A dynamic ordering policy for a stochastic inventory problem with cash constraints

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Abstract

This paper investigates a stochastic inventory management problem in which a cash-constrained small retailer periodically purchases a product and sells it to customers while facing non-stationary demand. In each period, the retailer's available cash restricts the maximum quantity that can be ordered. There is a fixed ordering cost incurred when an order is issued by the retailer. We introduce a heuristic $(s, C(x), S)$ policy inspired by numerical findings and by a structural analysis. The policy operates as follows: when the initial inventory x is less than s and the initial cash is greater than the state-dependent value $C(x)$, the retailer should order a quantity that brings inventory as close to S as possible; otherwise, the retailer should not order. We first determine the values of the controlling parameters s, $C(x)$ and S via the results of stochastic dynamic programming and test their performance in an extensive computational study. The results show that the $(s, C(x), S)$ policy performs well, with a maximum optimality gap of less than 1%, and an average gap of approximately 0.03%. We then develop a simple and time-efficient heuristic method for computing policy $(s, C(x), S)$ by solving a mixed-integer linear programming problem: the average gap for this heuristic is less than 1% on our test bed.

Keywords: stochastic inventory; non-stationary demand; cash-flow constraint; $(s, C(x), S)$ policy

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1 Introduction

Cash flow management is key to the survival and growth of many businesses. This issue is particularly important for small-to-medium-sized enterprises (SMEs), including start-up companies and small retailers. Smirat and Yousef (2016) sampled SMEs in Jordan and found that cash management practices had influence on the financial performance of those firms. A report compiled by the research firm CB Insights found that 29% of start-ups failed because of cash crisis (CBInsights, 2018). For some small retailers, and particularly for so-called nanostores (Boulaksil and van Wijk, 2018) in developing countries, the maximum number of products they can purchase depends on their available cash. These retailers face greater difficulties than larger business entities when it comes to financing. It is therefore key for such small retailers to consider cash constraints during business operations.

An important class of inventory control problems considers a single-item at a single-stocking location subject to stochastic demand and fixed ordering/production costs. This problem has been widely investigated since the 1950s (Scarf, 1960; Wagner and Whitin, 1958). An interesting stream in the associated literature focuses on cash constraints. Cash constraints are different from order quantity capacity constraints, since cash flow is dynamic and related to the retailer's revenues and costs, while order quantity capacity constraints simply impose a fixed upper bound on the ordering quantity.

To the best of our knowledge, existing works on multi-item stochastic inventory problems considering cash constraints either do not provide a solution method (e.g. Bi et al., 2020) or are solved by simulation due to the complexity of the problem in a multi-period setting (e.g. Boulaksil and van Wijk, 2018). This motivates our study on a single-item multi-period problem. Another difference between our paper and existing works is the embedding of a fixed ordering cost. In real-life transactions, fixed ordering costs do exist for some retailers. For example, some small online retailers on Chinese E-commerce platforms such as Taobao.com of Alibaba Group usually purchase from their suppliers every few weeks. The transportation cost and some other expenses of each procurement can be viewed as a fixed ordering cost because they are not related to the order quantity.

As pointed out by Graves (1999), one major theme in the continuing development of inventory theory is to incorporate more realistic assumptions into inventory models. In this work, we consider the cash-constrained single-item single-stocking location stochastic lot sizing problem under non-stationary stochastic demand and fixed ordering costs. To the best of our knowledge, the optimal policy of this problem has not been investigated before and efficient approaches for computing (near-)optimal policy parameters are not available in the literature. We make the following contributions to the literature on stochastic lot sizing.

- We investigate a cash-constrained stochastic inventory system with fixed ordering cost for the retailer.
- We numerically investigate the characteristics of the optimal ordering policy, and we prove some structural properties.
- Inspired by the above characteristics and structural properties, we introduce a heuristic control policy, which we name $(s, C(x), S)$ policy, that operates as follows: if the initial inventory is less than s and initial cash is above $C(x)$, the inventory level should be replenished to as close to S as possible; otherwise, one should not order.
- We introduce a stochastic dynamic programming approach for computing the values of s, $C(x)$ and S. A comprehensive numerical study shows that the average optimality gap for this policy is very small, and the maximum gap observed is smaller than 1%; these findings confirm that the policy is near-optimal.
- We finally develop an efficient algorithm to obtain the values of s, $C(x)$ and S via mixed-integer linear programming (MILP). The method can solve realistic instances in fractions of a second, and the average optimality gap on our test bed is less than 1%.

The rest of this work is structured as follows. Section 2 reviews the related literature in detail. Section 3 describes the problem setting and formulates a stochastic dynamic programming model. Section 4 summarizes several characteristics of the optimal ordering pattern. We present a heuristic ordering policy and their computation methods in Section 5. A computational study and its results are detailed in Section 6. Finally, Section 7 draws conclusions and outlines future research directions.

2 Literature review

An important part of supply chain management is financial flow management (Cooper et al., 1997). In the context of inventory control, several works consider the Net Present Value (NPV) of the cash flow. For example, Wu et al. (2016) made a discounted cash-flow analysis for inventory models with deteriorating items and partial trade credits. Working capital is another aspect that has been investigated by many scholars in recent years. Bian et al. (2018) took working capital requirements into account in a single-item lot-sizing problem. Luo and Shang (2019) built a working capital maximization model for an inventory problem with trade credit and cash-related costs. There are also some works that address cash flow and maximize the long-term survival probability for start-up firms (Archibald et al., 2002; Tanrısever et al., 2012).

Although the aforementioned works have considered cash flow, the available cash in their models does not affect the ordering quantity decisions, hence these works do not consider cash constraints. Among works considering cash constraint, Buzacott and Zhang (2004) noted the importance of jointly considering operational and financial decisions by analyzing a cash-constrained newsvendor model with financing behavior. Dada and Hu (2008) built a Stackelberg model between a bank, manufacturer and cash-constrained retailer. Raghavan and Mishra (2011) considered a two-level supply chain with a retailer and manufacturer both facing cash constraints. Kouvelis and Zhao (2012) gave a detailed discussion of optimal trade credit contracts in a game model. Moussawi-Haidar and Jaber (2013) developed a model for a cash-constrained retailer under delay payments. Tunca and Zhu (2018) discussed the role and efficiency of buyer intermediation in supplier financing through a game-theoretical model. Zhao and Huchzermeier (2019) examined advance payment discount and buyer-backed purchase order financing. Jin et al. (2019) discussed non-collaborative and collaborative financing. Yuan et al. (2020) integrated supply risk in a cash constrained problem.

Most of the literature focusing on multi-period stochastic inventory problems with cash constraints consider single-item problems. Chao et al. (2008) investigated a multi-period single-item self-financing newsvendor problem. They proved the optimal ordering pattern is an approximate base stock policy and presented a simple algorithm to solve the problem for stationary demands. Gong et al. (2014) extended this by considering short-term financing. Katehakis et al. (2016) analyzed non-stationary demand processes and time-varying interests. Boulaksil and van Wijk (2018) formulated a cash-constrained stochastic inventory model with consumer loans and supplier credits for nanostores. Bi et al. (2020) addressed short-term financing in an inventory problem of multi-item setting. Li et al. (2020) considered a two-stage inventory management problem with financing under demand updates.

Along with cash constraints, no fixed ordering cost is embedded in the aforementioned works. The existence of fixed ordering cost makes it difficult to analyze the structure of the optimal ordering policy. Chen and Zhang (2019) investigated cash flow constraints in a lot-sizing problem. However, their system operates under deterministic demand. Next, we will review some pioneering works dealing with the stochastic inventory problem with fixed ordering costs, which inspires us to develop the ordering policy in this paper.

Without a capacity constraint, Scarf (1960) proved that the optimal ordering policy for the general single-item stochastic lot-sizing problem is (s, S), where s denotes the reorder point and S is the order-up-to level. The optimality is proved through a property called K-convexity. Optimal ordering patterns have not been fully characterized for the capacitated stochastic lot-sizing problem. Shaoxiang and Lambrecht (1996) proved that with stationary demand and fixed capacity, the optimal policy features an $\mathbf{X}-\mathbf{Y}$ band structure: when the initial inventory level is below $\mathbf{X}$, it is optimal to order at full capacity; when the initial inventory level is above $\mathbf{Y}$, do not order. Gallego and Scheller-Wolf (2000) further clarified this structure with CK-convexity and divided the optimal ordering policy into four regions: in two of these regions the optimal policy is completely specified, while it is partially specified in the other two. Shaoxiang (2004) extended this analysis to the infinite horizon model. Gallego and Toktay (2004), Chao and Zipkin (2008) found the optimal policy for a capacitated problem with special supply chain contracts. Özener et al. (2014) relaxed the capacitated problem with linear holding and penalty cost, Poisson demand, and proposed a heuristic (s, Δ) policy. Shi et al. (2014) developed approximation algorithms with worst-case performance guarantee for this problem. There are also some other related works such as Chan and Song (2003) and Yang et al. (2014). While most of the above works considered capacity constraints as static and exogenous, cash constraints are in fact more complex, because they are a function of operational decisions.

The computation of optimal (s, S) policy parameters has been considered prohibitively expensive for years. A number of existing works have attempted to solve this problem under stationary demands; see e.g. Federgruen and Zipkin (1984), Zheng and Federgruen (1991), Feng and Xiao (2000). For non-stationary demands, recent progress has been made by Bollapragada and Morton (1999) and Xiang et al. (2018). Nevertheless, all these works focus on the uncapacitated situation in which there are no upper bounds on ordering quantity.

Since fixed ordering costs and nonstationary demand are the distinguishing features of this research in addition to cash constraints, we review some literature related to these topics. Besides, MILP formulations in stochastic inventory control are also surveyed, as we adopt similar methods in our paper. Under fixed ordering costs, the multi-period inventory problems is often called the lot-sizing problems. Bookbinder and Tan (1988) proposed three strategies to solve the stochastic lot-sizing problem. Tarim and Kingsman (2004) proposed an MILP approach to compute the near-optimal policy parameters under service level constraints. Özen et al. (2012) considered the problem with dynamic fixed-ordering, holding costs and dynamic penalty costs and proved that the optimal policy is a base-stock policy. Rossi et al. (2015) applied a piecewise MILP approximation of the expected total cost function to handle a range of service level measures, as well as lost sales. Absi and van den Heuvel (2019) analyzed the worst case behavior of several Relax-and-Fix heuristics, which are often used to solve the MILP formulation of the lot-sizing problems. Zangwill (1969) developed a network approach for the backlogging model and a multi-echelon model of the lot-sizing problem. Armentano et al. (1999) represented the capacitated lot-sizing problem as a minimum cost network flow problem and proposed a branch-and-bound method to solve it. Aksen et al. (2003) developed a forward recursive dynamic programming algorithm based on network flow representation of the single item lot-sizing problem with immediate lost sales. Samouei et al. (2015) made use of the network approach to model a multi-echelon spare-part inventory system with back orders and quantity discount.

A classification of the aforementioned works in the context of our research is presented in Table 1.

Table 1: Comparisons with some other related literature.

Authors	Demand pattern	Multi items	Fixed ordering cost	Multi period	Capacity constraints
Armentano et al. (1999)	Deterministic	✓	✓	✓	fixed
Aksen et al. (2003)	Deterministic		✓	✓	
Bian et al. (2018)	Deterministic		✓	✓	
Chen and Zhang (2019)	Deterministic		✓	✓	cash
Archibald et al. (2015)	Stochastic			✓	
Luo and Shang (2019)	Stochastic			✓	
Buzacott and Zhang (2004)	Stochastic				cash
Tunca and Zhu (2018)	Stochastic				cash
Scarf (1960)	Stochastic		✓	✓	
Shaoxiang and Lambrecht (1996)	Stochastic		✓	✓	fixed
Gallego and Scheller-Wolf (2000)	Stochastic		✓	✓	fixed
Shaoxiang (2004)	Stochastic		✓	✓	fixed
Chao et al. (2008)	Stochastic			✓	cash
Gong et al. (2014)	Stochastic			✓	cash
Katehakis et al. (2016)	Stochastic			✓	cash
Bi et al. (2020)	Stochastic	✓		✓	cash
Boulaksil and van Wijk (2018)	Stochastic	✓		✓	cash
This work	Stochastic		✓	✓	cash

3 Problem setting

For convenience, the notation adopted in this paper is illustrated in Table 2. Other relevant notation will be introduced as needed.

Table 2: Notations adopted in our paper.

Notations	Description
Indices	
n	Period index, the length of the planning horizon is N
Problem parameters	
R_0	Initial cash in the first period
x_0	Initial inventory in the first period
x_n	End-of-period inventory in period n
R_n	End-of-period cash in period n
p	Selling price
K	Fixed ordering cost
c	Unit variable ordering cost
γ	Unit salvage value for remnant inventory in the final period
$B(R)$	A function of a period initial cash R, representing the cash constraint
Random variables	
ξ_n	Random demand in period n, with probability density function $\phi_n(\xi)$, cumulative distribution function $\Phi_n(\xi)$
Decision variables	
Q_n	Order quantity in period n
y_n	Inventory level after ordering in period n, i.e. $y_n = x_{n-1} + Q_n$

3.1 Problem description

We consider the single-item, single-stocking location, nonstationary stochastic lot sizing problem under cash constraints. The planning horizon comprises N periods; customer demand in each period is non-negative, discrete, and independently distributed from period to period. Demand in each period n is non-stationary and it is represented by a variable ξ_n with probability density function $\phi_n(\xi)$ and cumulative distribution function $\Phi_n(\xi)$. Since we focus on a retail setting, we operate under "lost sales," and hence we assume that any unmet demand is lost. The lost sales assumption is adopted by many cash constrained inventory problems, e.g. Chao et al. (2008), Katehakis et al. (2016), Bi et al. (2020). Suppliers require immediate payment and they supply items immediately, i.e. the order delivery lead time is zero. Excess stock at the end of a period is transferred to the next period, and the selling back of excess stock is not allowed.

An overhead cost per period incurred irrespective of the activity of the firm may be easily embedded in our model by subtracting this cost from the initial capital available in each period; or equivalently, by making sure the firm has sufficient initial cash to cover operational costs over the planning horizon. Without loss of generality, to keep the model and the discussion as simple as possible, we disregard this cost component.[1]

Let x_{n-1} (resp. $R_{n-1} \geq 0$) denote the inventory (resp. cash) available at the end of period $n-1$, or equivalently, at the beginning of period n. Moreover, let the initial inventory (resp. cash) at the beginning of the planning horizon be x_0 (resp. $R_0 \geq 0$). At the beginning of period n, the retailer must decide the period's order quantity Q_n, where $Q_n \geq 0$; for convenience, we let $y_n = x_{n-1} + Q_n$, where y_n denotes the "order-up-to level." The inventory conservation constraint at the retailer can then be expressed as

$$x_n = (y_n - \xi_n)^+, \tag{1}$$

where $(x)^+$ denotes $\max\{x, 0\}$.

A fixed ordering cost K is charged to the retailer when ordering from its suppliers; there is also a variable ordering cost c charged on every ordered unit.

The inventory holding cost charged on every item unit carried from one period to the next is assumed to be negligible. Inventory holding cost is generally described as including two key components (see e.g. Zipkin, 2000): one is the direct costs associated with inventory itself (e.g physical handling cost), and the other is the financing cost related to interest rate. For cash-constrained small retailers, physical handling cost such as space rental costs, utility bills etc, are fixed per period and not proportional to inventory; hence they are captured by the aforementioned overhead cost per period, which we argued can be disregarded without loss of generality in our model. Finally, as discussed by Archibald et al. (2002), opportunity costs associated with holding extra inventory are already reflected in the lower capital available to the firm, which is one of our state variables; we therefore argue that per unit holding costs can be safely disregarded without affecting the general message of our work. For the sake of completeness, the effect of a positive inventory holding cost related with the inventory level is investigated in Appendix E.

In each period n, the initial cash is R_{n-1} and the initial inventory is x_{n-1}, the ordering quantity is $y_n - x_{n-1}$. Define $\delta(y_n - x_{n-1})$ as a unit step function to determine whether the retailer places an order in period n or not. The total ordering cost in period n is $K\delta(y_n - x_{n-1}) + c(y_n - x_{n-1})$, which includes the fixed ordering cost and the variable ordering cost. Because of the cash constraint, when the retailer purchases items, its available cash must be greater than the ordering costs in this period. This is captured by the following constraint

$$K\delta(y_n - x_{n-1}) + c(y_n - x_{n-1}) \leq R_{n-1}. \tag{2}$$

[1] Note that in this work we are not concerned with bankruptcy and we assume the level of cash available at the end of each period remains positive; for a discussion on maximising a firm's survival probability see (Archibald et al., 2002).

The selling price of an item is p, and the retailer receives payments only when its items are delivered to the customers. The sales quantity in period n is $\min\{\xi_n, y_n\}$, and revenue is thus $p\min\{\xi_n, y_n\}$. End-of-period cash R_n for period n is defined as the period's initial cash balance R_{n-1}, plus the period's revenue, minus the period's total ordering costs $K\delta(y_n - x_{n-1}) + c(y_n - x_{n-1})$. Any inventory left at the end of the planning horizon has a salvage value γ per unit. Therefore, the full expression of R_n is given by Eq. (3).

$$R_n = \begin{cases} R_{n-1} + p\min\{\xi_n, y_n\} - [K\delta(y_n - x_{n-1}) + c(y_n - x_{n-1})] & n \leq N \\ R_n + \gamma x_N & n = N + 1. \end{cases} \tag{3}$$

Note that it is reasonable to assume $0 \leq \gamma < c < p$. Our aim is to find a replenishment plan that maximizes the expected cash increment at the end of the planning horizon, i.e., $E(R_{N+1}) - R_0$.

The main difference between our problem and previous inventory literature about cash constraint in inventory problems, such as Chao et al. (2008) and Gong et al. (2014), is that we consider a fixed ordering cost K for the retailer. Another difference is that we do not account for the deposit interest of each period's cash balance because the deposit interest from the bank is generally very small compared with the retailer's transaction volume and usually does not exert an effect on its operational decisions.

3.2 Stochastic dynamic programming modeling

We next formulate our problem as a stochastic dynamic program (Bellman, 1957).

States. The system state at the beginning of period n is represented by the initial inventory x_{n-1} and the initial cash quantity R_{n-1}.

Actions. The action in period n is the order-up-to level y_n (or equivalently, the order quantity $Q_n = y_n - x_{n-1}$), given initial inventory x_{n-1} and initial cash quantity R_{n-1}. From the cash constraint (2), we use a function $B(R)$ to show the upper bound for the ordering quantity.

$$B(R) = \max\left\{0, \frac{R - K}{c}\right\}; \tag{4}$$

therefore the bounds for y_n are represented by $x_{n-1} \leq y_n \leq x_{n-1} + B(R_{n-1})$.

State transition function. The state transition function for inventory is given by inventory flow equation (1), while the state transition function for cash is presented by cash flow equation (3).

Immediate profit. For convenience, we define the concave one-period expected revenue from sales as follows,

$$L_n(y) = \mathrm{E}[p\min\{\xi_n, y\}], \tag{5}$$

where E[] denotes the expectation operator. The immediate profit for period n is the expected cash increment $\Delta R_n(x, y)$ when the initial inventory of period n is x, and the order-up-to-level is y, this is expressed as

$$\Delta R_n(x, y) = L_n(y) - \left[K\delta(y - x) + c(y - x)\right]. \tag{6}$$

Note that this expression is independent of the initial cash available at the beginning of period n.

Optimality Equation. Define $F_n(x, R)$ as the maximum expected total cash increment during periods $n, n + 1, \ldots, N$. The optimality equation for the problem is expressed as:

$$F_n(x, R) = \max_{x \le y \le x + B(R)} \Delta R_n(x, y) + \mathrm{E}[F_{n+1}((y - \xi_n)^+, \Delta R_n(x, y) + R)]; \tag{7}$$

with boundary condition $F_{N+1}(x_N, R_N) = \gamma x_N$.

4 Ordering policy

We next introduce the $(s, C(x), S)$ policy, a novel heuristic policy that operates as follows: if the initial inventory is less than s and initial cash is above $C(x)$, the inventory level should be replenished to as close to S as possible; otherwise, one should not order.

More formally, the $(s, C(x), S)$ policy takes the following form:

$$Q(x, R) = \begin{cases} \min\{B(R), S - x\} & \text{if } x < s \text{ and } R > C(x) \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

$Q(x, R)$ is the ordering quantity given an available inventory x and cash R. The values of s, $C(x)$, S are time dependent. In addition to being time dependent, the values of C are state-dependent and related to initial inventory x. For the last period N, closed form expressions of s, $C(x)$ and S can be obtained; this is discussed in Appendix A.

The structure of this policy is motivated by numerical findings presented in Section 4.1 and by some structural observations presented in Section 4.2.

4.1 Numerical findings

A small numerical example is provided below to illustrate some characteristics of the optimal policy.

Example. There are 3 periods; demand in each period $t = 1, \ldots, 3$ follow a Poisson distribution with rates $\lambda_t = \{2, 3, 8\}$; fixed ordering cost is $K = 10$, unit variable ordering cost $c = 1$, selling price $p = 8$ and unit salvage value $\gamma = 0.5$.

The optimal ordering quantities $Q^*(x, R)$ for different initial inventory x and cash R in the first period are given in Table 3; for instance, suppose the initial inventory x is 1 and the initial cash R is 20, then, the optimal ordering quantity is 10 units. The following observations stem from this numerical case:

- The optimal ordering policy is not of the (s, S) type.
- When the initial inventory is very large or initial cash is very small, the optimal ordering quantity is always zero. For example, when $x \geq 5$ or $R \leq 11$, $Q^*(x, R) = 0$. Furthermore, the ordering threshold of R is different for different initial inventory. When $x = 0$, it is optimal to not order when $R \leq 11$; when $x = 3$, it is optimal to not order when $R \leq 22$.
- When ordering, it is optimal for the retailer to order as close to some target inventory level as possible. In this case, the target inventory level is 19.
- As shown by the box in Table 3, it may be better to not order when initial cash is not large enough, although it is better to order for a smaller initial cash level. This is because sometimes the retailer would attain a greater expected profit by just selling its available inventory in the current period, and by placing a larger order in one of the next periods. Note that our proposed heuristic ordering policy does not capture this scenario, but still performs well because this ordering pattern is rare in practice.

To further clarify these properties, we introduce two new functions: $G_n(x, R)$ and $J_n(x, R)$. Let

$$G_n(y, R) \triangleq -cy + L_n(y) + \mathrm{E}[F_{n+1}((y - \xi_n)^+, R + p \min\{\xi_n, y\})]. \tag{9}$$

Rewrite

$$\begin{aligned} F_n(x, R) &= cx + \max\{G_n(x, R), \max_{x \leq y \leq B(R)} G_n(y, R - c(y - x) - K) - K\}, \\ &= cx + G_n(x, R) + \max\{0, \max_{x \leq y \leq x + B(R)} G_n(y, R - c(y - x) - K) - G_n(x, R) - K\}. \end{aligned}$$

Define

$$J_n(x, R) \triangleq \max_{x \leq y \leq x + B(R)} G_n(y, R - c(y - x) - K) - G_n(x, R) - K. \tag{10}$$

Then,

$$F_n(x, R) = cx + G_n(x, R) + \max\{0, J_n(x, R)\}. \tag{11}$$

Table 3: Optimal ordering quantity for different x and R at period 1

	$Q^*(x,R)$	R																				
	↘	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
	10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x	6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	14	15	15	15	15	15	15
	3	0	0	0	0	0	0	0	0	0	0	0	0	0	13	14	15	16	16	16	16	16
	2	0	0	0	0	0	0	0	0	0	0	0	0	12	13	14	15	16	17	17	17	17
	1	0	0	0	0	4	5	0	0	0	0	10	11	12	13	14	15	16	17	18	18	18
	0	0	0	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	19

This equation means the retailer should not order if $J_n(x,R) \leq 0$; otherwise, it should order up to a level y, where $x < y \leq x + B(R)$, which maximizes $G_n(y, R - c(y-x) - K)$:

$$y^*(x,R) = \begin{cases} x & J_n(x,R) \leq 0 \\ \arg\max_{x<y\leq x+B(R)} G_n(y, R - c(y-x) - K) & \text{otherwise.} \end{cases} \tag{12}$$

An example of how to obtain $y^*(x,R)$ by the values of $G_n(x,R)$ is given in Appendix B. A contour plot of $J_1(x,R)$ is shown in Figure 1. The retailer should order in the hatched area ($J_1(x,R) > 0$) and not order in other area ($J_1(x,R) \leq 0$). When initial inventory x is larger than s, it is better to not order. If the retailer's initial inventory x_0 is less than s, but its initial cash R is smaller than the threshold $C(x_0)$, it is better not to order either.

4.2 Structural properties

The numerical findings in the previous section lead to the following results.

Lemma 1 *For any period n and fixed x, $F_n(x,R)$ and $G_n(x,R)$ are both non-decreasing in R.*

Proof. We provide a proof by induction for $F_n(x,R)$; the proof for $G_n(x,R)$ is constructed in a similar way.

When $n = N+1$, $F_{N+1}(x,R) = \gamma x$, hence it is non-decreasing in R. Now, suppose this property holds for period $n+1$; we prove the results for period n.

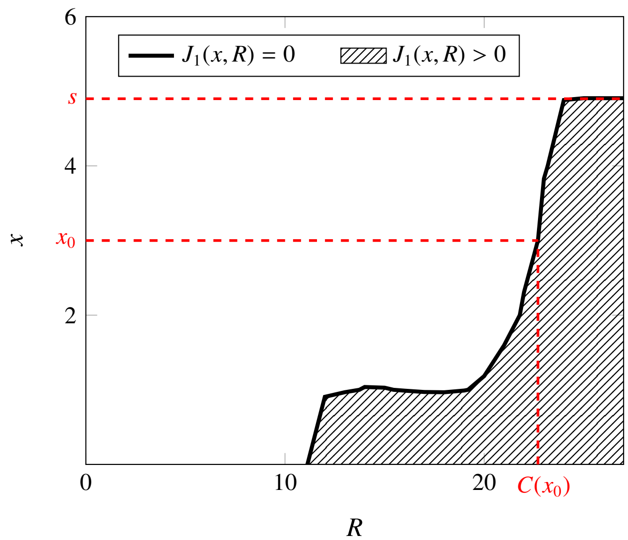

Figure 1: s and $C(x)$ in a contour plot of $J_1(x, R) = 0$

Consider two cash levels, R_1 and R_2, such that $R_1 < R_2$. Let the initial inventory be x, the domain of the order-up-to-level y when the initial cash level is R_1 be $[x, y_1]$, and the domain of the order-up-to-level y when the initial cash level is R_2 be $[x, y_2]$. Let the optimal order-up-to-level for initial cash R_1 be $y_1^* \in [x, y_1] \subseteq [x, y_2]$. Recall that the expression in Eq. (6) is independent of the initial cash level R. Since $R_1 < R_2$, in Eq. (7), $F_{n+1}((y_1^* - \xi)^+, R_2 + \Delta R(x, y_1^*)) \geq F_{n+1}((y_1^* - \xi)^+, R_1 + \Delta R(x, y_1^*))$; therefore $F_n(x, R_2) \geq F_n(x, R_1)$. $\square$

Note that when R is sufficiently large, there is no ordering quantity constraint, and the problem becomes similar to the traditional lot-sizing problem in Scarf (1960). In this situation, for fixed x, $F_n(x, R)$ and $G_n(x, R)$ are constant and there exists an inventory level s below which we always order.

Since there is no inventory holding cost in our problem, if the retailer has enough cash, a single order can be issued in period n to meet all subsequent demand and save fixed ordering costs. Therefore, to compute S, we leverage a newsvendor problem approximation over periods $\{n, \ldots, N\} \triangleq n \sim N$. The expected profit is

$$L'_{n\sim N}(x) = \int [p \min\{\xi, y\} - cx + \gamma(x - \xi)^+] \phi_{n\sim N}(\xi) d\xi,$$

where $\phi_{n\sim N}$ is the probability density function of the total demand over periods $n \sim N$. $L'_{n\sim N}(x)$ is a concave function, which reaches its maximum value at

$$x = \Phi^{-1}_{n\sim N}\left(\frac{p - c}{p - \gamma}\right). \tag{13}$$

We employ Eq. (13) to determine an order-up-to level S for each period n. If the retailer's available cash is not sufficient to order up to S, the retailed should order as close to S as possible. Note that when there is inventory holding cost, we adopt another heuristic step to compute S; this is described in Appendix E.

A second heuristic step is to approximate the boundary of the contour plot at $J(x, R) = 0$ by means of cutting off the bulge in the plot (the red dashed line in Figure 2). This approximation ensures uniqueness of cash threshold $C(x)$, when $x < s$. The resulting boundary for the numerical example in Section 4.1 after this heuristic step is shown in Table 4. Note that the process of finding $C(x)$ in the SDP method of Section 5.1 reflects this heuristic approach.

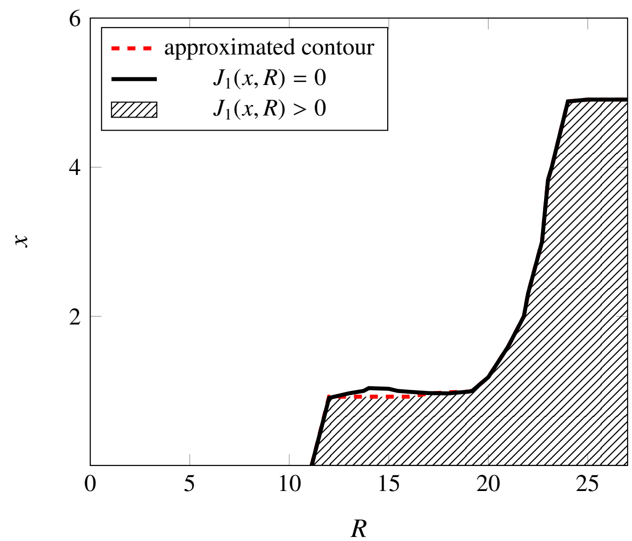

Figure 2: Approximation of the contour plot of $J(x, R)$

5 Computation for s, $C(x)$ and S

To compute values of s, $C(x)$ and S, we provide two methods here: the first leverages the results of stochastic dynamic programming (SDP); the second approximates the problem via mixed-integer linear programming.

Table 4: Ordering quantity after heuristic handling for $C(x)$

$Q^*(x,R)$	R																				
	↘	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
	10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x	6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	14	15	15	15	15	15	15
	3	0	0	0	0	0	0	0	0	0	0	0	0	0	13	14	15	16	16	16	16	16
	2	0	0	0	0	0	0	0	0	0	0	0	0	12	13	14	15	16	17	17	17	17
	1	0	0	0	0	0	0	0	0	0	0	10	11	12	13	14	15	16	17	18	18	18
	0	0	0	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	19

5.1 An SDP approximation of s, $C(x)$, and S

Let $Q^*_n(x, R)$ represent the optimal ordering quantity for initial inventory x and initial cash R at period n in the SDP results[2]. The default values for s and S are zero, the default values for C are fixed ordering cost K.

For period $n = 1$, S is computed by Eq. (13); if $Q^*_1(x, R) > 0$, $s = x + 1$, and $C(x) = 0$; if $Q^*_1(x, R) = 0$, $s = x$, and $C(x)$ can be fixed arbitrarily. When $n = N$, the values of s, S and $C(x)$ can be obtained by Eq. (44), (42), (45), respectively. For $1 < n < N$, the method for obtaining s, S, C is given below.

Find s. s is the minimum initial inventory level that results in not ordering in the SDP results.

Find S. In a given period, when $x < s$, S is computed by Eq. (13).

Find $C(x)$. For initial inventory x in a given period, and $x < s$, $C(x)$ is the maximum value of initial cash that results in not ordering in the SDP results.

The pseudo-code of this method is given in Algorithm 1 of Appendix C. Since this method is based on the results of SDP, it enumerates all the states in each period. This is very time-consuming when the planning horizon is large. For example, in our numerical tests, if the length of the planning horizon is 15, it may take more than one hour to compute one instance. So this method is only suitable for short planning horizons.

[2]Note that since we assume the cash state is always positive in the problem, a very large penalty cost is assigned to the model if encountering negative cash state in the results of SDP.

5.2 A mixed-integer linear programming approximation of s, $C(x)$, and S

This approach here presented is similar to the "static-dynamic uncertainty" strategy in (Bookbinder and Tan, 1988). Replenishment periods are first determined by leveraging an MILP model; values of s, S and $C(x)$ for each period are then computed by approximating a newsvendor problem for each replenishment cycle.

We next formulate our problem as a stochastic program, in which z_n is a binary variable representing whether an order in period n is placed or not.

$$\max \quad \mathrm{E}(R_N + \gamma x_N^+ - R_0) \tag{14}$$

$$\text{s.t.} \quad n = 1, 2, \ldots, N$$

$$x_n = x_{n-1}^+ + Q_t - \xi_n \tag{15}$$

$$x_n^+ = \max\{x_{n-1}^+ + Q_n - \xi_n, 0\} \tag{16}$$

$$x_n = x_n^+ - x_n^- \tag{17}$$

$$R_n = R_{n-1} + p\min\{x_{n-1}^+ + Q_n, \xi_n\} - cQ_n - Kz_n \tag{18}$$

$$cQ_n + Kz_n \leq R_{n-1} \tag{19}$$

$$z_n = \begin{cases} 1 & Q_n > 0 \\ 0 & Q_n = 0 \end{cases} \tag{20}$$

$$R_n, x_n^+, x_n^- \geq 0 \tag{21}$$

Objective function (14) maximizes the expected cash increment in the planning horizon. Constraint (15) is the inventory flow equation. Constraint (16) makes sure that the unmet demand is lost. Constraint (17) shows the relationship between x_n, x_n^+ and x_n^-. Constraint (18) is the cash flow function. Constraint (19) is the cash constraint of the problem. Constraint (20) sets z_n to one if ordering items in this period and sets z_n to zero if not ordering. Constraint (21) ensures the non-negativity of R_n, x_n^+ and x_n^-.

Next, we reformulate the model by replacing Q_n with the order-up-to level S_n, defined as $S_t = I_{t-1}^+ + Q_t$.

$$\begin{aligned}
\max \quad & \mathrm{E}(R_N + \gamma x_N^+ - R_0) && (14)\\
\text{s.t.} \quad & n = 1, 2, \ldots, N && \\
& x_n = S_n - \xi_n && (22)\\
& x_n^+ = \max\{S_n - \xi_n, 0\} && (23)\\
& x_n = x_n^+ - x_n^- && (24)\\
& R_n = R_{n-1} + p\min\{S_n, \xi_n\} - c(S_n - x_{n-1}^+) - Kz_n && (25)\\
& c(S_n - x_{n-1}^+) + Kz_n \leq R_{n-1} && (26)\\
& z_n = \begin{cases} 1 & \text{if } S_n > x_{n-1}^+ \\ 0 & \text{otherwise} \end{cases} && (27)\\
& R_n, S_n, x_n^+, x_n^- \geq 0 && (28)
\end{aligned}$$

By following Bookbinder and Tan's "static-dynamic uncertainty" strategy — which establishes the ordering periods and the associated order-up-to levels at the beginning of the planning horizon — and thus by fixing z_n and S_n at the beginning of the planning horizon, we now transform the above models into an equivalent mixed integer linear programming model that can be solved using an off-the-shelf solver.

After taking expectations, ξ_n, x_n, x_n^+, x_n^- R_n are represented by $\tilde{d}_n$, $\tilde{x}_n$, $\tilde{x}_n{}^+$, $\tilde{x}_n{}^-$, $\tilde{R}_n$, respectively (z_n and S_n are fixed by the assumption of "static-dynamic uncertainty" strategy). Objective function (14) translates into (29); constraint (22) is transformed into (30); constraint (23) is transformed into (31); constraint (24) is transformed into (32); constraint (25) is transformed into (33), in which $S_n - \tilde{x}_n^+ = S_n - \max\{S_n - \tilde{d}_n, 0\} = \min\{S_n, \tilde{d}_n\}$; constraint (26) is transformed into (34); constraint (27) is transformed into (35); constraint (28) is transformed into (36).

$$
\begin{aligned}
\max \quad & \tilde{R}_N + \gamma \tilde{x_N}^+ - R_0 && (29)\\
\text{s.t.} \quad & n = 1, 2, \ldots, N \\
& \tilde{x}_n = S_n - \tilde{d}_n && (30)\\
& \tilde{x}_n^+ = \max\{S_n - \tilde{d}_n, 0\} && (31)\\
& \tilde{x}_n = \tilde{x}_n^+ - \tilde{x}_n^- && (32)\\
& \tilde{R}_n = \tilde{R}_{n-1} + p(S_n - \tilde{x}_n^+) - K z_n - c(S_n - \tilde{x}_{n-1}^+), && (33)\\
& K z_n + c(S_n - x_n^+) \le \tilde{R}_{n-1}, && (34)\\
& z_n = \begin{cases} 1 & \text{if } S_n > \tilde{x}_{n-1}^+ \\ 0 & \text{otherwise} \end{cases} && (35)\\
& \tilde{R}_n, S_n, \tilde{x}_n^+, \tilde{x}_n^- \ge 0. && (36)
\end{aligned}
$$

Note that in the above MILP model, the non-linear first-order loss function in Eq. (31) can be piecewise linearized by leveraging the approach discussed in (Rossi et al., 2015). The key purpose of this MILP model is to obtain a heuristic ordering plan by determining the values of binary variables z_n.

Once values of binary variables z_n have been obtained, we must compute values of s, S and $C(x)$ for each period. For the last period in the planning horizon, S, s and C are obtained as shown in Appendix A, i.e. Eq. (42), Eq. (44) and Eq. (45). For $n < N$, we compute them as follows.

Find S. Similar to the SDP method, S_n is computed by Eq. (13)

Find s. To compute s_n, we proceed as follows. Ordering cycles can be obtained from the values of z_n in the MILP model. For example, if $z = \{1, 0, 0, 1, 0\}$, there are two ordering cycles in this ordering plan: period 1∼3 and period 4∼5, as shown in Figure 3. Let the first period of an ordering cycle be n_1, and the last

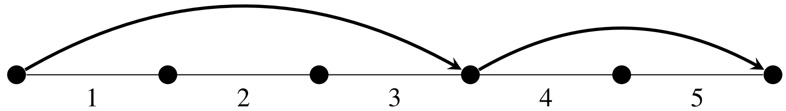

Figure 3: A sketch of two consecutive ordering cycles

period of an ordering cycle be n_2 ($n_1 < n_2 \le N$); moreover, let $\phi_{n_1\sim n_2}$ and $\Phi_{n_1\sim n_2}$ be the probability density function and the cumulative distribution function of $d_{n_1}+\ldots+d_{n_2}$, respectively. In a replenishment cycle, the expected cash increment without fixed ordering cost from period i ($n_1 \le i \le n_2$) to n_2 can be approximated via a newsvendor model, that is:

$$L_{i\sim n_2}(y) = \begin{cases} \int \left[p\min\{\xi, y\} - cy\right]\phi_{i\sim n_2}(\xi)d\xi & n_2 < N \\ \int \left[p\min\{\xi, y\} - cy + \gamma(y-\xi)^+\right]\phi_{i\sim n_2}(\xi)d\xi & n_2 = N. \end{cases} \tag{37}$$

Since $L_{i\sim n_2}(y)$ is concave, it reaches its maximum at $y^*_{i\sim n_2} \triangleq \Phi^{-1}_{i\sim n_2}\left(\frac{p-c}{p}\right)$, where $n_2 < N$; or $y^*_{i\sim n_2} \triangleq \Phi^{-1}_{i\sim n_2}\left(\frac{p-c}{p-\gamma}\right)$, where $n_2 = N$. For $n_1 \le i \le n_2$, let

$$S'_i \triangleq \min\{\tilde{x}_{i-1} + B(\tilde{R}_{i-1}), y^*_{i\sim n_2}\}, \tag{38}$$

where $\tilde{x}_{i-1}$ and $\tilde{R}_{i-1}$ are obtained from the MILP model, and $\tilde{x}_{i-1} + B(\tilde{R}_{i-1})$ is the approximate maximum order-up-to level for period i. The relation of S'_i, y^* and $L_{i\sim n_2}$ for a numerical instance is shown in Figure 4, where the red line represents $L_{i\sim n_2}(y)$; and in this case $S'_i = \tilde{x}_{i-1} + B(\tilde{R}_{i-1})$.

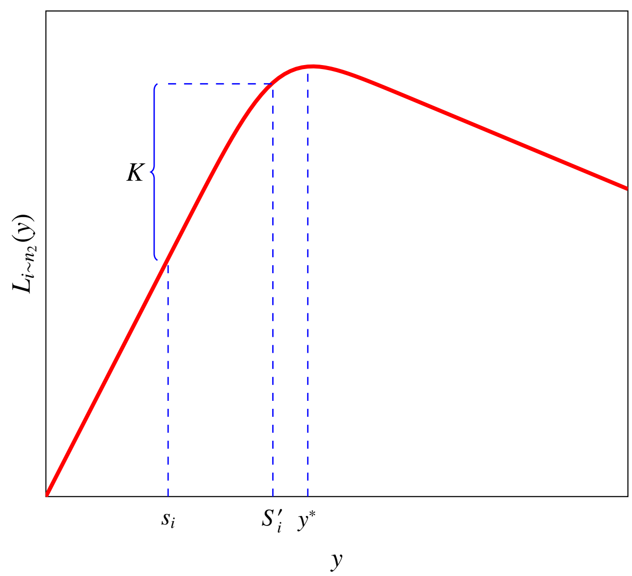

Figure 4: s_i, S'_i, y^* and $L_{i\sim n_2}$

In an ordering cycle $n_1 \sim n_2$, we use the following heuristic step to compute s_i,

$$s_i \triangleq \min\{y \mid L_{i\sim n_2}(y) \ge L_{i\sim n_2}(S'_i) - K, 0 \le y \le S'_i\}, \tag{39}$$

where $n_1 \le i \le n_2$. The idea behind Eq. (39) is that s_i is the smallest y such that the gap between $L_{i\sim n_2}(y)$ and $L_{i\sim n_2}(S_i')$ is larger than the fixed ordering cost K. This is also shown in Figure 4.

Find $C(x)$. Consider period i, where $n_1 \le i \le n_2$; when initial inventory $x < s_i$, $C_i(x)$ is computed by the following heuristic equation:

$$C_i(x) \triangleq \begin{cases} \max\{R \mid L_i(x) \le L_i(x + B(R)) - K, 0 \le x \le S\} & L_i(S) \ge K \\ M & \text{otherwise.} \end{cases} \tag{40}$$

Note that $C_i(x)$ is related to initial inventory x. In Eq. (40), $L_i(\cdot)$ is the one-period expected cash increment for period i, and S is its maximum point, which is $\Phi_i^{-}\left(\frac{p-c}{p}\right)$. The relationship between $C_i(x)$ and x is shown in Figure 5. Eq. (40) suggests that, if the one-period maximum cash increment is larger than the fixed ordering cost, $C_i(x)$ is the maximum cash such that the one-period expected cash increment minus the full capacity ordering costs is larger than the cost of not ordering, otherwise the retailer should not order and $C_i(x) = M$.

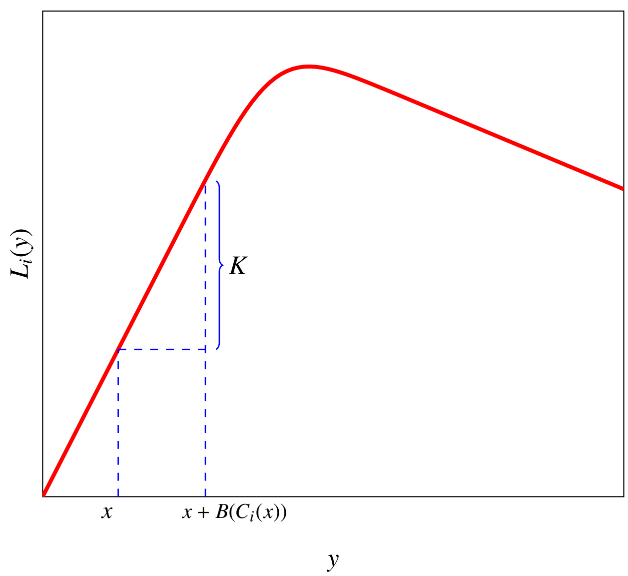

Figure 5: $C_i(x)$

The pseudo-code for this method is shown in Algorithm 2 of Appendix C. This method is considerably faster than our previous SDP-based heuristic. For numerical instances over 20 periods, this MILP heuristic can obtain values of s, $C(x)$ and S in seconds.

Example. To further illustrate the policy and our computational methods, we present a numerical example: there are 4 periods, and the demands in each period follow a Poisson distribution. The mean demand in each period is [20, 12, 10, 14]. Other relevant parameter values are fixed ordering cost $K = 24$, unit variable ordering cost $c = 1$, selling price $p = 4$, unit salvage value $\gamma = 0$, initial inventory $x = 0$, and initial cash $R = 33$. The results of different methods are given in Table 5. As Table 5 shows, the SDP method is near-optimal, while the MILP method has an optimality gap 4.52%.

Table 5: An illustrative example of computing s, $C(x)$, and S

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	Optimality gap
SDP method	$s = 1$	$s = 1$	$s = 4$	$s = 5$	0.00%
	$C(x = 0) = 0$	$C(x = 0) = 32$	$C(x = 0) = 32$	$C(x = 0) = 32$	
			$C(x = 1) = 34$	$C(x = 1) = 32$	
			$C(x = 2) = 36$	$C(x = 2) = 32$	
			$C(x = 3) = 35$	$C(x = 3) = 32$	
				$C(x = 4) = 33$	
	$S = 61$	$S = 40$	$S = 27$	$S = 16$	
MIP method	$s = 1$	$s = 3$	$s = 14$	$s = 5$	4.52%
	$C(x = 0) = 32$	$C(x = 0) = 32$	$C(x = 0) = 33$	$C(x = 0) = 32$	
		$C(x = 1) = 32$	$C(x = 1) = M$	$C(x = 1) = 32$	
		$C(x = 2) = 33$	$C(x = 2) = M$	$C(x = 2) = 32$	
			$\vdots$	$C(x = 3) = 32$	
			$C(x = 13) = M$	$C(x = 4) = 33$	
	$S = 61$	$S = 40$	$S = 27$	$S = 16$	

6 Computational study

In this section, we present an extensive computational study on large test beds to investigate the effectiveness of the ordering policy and computational methods we proposed.

6.1 Test beds

The test beds are adopted from Rossi et al. (2015) with some modifications. There are 10 demand patterns for numerical analysis: 2 life cycle patterns (LCY1 and LCY2), 2 sinusoidal patterns (SIN1 and SIN2), 1 stationary pattern (STA), 1 random pattern (RAND), and 4 empirical patterns (EMP1, EMP2, EMP3, EMP4). To ensure that SDP can solve these instances in reasonable time, we rescale the original demand values, select 10 successive periods for testing, and make cash and inventory be integers. The demand patterns are shown in Figure 6. Expected demands for different patterns are given in Table 8 in Appendix D.

Fixed ordering cost K takes values in (10, 15, 20); unit variable ordering cost c is 1, and unit salvage value $\gamma = 0.5$; price p takes values in (5, 6, 7), which results in product margin taking values in (4, 5, 6); and initial cash capacity takes values in (3, 5, 7), which means initial cash B_0 is sufficient to order 3, 5 or 7 items. For example, if initial cash capacity is 5, initial cash $B_0 = K + 5c$. The initial cash capacity settings guarantee that the retailer does not have enough cash in the first period, thus ensuring these instances do not represent uncapacitated cases. Demands follow a Poisson distribution and are independent in each period. Therefore, there are 270 numerical cases in total for each ordering policy.

The computational studies are coded in Java and run on a desktop computer with an Intel (R) Core (TM) i5-7500 CPU, at 3.40 GHz, 8 GB of RAM, and 64-bit Windows 7 operating system.

6.2 Results analysis

We compare the SDP and MILP heuristics with the optimal policy obtained by stochastic dynamic programming (SDP). After the values of s, $C(x)$ and S of the different methods are obtained, we simulate 100,000 runs to obtain the expected final values and compare the gaps with respect to SDP. The average optimality gap (AVG) and maximum optimal gap (MAX) are adopted as performance metrics.

As shown in Table 6, the $(s, C(x), S)$ policy performs very well in terms of both the maximum gap (0.95%) and the average gap (0.03%). Our MILP heuristic has a maximum gap of approximately 13.05% and an average gap of approximately 0.74%. Among all 270 numerical cases in the test bed, there are no instances for which the SDP heuristic features an optimality gap greater than 1%. For the MILP method, 43 numerical cases have a gap exceeding 1%.

When fixed ordering cost, product margin, or cash capacity increase, the numerical tests show that the performance of both SDP method and MILP method do not change much. However, demand patterns may have a large effect on the performance of the MILP methods. The maximum gaps and average gaps all grow large in demand pattern SIN1, SIN2, EMP2 for the MILP method. For example, a numerical case with maximum gap 13.05% falls in the SIN2 demand pattern, which features large fluctuation with mean demand 15.7 in the first period and mean demand 2 in the forth period. In this case, the large optimality gap originates from the heuristic step in Eq. (40) to compute the values of $C(x)$. In the MILP method, $C(x)$ is set to a large number (M) for period 4, which means the retailer should not order in this period. However, in an optimal policy, the retailer should order for some states in period 4. Our results show that demand patterns

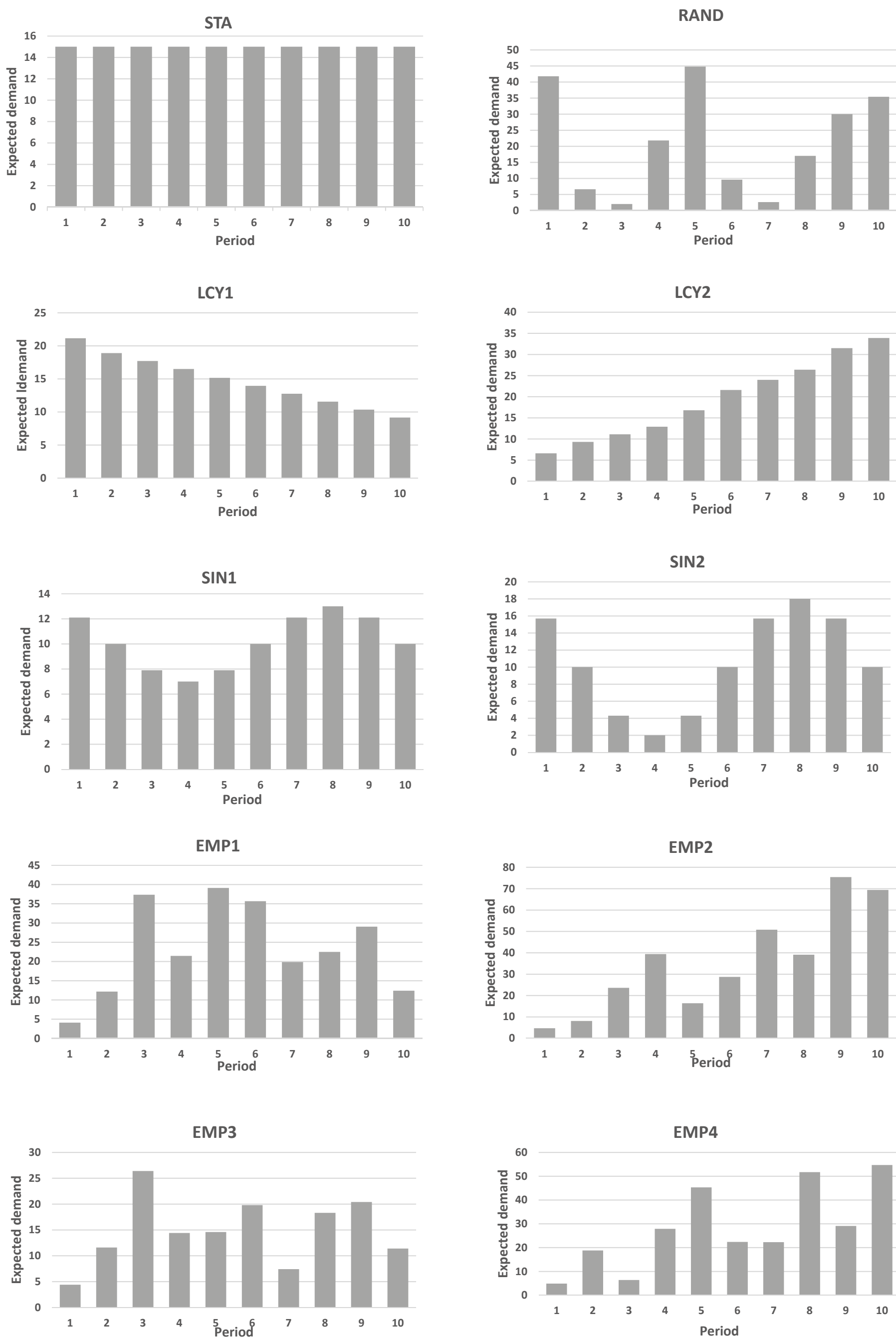

Figure 6: Demand patterns in our computational analysis

Table 6: Computational study results of different computation methods.

	$(s, C(x), S)$ by SDP		$(s, C(x), S)$ by MILP		Cases
	MAX	AVG	MAX	AVG	
Fixed cost					
10	0.19%	0.03%	6.13%	0.75%	90
15	0.62%	0.03%	12.13%	0.73%	90
20	0.95%	0.02%	13.05%	0.73%	90
Margin					
4	0.58%	0.03%	12.13%	0.71%	90
5	0.63%	0.02%	13.05%	0.69%	90
6	0.95%	0.03%	6.06%	0.80%	90
Cash capacity					
3	0.19%	0.01%	5.81%	0.29%	90
5	0.96%	0.04%	13.05%	1.05%	90
7	0.58%	0.03%	7.58%	0.87%	90
Demand pattern					
STA	0.08%	0.02%	1.33%	0.35%	27
LC1	0.10%	0.02%	1.59%	0.39%	27
LC2	0.10%	0.02%	1.22%	0.48%	27
SIN1	0.58%	0.07%	3.98%	0.93%	27
SIN2	0.96%	0.10%	13.05%	1.98%	27
RAND	0.05%	0.01%	1.03%	0.30%	27
EMP1	0.06%	0.01%	0.56%	0.31%	27
EMP2	0.00%	0.00%	6.13%	2.00%	27
EMP3	0.08%	0.02%	1.29%	0.34%	27
EMP4	0.01%	0.00%	0.54%	0.27%	27
Gap over 1%	0 case		43 cases		
Avg Time	220 s		0.03 s		
General	0.95%	0.03%	13.05%	0.74%	270

SIN1 and SIN2 also create difficulties for the SDP heuristic. Conversely, for the stationary demand pattern (STA), both methods show good performance with very low maximum gap and average gap.

In terms of computational efficiency, the average running time for the SDP heuristic exceeds 220 seconds. Conversely, the MILP heuristic is very fast, with an average running time of less than 1 second.

Finally, we conducted some numerical tests for the $(s, C(x), S)$ policy when a positive inventory holding cost is considered. The heuristic techniques to compute S are slightly modified in this situation. Details of the results and descriptions about the new heuristic steps are shown in Appendix E. The policy still shows good performance under a positive inventory holding cost.

7 Conclusions

Cash flow management is a key concern for many small businesses. In this paper, we considered the cash-constrained stochastic lot sizing problem under fixed ordering cost and nonstationary demand. We investigated numerical as well as structural properties of the optimal policy of this problem. Inspired by this analysis, we introduced a heuristic $(s, C(x), S)$ ordering policy for this problem. Our computational study demonstrates that the $(s, C(x), S)$ policy features very small optimality gaps. To efficiently compute near-optimal policy parameters, we developed an MILP-based heuristic method. The method computes values of s, $C(x)$ and S for real-sized instances in negligible time. Our computational study shows that this method features narrow average optimality gaps (0.74%). Future research may consider several directions: one possible direction is to further characterize the structure of the optimal ordering policy for this problem; a second possible direction is to investigate cash constraints in a multi-item setting.

Acknowledgments

The research presented in this paper is supported by Chongqing Social Science Planning Program under Project 2020BS49, the Fundamental Research Funds for the Central Universities of China under Project SWU1909738 and the Research Funds of the Research Institute of Intelligent Finance and Platform Economics, Southwest University under Project 20YJ0105.

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Appendix A Computing s, $C(x)$ and S for the last period

When $n = N$,

$$F_N(x, R) = \max_{x \le y \le x+B(R)} \left\{ L(y) - c(y-x) - K\delta(y-x) + \gamma(y-\xi)^+ \right\}. \tag{41}$$

Recall that $L(y) = p \min\{\xi, y\}$. Let $L'(y) = L(y) - c(y-x) + \gamma(y-\xi)^+$. We draw $L'(y)$ of a numerical example in Figure 7. Since $L'(y)$ is a concave function, it reaches its max at:

$$S = \Phi_N^{-1}\left(\frac{p-c}{p-\gamma}\right). \tag{42}$$

By Eq. (41), the optimal decision for period N is

$$y_N^*(x, R) = \begin{cases} y^* & \text{if } L'(x) < \max_{x<y\le B(R)} L'(y) - K \\ x & \text{if } L'(x) \ge \max_{x<y\le B(R)} L'(y) - K. \end{cases} \tag{43}$$

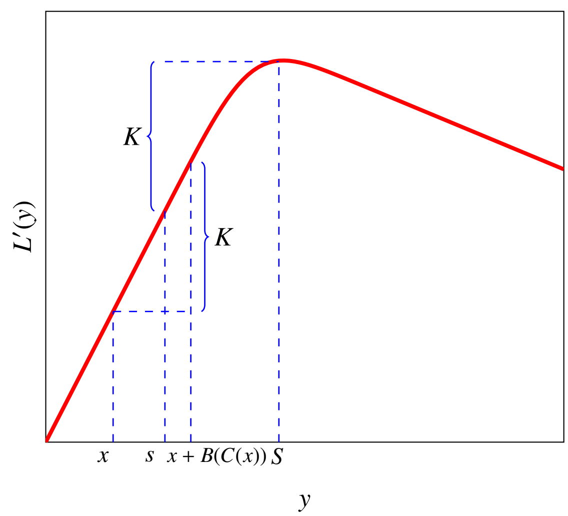

Figure 7: s, $C(x)$ and S in the last period N

By the concavity of $L'(y)$, we can obtain an **s** threshold for period N (Figure 7) as follows

$$s \triangleq \min\{y \mid L'(y) \ge L'(S) - K\}. \tag{44}$$

When $x < s$, there exists a **C** threshold. When $R \le C(x)$, it is optimal for the retailer to not order because the expected profit cannot cover the ordering costs. The **C** threshold given by Eq. (45) is the cash level for which the retailer's expected profit of not ordering plus fixed ordering cost exceeds its expected profit of ordering full capacity. Recall that $B(R)$ means the maximum ordering quantity under initial cash R, then

$$C(x) \triangleq \max\{R \mid L'(x) + K \geq L'(x + B(R))\}. \tag{45}$$

For an initial inventory level $x < s$, its minimum cash requirement for ordering is $C(x)$. Its maximum order-up-to level under initial cash $C(x)$ is $x + B(C(x))$. The relation of s, S, $C(x)$ and $L'(y)$ is shown in Figure 7. Since $L'(y)$ is increasing when $y < S$, it is optimal for the retailer to replenish its inventory level as close to S as possible when $x < s < S$ and $R > C(x)$.

Appendix B Computing $y^*(x, R)$ from the values of $G_n(x, R)$

Table 7: $G_1(x, R)$ for the numerical example in Table 3

$x \mid R$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	28.8	43.5	48.0	51.1	53.0	54.4	55.3	56.1	57.6
1	7.0	7.0	7.0	7.0	35.8	50.5	55.0	58.1	60.0	61.4	62.3	63.1	64.6	66.5	68.3	69.9	70.9	71.5	71.7	71.7	71.7
2	59.6	62.0	63.7	65.1	66.7	68.2	69.9	71.6	72.7	73.4	73.8	74.3	74.9	75.5	76.0	76.3	76.5	76.5	76.5	76.5	76.5
3	72.4	73.1	73.6	74.1	74.7	75.3	75.8	76.2	76.3	76.5	76.9	77.5	78.1	78.6	78.9	79.1	79.2	79.2	79.2	79.2	79.2
4	75.8	76.0	76.6	77.2	77.8	78.3	78.6	78.8	78.9	78.9	78.9	79.2	79.7	80.1	80.2	80.3	80.3	80.3	80.3	80.3	80.3
5	78.6	78.7	78.7	79.1	79.5	79.9	80.1	80.1	80.1	80.1	80.1	80.2	80.5	80.7	80.7	80.7	80.7	80.7	80.7	80.7	80.7
6	80.0	80.1	80.1	80.1	80.4	80.6	80.7	80.7	80.7	80.7	80.7	80.7	80.8	80.9	80.9	80.9	80.9	80.9	80.9	80.9	80.9
7	80.7	80.7	80.7	80.7	80.9	80.9	80.9	80.9	80.9	80.9	80.9	80.9	80.9	80.9	80.9	80.9	80.9	80.9	80.9	80.9	80.9
8	81.0	81.0	81.0	81.0	81.0	81.0	81.0	81.0	81.0	81.0	81.0	81.0	81.0	81.0	81.0	81.0	81.0	81.0	81.0	81.0	81.0
9	81.2	81.2	81.2	81.2	81.2	81.2	81.2	81.2	81.2	81.2	81.2	81.2	81.2	81.2	81.2	81.2	81.2	81.2	81.2	81.2	81.2
10	81.6	81.6	81.6	81.6	81.6	81.6	81.6	81.6	81.6	81.6	81.6	81.6	81.6	81.6	81.6	81.6	81.6	81.6	81.6	81.6	81.6
11	82.4	82.4	82.4	82.4	82.4	82.4	82.4	82.4	82.4	82.4	82.4	82.4	82.4	82.4	82.4	82.4	82.4	82.4	82.4	82.4	82.4

Consider Table 7, to illustrate the computation of the optimal order-up-to-level we arbitrarily choose three pairs (x, R). For a given (x, R) pair, the optimal order-up-to-level $y^*(x, R)$ can be obtained via Eq. (12):

- when $x = 1$, $R = 15$, $G_1(x, R) = 69.9$. Order-up-to levels y are: 1, 2, 3, 4, 5, 6 and the corresponding values of $G_1(x, R)$, $G_1(y, R - K - c(y - x))$ are highlighted in light-blue: 69.9, 66.7, 74.1, 76.6, 78.7, 80.0. By Eq. (12), $\max\{69.9,\ 66.7 - 10,\ 74.1 - 10,\ 76.6 - 10,\ 78.7 - 10,\ 80.0 - 10\} = 70$, and the optimal order-up-to-level for $x = 1$, $R = 15$ is 6;

- when $x = 1$, $R = 17$, the values of $G_1(x, R)$ and $G_1(y, R - K - c(y - x))$ are the data filled with light-green color. By Eq. (12), the optimal order-up-to level is 1;

- when $x = 1$, $R = 20$, $G_1(x, R)$ and $G_1(y, R - K - c(y - x))$ are highlighted in light-red. The optimal order-up-to level is 11.

Appendix C Algorithms to compute s, $C(x)$ and S

The following algorithm can be used to obtain s, $C(x)$, S for each period from the results of stochastic dynamic programming (SDP). For the last period, s, $C(x)$, S can be computed in closed form; for other periods, s is the minimum initial inventory level that results in not ordering, $C(x)$ is the maximum initial cash level that results in not ordering when $x < s$ and S is computed by Eq. (13).

Algorithm 1: Obtaining s, C, S via SDP

Data: Optimal ordering quantity $Q_n^*(x, R)$ for each state x, R in each period n from SDP, total states for period n is represented by set Ω_n

Result: Values of $s, C(x), S$ in each period

Initialize: $s \leftarrow 0$, S is computed by Eq. (13), $C(x) \leftarrow K$.;
for $n = 1$ **to** N **do**
 if $n = 1$ **then**
  if $Q_1^*(x, R) > 0$ **then**
   $s \leftarrow x + 1$;
   $C(x) \leftarrow 0$;
  else
   $s \leftarrow x$;
   $C(x)$ can be fixed arbitrarily;
  end
 else if $n = N$ **then**
  s is computed by Eq. (44), S is computed by Eq. (42), $C(x)$ is computed by Eq. (45);
 else
  $s \leftarrow \min\{x \mid Q_n^*(x, R) = 0,\ (x, R) \in \Omega_n\}$;
  $C(x) \leftarrow \max\{R \mid Q_n^*(x, R) = 0, x < s\}$;
 end
end

The following algorithm can be used to compute s, $C(x)$, S for each period by leveraging our MILP model and some heuristic steps. It begins bby generating a heuristic ordering plan from the MILP model, and then it computes s, $C(x)$, S for each ordering cycle obtained.

Algorithm 2: Obtaining s, C, S by MILP approximation

Data: parameter values of the problem

Result: Values of s, $C(x)$, S

```
Initialize: s ← 0,  S is computed by Eq. (13),  C(x) ← K;
Compute the MILP model and get a heuristic ordering plan represented by values of z_n;
foreach ordering cycle in the ordering plan do
    foreach period in the ordering cycle do
        compute s by Eq. (39);
        compute C(x) by Eq. (40);
    end
end
```

Appendix D Demand patterns in the test bed

Table 8: Expected demand values for different demand patterns.

Demand pattern	Expected demand values									
STA	15	15	15	15	15	15	15	15	15	15
LC1	21.15	18.9	17.7	16.5	15.15	13.95	12.75	11.55	10.35	9.15
LC2	6.6	9.3	11.1	12.9	16.8	21.6	24	26.4	31.5	33.9
SIN1	12.1	10	7.9	7	7.9	10	12.1	13	12.1	10
SIN2	15.7	10	4.3	2	4.3	10	15.7	18	15.7	10
RAND	41.8	6.6	2	21.8	44.8	9.6	2.6	17	30	35.4
EMP1	4.08	12.16	37.36	21.44	39.12	35.68	19.84	22.48	29.04	12.4
EMP2	4.7	8.1	23.6	39.4	16.4	28.7	50.8	39.1	75.4	69.4
EMP3	4.4	11.6	26.4	14.4	14.6	19.8	7.4	18.3	20.4	11.4
EMP4	4.9	18.8	6.4	27.9	45.3	22.4	22.3	51.7	29.1	54.7

Appendix E Numerical tests under a positive holding cost

When including inventory holding cost in the problem, we make the following modifications to the SDP method and MILP method about computing S.

Find S in SDP method. When $x < s$, there are different states (x, R), and each state (x, R) is associated with an order-up-to level. Since there may be more than one order-up-to level, and the retailer orders as close to S as possible in policy $(s, C(x), S)$, we adopt a new heuristic step and select the most frequent order-up-to level for all the states in this period as S.

Find S in MILP method. Compute S by Eq. (38).

In the numerical tests, unit holding cost takes values in {0.5, 1} and there are 540 cases in total. In Table 9, the maximum gap for the SDP heuristic is 0.77%, while it is 6.40% for the MILP heuristic. The average gap for the SDP heuristic is 0.02%, while it is 0.92% for the MILP heuristic. This demonstrates the robustness of the $(s, C(x), S)$ policy when a positive holding cost is considered.

Table 9: Computational study results with holding cost.

	$(s, C(x), S)$ by SDP		$(s, C(x), S)$ by MILP		Cases
	MAX	AVG	MAX	AVG	
Fixed cost					
10	0.77%	0.02%	3.49%	0.79%	180
15	0.75%	0.03%	5.72%	0.94%	180
20	0.52%	0.03%	6.40%	1.02%	180
Margin					
4	0.77%	0.03%	6.40%	1.11%	180
5	0.47%	0.03%	6.38%	0.99%	180
6	0.39%	0.01%	3.16%	0.66%	180
Cash capacity					
3	0.27%	0.01%	2.80%	0.33%	180
5	0.77%	0.04%	6.39%	1.02%	180
7	0.53%	0.03%	6.40%	1.41%	180
Unit holding cost					
0.5	0.75%	0.02%	6.09%	0.83%	270
1	0.77%	0.03%	6.40%	1.01%	270
Demand pattern					
STA	0.34%	0.01%	1.81%	0.53%	54
LC1	0.44%	0.02%	1.62%	0.66%	54
LC2	0.01%	0.01%	1.38%	0.49%	54
SIN1	0.01%	0.01%	6.40%	1.47%	54
SIN2	0.47%	0.04%	5.72%	1.28%	54
RAND	0.77%	0.12%	2.03%	0.49%	54
EMP1	0.00%	0.00%	6.39%	1.56%	54
EMP2	0.07%	0.01%	2.20%	0.47%	54
EMP3	0.01%	0.00%	5.78%	0.34%	54
EMP4	0.53%	0.05%	4.84%	0.82%	54
Gap over 1%	0 case		156 cases		
Avg Time	309 s		0.03 s		
General	0.77%	0.02%	6.40%	0.92%	540