e_{2,z}, \cdots, e_{n_y,z}\}. \end{aligned}$$

Remark 1 *It is worth mentioning that the introduction of the probabilistic uniform quantizer and the BES would give rise to the quantization error e_z and the flipping error Λ_z, which constitute the main sources of the transmission errors and bring about some difficulties in the design of QF algorithm. To this end, by resorting to the term $\mathbb{E}[\tilde{q}(y_{i,z}, S)]$, we have established an explicit relationship among the restored signal $\tilde{y}_z$, the original signal y_z, the quantization error e_z and the flipping error Λ_z.*

3 The Quadratic Filtering Problem

In this section, the QF problem is investigated for the linear non-Gaussian systems described by (9). To begin with, we shall construct an augmented system by using the quadratic Kronecker powers of the state and measurement vectors.

Based on the definition of $x_{z+1}^{[2]}$ and the properties of Kronecker algebra, we have

$$\begin{aligned} x_{z+1}^{[2]} =& \Gamma_z^{[2]} x_z^{[2]} + (I + K_{n_x,n_x})(\Gamma_z x_z \otimes w_z) + w_z^{[2]} \\ =& \Gamma_z^{[2]} x_z^{[2]} + \varphi_{w_z}^{(2)} + \tilde{w}_z^{(2)} \end{aligned} \tag{10}$$

where

$$\tilde{w}_z^{(2)} \triangleq (I + K_{n_x,n_x})(\Gamma_z x_z \otimes w_z) + w_z^{[2]} - \varphi_{w_z}^{(2)}.$$

Similarly, we can obtain that

$$y_z^{[2]} = D_z^{[2]} x_z^{[2]} + (I + K_{n_y,n_y})(D_z x_z \otimes v_z) + v_z^{[2]} \quad (11)$$

and

$$\begin{aligned}\tilde{y}_z^{[2]} =& \Upsilon^{[2]} y_z^{[2]} + \Upsilon^{[2]} e_z^{[2]} + \Lambda_z^{[2]} + (I + K_{n_y,n_y}) \\ & \times [\Upsilon^{[2]}(y_z \otimes e_z) + (\Upsilon y_z \otimes \Lambda_z) + (\Upsilon e_z \otimes \Lambda_z)] \\ =& \Upsilon^{[2]} D_z^{[2]} x_z^{[2]} + \Upsilon^{[2]} \varphi_{v_z}^{(2)} + \varphi_{\Lambda_z}^{(2)} + \tilde{v}_z^{(2)} \end{aligned} \quad (12)$$

where

$$\begin{aligned} v_z^{(2)} \triangleq& \Upsilon^{[2]}(I + K_{n_y,n_y})(D_z x_z \otimes v_z) + \Upsilon^{[2]} v_z^{[2]} \\ & + \Upsilon^{[2]} e_z^{[2]} + \Lambda_z^{[2]} + (I + K_{n_y,n_y})[\Upsilon^{[2]}(y_z \otimes e_z) \\ & + (\Upsilon y_z \otimes \Lambda_z) + (\Upsilon e_z \otimes \Lambda_z)], \\ \tilde{v}_z^{(2)} \triangleq& v_z^{(2)} - \Upsilon^{[2]} \varphi_{v_z}^{(2)} - \varphi_{\Lambda_z}^{(2)}. \end{aligned}$$

On the other hand, it follows from (9) that

$$\begin{aligned}\tilde{y}_z =& \Upsilon y_z + \Upsilon e_z + \Lambda_z \\ =& \Upsilon D_z x_z + \tilde{v}_z \end{aligned} \quad (13)$$

where $\tilde{v}_z \triangleq \Upsilon v_z + \Upsilon e_z + \Lambda_z$.

By aggregating the state/measurement vectors and the corresponding quadratic Kronecker powers, we construct the following augmented vectors

$$\mathcal{X}_z \triangleq \begin{bmatrix} x_z \\ x_z^{[2]} \end{bmatrix}, \mathcal{Y}_z \triangleq \begin{bmatrix} y_z \\ y_z^{[2]} \end{bmatrix}, \tilde{\mathcal{Y}}_z \triangleq \begin{bmatrix} \tilde{y}_z \\ \tilde{y}_z^{[2]} \end{bmatrix}. \quad (14)$$

Then, the dynamics of the corresponding augmented system can be expressed as

$$\begin{cases} \mathcal{X}_{z+1} = \tilde{\Gamma}_z \mathcal{X}_z + U_z + W_z \\ \tilde{\mathcal{Y}}_z = \tilde{D}_z \mathcal{X}_z + T_z + \tilde{V}_z \end{cases} \quad (15)$$

where

$$\tilde{\Gamma}_z \triangleq \begin{bmatrix} \Gamma_z & 0 \\ 0 & \Gamma_z^{[2]} \end{bmatrix}, U_z \triangleq \begin{bmatrix} 0 \\ \varphi_{w_z}^{(2)} \end{bmatrix},$$

$$W_z \triangleq \begin{bmatrix} w_z \\ \tilde{w}_z^{(2)} \end{bmatrix}, \tilde{D}_z \triangleq \begin{bmatrix} \Upsilon D_z & 0 \\ 0 & \Upsilon^{[2]} D_z^{[2]} \end{bmatrix},$$

$$T_z \triangleq \begin{bmatrix} 0 \\ \Upsilon^{[2]} \varphi_{v_z}^{(2)} + \varphi_{\Lambda_z}^{(2)} \end{bmatrix}, \tilde{V}_z \triangleq \begin{bmatrix} \tilde{v}_z \\ \tilde{v}_z^{(2)} \end{bmatrix}.$$

According to the available measurements $\tilde{\mathcal{Y}}_z$, we design the following quadratic filter:

$$\begin{cases} \hat{\mathcal{X}}_{z+1|z} = \tilde{\Gamma}_z \hat{\mathcal{X}}_{z|z} + U_z \\ \hat{\mathcal{X}}_{z+1|z+1} = \hat{\mathcal{X}}_{z+1|z} + F_{z+1}[\tilde{\mathcal{Y}}_{z+1} - \tilde{D}_{z+1} \hat{\mathcal{X}}_{z+1|z} - T_{z+1}] \end{cases} \quad (16)$$

where $\hat{\mathcal{X}}_{z+1|z}$ and $\hat{\mathcal{X}}_{z+1|z+1}$ denote, respectively, the one-step prediction and the state estimate at time instant $z+1$, and F_{z+1} is the filter gain matrix to be determined.

Remark 2 *It should be pointed out that, based on the augmented measurements $\tilde{\mathcal{Y}}_z$, the QF algorithm can achieve higher filtering accuracy than traditional filters with $\tilde{y}_z$ only. Nevertheless, the co-existence of the quantization errors and random bit errors as well as the induced unknown term $\mathbb{E}[e_z^{[2]}]$ renders it really difficult to design the optimal quadratic filter. As such, in this paper, we turn to propose a suboptimal recursive quadratic filter of the structure (16). It is clear that the innovation term $\tilde{\mathcal{Y}}_{z+1} - \tilde{D}_{z+1}\hat{\mathcal{X}}_{z+1|z} - T_{z+1}$ takes into account the influence of the probabilistic uniform quantization as well as the random bit errors. Meanwhile, according to the relationship between x_z and $\mathcal{X}_z$, the estimate $\hat{x}_{z|z}$ of the original plant state can be easily obtained by extracting the first n_x entries of the estimate $\hat{\mathcal{X}}_{z|z}$, i.e., $\hat{x}_{z|z} = [I_{n_x}, 0]\hat{\mathcal{X}}_{z|z}$.*

To facilitate the subsequent developments, the prediction error and filtering error as well as their corresponding error covariances are, respectively, defined as

$$\begin{aligned} \tilde{\mathcal{X}}_{z+1|z} \triangleq& \mathcal{X}_{z+1} - \hat{\mathcal{X}}_{z+1|z}, \\ \tilde{\mathcal{X}}_{z+1|z+1} \triangleq& \mathcal{X}_{z+1} - \hat{\mathcal{X}}_{z+1|z+1}, \\ \mathcal{O}_{z+1|z} \triangleq& \mathbb{E}[\tilde{\mathcal{X}}_{z+1|z} \tilde{\mathcal{X}}_{z+1|z}^T], \\ \mathcal{O}_{z+1|z+1} \triangleq& \mathbb{E}[\tilde{\mathcal{X}}_{z+1|z+1} \tilde{\mathcal{X}}_{z+1|z+1}^T]. \end{aligned}$$

The primary objective of this paper is to propose a suboptimal quadratic filter of form (16) for augmented linear discrete-time systems (15) such that, in the presence of non-Gaussian noises and BES, the following requirements are simultaneously satisfied:

1) the FEC has an upper bound $\bar{\mathcal{O}}_{z+1|z+1}$, i.e., $\mathcal{O}_{z+1|z+1} \leq \bar{\mathcal{O}}_{z+1|z+1}$;

2) the obtained upper bound $\bar{\mathcal{O}}_{z+1|z+1}$ is minimized by properly designing the filter gain F_{z+1}; and

3) sufficient conditions are provided to ensure the boundedness of the FEC.

4 Main Results

In this section, some preliminary lemmas are first derived on the statistics of e_z, Λ_z, W_z and $\tilde{V}_z$. Then, an upper bound is given on the FEC $\mathcal{O}_{z+1|z+1}$ by resorting to two Riccati-like difference equations, and such a bound is subsequently minimized by selecting the gain matrix F_{z+1}. In addition, sufficient conditions are provided for the boundedness of the FEC.

4.1 Preliminary Lemmas

Lemma 1 *The second- and fourth-order moments of the quantization error $e_{i,z}$ satisfy the following relationships:*

$$\begin{aligned} \mathbb{E}[e_{i,z}^2] =& (\delta_{i,z} - \delta_{i,z}^2) h^2 \leq \frac{1}{4} h^2, \\ \mathbb{E}[e_{i,z}^4] =& [(1 - \delta_{i,z})\delta_{i,z}^4 + (1 - \delta_{i,z})^4 \delta_{i,z}] h^4 \leq \chi_1 h^4, \end{aligned} \quad (17)$$

where $\chi_1 \triangleq \max_{\delta \in [0,1]}[(1 - \delta)\delta^4 + (1 - \delta)^4 \delta]$.

Proof: The proof follows directly from (3) and the definition of mathematical expectation, and is thus omitted here for conciseness. ■

Lemma 2 *Define* $\tilde{\alpha}_{j,z} \triangleq \alpha_{j,z} - \bar{\alpha}$. *Then, we have the following results:*

$$\begin{aligned}\mathbb{E}[\tilde{\alpha}_{j,z}] =&0,\\ \mathbb{E}[\tilde{\alpha}_{j,z}^2] =&\bar{\alpha} - \bar{\alpha}^2,\\ \mathbb{E}[\tilde{\alpha}_{j,z}^3] =&\bar{\alpha} - 3\bar{\alpha}^2 + 2\bar{\alpha}^3,\\ \mathbb{E}[\tilde{\alpha}_{j,z}^4] =&\bar{\alpha} - 4\bar{\alpha}^2 + 6\bar{\alpha}^3 - 3\bar{\alpha}^4.\end{aligned} \tag{18}$$

Proof: The proof follows directly from the properties of the random variables $\alpha_{j,z}$, and is thus omitted here for simplicity. ■

Lemma 3 *The second-, third- and fourth-order moments of the random variables* $\lambda_{i,z}$ *are given as follows:*

$$\begin{aligned}\mathbb{E}[\lambda_{i,z}^2] =&\bar{\alpha}(1-\bar{\alpha})h^2\frac{2^{2S}-1}{3} \triangleq \chi_2,\\ \mathbb{E}[\lambda_{i,z}^3] =&(\bar{\alpha} - 3\bar{\alpha}^2 + 2\bar{\alpha}^3)\sum_{j=1}^{S}(1-2b_{j,z}^{(i)})^3 2^{3(j-1)}h^3 \triangleq \chi_3,\\ \mathbb{E}[\lambda_{i,z}^4] =&(\bar{\alpha} - 4\bar{\alpha}^2 + 6\bar{\alpha}^3 - 3\bar{\alpha}^4)h^4\frac{2^{4S}-1}{15}\\ &+ 6\bar{\alpha}^2(1-\bar{\alpha})^2h^4\sum_{1\le j<s\le S}2^{2(j+s-2)} \triangleq \chi_4.\end{aligned} \tag{19}$$

Proof: See Appendix A. ■

So far, we have obtained the second- and fourth-order moments of $e_{i,z}$ and $\lambda_{i,z}$ via Lemmas 1-3. In what follows, the statistics of the augmented noises W_z and $\tilde{V}_z$ are summarized in Lemma 4.

Lemma 4 *Let* $Q_{W_z} \triangleq \mathbb{E}[W_zW_z^T]$, $Q_{\tilde{V}_z} \triangleq \mathbb{E}[\tilde{V}_z\tilde{V}_z^T]$, *and* $\bar{Q}_{\tilde{V}_z}$ *be an upper bound of* $Q_{\tilde{V}_z}$. *Then, it follows that*

$$\begin{aligned}Q_{W_z} =&\begin{bmatrix}\mathrm{st}(\varphi_{w_z}^{(2)}) & \mathrm{st}(\varphi_{w_z}^{(3)})\\ \mathrm{st}^T(\varphi_{w_z}^{(3)}) & Q_{W_z}^{(2,2)}\end{bmatrix},\\ \bar{Q}_{\tilde{V}_z} =&\begin{bmatrix}\Upsilon\big(\mathrm{st}(\varphi_{v_z}^{(2)}) + \chi_5 I_{n_y}\big)\Upsilon^T + \chi_2 I_{n_y} & \bar{Q}_{\tilde{V}_z}^{(1,2)}\\ (\bar{Q}_{\tilde{V}_z}^{(1,2)})^T & \bar{Q}_{\tilde{V}_z}^{(2,2)}\end{bmatrix},\end{aligned} \tag{20}$$

where

$$\begin{aligned}Q_{W_z}^{(2,2)} \triangleq&(I + K_{n_x,n_x})(\Gamma_z\mathrm{st}(\varphi_{x_z}^{(2)})\Gamma_z^T \otimes \mathrm{st}(\varphi_{w_z}^{(2)}))\\ &\times (I + K_{n_x,n_x})^T + \mathrm{st}(\varphi_{w_z}^{(4)}) - \varphi_{w_z}^{(2)}(\varphi_{w_z}^{(2)})^T,\\ \chi_5 \triangleq&\frac{h^2}{4}n_y + n_y\chi_1 h^4 + n_y(n_y-1)\frac{h^4}{16},\\ \bar{Q}_{\tilde{V}_z}^{(1,2)} \triangleq&\Upsilon\mathrm{st}(\varphi_{v_z}^{(3)})(\Upsilon^{[2]})^T + \mathrm{st}(\varphi_{\Lambda_z}^{(3)}),\\ Q_{\tilde{V}_z}^{(2,2)} \triangleq&\Upsilon^{[2]}(I + K_{n_y,n_y})(D_z\mathrm{st}(\varphi_{x_z}^{(2)})D_z^T \otimes \mathrm{st}(\varphi_{v_z}^{(2)}))\\ &\times (I + K_{n_y,n_y})^T(\Upsilon^{[2]})^T + \Upsilon^{[2]}\mathrm{st}(\varphi_{v_z}^{(4)})(\Upsilon^{[2]})^T\\ &+ (I + K_{n_y,n_y})\Big\{\Upsilon^{[2]}\Big[(D_z\mathrm{st}(\varphi_{x_z}^{(2)})D_z^T + \mathrm{st}(\varphi_{v_z}^{(2)}))\\ &\otimes \frac{1}{4}h^2 I_{n_y}\Big](\Upsilon^{[2]})^T + \Big[\Upsilon(D_z\mathrm{st}(\varphi_{x_z}^{(2)})D_z^T + \mathrm{st}(\varphi_{v_z}^{(2)}))\Upsilon^T\\ &\otimes \chi_2 I_{n_y}\Big] + (\Upsilon\frac{1}{4}h^2 I_{n_y}\Upsilon^T \otimes \chi_2 I_{n_y})\Big\}(I + K_{n_y,n_y})^T\\ &- \Upsilon^{[2]}\varphi_{v_z}^{(2)}(\varphi_{v_z}^{(2)})^T(\Upsilon^{[2]})^T - \varphi_{\Lambda_z}^{(2)}(\varphi_{\Lambda_z}^{(2)})^T,\\ \bar{Q}_{\tilde{V}_z}^{(2,2)} \triangleq&Q_{\tilde{V}_z}^{(2,2)} + \mathrm{st}(\varphi_{\Lambda_z}^{(4)}) + \chi_5\Upsilon^{[2]}I_{n_y^2}(\Upsilon^{[2]})^T.\end{aligned} \tag{21}$$

Proof: See Appendix B. ■

Remark 3 *It should be emphasized that, by using the augmentation technique, the QF problem for the system (9) has been transformed into a recursive linear filtering problem for the augmented system (15). Therefore, the statistical characteristics (e.g. covariances) of the augmented noises would play a crucial role in the filter design. Nevertheless, due to the fact that the exact values of* $\mathbb{E}[e_ze_z^T]$ *and* $\mathbb{E}[e_z^{[2]}(e_z^{[2]})^T]$ *are unknown, it is literally impossible to establish an explicit expression for the covariance matrix* $Q_{\tilde{V}_z}$. *As such, by utilizing the elementary inequality* $\mathbb{E}[\beta_z\beta_z^T] \le \mathbb{E}[\beta_z^T\beta_z]I$ *for any* $\beta_z \in \mathbb{R}^n$, *we have derived an upper bound for* $Q_{\tilde{V}_z}$ *in Lemma 4.*

Lemma 5 *The state covariance matrix*

$$\Pi_{z+1} \triangleq \mathbb{E}[\mathcal{X}_{z+1}\mathcal{X}_{z+1}^T]$$

satisfies the following recursion:

$$\begin{aligned}\Pi_{z+1} =&\tilde{\Gamma}_z\Pi_z\tilde{\Gamma}_z^T + U_zU_z^T + Q_{W_z}\\ &+ \tilde{\Gamma}_zH_zU_z^T + U_zH_z^T\tilde{\Gamma}_z^T\end{aligned} \tag{22}$$

where

$$H_z \triangleq \mathbb{E}[\mathcal{X}_z] = \begin{bmatrix}0\\ \varphi_{x_z}^{(2)}\end{bmatrix}.$$

Proof: See Appendix C. ■

Remark 4 *It is worth mentioning that the state covariance matrix* Π_{z+1} *can be partitioned by*

$$\Pi_{z+1} = \begin{bmatrix}\Pi_{11,z+1} & \Pi_{12,z+1}\\ \Pi_{12,z+1}^T & \Pi_{22,z+1}\end{bmatrix}$$

where each block is specified as follows:

$$\begin{aligned}\Pi_{11,z+1} \triangleq&\mathbb{E}[x_{z+1}x_{z+1}^T] = \mathrm{st}(\varphi_{x_{z+1}}^{(2)}),\\ \Pi_{12,z+1} \triangleq&\mathbb{E}[x_{z+1}(x_{z+1}^{[2]})^T] = \mathrm{st}(\varphi_{x_{z+1}}^{(3)}),\\ \Pi_{22,z+1} \triangleq&\mathbb{E}[x_{z+1}^{[2]}(x_{z+1}^{[2]})^T] = \mathrm{st}(\varphi_{x_{z+1}}^{(4)}).\end{aligned}$$

Therefore, the terms $\varphi_{x_{z+1}}^{(2)}$, $\varphi_{x_{z+1}}^{(3)}$, *and* $\varphi_{x_{z+1}}^{(4)}$ *can be obtained via Lemma 5.*

4.2 Analysis on Filtering Error Covariance Matrices

In the subsection, an upper bound is first derived on the FEC, and the obtained bound is then minimized by appropriately choosing the filter gain matrix. To begin with, let us introduce the following notation:

$$\begin{aligned}\tilde{Q}^{(2,2)}_{\tilde{V}_{z+1}} \triangleq & \Upsilon^{[2]}(\mathrm{st}(\varphi^{(4)}_{v_{z+1}}) - \varphi^{(2)}_{v_{z+1}}(\varphi^{(2)}_{v_{z+1}})^T)(\Upsilon^{[2]})^T \\ & + \mathrm{st}(\varphi^{(4)}_{\Lambda_{z+1}}) - \varphi^{(2)}_{\Lambda_{z+1}}(\varphi^{(2)}_{\Lambda_{z+1}})^T \\ & + \Upsilon^{[2]}(n_y\chi_1 h^4 + n_y(n_y-1)\frac{h^4}{16})(\Upsilon^{[2]})^T, \\ \tilde{Q}^{(2)}_{\tilde{V}_{z+1}} = & \begin{bmatrix} 0 & 0 \\ 0 & \tilde{Q}^{(2,2)}_{\tilde{V}_{z+1}} \end{bmatrix}.\end{aligned}$$

Theorem 1 *For a given positive scalar $\mu_{1,z+1}$, assume that there exist two solutions $\bar{\mathcal{O}}_{z+1|z}$ and $\bar{\mathcal{O}}_{z+1|z+1}$ to the following recursions:*

$$\bar{\mathcal{O}}_{z+1|z} = \tilde{\Gamma}_z \bar{\mathcal{O}}_{z|z}\tilde{\Gamma}_z^T + Q_{W_z} \tag{23}$$

and

$$\begin{aligned}\bar{\mathcal{O}}_{z+1|z+1} = & (1+\mu_{1,z+1})(I - F_{z+1}\tilde{D}_{z+1})\bar{\mathcal{O}}_{z+1|z} \\ & \times (I - F_{z+1}\tilde{D}_{z+1})^T + F_{z+1}\bar{Q}_{\tilde{V}_{z+1}}F^T_{z+1} \\ & + \mu^{-1}_{1,z+1}F_{z+1}\tilde{Q}^{(2)}_{\tilde{V}_{z+1}}F^T_{z+1}\end{aligned} \tag{24}$$

with the initial condition $\mathcal{O}_{0|0} = \bar{\mathcal{O}}_{0|0} > 0$. Then, $\bar{\mathcal{O}}_{z+1|z}$ and $\bar{\mathcal{O}}_{z+1|z+1}$ are, respectively, the upper bounds of $\mathcal{O}_{z+1|z}$ and $\mathcal{O}_{z+1|z+1}$, i.e.,

$$\mathcal{O}_{z+1|z} \le \bar{\mathcal{O}}_{z+1|z}, \mathcal{O}_{z+1|z+1} \le \bar{\mathcal{O}}_{z+1|z+1}.$$

Proof: Let us prove this theorem by mathematical induction. From the initial condition, it follows that $\mathcal{O}_{0|0} \le \bar{\mathcal{O}}_{0|0}$. Suppose that $\mathcal{O}_{z|z} \le \bar{\mathcal{O}}_{z|z}$. In what follows, we will prove $\mathcal{O}_{z+1|z+1} \le \bar{\mathcal{O}}_{z+1|z+1}$.

First, it is obtained from (15) and (16) that

$$\tilde{\mathcal{X}}_{z+1|z} = \tilde{\Gamma}_z\tilde{\mathcal{X}}_{z|z} + W_z \tag{25}$$

which, together with the definition of prediction error covariance, yields

$$\begin{aligned}\mathcal{O}_{z+1|z} = & \tilde{\Gamma}_z\mathcal{O}_{z|z}\tilde{\Gamma}_z^T + \mathbb{E}[W_zW_z^T] \\ & + \mathbb{E}[\tilde{\Gamma}_z\tilde{\mathcal{X}}_{z|z}W_z^T] + \mathbb{E}[W_z\tilde{\mathcal{X}}^T_{z|z}\tilde{\Gamma}_z^T].\end{aligned} \tag{26}$$

Defining $G_{1,z} \triangleq \mathbb{E}[\tilde{\Gamma}_z\tilde{\mathcal{X}}_{z|z}W_z^T]$, the recursion of $\mathcal{O}_{z+1|z}$ satisfies:

$$\mathcal{O}_{z+1|z} = \tilde{\Gamma}_z\mathcal{O}_{z|z}\tilde{\Gamma}_z^T + Q_{W_z} + G_{1,z} + G_{1,z}^T \tag{27}$$

Bearing in mind that the initial state x_0, w_z and v_z are uncorrelated with zero mean, we can obtain that $G_{1,z}$ is zero, which further indicates that

$$\mathcal{O}_{z+1|z} = \tilde{\Gamma}_z\mathcal{O}_{z|z}\tilde{\Gamma}_z^T + Q_{W_z} \tag{28}$$

Then, it follows from the condition $\mathcal{O}_{z|z} \le \bar{\mathcal{O}}_{z|z}$ that $\mathcal{O}_{z+1|z} \le \bar{\mathcal{O}}_{z+1|z}$.

Similarly, the filtering error $\tilde{\mathcal{X}}_{z+1|z+1}$ and the FEC $\mathcal{O}_{z+1|z+1}$ are, respectively, calculated as follows:

$$\begin{aligned}\tilde{\mathcal{X}}_{z+1|z+1} = & \tilde{\mathcal{X}}_{z+1|z} - F_{z+1}(\tilde{D}_{z+1}\tilde{\mathcal{X}}_{z+1|z} + \tilde{V}_{z+1}) \\ = & (I - F_{z+1}\tilde{D}_{z+1})\tilde{\mathcal{X}}_{z+1|z} - F_{z+1}\tilde{V}_{z+1}\end{aligned} \tag{29}$$

and

$$\begin{aligned}\mathcal{O}_{z+1|z+1} = & (I - F_{z+1}\tilde{D}_{z+1})\mathcal{O}_{z+1|z}(I - F_{z+1}\tilde{D}_{z+1})^T \\ & - \mathbb{E}[(I - F_{z+1}\tilde{D}_{z+1})\tilde{\mathcal{X}}_{z+1|z}\tilde{V}^T_{z+1}F^T_{z+1}] \\ & - \mathbb{E}[F_{z+1}\tilde{V}_{z+1}\tilde{\mathcal{X}}^T_{z+1|z}(I - F_{z+1}\tilde{D}_{z+1})^T] \\ & + F_{z+1}\mathbb{E}[\tilde{V}_{z+1}\tilde{V}^T_{z+1}]F^T_{z+1}.\end{aligned} \tag{30}$$

Next, we define $G_{2,z+1} \triangleq \mathbb{E}[-(I - F_{z+1}\tilde{D}_{z+1})\tilde{\mathcal{X}}_{z+1|z}\tilde{V}^T_{z+1}F^T_{z+1}]$, and obtain the recursion of $\mathcal{O}_{z+1|z+1}$ as follows:

$$\begin{aligned}\mathcal{O}_{z+1|z+1} = & (I - F_{z+1}\tilde{D}_{z+1})\mathcal{O}_{z+1|z}(I - F_{z+1}\tilde{D}_{z+1})^T \\ & + F_{z+1}Q_{\tilde{V}_{z+1}}F^T_{z+1} + G_{2,z+1} + G^T_{2,z+1}.\end{aligned} \tag{31}$$

Now, by means of the recursion of $\mathcal{O}_{z+1|z+1}$, we are going to prove that $\mathcal{O}_{z+1|z} \le \bar{\mathcal{O}}_{z+1|z}$.

Let us define

$$\tilde{\tilde{v}}^{(2)}_{z+1} \triangleq \Upsilon^{[2]}(v^{[2]}_{z+1} - \varphi^{(2)}_{v_{z+1}}) + \Lambda^{[2]}_{z+1} - \varphi^{(2)}_{\Lambda_{z+1}} + \Upsilon^{[2]}e^{[2]}_{z+1}.$$

Then, the term $G_{2,z+1}$ can be rewritten as

$$G_{2,z+1} = \mathbb{E}\{-(I - F_{z+1}\tilde{D}_{z+1})\tilde{\mathcal{X}}_{z+1|z}[0, (\tilde{\tilde{v}}^{(2)}_{z+1})^T]F^T_{z+1}\}. \tag{32}$$

Moreover, there exists a positive number $\mu_{1,z+1} > 0$ such that

$$\begin{aligned}& \mathbb{E}\Big\{\Big(\mu^{\frac{1}{2}}_{1,z+1}(I - F_{z+1}\tilde{D}_{z+1})\tilde{\mathcal{X}}_{z+1|z} \\ & \quad + \mu^{-\frac{1}{2}}_{1,z+1}F_{z+1}\begin{bmatrix} 0 & (\tilde{\tilde{v}}^{(2)}_{z+1})^T\end{bmatrix}^T\Big) \\ & \times\Big(\mu^{\frac{1}{2}}_{1,z+1}(I - F_{z+1}\tilde{D}_{z+1})\tilde{\mathcal{X}}_{z+1|z} \\ & \quad + \mu^{-\frac{1}{2}}_{1,z+1}F_{z+1}\begin{bmatrix} 0 & (\tilde{\tilde{v}}^{(2)}_{z+1})^T\end{bmatrix}^T\Big)^T\Big\} \\ = & \mu_{1,z+1}(I - F_{z+1}\tilde{D}_{z+1})\mathcal{O}_{z+1|z}(I - F_{z+1}\tilde{D}_{z+1})^T \\ & + \mu^{-1}_{1,z+1}F_{z+1}\tilde{Q}^{(2)}_{\tilde{V}_{z+1}}F^T_{z+1} - G_{2,z+1} - G^T_{2,z+1} \ge 0.\end{aligned} \tag{33}$$

Rearranging the above inequality, we have

$$\begin{aligned}& G_{2,z+1} + G^T_{2,z+1} \\ \le & \mu_{1,z+1}(I - F_{z+1}\tilde{D}_{z+1})\mathcal{O}_{z+1|z}(I - F_{z+1}\tilde{D}_{z+1})^T \\ & + \mu^{-1}_{1,z+1}F_{z+1}\tilde{Q}^{(2)}_{\tilde{V}_{z+1}}F^T_{z+1}.\end{aligned} \tag{34}$$

Substituting (34) into (31) yields

$$\begin{aligned}\mathcal{O}_{z+1|z+1} \le & (1+\mu_{1,z+1})(I - F_{z+1}\tilde{D}_{z+1})\mathcal{O}_{z+1|z} \\ & \times (I - F_{z+1}\tilde{D}_{z+1})^T + F_{z+1}\bar{Q}_{\tilde{V}_{z+1}}F^T_{z+1}\end{aligned}$$

$$+\mu_{1,z+1}^{-1}F_{z+1}\tilde{Q}^{(2)}_{\tilde{V}_{z+1}}F_{z+1}^T. \tag{35}$$

By using the mathematical induction method, we arrive at the conclusion that $\mathcal{O}_{z+1|z+1} \le \bar{\mathcal{O}}_{z+1|z+1}$. ■

Theorem 2 *If the filter gain matrix is chosen as*

$$F_{z+1} = (1+\mu_{1,z+1})\bar{\mathcal{O}}_{z+1|z}\tilde{D}_{z+1}^T\Sigma_{z+1}^{-1} \tag{36}$$

then the minimal upper bound matrix $\bar{\mathcal{O}}_{z+1|z+1}$ satisfies

$$\begin{aligned}\bar{\mathcal{O}}_{z+1|z+1} =&(1+\mu_{1,z+1})\bar{\mathcal{O}}_{z+1|z} - (1+\mu_{1,z+1})^2\\ &\times \bar{\mathcal{O}}_{z+1|z}\tilde{D}_{z+1}^T\Sigma_{z+1}^{-1}\tilde{D}_{z+1}\bar{\mathcal{O}}_{z+1|z}\end{aligned} \tag{37}$$

where

$$\begin{aligned}\Sigma_{z+1} \triangleq&(1+\mu_{1,z+1})\tilde{D}_{z+1}\bar{\mathcal{O}}_{z+1|z}\tilde{D}_{z+1}^T\\ &+\bar{Q}_{\tilde{V}_{z+1}} + \mu_{1,z+1}^{-1}\tilde{Q}^{(2)}_{\tilde{V}_{z+1}}.\end{aligned} \tag{38}$$

Proof: The upper bound matrix $\bar{\mathcal{O}}_{z+1|z+1}$ in (24) can be reformulated as

$$\begin{aligned}\bar{\mathcal{O}}_{z+1|z+1} =&(1+\mu_{1,z+1})\bar{\mathcal{O}}_{z+1|z} + F_{z+1}\Sigma_{z+1}F_{z+1}^T\\ &-(1+\mu_{1,z+1})\bar{\mathcal{O}}_{z+1|z}\tilde{D}_{z+1}^TF_{z+1}^T\\ &-(1+\mu_{1,z+1})F_{z+1}\tilde{D}_{z+1}\bar{\mathcal{O}}_{z+1|z}\\ =&(1+\mu_{1,z+1})\bar{\mathcal{O}}_{z+1|z}\\ &+[F_{z+1}-(1+\mu_{1,z+1})\bar{\mathcal{O}}_{z+1|z}\tilde{D}_{z+1}^T\Sigma_{z+1}^{-1}]\Sigma_{z+1}\\ &\times[F_{z+1}-(1+\mu_{1,z+1})\bar{\mathcal{O}}_{z+1|z}\tilde{D}_{z+1}^T\Sigma_{z+1}^{-1}]^T\\ &-(1+\mu_{1,z+1})^2\bar{\mathcal{O}}_{z+1|z}\tilde{D}_{z+1}^T\Sigma_{z+1}^{-1}\tilde{D}_{z+1}\bar{\mathcal{O}}_{z+1|z}.\end{aligned} \tag{39}$$

Obviously, the upper bound $\bar{\mathcal{O}}_{z+1|z+1}$ is minimized by choosing $F_{z+1} = (1+\mu_{1,z+1})\bar{\mathcal{O}}_{z+1|z}\tilde{D}_{z+1}^T\Sigma_{z+1}^{-1}$. The proof is now complete. ■

Remark 5 *Note that the dynamics of the prediction and filtering error covariances have been established in (23) and (24), which are dependent on the statistical characteristics of the quantization errors, random bit errors as well as the augmented noises. Nevertheless, owing mainly to the occurrence of the cross-term $G_{2,z+1}$, it would be really hard to get the accurate value of the covariance matrix. As such, we have resorted to the elementary inequality (as shown in (33)) and obtained the upper bound on the FEC. Subsequently, the obtained upper bound $\bar{\mathcal{O}}_{z+1|z+1}$ is minimized by choosing the proper F_{z+1} in Theorem 2.*

Remark 6 *In comparison to most existing QF results (see e.g. [37]), the consideration of the BESs in this paper brings some challenges in designing the QF algorithm, in which the impacts from quantization error, flipping error and non-Gaussian noises on the filtering design have been analyzed. Note that the main contributions of [37] are to propose a quadratic filter for the linear non-Gaussian systems with the time-correlated fading channels, and discuss the relationships between the high-order moments of the fading channel coefficients and the locally minimized upper bound on the estimation error covariance.*

4.3 Boundedness Analysis

In this subsection, the boundedness of the QF error covariance is analyzed for the proposed quadratic filter. Before proceeding further, we need the following assumption:

Assumption 1 *There exist positive real numbers $\underline{r}$, $\bar{r}$, $\underline{\mu}_1$, $\bar{\mu}_1$, $\underline{d}$, $\bar{d}$, $\underline{q}_w$, $\bar{q}_w$, $\underline{q}_{\tilde{v}}$, $\bar{q}_{\tilde{v}}$ and $\tilde{q}_{\tilde{v}}$, such that the following inequalities hold*

$$\begin{aligned}&\underline{r}I \le \tilde{\Gamma}_z^T\tilde{\Gamma}_z, \tilde{\Gamma}_z\tilde{\Gamma}_z^T \le \bar{r}I, \underline{\mu}_1 \le \mu_{1,z} \le \bar{\mu}_1\\ &\underline{d}I \le \tilde{D}_z^T\tilde{D}_z \le \bar{d}I, \underline{q}_wI \le Q_{W_z} \le \bar{q}_wI,\\ &\underline{q}_{\tilde{v}}I \le \bar{Q}_{\tilde{V}_z} \le \bar{q}_{\tilde{v}}I, \tilde{Q}^{(2)}_{\tilde{V}_{z+1}} \le \tilde{q}_{\tilde{v}}I.\end{aligned}$$

Theorem 3 *Consider the stochastic augmented system (15) with the quadratic filter (16) and filter gain (36). Under Assumption 1, if*

$$(1+\bar{\mu}_1)\bar{r} < 1 \tag{40}$$

then the upper bound matrix $\bar{\mathcal{O}}_{z+1|z+1}$ satisfies

$$\underline{\theta}I \le \bar{\mathcal{O}}_{z+1|z+1} \le \bar{\theta}I \tag{41}$$

where

$$\begin{aligned}\underline{\theta} \triangleq&(\underline{q}_w^{-1} + \bar{d}\underline{q}_{\tilde{v}}^{-1})^{-1}, \theta_0 \triangleq \lambda_{\max}(\bar{\mathcal{O}}_{0|0}),\\ \bar{\theta} \triangleq&(1+\bar{\mu}_1)[\theta_0\bar{r} + \frac{\bar{q}_w}{1-(1+\bar{\mu}_1)\bar{r}}].\end{aligned}$$

Proof: Utilizing the matrix inverse lemma and (37), it is not difficult to verify that

$$\begin{aligned}\bar{\mathcal{O}}_{z+1|z+1} =&\Big[(1+\mu_{1,z+1})^{-1}\bar{\mathcal{O}}_{z+1|z}^{-1} + \tilde{D}_{z+1}^T(\bar{Q}_{\tilde{V}_{z+1}}\\ &+\mu_{1,z+1}^{-1}\tilde{Q}^{(2)}_{\tilde{V}_{z+1}})^{-1}\tilde{D}_{z+1}\Big]^{-1}\\ \ge&\Big[(1+\mu_{1,z+1})^{-1}\bar{\mathcal{O}}_{z+1|z}^{-1} + \tilde{D}_{z+1}^T\bar{Q}_{\tilde{V}_{z+1}}^{-1}\tilde{D}_{z+1}\Big]^{-1}\end{aligned} \tag{42}$$

which implies that

$$\bar{\mathcal{O}}_{z+1|z+1}^{-1} \le (1+\mu_{1,z+1})^{-1}\bar{\mathcal{O}}_{z+1|z}^{-1} + \tilde{D}_{z+1}^T\bar{Q}_{\tilde{V}_{z+1}}^{-1}\tilde{D}_{z+1}. \tag{43}$$

Noting that

$$\bar{\mathcal{O}}_{z+1|z} \ge Q_{W_z} \ge \underline{q}_wI \tag{44}$$

it then follows from (43)-(44) and Assumption 1 that

$$\bar{\mathcal{O}}_{z+1|z+1}^{-1} \le \underline{q}_w^{-1}I + \bar{d}\underline{q}_{\tilde{v}}^{-1}I \tag{45}$$

thus we immediately obtain $\bar{\mathcal{O}}_{z+1|z+1} \ge \underline{\theta}I$.

In order to prove the conclusion that $\bar{\mathcal{O}}_{z+1|z+1} \le \bar{\theta}I$, we first show $\bar{\mathcal{O}}_{z+1|z+1} \le \theta_{z+1}I$ by the mathematical induction method, where

$$\theta_{z+1} \triangleq \theta_0[(1+\bar{\mu}_1)\bar{r}]^{z+1} + (1+\bar{\mu}_1)\bar{q}_w\sum_{i=0}^{z}[(1+\bar{\mu}_1)\bar{r}]^i.$$

Obviously, one has

$$\bar{\mathcal{O}}_{0|0} \le \lambda_{\max}(\bar{\mathcal{O}}_{0|0})I \tag{46}$$

Assume that $\bar{\mathcal{O}}_{z|z} \le \theta_z$ holds, we are now ready to obtain $\bar{\mathcal{O}}_{z+1|z+1} \le \theta_{z+1}$.

Based on (23) and (37), we can see that

$$\begin{aligned}\bar{\mathcal{O}}_{z+1|z+1} \le &(1+\bar{\mu}_1)[\tilde{\Gamma}_z \bar{\mathcal{O}}_{z|z} \tilde{\Gamma}_z^T + Q_{W_z}] \\ \le &(1+\bar{\mu}_1)[\theta_z \bar{r} I + \bar{q}_w I] \\ = &\theta_{z+1}.\end{aligned} \tag{47}$$

It is clear that if $(1+\bar{\mu}_1)\bar{r} < 1$, then $\bar{\mathcal{O}}_{z+1|z+1} \le \bar{\theta} I$ holds, which completes this proof. ■

If $(1+\bar{\mu}_1)\bar{r} \ge 1$, a sufficient condition is provided in the following theorem to ensure the boundedness of the error covariance matrix.

Theorem 4 *Under Assumption 1, suppose that there exist two positive scalars $\bar{\theta}_\varepsilon$ and l such that the following inequalities:*

$$\sum_{i=z-l+1}^{z+1} \kappa_1^{z+1-i} \Phi^T(i, z+1)\kappa_2 \Phi(i, z+1) \ge \bar{\theta}_\varepsilon^{-1} I, z \ge l-1 \tag{48}$$

$$\theta_{z+1} \le \bar{\theta}_\varepsilon, 0 \le z < l-1 \tag{49}$$

hold for all $z \ge 0$, then the upper bound matrix $\bar{\mathcal{O}}_{z+1|z+1}$ satisfies

$$\underline{\theta} I \le \bar{\mathcal{O}}_{z+1|z+1} \le \bar{\theta}_\varepsilon I \tag{50}$$

where $\Phi(i, z+1) \triangleq \tilde{\Gamma}_i^{-1} \tilde{\Gamma}_{i+1}^{-1} \cdots \tilde{\Gamma}_z^{-1} (i < z)$, $\Phi(z+1, z+1) \triangleq I$ and

$$\begin{aligned}\kappa_1 \triangleq &(1+\bar{\mu}_1)^{-1}(1+\underline{r}^{-1}\bar{q}_w \underline{\theta}^{-1})^{-1}, \\ \kappa_2 \triangleq &\underline{d}(\bar{q}_{\tilde{v}} + \underline{\mu}_1^{-1} \tilde{q}_{\tilde{v}})^{-1} I\end{aligned}$$

Proof: In light of (23), we have

$$\begin{aligned}\bar{\mathcal{O}}_{z+1|z} = &\tilde{\Gamma}_z \left(\bar{\mathcal{O}}_{z|z} + \tilde{\Gamma}_z^{-1} Q_{W_z} \tilde{\Gamma}_z^{-T}\right)\tilde{\Gamma}_z^T \\ \le &\tilde{\Gamma}_z \left(\bar{\mathcal{O}}_{z|z} + \underline{r}^{-1}\bar{q}_w \underline{\theta}^{-1} \bar{\mathcal{O}}_{z|z}\right)\tilde{\Gamma}_z^T \\ \le &(1+\underline{r}^{-1}\bar{q}_w \underline{\theta}^{-1})\tilde{\Gamma}_z \bar{\mathcal{O}}_{z|z} \tilde{\Gamma}_z^T.\end{aligned} \tag{51}$$

Combining (42) and (51) yields

$$\begin{aligned}\bar{\mathcal{O}}_{z+1|z+1}^{-1} = &(1+\mu_{1,z+1})^{-1} \bar{\mathcal{O}}_{z+1|z}^{-1} + \tilde{D}_{z+1}^T (\bar{Q}_{\tilde{V}_{z+1}} \\ &+ \mu_{1,z+1}^{-1} \tilde{Q}_{\tilde{V}_{z+1}}^{(2)})^{-1} \tilde{D}_{z+1} \\ \ge &\kappa_1 \tilde{\Gamma}_z^{-T} \bar{\mathcal{O}}_{z|z}^{-1} \tilde{\Gamma}_z^{-1} + \kappa_2 \\ \ge &\cdots \\ \ge &\kappa_1^{l+1} \Phi^T(z-l, z+1) \bar{\mathcal{O}}_{z-l|z-l}^{-1} \Phi(z-l, z+1) \\ &+ \sum_{i=z-l+1}^{z+1} \kappa_1^{z+1-i} \Phi^T(i, z+1)\kappa_2 \Phi(i, z+1) \\ \ge &\sum_{i=z-l+1}^{z+1} \kappa_1^{z+1-i} \Phi^T(i, z+1)\kappa_2 \Phi(i, z+1) \\ \ge &\bar{\theta}_\varepsilon^{-1} I\end{aligned} \tag{52}$$

which indicates $\bar{\mathcal{O}}_{z+1|z+1} \le \bar{\theta}_\varepsilon I$. The proof is complete. ■

Remark 7 *Up to now, we have made one of the first attempts to cope with the QF problem for linear non-Gaussian systems with the BESs. It should be emphasized that, during the design of quadratic filter, we have taken into consideration all important aspects including higher-order moments of the quantization errors, random bit errors and non-Gaussian noises, the flip probability and the second-order Kronecker power of state/measurement matrices. Compared with the available results on the QF problem, the main results obtained in this paper have the following features: 1) a unified framework is established to simultaneously handle the effects of the quantization errors, random bit errors and non-Gaussian noises; 2) a new QF approach is presented with the ability to improve the filtering accuracy when compared with the traditional linear filter; and 3) sufficient conditions are provided to ensure the boundedness of the QF error covariance.*

5 An Illustrative Example

In this section, we provide a numerical example to demonstrate the usefulness of the designed QF scheme.

The parameters of a linear discrete time-varying system are given by:

$$\Gamma_z = \begin{bmatrix} 0.45 & 0.3 + 0.4\sin(0.1z) \\ 0.15 & 0.29 + 0.3\cos(0.1z) \end{bmatrix},$$

$$D_z = \begin{bmatrix} 0.20 & 0.95 \\ -0.10 & 0.20 \end{bmatrix}.$$

In the simulation, we assume that x_0 obeys the Gaussian distribution with $\mathbb{E}(x_0) = 0$ and $\mathrm{Cov}(x_0) = 10^{-2} I_2$. Other parameters are chosen as $h = 0.2$, $S = 5$, $L = 3.1$, $\mu_{1,z} = 0.5$, and $\bar{\alpha} = 0.001$. Meanwhile, the non-Gaussian noises $w_z = [w_{z,1},\ w_{z,2}]^T$ and $v_z = [v_{z,1},\ v_{z,2}]^T$ are generated by the following sequences:

$$\begin{aligned}&w_{1,z} = -1.4\zeta_{w_{1,z}} + 0.6(1-\zeta_{w_{1,z}}), \\ &w_{2,z} = 0.1\zeta_{w_{2,z}} - 0.1(1-\zeta_{w_{2,z}}), \\ &v_{1,z} = 1.5\zeta_{v_{1,z}} - 0.5(1-\zeta_{v_{1,z}}), \\ &v_{2,z} = -2\zeta_{v_{2,z}} + 0.5(1-\zeta_{v_{2,z}}),\end{aligned}$$

where $\zeta_{w_{1,z}}$, $\zeta_{w_{2,z}}$, $\zeta_{v_{1,z}}$ and $\zeta_{v_{2,z}}$ are independent Bernoulli random variables satisfying

$$\begin{aligned}&\mathbb{P}(\zeta_{w_{1,z}} = 1) = 0.3,\ \mathbb{P}(\zeta_{w_{2,z}} = 1) = 0.5, \\ &\mathbb{P}(\zeta_{v_{1,z}} = 1) = 0.25,\ \mathbb{P}(\zeta_{v_{2,z}} = 1) = 0.2.\end{aligned}$$

The corresponding high-order moments of the non-Gaussian noises are given in Table 1.

The trajectories of the actual states, quadratic filter and the recursive filter (that only utilizes the information of

Table 1
The 2nd, 3rd and 4th-order moments of $w_{i,z}$ and $v_{i,z}$ ($i = 1, 2$).

	$\mathbb{E}\{(\cdot)^2\}$	$\mathbb{E}\{(\cdot)^3\}$	$\mathbb{E}\{(\cdot)^4\}$
$w_{1,z}$	0.8400	-0.6720	1.2432
$w_{2,z}$	0.0100	0.0000	10^{-4}
$v_{1,z}$	0.7500	0.7500	1.3125
$v_{2,z}$	1.0000	-1.5000	3.2500

$\tilde{y}_z$) are depicted in Figs. 1-2, which illustrate that the presented quadratic filter has a good tracking performance. To evaluate the QF performance, the trajectories of the mean square errors (MSEs) and the upper bounds of the FEC are displayed in Figs. 3-4. It can be seen that, the curves of MSEs always stay below those of their corresponding upper bounds, thereby demonstrating the correctness of our theoretical results.

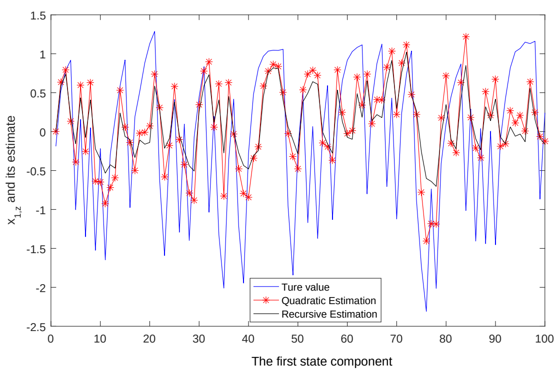

Fig. 1. The trajectories of $x_{1,z}$ and $\hat{x}_{1,z}$.

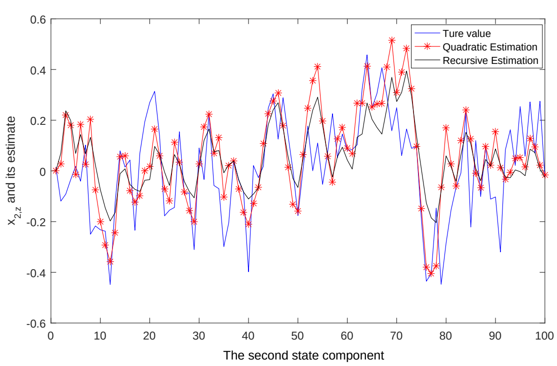

Fig. 2. The trajectories of $x_{2,z}$ and $\hat{x}_{2,z}$.

To further show the superiority of the proposed QF method in improving the filtering accuracy, we depict the MSE curves of the quadratic filter and the recursive filter in Figs. 5-6. It is obvious that the filtering accuracy of the developed quadratic filter is higher than that of the latter.

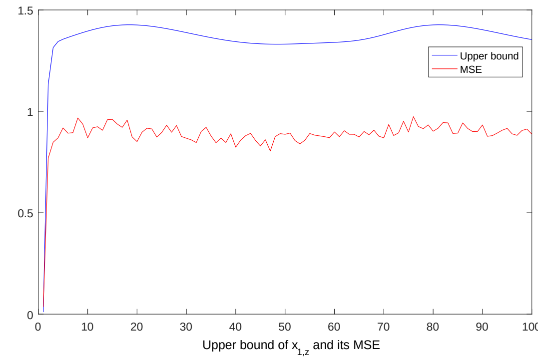

Fig. 3. The MSE of $x_{1,z}$ and its upper bound.

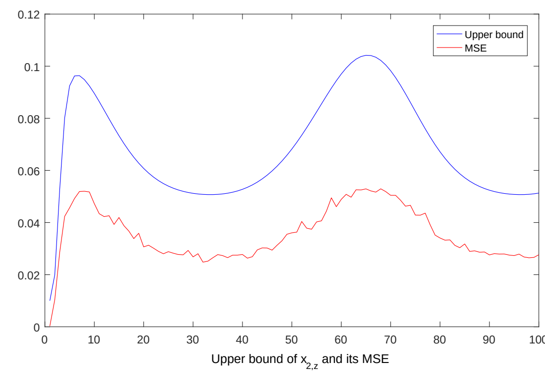

Fig. 4. The MSE of $x_{2,z}$ and its upper bound.

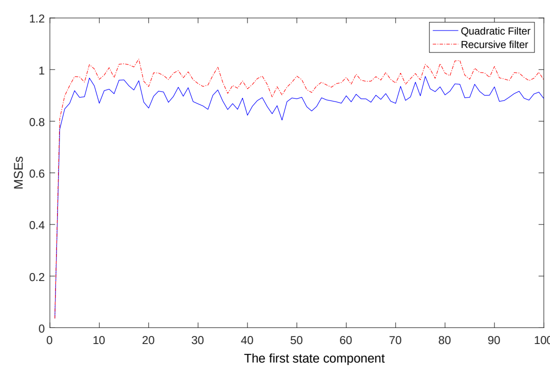

Fig. 5. The MSEs of the quadratic filter and the recursive filter.

6 Conclusions

In this paper, the QF problem has been studied for an array of linear discrete-time systems with non-Gaussian noises and BESs. A new measurement model has been constructed to take into account the probabilistic quantization errors as well as the random bit errors. By means of the augmentation method, the original non-Gaussian system has been converted into a linear one, which is composed of the quadratic functions of the

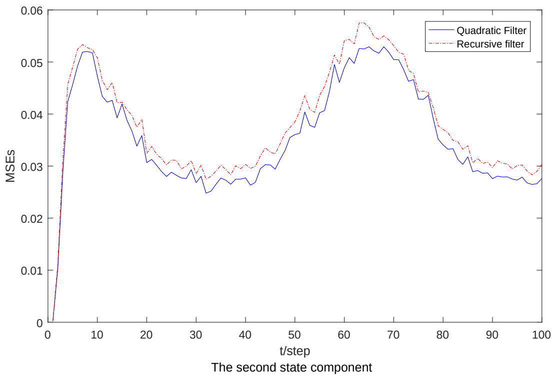

Fig. 6. The MSEs of the quadratic filter and the recursive filter.

state and measurement vectors. Accordingly, the statistical properties of the quantization errors, random bit errors and augmented non-Gaussian noises have been analyzed, based on which an upper bound of the FEC has been established in terms of the solutions to two Riccati-like difference equations. The filter gain is then determined by minimizing the obtained upper bound. Moreover, sufficient conditions have been derived to guarantee the boundedness of the QF error covariance. Finally, a numerical simulation has been conducted to show the effectiveness of the designed QF algorithm. Our future work would include the extension of our developed results to more general systems, such as sensor networks and complex networks.

7 Appendix

7.1 *Proof of Lemma 3*

Proof: From (7), it is easy to verify that

$$\begin{aligned}\lambda_{i,z} =& \tilde{q}_z(y_{i,z}, S) - \mathbb{E}[\tilde{q}_z(y_{i,z}, S)]\\ =& \sum_{j=1}^{S}[\tilde{b}_{j,z}^{(i)} - \mathbb{E}(\tilde{b}_{j,z}^{(i)})]2^{j-1}h\\ =& \sum_{j=1}^{S}\tilde{\alpha}_{j,z}(1-2b_{j,z}^{(i)})2^{j-1}h. \end{aligned} \tag{53}$$

Noting that $\mathbb{E}(\tilde{\alpha}_{j,z}) = 0$, one has $\mathbb{E}(\lambda_{i,z}) = 0$.

From (53) and the fact that $\mathbb{E}(\tilde{\alpha}_{s,z}\tilde{\alpha}_{j,z}) = 0$ $(s \neq j)$, we have

$$\begin{aligned}\mathbb{E}[\lambda_{i,z}^2] =& \sum_{j=1}^{S}\mathbb{E}(\tilde{\alpha}_{j,z}^2)(1-2b_{j,z}^{(i)})^2 2^{2(j-1)}h^2\\ =& \bar{\alpha}(1-\bar{\alpha})h^2\sum_{j=1}^{S}(1-2b_{j,z}^{(i)})^2 2^{2(j-1)}. \end{aligned} \tag{54}$$

Since the value of $b_{j,z}^{(i)}$ is 0 or 1, we have $(1-2b_{j,z}^{(i)})^2 = 1$. Then, it follows that

$$\mathbb{E}[\lambda_{i,z}^2] = \bar{\alpha}(1-\bar{\alpha})h^2\sum_{j=1}^{S}2^{2(j-1)} \triangleq \chi_2. \tag{55}$$

Noticing that $\mathbb{E}(\tilde{\alpha}_{i,z}^2\tilde{\alpha}_{j,z}) = 0 (i \neq j)$, one has

$$\begin{aligned}\mathbb{E}[\lambda_{i,z}^3] =& \sum_{j=1}^{S}\mathbb{E}(\tilde{\alpha}_{j,z}^3)(1-2b_{j,z}^{(i)})^3 2^{3(j-1)}h^3\\ =& (\bar{\alpha} - 3\bar{\alpha}^2 + 2\bar{\alpha}^3)\sum_{j=1}^{S}(1-2b_{j,z}^{(i)})^3 2^{3(j-1)}h^3\\ \triangleq& \chi_3 \end{aligned} \tag{56}$$

and

$$\begin{aligned}\mathbb{E}[\lambda_{i,z}^4] =& \sum_{j=1}^{S}\mathbb{E}(\tilde{\alpha}_{j,z}^4)(1-2b_{j,z}^{(i)})^4 2^{4(j-1)}h^4\\ &+ 6\sum_{1\le j<s\le S}\mathbb{E}(\tilde{\alpha}_{j,z}^2\tilde{\alpha}_{s,z}^2)(1-2b_{j,z}^{(i)})^2\\ &\times (1-2b_{s,z}^{(i)})^2 2^{2(j-1)}2^{2(s-1)}h^4\\ =& (\bar{\alpha} - 4\bar{\alpha}^2 + 6\bar{\alpha}^3 - 3\bar{\alpha}^4)h^4\sum_{j=1}^{S}2^{4(j-1)}\\ &+ 6\bar{\alpha}^2(1-\bar{\alpha})^2h^4\sum_{1\le j<s\le S}2^{2(j-1)}2^{2(s-1)}\\ \triangleq& \chi_4. \end{aligned} \tag{57}$$

The proof is now complete. ∎

7.2 *Proof of Lemma 4*

Proof: Recalling the expression of W_z, we know that

$$Q_{W_z} = \begin{bmatrix} \mathbb{E}[w_z w_z^T] & \mathbb{E}[w_z(\tilde{w}_z^{(2)})^T] \\ \mathbb{E}[\tilde{w}_z^{(2)}w_z^T] & \mathbb{E}[\tilde{w}_z^{(2)}(\tilde{w}_z^{(2)})^T] \end{bmatrix}. \tag{58}$$

It then follows from the properties of w_z that

$$\begin{aligned}\mathbb{E}[w_z w_z^T] =& \mathrm{st}(\varphi_{w_z}^{(2)})\\ \mathbb{E}[w_z(\tilde{w}_z^{(2)})^T] =& \mathbb{E}\{w_z[(I+K_{n_x,n_x})(\Gamma_z x_z \otimes w_z)\\ &+ w_z^{[2]} - \mathrm{st}(\varphi_{w_z}^{(2)})]^T\}\\ =& \mathrm{st}(\varphi_{w_z}^{(3)}). \end{aligned} \tag{59}$$

For the term $\mathbb{E}[\tilde{w}_z^{(2)}(\tilde{w}_z^{(2)})^T]$ in (58), we can obtain that

$$\begin{aligned}Q_{w_z}^{(2)} \triangleq& \mathbb{E}[\tilde{w}_z^{(2)}(\tilde{w}_z^{(2)})^T]\\ =& (I+K_{n_x,n_x})(\Gamma_z\mathrm{st}(\varphi_{x_z}^{(2)})\Gamma_z^T \otimes \mathrm{st}(\varphi_{w_z}^{(2)}))\\ &\times (I+K_{n_x,n_x})^T + \mathrm{st}(\varphi_{w_z}^{(4)}) + \varphi_{w_z}^{(2)}(\varphi_{w_z}^{(2)})^T\\ &- \mathbb{E}[w_z^{[2]}(\varphi_{w_z}^{(2)})^T] - \mathbb{E}[\varphi_{w_z}^{(2)}(w_z^{[2]})^T]\\ =& (I+K_{n_x,n_x})(\Gamma_z\mathrm{st}(\varphi_{x_z}^{(2)})\Gamma_z^T \otimes \mathrm{st}(\varphi_{w_z}^{(2)}))\\ &\times (I+K_{n_x,n_x})^T + \mathrm{st}(\varphi_{w_z}^{(4)}) - \varphi_{w_z}^{(2)}(\varphi_{w_z}^{(2)})^T. \end{aligned} \tag{60}$$

Similarly, based on the expression of $\tilde{V}_z$, we get

$$Q_{\tilde{V}_z} = \begin{bmatrix} \mathbb{E}[\tilde{v}_z\tilde{v}_z^T] & \mathbb{E}[\tilde{v}_z(\tilde{v}_z^{(2)})^T] \\ \mathbb{E}[\tilde{v}_z^{(2)}\tilde{v}_z^T] & \mathbb{E}[\tilde{v}_z^{(2)}(\tilde{v}_z^{(2)})^T] \end{bmatrix}. \tag{61}$$

It is not difficult to see that

$$\begin{aligned}&\mathbb{E}[\tilde{v}_z\tilde{v}_z^T]\\=&\Upsilon\mathbb{E}[v_zv_z^T]\Upsilon^T+\Upsilon\mathbb{E}[e_ze_z^T]\Upsilon^T+\mathbb{E}[\Lambda_z\Lambda_z^T]\\&+\mathrm{Sym}\{\mathbb{E}[\Upsilon v_ze_z^T\Upsilon^T]+\mathbb{E}[\Upsilon v_z\Lambda_z^T]+\mathbb{E}[\Upsilon e_z\Lambda_z^T]\}\\=&\Upsilon\mathrm{st}(\varphi_{v_z}^{(2)})\Upsilon^T+\Upsilon\mathbb{E}[e_ze_z^T]\Upsilon^T+\mathbb{E}[\Lambda_z\Lambda_z^T].\end{aligned}\tag{62}$$

Since Λ_z is uncorrelated with random variables x_0, e_z and v_z, we have

$$\begin{aligned}&\mathbb{E}[(\Upsilon v_z+\Upsilon e_z+\Lambda_z)(D_zx_z\otimes v_z)^T]=0,\\&\mathbb{E}[(\Upsilon v_z+\Upsilon e_z+\Lambda_z)\big(\Upsilon^{[2]}(y_z\otimes e_z)\\&\ +(\Upsilon y_z\otimes\Lambda_z)+(\Upsilon e_z\otimes\Lambda_z)\big)]=0,\\&\mathbb{E}[(\Upsilon v_z+\Upsilon e_z+\Lambda_z)(\Upsilon^{[2]}\varphi_{v_z}^{(2)})^T]=0.\end{aligned}$$

Accordingly, we can obtain that

$$\begin{aligned}\mathbb{E}[\tilde{v}_z(\tilde{v}_z^{(2)})^T]=&\mathbb{E}\Big\{(\Upsilon v_z+\Upsilon e_z+\Lambda_z)\Big[\Upsilon^{[2]}(I+K_{n_y,n_y})\\&\times(D_zx_z\otimes v_z)+\Upsilon^{[2]}v_z^{[2]}+\Upsilon^{[2]}e_z^{[2]}+\Lambda_z^{[2]}\\&+(I+K_{n_y,n_y})\big(\Upsilon^{[2]}(y_z\otimes e_z)+(\Upsilon y_z\otimes\Lambda_z)\\&+(\Upsilon e_z\otimes\Lambda_z)\big)-\Upsilon^{[2]}\varphi_{v_z}^{(2)}-\varphi_{\Lambda_z}^{(2)}\Big]^T\Big\}\\=&\Upsilon\mathrm{st}(\varphi_{v_z}^{(3)})(\Upsilon^{[2]})^T+\Upsilon\mathbb{E}[e_z(e_z^{[2]})^T](\Upsilon^{[2]})^T\\&+\mathbb{E}[\Lambda_z(\Lambda_z^{[2]})^T].\end{aligned}\tag{63}$$

On the other hand, we get

$$\begin{aligned}&\mathbb{E}[v_z^{(2)}(v_z^{(2)})^T]\\=&\Upsilon^{[2]}(I+K_{n_y,n_y})(D_z\mathrm{st}(\varphi_{x_z}^{(2)})D_z^T\otimes\mathrm{st}(\varphi_{v_z}^{(2)}))\\&\times(I+K_{n_y,n_y})^T(\Upsilon^{[2]})^T+\Upsilon^{[2]}\mathrm{st}(\varphi_{v_z}^{(4)})(\Upsilon^{[2]})^T\\&+\mathbb{E}[\Upsilon^{[2]}e_z^{[2]}(e_z^{[2]})^T(\Upsilon^{[2]})^T]+\mathbb{E}[\Lambda_z^{[2]}(\Lambda_z^{[2]})^T]\\&+(I+K_{n_y,n_y})\Big[\Upsilon^{[2]}(y_zy_z^T\otimes e_ze_z^T)(\Upsilon^{[2]})^T\\&+(\Upsilon y_zy_z^T\Upsilon^T\otimes\Lambda_z\Lambda_z^T)+(\Upsilon e_ze_z^T\Upsilon^T\otimes\Lambda_z\Lambda_z^T)\Big]\\&\times(I+K_{n_y,n_y})^T+\mathrm{Sym}\Big\{\mathbb{E}[\Upsilon^{[2]}v_z^{[2]}(e_z^{[2]})^T(\Upsilon^{[2]})^T]\\&+\Upsilon^{[2]}v_z^{[2]}(\Lambda_z^{[2]})^T+\mathbb{E}[\Upsilon^{[2]}e_z^{[2]}(\Lambda_z^{[2]})^T]\Big\}\end{aligned}\tag{64}$$

and

$$\begin{aligned}&\mathbb{E}[\tilde{v}_z^{(2)}(\tilde{v}_z^{(2)})^T]\\=&\mathbb{E}[v_z^{(2)}(v_z^{(2)})^T]-(\Upsilon^{[2]}\varphi_{v_z}^{(2)}+\varphi_{\Lambda_z}^{(2)})(\Upsilon^{[2]}\varphi_{v_z}^{(2)}+\varphi_{\Lambda_z}^{(2)})^T\\&-\mathrm{Sym}\Big\{\mathbb{E}[\Upsilon^{[2]}e_z^{[2]}(\Upsilon^{[2]}\varphi_{v_z}^{(2)}+\varphi_{\Lambda_z}^{(2)})^T]\Big\}.\end{aligned}\tag{65}$$

Moreover, it is not hard to verify that

$$\begin{aligned}\mathbb{E}[e_z(e_z)^T]=&\mathrm{diag}\{\mathbb{E}[e_{1,z}^2],\cdots,\mathbb{E}[e_{n_y,z}^2]\}\le\frac{1}{4}h^2I_{n_y},\\\mathbb{E}[\Lambda_z\Lambda_z^T]=&\mathrm{diag}\{\mathbb{E}[\lambda_{1,z}^2],\cdots,\mathbb{E}[\lambda_{n_y,z}^2]\}=\chi_2I_{n_y},\\\mathbb{E}[y_zy_z^T]=&D_z\mathrm{st}(\varphi_{x_z}^{(2)})D_z^T+\mathrm{st}(\varphi_{v_z}^{(2)}).\end{aligned}\tag{66}$$

Then, substituting (66) into (65) yields

$$\begin{aligned}&\mathbb{E}[\tilde{v}_z^{(2)}(\tilde{v}_z^{(2)})^T]\\\le&Q_{\tilde{V}_z}^{(2,2)}+\mathbb{E}[\Upsilon^{[2]}e_z^{[2]}(e_z^{[2]})^T(\Upsilon^{[2]})^T]+\mathbb{E}[\Lambda_z^{[2]}(\Lambda_z^{[2]})^T].\end{aligned}\tag{67}$$

As a result, combining (61)-(67), one has

$$\begin{aligned}Q_{\tilde{V}_z}\le&\begin{bmatrix}\Upsilon\mathrm{st}(\varphi_{v_z}^{(2)})\Upsilon^T&\Upsilon\mathrm{st}(\varphi_{v_z}^{(3)})(\Upsilon^{[2]})^T\\\Upsilon^{[2]}\mathrm{st}^T(\varphi_{v_z}^{(3)})\Upsilon^T&Q_{\tilde{V}_z}^{(2,2)}\end{bmatrix}\\&+\begin{bmatrix}\Upsilon&0\\0&\Upsilon^{[2]}\end{bmatrix}\begin{bmatrix}\mathbb{E}[e_ze_z^T]&\mathbb{E}[e_z(e_z^{[2]})^T]\\\mathbb{E}[e_z^{[2]}e_z^T]&\mathbb{E}[e_z^{[2]}(e_z^{[2]})^T]\end{bmatrix}\\&\times\begin{bmatrix}\Upsilon^T&0\\0&(\Upsilon^{[2]})^T\end{bmatrix}\\&+\begin{bmatrix}\mathbb{E}[\Lambda_z\Lambda_z^T]&\mathbb{E}[\Lambda_z(\Lambda_z^{[2]})^T]\\\mathbb{E}[\Lambda_z^{[2]}\Lambda_z^T]&\mathbb{E}[\Lambda_z^{[2]}(\Lambda_z^{[2]})^T]\end{bmatrix}.\end{aligned}\tag{68}$$

For the second term in (68), we can see that

$$\begin{aligned}&\mathbb{E}\Big\{\begin{bmatrix}e_z\\e_z^{[2]}\end{bmatrix}\begin{bmatrix}e_z^T&(e_z^{[2]})^T\end{bmatrix}\Big\}\\\le&\Big\{\mathbb{E}[e_z^Te_z]+\mathbb{E}[(e_z^{[2]})^Te_z^{[2]}]\Big\}I_{n_y(n_y+1)}\\\le&\Big(\frac{h^2}{4}n_y+n_y\chi_1h^4+n_y(n_y-1)\frac{h^4}{16}\Big)I_{n_y(n_y+1)}\\\triangleq&\chi_5I_{n_y(n_y+1)}.\end{aligned}\tag{69}$$

Recalling (19) and the expression of Λ_z, we obtain

$$\begin{bmatrix}\mathbb{E}[\Lambda_z\Lambda_z^T]&\mathbb{E}[\Lambda_z(\Lambda_z^{[2]})^T]\\\mathbb{E}[\Lambda_z^{[2]}\Lambda_z^T]&\mathbb{E}[\Lambda_z^{[2]}(\Lambda_z^{[2]})^T]\end{bmatrix}=\begin{bmatrix}\chi_2I_{n_y}&\mathrm{st}(\varphi_{\Lambda_z}^{(3)})\\\mathrm{st}^T(\varphi_{\Lambda_z}^{(3)})&\mathrm{st}(\varphi_{\Lambda_z}^{(4)})\end{bmatrix}.\tag{70}$$

Substituting (69)-(70) into (68) yields

$$\begin{aligned}Q_{\tilde{V}_z}\le&\begin{bmatrix}\Upsilon\mathrm{st}(\varphi_{v_z}^{(2)})\Upsilon^T+\chi_2I_{n_y}&\bar{Q}_{\tilde{V}_z}^{(1,2)}\\(\bar{Q}_{\tilde{V}_z}^{(1,2)})^T&Q_{\tilde{V}_z}^{(2,2)}+\mathrm{st}(\varphi_{\Lambda_z}^{(4)})\end{bmatrix}\\&+\begin{bmatrix}\Upsilon&0\\0&\Upsilon^{[2]}\end{bmatrix}\chi_5I_{n_y(n_y+1)}\begin{bmatrix}\Upsilon^T&0\\0&(\Upsilon^{[2]})^T\end{bmatrix}.\end{aligned}\tag{71}$$

Reorganizing the above formula, we can arrive at (20). The proof is now complete. ■

7.3 Proof of Lemma 5

Proof: Utilizing (15), we can obtain that

$$\begin{aligned}\Pi_{z+1}=&\tilde{\Gamma}_z\Pi_z\tilde{\Gamma}_z^T+U_zU_z^T+\mathbb{E}[W_zW_z^T]+\mathrm{Sym}\Big\{\tilde{\Gamma}_z\mathbb{E}[\mathcal{X}_zU_z^T]\\&+\tilde{\Gamma}_z\mathbb{E}[\mathcal{X}_zW_z^T]+\mathbb{E}[U_zW_z^T]\Big\}.\end{aligned}\tag{72}$$

Noticing that $\mathbb{E}(W_z)=0$ and U_z is uncorrelated with W_z and X_z, we have

$$\mathbb{E}[\mathcal{X}_zU_z^T]=H_zU_z^T,\ \mathbb{E}[\mathcal{X}_zW_z^T]=0,\ \mathbb{E}[U_zW_z^T]=0.\tag{73}$$

Substituting (73) into (72) yields (22), which completes the proof. ■

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