STABILITY PROPERTIES FOR PARAMETRIC LINEAR PROGRAMS UNDER DATA AMBIGUITIES*

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Abstract. In this paper, we study a new parametric robust linear problem (PRLP, for short) whose data are allowed to be perturbed not only on the objective and constraint functions but also on the size of the uncertainty sets. Using a dual approach, we examine the stability and sensitivity properties of (PRLP) by looking at how the behaviors of its optimal value function and solution map change accordingly the change of the parameters. More precisely, we examine the closedness, lower and upper semicontinuity of the solution map of and the lower and upper semicontinuity as well as Lipschitz property of the optimal value function of (PRLP) varying around a reference parameter. In this way, we obtain the nonemptiness and boundedness of the solution sets and a characterization for the Lipschitz continuity of the optimal value function for semi-infinite linear programs when fixing the corresponding index sets.

Key words. Parametric linear program, uncertain optimization, semi-infinite programming, stability, solution map.

AMS subject classifications. 65K10, 49K99, 90C46, 90C29

1. Introduction. Real-life optimization models are often complicated due to noisy information, ambiguity and uncertainty input factors [4,5,23]. Therefore, *solution stability or sensitivity* has been received a lot of attention from the researchers and practitioners because it allows one to first evaluate and predict the variability of solutions in dynamic and fluctuated scenarios and settings, and then provide and suggest stable feasible options or robust practical/optimal solutions for the decision makers (see e.g., [1, 2, 7, 9, 22, 26, 30–32] and the references therein). A common way to study the stability and sensitivity is to examine functional behaviors such as lower and upper semicontinuity, calmness and Lipschitz properties of the solution map and the optimal value function of a parametric optimization problems [1, 2, 6, 18, 19, 24, 27]. Another way is to explore the change of optimality conditions, relaxations and duality relations [8, 13, 14] or the preservation of nonemptiness/radius of feasibility/solution [16, 17, 34] with respect to the perturbations of corresponding parameters.

In most applications in optimization under data uncertainty [23], uncertainty sets are used to describe a trade-off level of knowledge between known data and complete lack of input factors in the scenarios and formulations. In fact, the relationships and degrees of the known and unknown knowledge are relatively varying accordingly time and optimization process after receiving updated information and so the ambiguity or uncertainty sets need also to be adjusted with respect to the changes of corresponding scenarios and settings.

In this paper, we propose below a new general parametric linear programming problem, where the problem data are allowed to be perturbed not only on the objective and constraint functions but also on the size of the uncertainty sets. To proceed, let $\mathcal{A}_j : \mathbb{R}^s \to \mathbb{R}^n$ and $\mathcal{B}_j : \mathbb{R}^s \to \mathbb{R}, j = 1, \ldots, q$ be *parametric* affine mappings given respectively by

$$\mathcal{A}_j(u_j) := a_j^0 + \sum_{i=1}^{s} u_j^i a_j^i,\ \mathcal{B}_j(u_j) := b_j^0 + \sum_{i=1}^{s} u_j^i b_j^i \ \text{ for } \ u_j := (u_j^1, \ldots, u_j^s) \in \mathbb{R}^s \tag{1.1}$$

with $a_j^i \in \mathbb{R}^n, b_j^i \in \mathbb{R}, i = 0, 1, \ldots, s, j = 1, \ldots, q$. In what follows, we denote $\mathcal{A}_j := (a_j^i)_{i=0,1,\ldots,s} \in \mathbb{R}^{(s+1)n}, \mathcal{B}_j := (b_j^i)_{i=0,1,\ldots,s} \in \mathbb{R}^{s+1}, j = 1, \ldots, q$, for simplicity. We consider *perturbed* uncertainty

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sets defined in the forms of *spectrahedra* (cf. [29, 33]) as

$$U_j^\alpha := \{u_j := (u_j^1, \dots, u_j^s) \in \mathbb{R}^s \mid A_j^0 + \sum_{i=1}^s u_j^i A_j^i + \alpha I_j \succeq 0\},\ j = 1, \dots, q, \tag{1.2}$$

where $\alpha \in \mathbb{R}_+ := [0, +\infty)$ is a *parameter*, $A_j^i, i = 0, 1, \dots, s$, are symmetric $(m_j \times m_j)$ matrices with $m_j \in \mathbb{N} := \{1, 2, \dots\}$, and I_j is an identity $(m_j \times m_j)$ matrix for each $j \in \{1, \dots, q\}$. The linear matrix inequalities $A_j^0 + \sum_{i=1}^s u_j^i A_j^i + \alpha I_j \succeq 0, j = 1, \dots, q$ signify that the matrices $A_j^0 + \sum_{i=1}^s u_j^i A_j^i + \alpha I_j, j = 1, \dots, q$ are positive semi-definite. Throughout this paper, by choosing a suitable interval Λ with $0 \in \Lambda \subset \mathbb{R}_+$, we assume that the parametric uncertainty sets $U_j^\alpha, j = 1, \dots, p$ are *nonempty* and *compact* for all $\alpha \in \Lambda$.

Let $Z := \mathbb{R}^n \times \mathbb{R}^{(s+1)nq} \times \mathbb{R}^{(s+1)q} \times \Lambda$ be a subset of the normed space $\mathcal{Z} := \mathbb{R}^n \times \mathbb{R}^{(s+1)nq} \times \mathbb{R}^{(s+1)q} \times \mathbb{R}$, which is equipped with a norm defined by

$$||p - \widetilde{p}|| := \max\{||c - \widetilde{c}||, \max_{u_j \in U_j^\alpha \cup U_j^{\widetilde{\alpha}}} \{||\mathcal{A}_j(u_j) - \widetilde{\mathcal{A}}_j(u_j)||, |\mathcal{B}_j(u_j) - \widetilde{\mathcal{B}}_j(u_j)|, j = 1, \dots, q\}, |\alpha - \widetilde{\alpha}|\}$$

for any $p := (c, \mathcal{A}, \mathcal{B}, \alpha) \in \mathcal{Z}$ and $\widetilde{p} := (\widetilde{c}, \widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\alpha}) \in \mathcal{Z}$, where $\mathcal{A} := (\mathcal{A}_j)_{j=1,\dots,q} \in \mathbb{R}^{(s+1)nq}$ and $\mathcal{B} := (\mathcal{B}_j)_{j=1,\dots,q} \in \mathbb{R}^{(s+1)q}$. The metric space Z can be regarded as a *parametric space* for the problem (PRLP) defined below.

The *parametric robust* linear program (PRLP) is defined as follows: for each *parameter* $p := (c, \mathcal{A}, \mathcal{B}, \alpha) \in Z$, we have a *robust* linear optimization problem

$$\inf_{x \in \mathbb{R}^n} \{c^\top x \mid \mathcal{A}_j(u_j)^\top x \le \mathcal{B}_j(u_j),\ \forall u_j \in U_j^\alpha,\ j = 1, \dots, q\}, \tag{RP$_p$}$$

which is the robust counterpart of the *uncertain* linear optimization problem

$$\inf_{x \in \mathbb{R}^n} \{c^\top x \mid \mathcal{A}_j(u_j)^\top x \le \mathcal{B}_j(u_j),\ j = 1, \dots, q\}, \tag{P$_p$}$$

where $u_j, j = 1, \dots, q$ are *uncertain* and they belong to the prescribed *uncertainty* sets U_j^α.

Observed that the *parametric* robust linear program (PRLP) is actually a family of (standard) robust linear optimization problems in which each individual robust linear optimization problem (RP$_p$) is specified at a corresponding parameter p. In this setting, the *optimal value function* $v : Z \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ and the *solution map* $S : Z \rightrightarrows \mathbb{R}^n$ of problem (PRLP) are defined respectively by

$$v(p) : = \inf\{c^\top x \mid x \in C(p)\},\ p \in Z, \tag{1.3}$$

$$S(p) : = \{x \in C(p) \mid c^\top x = v(p)\},\ p \in Z, \tag{1.4}$$

where $C(p) := \{x \in \mathbb{R}^n \mid \mathcal{A}_j(u_j)^\top x \le \mathcal{B}_j(u_j),\ \forall u_j \in U_j^\alpha,\ j = 1, \dots, q\}$ is the set of all robust feasible points of (P$_p$).

It is worth emphasizing here that the parametric robust linear program (PRLP) handles flexibly dynamic decision-making processes under data uncertainties by allowing decision variables to evolve over time and optimization procedures with respect to an altering size of uncertainty sets after the updated information of uncertainty data is revealed. This is often seen in practice, for instance, the cost of producing a product is just an estimated value residing in a prescribed uncertainty set until the product is actually made [21], and so the production decision maker needs to possibly adjust the investment strategy in a new/updated range of uncertainty scenarios according to the actual production cost. Moreover, the parametric robust linear program (PRLP) makes use of the spectrahedral structures in (1.2) as parametric uncertainty ranges that empowers us not only to cover a large spectrum of popular uncertain optimization problems involving ellipsoidal, ball, polytopic and box uncertainty data in robust optimization [4, 23] but also to employ the solution characterization and strong dualization in linear semi-infinite programming [11] to examine the stability and sensitivity from an innovative dual point of view.

Relationships to robust linear programs with index set parameter. Let us pay attention to a robust linear program with index set parameter called (PU) (see [10,12,17]) as follows: for each parameter $\alpha \in \mathbb{R}_+$, one has a robust linear program:

$$(\mathrm{PR}_\alpha) \qquad \inf_{x\in\mathbb{R}^n} \{c^\top x \mid a_j^\top x - b_j \le 0,\ \forall (a_j, b_j) \in \widetilde{U}_j^\alpha,\ j = 1,\dots,q\},$$

where $c \in \mathbb{R}^n$ is fixed, and $\widetilde{U}_j^\alpha \subset \mathbb{R}^{n+1}, j = 1,\dots,q$, are uncertainty sets. Note that (PR_α) is the robust counterpart of the *uncertain* linear optimization problem

$$(\mathrm{P}_\alpha) \qquad \inf_{x\in\mathbb{R}^n} \{c^\top x \mid a_j^\top x - b_j \le 0,\ j = 1,\dots,q\},$$

where $(a_j, b_j), j = 1,\dots,q$ are *uncertain* residing in $\widetilde{U}_j^\alpha$. These uncertainty sets $\widetilde{U}_j^\alpha, j = 1,\dots,q$ are defined by

$$\widetilde{U}_j^\alpha := (\overline{a}_j, \overline{b}_j) + (\alpha_0 + \alpha)B,\ j = 1,\dots,q,$$

where $(\overline{a}_j, \overline{b}_j) \in \mathbb{R}^{n+1}, \alpha_0 \in \mathbb{R}_+ \setminus \{0\}$ are fixed and $B \subset \mathbb{R}^{n+1}$ is a spectrahedron described by

$$B := \big\{y := (y^1,\dots,y^{n+1}) \in \mathbb{R}^{n+1} \mid A^0 + \sum_{i=1}^{n+1} y^i A^i \succeq 0\big\} \tag{1.5}$$

with $A^i, i = 0,1,\dots,n+1$ being symmetric $(l \times l)$ matrices with $l \in \mathbb{N}$. Moreover, the set B is assumed to be compact with $0_{n+1} \in \mathrm{int}B$. The last relation ensures that the matrix A^0 in (1.5) can be chosen to be positive definite and consequently, it can be congruently transformed to an identity $(l \times l)$ matrix I (see e.g., [25, Page 252]). So, there is no loss of generality in assuming that

$$B := \big\{y := (y^1,\dots,y^{n+1}) \in \mathbb{R}^{n+1} \mid I + \sum_{i=1}^{n+1} y^i A^i \succeq 0\big\}. \tag{1.6}$$

It can be easily seen that the problem (PU) is a particular case of the problem (PRLP). To see this, just first take $s := n+1, a_j^0 := \overline{a}_j, a_j^i := e^i, i = 1,\dots,n, a_j^{n+1} := 0, j = 1,\dots,q$, where $e^i \in \mathbb{R}^n$ is a vector whose ith component is one and all others are zero, and $b_j^0 := \overline{b}_j, b_j^i := 0, i = 1,\dots,n, b_j^{n+1} := 1, j = 1,\dots,q$. Then, the mappings $\mathcal{A}_j : \mathbb{R}^{n+1} \to \mathbb{R}^n$ and $\mathcal{B}_j : \mathbb{R}^{n+1} \to \mathbb{R}, j = 1,\dots,q$ in (1.1) are described by

$$\mathcal{A}_j(u) := \overline{a}_j + (u^1,\dots,u^n),\ \mathcal{B}_j(u) := \overline{b}_j + u^{n+1} \ \text{ for }\ u := (u^1,\dots,u^{n+1}) \in \mathbb{R}^{n+1}, \tag{1.7}$$

where $u_j := u$ for all $j = 1,\dots,q$. Next, by choosing $A_j^0 := \alpha_0 I, I_j := I$ and $A_j^i := A^i$ for $i = 1,\dots,n+1, j = 1,\dots,q$, the spectrahedra in (1.2) become

$$U_j^\alpha = \big\{u := (u^1,\dots,u^{n+1}) \in \mathbb{R}^{n+1} \mid (\alpha_0 + \alpha)I + \sum_{i=1}^{n+1} u^i A^i \succeq 0\big\},\ j = 1,\dots,q, \tag{1.8}$$

and then, it holds that $(\alpha_0 + \alpha)B = U_j^\alpha$ for all $j = 1,\dots,q$.

Now, we can check that the problem (PR_α) is the same as the following one

$$\inf_{x\in\mathbb{R}^n} \{c^\top x \mid \mathcal{A}_j(u)^\top x \le \mathcal{B}_j(u),\ \forall u \in U_j^\alpha,\ j = 1,\dots,q\},$$

where $\mathcal{A}_j, \mathcal{B}_j, j = 1,\dots,q$ are given in (1.7) and $U_j^\alpha, j = 1,\dots,q$ are given in (1.8). So, we can employ some results established in [10,12,17] about computing the so-called *radius of robust feasibility* for (PU), which is defined by

$$\rho := \sup\{\alpha \in \mathbb{R}_+ \mid (\mathrm{PR}_\alpha) \text{ is feasible}\}, \tag{1.9}$$

to derive the corresponding results for this our particular problem. In this vein, we will know the maximal 'size' of the parameter $\alpha \in \mathbb{R}_+$ that the robust feasibility of our problem is guaranteed. Note here that the problem (PU) depends *only* on the index set parameter α.

In this work, we allow the problem (PRLP) to be perturbed not only on the parameter α but also other parameters as described by p above. Our main purpose is to study, for the first time, the stability and sensitivity properties of problem (PRLP) in the sense that we want to explore how the behaviors of the optimal value function v of (PRLP) in (1.3) and the solution map S of (PRLP) in (1.4) change accordingly when the parameter p varies nearly a reference point in the parametric space Z. More precisely, we examine the closedness, lower and upper semicontinuity of the solution map S as well as the lower/upper semicontinuity and order-Lipschitz continuity of the optimal value function v of (PRLP) depending on the parameter p near a reference point.

We employ a different approach from those existing in the literature called *dual approach* that exploits solution characterization and strong dualization in linear semi-infinite programming (see e.g., [11]) to establish and verify the stability and sensitivity results. As by-product, we obtain the nonemptiness and boundedness of the solution sets and a characterization for the Lipschitz continuity of the optimal value function for semi-infinite linear programs by *fixing* the index sets (see e.g., [1, 2]). It is worth emphasizing here that the study of the above stability and sensitivity properties is motivated also by some earlier works on the investigation of such properties for the class of semi-infinite linear programs [2] or of uncertain linear programs [3]. So there is the parallelism between our paper and [2] or [3]. However, by using a different dual approach to deal with a more general class of parametric robust linear programs involving *perturbed* uncertainty sets, the results obtained in this paper would not be derived from those in [2] and [3]. The interested reader is referred to [30] for a primal approach to the study of solution stability of a robust optimization problem involving *fixed* uncertainty sets.

The organization of the paper is as follows. Section 2 is devoted to providing preliminaries on some stability properties of set-valued analysis and a characterization of solution via dual data for the problem (PRLP) at a given parameter. In Section 3, we examine the closedness, sensitivity and upper/lower semicontinuity of the solution map S in (1.4). Section 4 presents the upper/lower semicontinuity and order-Lipschitz continuity of the optimal value function v in (1.3). The last section provides conclusions and research perspectives.

2. Preliminary and Auxiliary Results. In this section, we recall some stability properties of a set-valued map from set-valued analysis and provide several auxiliary results related to the preservation of the Slater condition under a perturbation as well as a characterization of solution via dual data for the problem (PRLP) at a given parameter. The space $\mathbb{R}^n$, $n \in \mathbb{N}$, is equipped with a norm $\|\cdot\|$ and the inner product defined by $\langle z, w\rangle := z^\top w$ for $z, w \in \mathbb{R}^n$.

Let X be a metric space. A set-valued map $F : X \rightrightarrows \mathbb{R}^m$ is called *closed* at $\overline{x} \in X$ if for all sequences $\{x^k\} \subset X$ and $\{y^k\} \subset \mathbb{R}^m$ satisfying $y^k \in F(x^k), x^k \to \overline{x}, y^k \to \overline{y}$ as $k \to \infty$, one has $\overline{y} \in F(\overline{x})$. A set-valued map $F : X \rightrightarrows \mathbb{R}^m$ is called *lower semicontinuous* at $\overline{x} \in X$ if for each open set $V \subset \mathbb{R}^m$ such that $V \cap F(\overline{x}) \neq \emptyset$, there exists an open set $U \subset X$, containing $\overline{x}$, such that $V \cap F(x) \neq \emptyset$ for each $x \in U$. A set-valued map $F : X \rightrightarrows \mathbb{R}^m$ is said to be *upper semicontinuous* at $\overline{x} \in X$ if for each open set $V \subset \mathbb{R}^m$ such that $F(\overline{x}) \subset V$, there exists an open neighborhood of $\overline{x}$ in X, say U, such that $F(x) \subset V$ for every $x \in U$. For Λ with $0 \in \Lambda \subset \mathbb{R}_+$, a function $\psi : X \times \Lambda \to \overline{\mathbb{R}}$ is *locally order-Lipschitz* continuous at $(\overline{x}, 0) \in X \times \Lambda$ if there exist a real number $\kappa > 0$ and a neighborhood V of $(\overline{x}, 0)$ in $X \times \Lambda$ such that

$$\psi(\widetilde{x}, \widetilde{\alpha}) - \psi(\check{x}, \check{\alpha}) \leq \kappa||(\widetilde{x}, \widetilde{\alpha}) - (\check{x}, \check{\alpha})||$$

for $(\widetilde{x}, \widetilde{\alpha}), (\check{x}, \check{\alpha}) \in V$ satisfying $\widetilde{\alpha} \leq \check{\alpha}$. A function $\phi : X \to \overline{\mathbb{R}}$ is *locally Lipschitz* continuous at $\overline{x} \in X$ if there exist a real number $\kappa > 0$ and a neighborhood V of $\overline{x}$ in X such that

$$|\phi(\widetilde{x}) - \phi(\check{x})| \leq \kappa||\widetilde{x} - \check{x}||$$

for $\widetilde{x}, \check{x} \in V$.

We say that the Slater condition holds for the problem (PRLP) at $p := (c, \mathcal{A}, \mathcal{B}, \alpha) \in Z$ if there is $\widehat{x} \in \mathbb{R}^n$ such that

$$\mathcal{A}_j(u_j)^\top \widehat{x} < \mathcal{B}_j(u_j),\ \forall u_j \in U_j^\alpha,\ j = 1, \dots, q. \tag{2.1}$$

The following lemma states that the Slater condition is locally preserved at a given parameter of problem (PRLP).

Lemma 2.1. *Let the Slater condition hold for the problem (PRLP) at $p := (c, \mathcal{A}, \mathcal{B}, 0) \in Z$, i.e., there exists $\widehat{x} \in \mathbb{R}^n$ such that*

$$\mathcal{A}_j(u_j)^\top \widehat{x} < \mathcal{B}_j(u_j),\ \forall u_j \in U_j^0,\ j = 1, \dots, q. \tag{2.2}$$

Then, we can find $\epsilon > 0$ such that

$$\widetilde{\mathcal{A}}_j(u_j)^\top \widehat{x} < \widetilde{\mathcal{B}}_j(u_j),\ \forall u_j \in U_j^{\widetilde{\alpha}},\ j = 1, \dots, q \tag{2.3}$$

for all $\widetilde{p} := (\widetilde{c}, \widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\alpha}) \in Z$ satisfying $||\widetilde{p} - p|| < \epsilon$.

Proof. Under the validation of the Slater condition in (2.2), we first show that there exist $\rho_0 > 0$ and $\gamma > 0$ such that

$$\mathcal{A}_j(u_j)^\top \widehat{x} \le \mathcal{B}_j(u_j) - \rho_0,\ \forall u_j \in U_j^{\widetilde{\alpha}},\ j = 1, \dots, q \tag{2.4}$$

for all $\widetilde{\alpha} \in [0, \gamma] \cap \Lambda$. If this is not the case, there exist $j \in \{1, \dots, q\}$ and sequences $\{\alpha_k\} \subset \Lambda$, $\{u_{jk}\} \subset U_j^{\alpha_k}$ such that $0 \le \alpha_k \le \frac{1}{k}$ and

$$\mathcal{A}_j(u_{jk})^\top \widehat{x} > \mathcal{B}_j(u_{jk}) - \frac{1}{k} \ \text{ for all } \ k \in \mathbb{N}. \tag{2.5}$$

We assert that the sequence $\{u_{jk}\}, k \in \mathbb{N}$, is bounded. Otherwise, by taking a subsequence if necessary, we may assume that $||u_{jk}|| \to +\infty$ as $k \to \infty$. Then, $\{\frac{u_{jk}}{||u_{jk}||}\}$ is a bounded sequence and thus has a convergent subsequence; without loss of generality, we assume that $\frac{u_{jk}}{||u_{jk}||} \to w := (w_j^1, \dots, w_j^s) \in \mathbb{R}^s$ with $||w|| = 1$ as $k \to \infty$.

The relation $u_{jk} := (u_{jk}^1, \dots, u_{jk}^s) \in U_j^{\alpha_k}, k \in \mathbb{N}$, means that

$$A_j^0 + \sum_{i=1}^s u_{jk}^i A_j^i + \alpha_k I_j \succeq 0, \tag{2.6}$$

and thus,

$$\frac{1}{||u_{jk}||} A_j^0 + \sum_{i=1}^s \frac{u_{jk}^i}{||u_{jk}||} A_j^i + \frac{\alpha_k}{||u_{jk}||} I_j \succeq 0$$

for k large enough, and thus, we arrive at

$$\sum_{i=1}^s w_j^i A_j^i \succeq 0 \tag{2.7}$$

by letting $k \to \infty$. Pick up $\overline{u}_j := (\overline{u}_j^1, \dots, \overline{u}_j^s) \in U_j^0$. It follows by definition that $A_j^0 + \sum_{i=1}^s \overline{u}_j^i A_j^i \succeq 0$ and then, due to (2.7),

$$A_j^0 + \sum_{i=1}^s (\overline{u}_j^i + t w_j^i) A_j^i = \left(A_j^0 + \sum_{i=1}^s \overline{u}_j^i A_j^i \right) + t \sum_{i=1}^s w_j^i A_j^i \succeq 0 \ \text{ for all } \ t > 0.$$

This guarantees that $\overline{u}_j + tw \in U_j^0$ for all $t > 0$, which contradicts the fact that U_j^0 is a compact set. So, the sequence $\{u_{jk}\}, k \in \mathbb{N}$, must be bounded.

By taking a subsequence if necessary, we may assume that $u_{jk} \to \widetilde{u}_j := (\widetilde{u}_j^1, \dots, \widetilde{u}_j^s) \in \mathbb{R}^s$ as $k \to \infty$. From (2.6) and (2.5), we conclude that $\widetilde{u}_j \in U_j^0$ and

$$\mathcal{A}_j(\widetilde{u}_j)^\top \widehat{x} \geq \mathcal{B}_j(\widetilde{u}_j),$$

which contradicts (2.2). So, the assertion in (2.4) is valid.

Now, let $0 < \rho < \rho_0$ and $0 < \epsilon < \min\left\{\gamma, \frac{\rho_0 - \rho}{||\widehat{x}||+1}\right\}$. Take any $\widetilde{p} := (\widetilde{c}, \widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\alpha}) \in Z$ satisfying $||\widetilde{p} - p|| < \epsilon$. In view of (2.4), we see that

$$\begin{aligned}
\widetilde{\mathcal{A}}_j(u_j)^\top \widehat{x} &= [\widetilde{\mathcal{A}}_j(u_j) - \mathcal{A}_j(u_j)]^\top \widehat{x} + \mathcal{A}_j(u_j)^\top \widehat{x} \\
&\leq \epsilon||\widehat{x}|| + \mathcal{B}_j(u_j) - \rho_0 \\
&= \epsilon||\widehat{x}|| + [\mathcal{B}_j(u_j) - \widetilde{\mathcal{B}}_j(u_j)] + \widetilde{\mathcal{B}}_j(u_j) - \rho_0 \\
&\leq \epsilon(||\widehat{x}|| + 1) - \rho_0 + \widetilde{\mathcal{B}}_j(u_j) \\
&\leq \widetilde{\mathcal{B}}_j(u_j) - \rho, \ \forall u_j \in U_j^{\widetilde{\alpha}}, \ j = 1, \dots, q,
\end{aligned}$$

showing that (2.3) holds true. The proof is complete. □

Let us consider a relaxation problem associated with the parameter $p := (c, \mathcal{A}, \mathcal{B}, \alpha) \in Z$ as follows:

$$\sup_{(\lambda_j^0, \lambda_j^i)} \Big\{ -\sum_{j=1}^q (\lambda_j^0 b_j^0 + \sum_{i=1}^s \lambda_j^i b_j^i) \mid c + \sum_{j=1}^q (\lambda_j^0 a_j^0 + \sum_{i=1}^s \lambda_j^i a_j^i) = 0, \ \lambda_j^0(A_j^0 + \alpha I_j) + \sum_{i=1}^s \lambda_j^i A_j^i \succeq 0,$$

$$(\mathrm{SDP}_p) \qquad \lambda_j^0 \in \mathbb{R}_+, \lambda_j^i \in \mathbb{R}, i = 1, \dots, s, j = 1, \dots, q \Big\}.$$

The following dual solution characterization for the problem (PRLP) at a given parameter can be derived from [11, Theorem 3.1].

Lemma 2.2. *Let the Slater condition hold for the problem (PRLP) at $p := (c, \mathcal{A}, \mathcal{B}, \alpha) \in Z$, and let $\overline{x} \in C(p) := \{x \in \mathbb{R}^n \mid \mathcal{A}_j(u_j)^\top x \leq \mathcal{B}_j(u_j), \ \forall u_j \in U_j^\alpha, \ j = 1, \dots, q\}$. Denote*

$$(\mathrm{DSC}) \qquad \begin{cases} \exists \lambda_j^0 \geq 0, \lambda_j^i \in \mathbb{R}, j = 1, \dots, q, i = 1, \dots, s \ \textit{such that} \\ c + \sum\limits_{j=1}^q (\lambda_j^0 a_j^0 + \sum_{i=1}^s \lambda_j^i a_j^i) = 0, \ c^\top \overline{x} + \sum\limits_{j=1}^q (\lambda_j^0 b_j^0 + \sum_{i=1}^s \lambda_j^i b_j^i) = 0, \\ \textit{and } \lambda_j^0(A_j^0 + \alpha I_j) + \sum_{i=1}^s \lambda_j^i A_j^i \succeq 0, j = 1, \dots, q. \end{cases}$$

Then, we have

$$\overline{x} \in S(p) \Leftrightarrow (\mathrm{DSC}) \Leftrightarrow c^\top \overline{x} = \max (\mathrm{SDP}_p).$$

3. Sensitivity and semicontinuity of the solution map. This section is devoted to presenting the closedness, sensitivity and upper/lower semicontinuity of the solution map S of (PRLP) defined by (1.4).

Theorem 3.1. **(Closedness and upper semicontinuity of the solution map)** *Let the Slater condition hold for the problem (PRLP) at $p := (c, \mathcal{A}, \mathcal{B}, 0) \in Z$, and let $S(p) \neq \emptyset$.*
(i) *The solution map $S : Z \rightrightarrows \mathbb{R}^n$ is closed at p.*
(ii) *If assume in addition that $S(p)$ is bounded, then the solution map S is upper semicontinuous at p.*

Proof. (i) Suppose that $x_k \in S(p_k), p_k := (c_k, \mathcal{A}_k, \mathcal{B}_k, \alpha_k) \in Z$ for all $k \in \mathbb{N}$ and $p_k \to p, x_k \to x \in \mathbb{R}^n$ as $k \to \infty$. We need to show that $x \in S(p)$.

For each $k \in \mathbb{N}$, the relation $x_k \in S(p_k)$ entails that $x_k \in C(p_k)$. Hence,

$$\mathcal{A}_{jk}(u_j)^\top x_k \leq \mathcal{B}_{jk}(u_j), \ \forall u_j \in U_j^{\alpha_k}, \ j = 1, \dots, q.$$

Note that $U_j^0 \subset U_j^{\alpha_k}$ for all $j = 1, \dots, q$ and $k \in \mathbb{N}$. By letting $k \to \infty$, we arrive at

$$\mathcal{A}_j(u_j)^\top x \leq \mathcal{B}_j(u_j), \ \forall u_j \in U_j^0, \ j = 1, \dots, q,$$

which shows that $x \in C(p)$.

Since $p_k \to p$ as $k \to \infty$ and the Slater condition holds for the problem (PRLP) at p, we conclude by Lemma 2.1 that there exists $k_0 \in \mathbb{N}$ such that the Slater condition holds for the problem (PRLP) at p_k for all $k \geq k_0$. Consider any $k \geq k_0$. By $x_k \in S(p_k)$, we employ Lemma 2.2 to assert that there exist $\lambda^0_{jk} \geq 0, \lambda^i_{jk} \in \mathbb{R}, j = 1, \ldots, q, i = 1, \ldots, s$ such that

$$\begin{cases} c_k + \sum\limits_{j=1}^{q}(\lambda^0_{jk}a^0_{jk} + \sum\limits_{i=1}^{s}\lambda^i_{jk}a^i_{jk}) = 0, \quad c_k^\top x_k + \sum\limits_{j=1}^{q}(\lambda^0_{jk}b^0_{jk} + \sum\limits_{i=1}^{s}\lambda^i_{jk}b^i_{jk}) = 0, \\ \lambda^0_{jk}(A^0_j + \alpha_k I_j) + \sum\limits_{i=1}^{s}\lambda^i_{jk}A^i_j \succeq 0, j = 1, \ldots, q. \end{cases} \tag{3.1}$$

We claim that there exists $M > 0$ such that

$$\sum_{j=1}^{q}(\lambda^0_{jk})^2 + \sum_{j=1}^{q}\sum_{i=1}^{s}(\lambda^i_{jk})^2 \leq M \ \text{ for all } \ k \geq k_0. \tag{3.2}$$

Assume on the contrary that (3.2) is not true. Then, by taking a sequence if necessary, we may assume that

$$\sum_{j=1}^{q}(\lambda^0_{jk})^2 + \sum_{j=1}^{q}\sum_{i=1}^{s}(\lambda^i_{jk})^2 \to +\infty \ \text{ as } \ k \to \infty.$$

For $k \in \mathbb{N}$ large enough, by putting $\widetilde{\lambda}^0_{jk} := \frac{\lambda^0_{jk}}{\sqrt{\sum_{j=1}^{q}(\lambda^0_{jk})^2 + \sum_{j=1}^{q}\sum_{i=1}^{s}(\lambda^i_{jk})^2}}, \widetilde{\lambda}^i_{jk} := \frac{\lambda^i_{jk}}{\sqrt{\sum_{j=1}^{q}(\lambda_{jk})^2 + \sum_{j=1}^{q}\sum_{i=1}^{s}(\lambda^i_{jk})^2}}, j = 1, \ldots, q, i = 1, \ldots, s$, it holds that $\sum_{j=1}^{q}(\widetilde{\lambda}^0_{jk})^2 + \sum_{j=1}^{q}\sum_{i=1}^{s}(\widetilde{\lambda}^i_{jk})^2 = 1$, and hence, we may assume that $\widetilde{\lambda}^0_{jk} \to \widetilde{\lambda}^0_j \geq 0$ and $\widetilde{\lambda}^i_{jk} \to \widetilde{\lambda}^i_j \in \mathbb{R}, j = 1, \ldots, q, i = 1, \ldots, s$ as $k \to \infty$ with $\sum_{j=1}^{q}(\widetilde{\lambda}^0_j)^2 + \sum_{j=1}^{q}\sum_{i=1}^{s}(\widetilde{\lambda}^i_j)^2 = 1$. By letting $\widetilde{\lambda}_k := \frac{1}{\sqrt{\sum_{j=1}^{q}(\lambda^0_{jk})^2 + \sum_{j=1}^{q}\sum_{i=1}^{s}(\lambda^i_{jk})^2}}$, we get by (3.1) that

$$\begin{cases} \widetilde{\lambda}_k c_k + \sum\limits_{j=1}^{q}(\widetilde{\lambda}^0_{jk}a^0_{jk} + \sum\limits_{i=1}^{s}\widetilde{\lambda}^i_{jk}a^i_{jk}) = 0, \quad \widetilde{\lambda}_k c_k^\top x_k + \sum\limits_{j=1}^{q}(\widetilde{\lambda}^0_{jk}b^0_{jk} + \sum\limits_{i=1}^{s}\widetilde{\lambda}^i_{jk}b^i_{jk}) = 0, \\ \widetilde{\lambda}^0_{jk}(A^0_j + \alpha_k I_j) + \sum\limits_{i=1}^{s}\widetilde{\lambda}^i_{jk}A^i_j \succeq 0, j = 1, \ldots, q. \end{cases} \tag{3.3}$$

Letting $k \to \infty$ in (3.3), we obtain that

$$\sum_{j=1}^{q}(\widetilde{\lambda}^0_j a^0_j + \sum_{i=1}^{s}\widetilde{\lambda}^i_j a^i_j) = 0, \quad \sum_{j=1}^{q}\widetilde{\lambda}^0_j b^0_j + \sum_{j=1}^{q}\sum_{i=1}^{s}\widetilde{\lambda}^i_j b^i_j = 0, \tag{3.4}$$

$$\widetilde{\lambda}^0_j A^0_j + \sum_{i=1}^{s}\widetilde{\lambda}^i_j A^i_j \succeq 0, j = 1, \ldots, q. \tag{3.5}$$

Consider any $j \in \{1, \ldots, q\}$. By the compactness of U^0_j, we assert by (3.5) that if $\widetilde{\lambda}^0_j = 0$, then $\widetilde{\lambda}^i_j = 0$ for all $i = 1, \ldots, s$. Assume on the contrary that $\widetilde{\lambda}^0_j = 0$ but there exists $i_0 \in \{1, \ldots, s\}$ with $\widetilde{\lambda}^{i_0}_j \neq 0$. In this case, (3.5) gives $\sum_{i=1}^{s}\widetilde{\lambda}^i_j A^i_j \succeq 0$. Let $\overline{u}_j := (\overline{u}^1_j, \ldots, \overline{u}^s_j) \in U^0_j$. It follows by definition that $A^0_j + \sum_{i=1}^{s}\overline{u}^i_j A^i_j \succeq 0$ and thus,

$$A^0_j + \sum_{i=1}^{s}(\overline{u}^i_j + t\widetilde{\lambda}^i_j)A^i_j = \left(A^0_j + \sum_{i=1}^{s}\overline{u}^i_j A^i_j\right) + t\sum_{i=1}^{s}\widetilde{\lambda}^i_j A^i_j \succeq 0 \ \text{ for all } \ t > 0.$$

This guarantees that $\overline{u}_j + t(\widetilde{\lambda}_j^1, \ldots, \widetilde{\lambda}_j^s) \in U_j^0$ for all $t > 0$, which contradicts the fact that U_j^0 is a compact set. So, our claim must be true. With this fact, we conclude from the equality $\sum_{j=1}^{q} (\widetilde{\lambda}_j^0)^2 + \sum_{j=1}^{q}\sum_{i=1}^{s} (\widetilde{\lambda}_j^i)^2 = 1$ that there exists $j \in \{1, \ldots, q\}$ such that $\widetilde{\lambda}_j^0 \neq 0$, i.e., $\sum_{j=1}^{q} \widetilde{\lambda}_j^0 \neq 0$.

Now, take $\widehat{u}_j := (\widehat{u}_j^1, \ldots, \widehat{u}_j^s) \in U_j^0$ and define $\widetilde{u}_j := (\widetilde{u}_j^1, \ldots, \widetilde{u}_j^s)$ with

$$\widetilde{u}_j^i := \begin{cases} \widehat{u}_j^i & \text{if } \widetilde{\lambda}_j^0 = 0, \\ \frac{\widetilde{\lambda}_j^i}{\widetilde{\lambda}_j^0} & \text{if } \widetilde{\lambda}_j^0 \neq 0, \ i = 1, \ldots, s. \end{cases}$$

Then, it follows that

$$A_j^0 + \sum_{i=1}^{s} \widetilde{u}_j^i A_j^i = \begin{cases} A_j^0 + \sum_{i=1}^{s} \widehat{u}_j^i A_j^i & \text{if } \widetilde{\lambda}_j^0 = 0, \\ \frac{1}{\widetilde{\lambda}_j^0}(\widetilde{\lambda}_j^0 A_j^0 + \sum_{i=1}^{s} \widetilde{\lambda}_j^i A_j^i) & \text{if } \widetilde{\lambda}_j^0 \neq 0, \end{cases}$$

which shows that $A_j^0 + \sum_{i=1}^{s} \widetilde{u}_j^i A_j^i \succeq 0$, and so, $\widetilde{u}_j \in U_j^0$.

Since the Slater condition holds for the problem (PRLP) at p, we find $\widehat{x} \in \mathbb{R}^n$ such that

$$\big(a_j^0 + \sum_{i=1}^{s} \widetilde{u}_j^i a_j^i\big)^\top \widehat{x} < b_j^0 + \sum_{i=1}^{s} \widetilde{u}_j^i b_j^i, \ j = 1, \ldots, q. \tag{3.6}$$

Due to $\sum_{j=1}^{q} \widetilde{\lambda}_j^0 \neq 0$, it follows that

$$\sum_{j=1}^{q} \widetilde{\lambda}_j^0 \big(a_j^0 + \sum_{i=1}^{s} \widetilde{u}_j^i a_j^i\big)^\top \widehat{x} < \sum_{j=1}^{q} \widetilde{\lambda}_j^0 \big(b_j^0 + \sum_{i=1}^{s} \widetilde{u}_j^i b_j^i\big),$$

or equivalently,

$$\big(\sum_{j=1}^{q} \widetilde{\lambda}_j^0 a_j^0 + \sum_{j=1}^{q}\sum_{i=1}^{s} \widetilde{\lambda}_j^i a_j^i\big)^\top \widehat{x} - \sum_{j=1}^{q} \widetilde{\lambda}_j^0 b_j^0 - \sum_{j=1}^{q}\sum_{i=1}^{s} \widetilde{\lambda}_j^i b_j^i < 0,$$

which contradicts (3.4). Therefore, we conclude that (3.2) is valid. With this fact, we may assume that $\lambda_{jk}^0 \to \lambda_j^0 \geq 0$ and $\lambda_{jk}^i \to \lambda_j^i \in \mathbb{R}, j = 1, \ldots, q, i = 1, \ldots, s$ as $k \to \infty$.

Finally, letting $k \to \infty$ in (3.1), we obtain

$$\begin{cases} c + \sum_{j=1}^{q} (\lambda_j^0 a_j^0 + \sum_{i=1}^{s} \lambda_j^i a_j^i) = 0, \quad c^\top x + \sum_{j=1}^{q} \lambda_j^0 b_j^0 + \sum_{j=1}^{q}\sum_{i=1}^{s} \lambda_j^i b_j^i = 0, \\ \lambda_j^0 A_j^0 + \sum_{i=1}^{s} \lambda_j^i A_j^i \succeq 0, j = 1, \ldots, q, \end{cases}$$

and thus, conclude by Lemma 2.2 that $x \in S(p)$. So, the proof of (i) is established.

(ii) Let $M > 0$ be such that the following implication holds

$$[x \in S(p)] \Longrightarrow [||x|| \leq M]. \tag{3.7}$$

Assume on the contrary that $S : Z \rightrightarrows \mathbb{R}^n$ is not upper semicontinuous at p. Then, there exist an open set $V \subset \mathbb{R}^n$ such that $S(p) \subset V$, and a sequence $p_k := (c_k, \mathcal{A}_k, \mathcal{B}_k, \alpha_k) \to p$ such that $S(p_k) \cap (\mathbb{R}^n \setminus V) \neq \emptyset$ for all $k \in \mathbb{N}$. Taking $x_k \in S(p_k) \cap (\mathbb{R}^n \setminus V)$ for all $k \in \mathbb{N}$, we assert that $\{x_k\}$ is bounded. Otherwise, we may assume that $||x_k|| \to +\infty$ as $k \to \infty$. Then, $\{\frac{x_k}{||x_k||}\}$ is a bounded sequence and thus has a convergent subsequence; without loss of generality, we assume that $\frac{x_k}{||x_k||} \to w \in \mathbb{R}^n$ with $||w|| = 1$.

Since $p_k \to p$ as $k \to \infty$ and the Slater condition holds for the problem (PRLP) at p, it follows by Lemma 2.1 that there exists $k_0 \in \mathbb{N}$ such that the Slater condition holds for the problem (PRLP) at p_k for all $k \geq k_0$; i.e., there exists $\widehat{x} \in \mathbb{R}^n$ such that

$$\mathcal{A}_{jk}(u_j)^\top \widehat{x} < \mathcal{B}_{jk}(u_j),\ \forall u_j \in U_j^{\alpha_k},\ j = 1, \dots, q,\ \forall k \geq k_0,$$

which guarantees that $\widehat{x} \in C(p_k)$ for all $k \geq k_0$. On the one hand, due to $x_k \in S(p_k)$, we see that $c_k^\top x_k \leq c_k^\top \widehat{x}$, and hence,

$$c_k^\top \left(\frac{x_k}{||x_k||} \right) \leq \frac{c_k^\top \widehat{x}}{||x_k||} \tag{3.8}$$

for k large enough. Letting $k \to \infty$ in (3.8), we obtain that

$$c^\top w \leq 0. \tag{3.9}$$

On the other hand, due to $x_k \in C(p_k)$, it holds that $\mathcal{A}_{jk}(u_j)^\top x_k \leq \mathcal{B}_{jk}(u_j),\ \forall u_j \in U_j^{\alpha_k},\ j = 1, \dots, q$, which in turn implies that

$$\mathcal{A}_{jk}(u_j)^\top \left(\frac{x_k}{||x_k||} \right) \leq \frac{1}{||x_k||} \mathcal{B}_{jk}(u_j),\ \forall u_j \in U_j^{\alpha_k},\ j = 1, \dots, q \tag{3.10}$$

for k large enough. Since $\mathcal{B}_{jk} \to \mathcal{B}_j, j = 1, \dots, q$ as $k \to \infty$, there exists $\lambda > 0$ such that $|\mathcal{B}_{jk}(u_j)| \leq \lambda$ for all $u_j \in U_j^0 \subset U_j^{\alpha_k}, j = 1, \dots, q$ and $k \in \mathbb{N}$. Therefore, by letting $k \to \infty$ in (3.10), we arrive at

$$\mathcal{A}_j(u_j)^\top w \leq 0,\ \forall u_j \in U_j^0,\ j = 1, \dots, q \tag{3.11}$$

Now, take $\overline{x} \in S(p)$. Then, $\overline{x} \in C(p)$, which means that $\mathcal{A}_j(u_j)^\top \overline{x} \leq \mathcal{B}_j(u_j),\ \forall u_j \in U_j^0,\ j = 1, \dots, q$. This together with (3.11) ensures that $\mathcal{A}_j(u_j)^\top(\overline{x} + \lambda w) = \mathcal{A}_j(u_j)^\top \overline{x} + \lambda \mathcal{A}_j(u_j)^\top w \leq \mathcal{B}_j(u_j),\ \forall u_j \in U_j^0,\ j = 1, \dots, q$ for every $\lambda > 0$. Hence, $\overline{x} + \lambda w \in C(p)$ for all $\lambda > 0$. So, we claim by (3.9) that

$$c^\top w = 0. \tag{3.12}$$

If this is not the case, i.e., $c^\top w < 0$, which entails that $c^\top(\overline{x} + \lambda w) = c^\top \overline{x} + \lambda c^\top w < c^\top \overline{x}$ for each $\lambda > 0$. It is a contradiction due to $\overline{x} \in S(p)$.

On account of (3.12), it is true that $c^\top(\overline{x} + \lambda w) = c^\top \overline{x}$, and thus, $\overline{x} + \lambda w \in S(p)$ for all $\lambda > 0$. Hence, due to (3.7), $||\overline{x} + \lambda w|| \leq M$ for all $\lambda > 0$, which is a contradiction.

So, the above chosen sequence $\{x_k\}$ must be bounded. Without loss of generality, we may assume that $x_k \to x \in \mathbb{R}^n$ as $k \to \infty$. On the one side, as $x_k \in \mathbb{R}^n \setminus V$ for all $k \in \mathbb{N}$, and the set $\mathbb{R}^n \setminus V$ is closed, it follows that $x \in \mathbb{R}^n \setminus V$. On the other side, by the closedness of the map S as shown in (i), we assert that $x \in S(p)$, which results in a contradiction due to the fact that $S(p) \subset V$. In conclusion, $S : Z \rightrightarrows \mathbb{R}^n$ is upper semicontinuous at p. The proof is complete. □

Remark 3.2. The boundedness of the solution set $S(p)$ imposed in Theorem 3.1(ii) can be verified via the corresponding relaxation (SDP_p) under the fulfilment of the Slater condition. In this setting, it is equivalent to show that there exists $M > 0$ such that

$$[x \in C(p), c^\top x = \max(\text{SDP}_p)] \Longrightarrow [||x|| \leq M]. \tag{3.13}$$

The following example shows that the conclusion of Theorem 3.1 may fail to hold if the Slater condition does not hold for the problem (PRLP) at a reference point.

Example 3.3. **(The importance of the Slater condition)** Let $\mathcal{A}_j : \mathbb{R} \to \mathbb{R}^2, \mathcal{B}_j : \mathbb{R} \to \mathbb{R}, j = 1, 2, 3$ be *parametric* affine mappings given respectively by

$$\mathcal{A}_j(u_j) := a_j^0 + u_j a_j,\ \mathcal{B}_j(u_j) := b_j^0 + u_j b_j \ \text{ for } \ u_j \in \mathbb{R},$$

where $a_j^0, a_j \in \mathbb{R}^2, b_j^0, b_j \in \mathbb{R}, j = 1, 2, 3$, and let $U_1^\alpha := [-1-\alpha, -1+\alpha], U_2^\alpha := [-\alpha, 1+\alpha]$ and $U_3^\alpha := [2-\alpha, 2+\alpha]$ be uncertainty sets. It is clear that the sets $U_j^\alpha, j = 1, 2, 3$, can be expressed in terms of the spectrahedral form (1.2) with

$$A_1^0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, A_2^0 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, A_3^0 := \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}, A_j^1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, j = 1, 2, 3.$$

We consider a parametric robust linear program (PRLP) in $\mathbb{R}^2$ as follows. For each

$$p := (c, \mathcal{A}, \mathcal{B}, \alpha) \in Z := \mathbb{R}^2 \times \mathbb{R}^{12} \times \mathbb{R}^6 \times \Lambda$$

with $\Lambda := [0, 2]$, one has a robust linear program:

$$(\text{E1}_p) \qquad \inf_{x \in \mathbb{R}^2} \{c^\top x \mid \mathcal{A}_j(u_j)^\top x \le \mathcal{B}_j(u_j),\ \forall u_j \in U_j^\alpha,\ j = 1, 2, 3\}.$$

Let us pay attention to the parameter of $\overline{p} := (\overline{c}, \overline{\mathcal{A}}, \overline{\mathcal{B}}, \overline{\alpha}) \in Z$, where $\overline{c} := (2, 0), \overline{\alpha} := 0$, $\overline{\mathcal{A}} := (\overline{\mathcal{A}}_j)_{j=1,2,3}$, $\overline{\mathcal{B}} := (\overline{\mathcal{B}}_j)_{j=1,2,3}$ with $\overline{\mathcal{A}}_1(u_1) := (2, 0), \overline{\mathcal{B}}_1(u_1) := 0, \overline{\mathcal{A}}_2(u_2) := u_2(-1, 1), \overline{\mathcal{B}}_2(u_2) := u_2, \overline{\mathcal{A}}_3(u_3) := (0, -2), \overline{\mathcal{B}}_3(u_3) := 0$ for $u_j \in \mathbb{R}, j = 1, 2, 3$. We can verify that $C(\overline{p}) = \{x := (x_1, x_2) \in \mathbb{R}^2 \mid -1 \le x_1 \le 0, 0 \le x_2 \le x_1 + 1\}$ and is depicted by the shaded set in Figure 3.1.

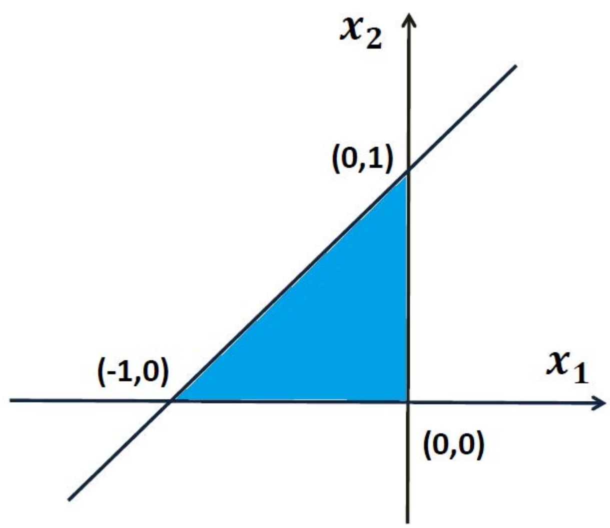

Fig. 3.1: The feasible set of $(\text{E1}_{\overline{p}})$.

So, the solution set of the problem $(\text{E1}_{\overline{p}})$ is $S(\overline{p}) = \{(-1, 0)\}$. Note that the Slater condition does not hold for the problem (PRLP) at $\overline{p}$ inasmuch as the strict inequalities $-u_2x_1 + u_2x_2 = \overline{\mathcal{A}}_2(u_2)^\top x < \overline{\mathcal{B}}_2(u_2) = u_2$ fails at $u_2 := 0 \in U_2^0$ for every $x := (x_1, x_2) \in \mathbb{R}^2$.

We show that the solution map $S : Z \rightrightarrows \mathbb{R}^2$ of (PRLP) is not closed at $\overline{p}$. To see this, let us choose a sequence $p_k := (c_k, \mathcal{A}_k, \mathcal{B}_k, \alpha_k) \in Z, k \ge 1$ with $c_k := (2, \frac{1}{k}), \alpha_k := \frac{1}{k}$ and $\mathcal{A}_{1k}(u_1) := (2, \frac{1}{k}), \mathcal{B}_{1k}(u_1) := \frac{1}{k}, u_1 \in \mathbb{R}, \mathcal{A}_{2k}(u_2) := u_2(-1, 1), u_2 \in \mathbb{R}, \mathcal{B}_{2k}(u_2) := \frac{k+1}{k}u_2 - \frac{1}{k}$ if $u_2 \in [\frac{1}{k+1}, +\infty), \mathcal{B}_{2k}(u_2) := 0$ if $u_2 \in (-\infty, \frac{1}{k+1}]$, and $\mathcal{A}_{3k}(u_3) := (\frac{1}{k}, -2), \mathcal{B}_{3k}(u_3) := \frac{1}{k}, u_3 \in \mathbb{R}$. It is true that $p_k \to \overline{p}$ as $k \to \infty$. Moreover, it is not hard to check that $C(p_k) = \{x := (x_1, x_2) \in \mathbb{R}^2 \mid x_1 = x_2, \frac{1}{1-2k} \le x_1 \le \frac{1}{1+2k}\}, k \ge 1$, and is depicted by the blue segment in Figure 3.2.

It is clear that the solution set of the problem (E1_{p_k}) is $S(p_k) = \{(\frac{1}{1-2k}, \frac{1}{1-2k})\}$ for $k \ge 1$. Since $(\frac{1}{1-2k}, \frac{1}{1-2k}) \to (0, 0)$ as $k \to \infty$, and $(0, 0) \notin S(\overline{p})$, which concludes that $S : Z \rightrightarrows \mathbb{R}^2$ is not closed at $\overline{p}$.

Now, we choose a neighborhood $\overline{V}$ of $S(\overline{p})$ such that $(a, a) \notin \overline{V}$ for all $a \in \mathbb{R}$. Then, $S(p_k) \not\subseteq V$ for all $k \ge 1$. Since $p_k \to \overline{p}$ as $k \to \infty$, we assert that $S : Z \rightrightarrows \mathbb{R}^2$ is not upper semicontinuous at $\overline{p}$ either, though $S(\overline{p})$ is obviously bounded.

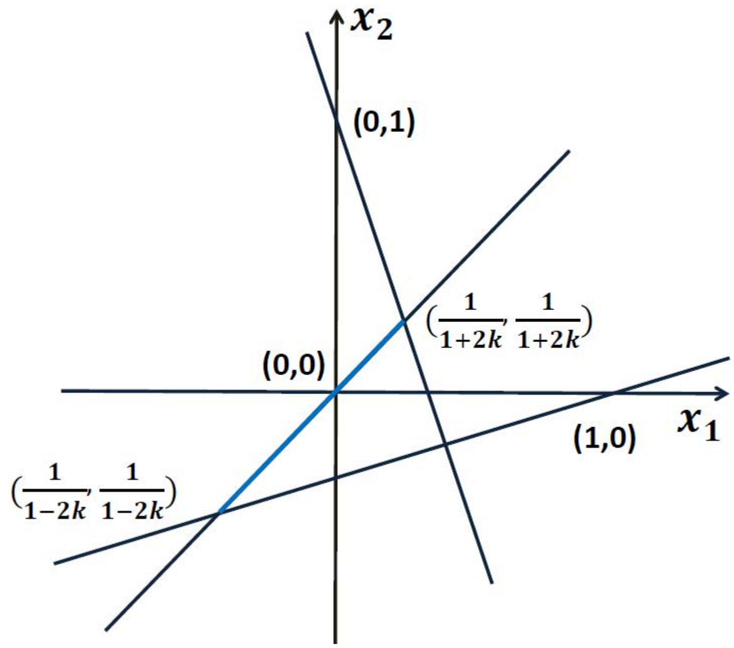

Fig. 3.2: The feasible set of $(\mathrm{E1}_{p_k})$.

The next example shows that the Slater condition and the nonemptiness of $S(p)$ are *not sufficient* for the upper semicontinuity of the solution map S at p. In other words, the conclusion of Theorem 3.1(ii) may fail if the boundedness of the solution set of the problem (PRLP) is violated at the reference point.

Example 3.4. **(The importance of the bounded condition)** Let $\mathcal{A}_j : \mathbb{R} \to \mathbb{R}^2, \mathcal{B}_j : \mathbb{R} \to \mathbb{R}, j = 1, 2$ be *parametric* affine mappings given respectively by

$$\mathcal{A}_j(u_j) := a_j^0 + u_j a_j,\ \mathcal{B}_j(u_j) := b_j^0 + u_j b_j \ \text{ for } \ u_j \in \mathbb{R},$$

where $a_j^0, a_j \in \mathbb{R}^2, b_j^0, b_j \in \mathbb{R}, j = 1, 2$, and let $U_1^\alpha := [-1-\alpha, -1+\alpha]$ and $U_2^\alpha := [1-\alpha, 1+\alpha]$ be uncertainty sets. It is clear that the sets $U_j^\alpha, j = 1, 2$ can be expressed in terms of the spectrahedral form (1.2) with

$$A_1^0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, A_2^0 := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, A_j^1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, j = 1, 2.$$

We consider a parametric robust linear program (PRLP) in $\mathbb{R}^2$ as follows. For each $p := (c, \mathcal{A}, \mathcal{B}, \alpha) \in Z := \mathbb{R}^2 \times \mathbb{R}^8 \times \mathbb{R}^4 \times \Lambda$ with $\Lambda := [0, 2]$, one has a robust linear program:

$$(\mathrm{E2}_p) \qquad \inf_{x \in \mathbb{R}^2} \{c^\top x \mid \mathcal{A}_j(u_j)^\top x \leq \mathcal{B}_j(u_j),\ \forall u_j \in U_j^\alpha,\ j = 1, 2\}.$$

Let us pay attention to the parameter of $\overline{p} := (\overline{c}, \overline{\mathcal{A}}, \overline{\mathcal{B}}, \overline{\alpha}) \in Z$ with $\overline{c} := (2, 0), \overline{\alpha} := 0$ and $\overline{\mathcal{A}}_1(u_1) := (2, 0), \overline{\mathcal{B}}_1(u_1) := 0, \overline{\mathcal{A}}_2(u_2) := (-2, 0), \overline{\mathcal{B}}_2(u_2) := 1$ for $u_j \in \mathbb{R}, j = 1, 2$. We can verify that $C(\overline{p}) = \{x := (x_1, x_2) \in \mathbb{R}^2 \mid -\frac{1}{2} \leq x_1 \leq 0, x_2 \in \mathbb{R}\}$, and the solution set of the problem $(\mathrm{E2}_{\overline{p}})$ is $S(\overline{p}) = \{-\frac{1}{2}\} \times \mathbb{R}$, which is clearly *unbounded*. Note that the Slater condition holds for the problem (PRLP) at $\overline{p}$ as the strict inequalities $\overline{\mathcal{A}}_j(u_j)^\top \widehat{x} < \overline{\mathcal{B}}_j(u_j), j = 1, 2$ are valid for $\widehat{x} := (-\frac{1}{4}, 0) \in \mathbb{R}^2$.

We claim that the solution map $S : Z \rightrightarrows \mathbb{R}^2$ of (PRLP) is not upper semicontinuous at $\overline{p}$. To see this, let us choose a sequence $p_k := (c_k, \mathcal{A}_k, \mathcal{B}_k, \alpha_k) \in Z, k \geq 1$ with $c_k := (2, \frac{1}{k}), \alpha_k := \frac{1}{k}$ and $\mathcal{A}_{1k}(u_1) := (2, \frac{1}{k}), \mathcal{B}_{1k}(u_1) := 0, \mathcal{A}_{2k}(u_2) := (-2, -\frac{1}{k}), \mathcal{B}_{2k}(u_2) := 1 + \frac{1}{k}$ for $u_1, u_2 \in \mathbb{R}$. It is true that $p_k \to \overline{p}$ as $k \to \infty$.

It can be verified that $C(p_k) = \{x := (x_1, x_2) \in \mathbb{R}^2 \mid -1 - \frac{1}{k} \leq 2x_1 + \frac{1}{k}x_2 \leq 0\}$, and hence, the solution set of the problem $(\mathrm{E2}_{p_k})$ is $S(p_k) = \{x := (x_1, x_2) \in \mathbb{R}^2 \mid 2x_1 + \frac{1}{k}x_2 = -1 - \frac{1}{k}\}$ for $k \geq 1$. Now, taking $x_k := (-\frac{1}{4}, -1 - \frac{k}{2}) \in S(p_k)$ for $k \geq 1$, and letting $\overline{V} := \{x := (x_1, x_2) \in \mathbb{R}^2 \mid x_1 < -\frac{1}{4}\}$, we see that $\overline{V}$ is an open set containing $S(\overline{p})$ but $x_k \notin \overline{V}$ for all $k \geq 1$. This confirms that $S : Z \rightrightarrows \mathbb{R}^2$ is not upper semicontinuous at $\overline{p}$.

THEOREM 3.5. **(Sensitivity and lower semicontinuity of the solution map)** *Let the Slater condition hold for the problem (PRLP) at $p := (c, \mathcal{A}, \mathcal{B}, 0) \in Z$, and let $S(p) \neq \emptyset$.*
(i) *If assume in addition that $S(p)$ is bounded, then there exists a neighborhood $\widetilde{W}$ of p such that $S(\widetilde{p}) \neq \emptyset$ for all $\widetilde{p} \in \widetilde{W}$.*
(ii) *The solution map $S : Z \rightrightarrows \mathbb{R}^n$ is lower semicontinuous at p if and only if there exists a unique $x \in C(p)$ such that $c^\top x = \max\,(\mathrm{SDP}_p)$.*

Proof. (i) Let $M > 0$ be such that the following implication holds

$$[x \in S(p)] \Longrightarrow [||x|| \leq M]. \tag{3.14}$$

Since the Slater condition holds for the problem (PRLP) at p, it follows by Lemma 2.1 that there exists a neighborhood W of p in Z such that the Slater condition holds for every $\widetilde{p} := (\widetilde{c}, \widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\alpha}) \in W$; i.e., there exists $\widehat{x} \in \mathbb{R}^n$ such that

$$\widetilde{\mathcal{A}}_j(u_j)^\top \widehat{x} < \widetilde{\mathcal{B}}_j(u_j),\ \forall u_j \in U_j^{\widetilde{\alpha}},\ j = 1, \ldots, q,$$

which guarantees that $\widehat{x} \in C(\widetilde{p})$ for all $\widetilde{p} \in W$.

Now, put $\Omega(\widetilde{p}) := \{x \in C(\widetilde{p}) \mid \widetilde{c}^\top x \leq \widetilde{c}^\top \widehat{x}\}$ for each $\widetilde{p} \in W$. We see that $\Omega(\widetilde{p})$ is a closed set and $\Omega(\widetilde{p}) \neq \emptyset$ for each $\widetilde{p} \in W$ due to $\widehat{x} \in \Omega(\widetilde{p})$.

We will prove that there exists another neighborhood $\widetilde{W}$ of p (assuming without loss of generality that $\widetilde{W} \subset W$) such that $\Omega(\widetilde{p})$ is bounded for all $\widetilde{p} \in \widetilde{W}$. If this is not the case, then we may assume that there exist a sequence $p_k := (c_k, \mathcal{A}_k, \mathcal{B}_k, \alpha_k) \in W, k \in \mathbb{N}$, and a sequence $x_k \in \Omega(p_k), k \in \mathbb{N}$, such that $p_k \to p$ and $||x_k|| \to +\infty$ as $k \to \infty$. Then, $\{\frac{x_k}{||x_k||}\}$ is a bounded sequence and thus has a convergent subsequence; without loss of generality, we assume that $\frac{x_k}{||x_k||} \to w \in \mathbb{R}^n$ with $||w|| = 1$ as $k \to \infty$. By $x_k \in \Omega(p_k), k \in \mathbb{N}$, it follows that

$$c_k^\top \left(\frac{x_k}{||x_k||}\right) \leq \frac{c_k^\top \widehat{x}}{||x_k||} \tag{3.15}$$

for k large enough. Letting $k \to \infty$ in (3.15), we obtain that

$$c^\top w \leq 0. \tag{3.16}$$

Besides, the relation $x_k \in \Omega(p_k)$ entails that $x_k \in C(p_k)$ for all $k \in \mathbb{N}$. Thus, it holds that $\mathcal{A}_{jk}(u_j)^\top x_k \leq \mathcal{B}_{jk}(u_j),\ \forall u_j \in U_j^{\alpha_k},\ j = 1, \ldots, q$, which in turn implies that

$$\mathcal{A}_{jk}(u_j)^\top \left(\frac{x_k}{||x_k||}\right) \leq \frac{1}{||x_k||}\mathcal{B}_{jk}(u_j),\ \forall u_j \in U_j^{\alpha_k},\ j = 1, \ldots, q$$

for k large enough. By letting $k \to \infty$, we arrive at

$$\mathcal{A}_j(u_j)^\top w \leq 0,\ \forall u_j \in U_j^0,\ j = 1, \ldots, q. \tag{3.17}$$

Now, take $\overline{x} \in S(p)$. Then, $\overline{x} \in C(p)$, which means that $\mathcal{A}_j(u_j)^\top \overline{x} \leq \mathcal{B}_j(u_j),\ \forall u_j \in U_j^0,\ j = 1, \ldots, q$. This together with (3.17) ensures that $\mathcal{A}_j(u_j)^\top(\overline{x} + \lambda w) = \mathcal{A}_j(u_j)^\top \overline{x} + \lambda \mathcal{A}_j(u_j)^\top w \leq \mathcal{B}_j(u_j),\ \forall u_j \in U_j^0,\ j = 1, \ldots, q$ for every $\lambda \geq 0$. So, $\overline{x} + \lambda w \in C(p)$ for all $\lambda \geq 0$. Moreover, by (3.16), $c^\top(\overline{x} + \lambda w) = c^\top \overline{x} + \lambda c^\top w \leq c^\top \overline{x}$ for all $\lambda \geq 0$. It confirms that $\overline{x} + \lambda w \in S(p)$ for all $\lambda \geq 0$. Hence, due to (3.14), $||\overline{x} + \lambda w|| \leq M$ for all $\lambda \geq 0$, which is absurd.

Consequently, there exists a neighborhood $\widetilde{W}$ of p such that $\widetilde{W} \subset W$ and $\Omega(\widetilde{p})$ is compact for all $\widetilde{p} \in \widetilde{W}$. Picking up any $\widetilde{p} := (\widetilde{c}, \widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\alpha}) \in \widetilde{W}$, we see that the function $\widetilde{c}^\top x$ attains its minimum over the compact set $\Omega(\widetilde{p})$; i.e., there exists $\widetilde{x} \in \Omega(\widetilde{p})$ such that $\widetilde{c}^\top \widetilde{x} \le \widetilde{c}^\top x$ for all $x \in \Omega(\widetilde{p})$. It follows that $\widetilde{c}^\top \widetilde{x} \le \widetilde{c}^\top x$ for all $x \in C(\widetilde{p})$. So, $\widetilde{x} \in S(\widetilde{p})$, or $S(\widetilde{p}) \neq \emptyset$.

(ii) $[\Longrightarrow]$ Assume that the solution map $S : Z \rightrightarrows \mathbb{R}^n$ is lower semicontinuous at p. We will prove that there exists a unique $x \in C(p)$ such that $c^\top x = \max(\text{SDP}_p)$. If this is not the case, we find $\overline{x}, \overline{y} \in C(p)$ such that $c^\top \overline{x} = c^\top \overline{y} = \max(\text{SDP}_p)$ and $\overline{x} \neq \overline{y}$. Letting $z := \overline{x} - \overline{y}$, we see that $z^\top \overline{x} - z^\top \overline{y} = z^\top(\overline{x} - \overline{y}) > 0$. Thus, we can find an open neighborhood V of $\overline{x}$ such that

$$z^\top x > z^\top \overline{y}, \quad \forall x \in V. \tag{3.18}$$

Put $p_k := (c_k, \mathcal{A}, \mathcal{B}, 0)$, where $c_k := c + \frac{1}{k} z$ for each $k \in \mathbb{N}$. We will show that

$$V \cap S(p_k) = \emptyset, \quad \forall k \in \mathbb{N}. \tag{3.19}$$

To see this, let us take an arbitrary $x \in V$ and consider any $k \in \mathbb{N}$. If $x \notin C(p_k)$, then it is evident that $x \notin S(p_k)$. If $x \in C(p_k)$, then, by (3.18), it follows that

$$c_k^\top(x - \overline{y}) = c^\top(x - \overline{y}) + \frac{1}{k} z^\top(x - \overline{y}) > 0, \tag{3.20}$$

where we bear in mind that $C(p) = C(p_k)$ for all $k \in \mathbb{N}$, and that $c^\top x \ge c^\top \overline{y}$ due to $\overline{y} \in S(p)$ as shown by Lemma 2.2. Since $\overline{y} \in C(p_k)$, we get by (3.20) that $x \notin S(p_k)$ and thus, (3.19) is valid. Since $p_k \to p$ as $k \to \infty$, we see that (3.19) contradicts the lower semicontinuity of S at p. So, our above assertion follows.

$[\Longleftarrow]$ Assume that there exists a unique $\overline{x} \in C(p)$ such that $c^\top \overline{x} = \max(\text{SDP}_p)$. Invoking Lemma 2.2, we have $S(p) = \{\overline{x}\}$, which shows that $S(p)$ is bounded. On the one hand, by (i), we can find a neighborhood $\widetilde{W}$ of p in Z such that $S(\widetilde{p}) \neq \emptyset$ for all $\widetilde{p} \in \widetilde{W}$. On the other hand, we assert by Theorem 3.1(ii) that the solution map $S : Z \rightrightarrows \mathbb{R}^n$ is upper semicontinuous at p.

Let V be an open set containing $\overline{x}$. The upper semicontinuity of S at p guarantees the existence of a neighborhood $\widetilde{W}_1$ of p in Z such that $S(\widetilde{p}) \subset V$ for all $\widetilde{p} \in \widetilde{W}_1$. Now, setting $W := \widetilde{W} \cap \widetilde{W}_1$, we see that W is a neighborhood of p and $S(\widetilde{p}) \cap V \neq \emptyset$ for all $\widetilde{p} \in W$. So, $S : Z \rightrightarrows \mathbb{R}^n$ is lower semicontinuous at p, which finishes the proof of the theorem. □

We close this section with a remark that the *sensitivity* for the problem (PRLP) in Theorem 3.5(i) is valid under the boundedness of the solution set at the reference point (cf. (3.14)). In general, we may not have such property if the boundedness of the solution set at the reference point is violated. To see this assertion, let us look back at the parametric robust linear program (PRLP) considered in Example 3.4. In this example, we have known that the solution set of the problem $(\text{E2}_{\overline{p}})$ is $S(\overline{p}) = \{-\frac{1}{2}\} \times \mathbb{R}$, where $\overline{p} := (\overline{c}, \overline{\mathcal{A}}, \overline{\mathcal{B}}, \overline{\alpha}) \in Z$ with $\overline{c} := (2, 0), \overline{\alpha} := 0$ and $\overline{\mathcal{A}}_1(u_1) := (2, 0), \overline{\mathcal{B}}_1(u_1) := 0, \overline{\mathcal{A}}_2(u_2) := (-2, 0), \overline{\mathcal{B}}_2(u_2) := 1$ for $u_j \in \mathbb{R}, j = 1, 2$, and the Slater condition holds for the problem (PRLP) at $\overline{p}$.

Let us pick up a sequence $p_k := (c_k, \mathcal{A}_k, \mathcal{B}_k, \alpha_k) \in Z, k \ge 1$ with $c_k := (2, 0), \alpha_k := \frac{1}{k}$ and $\mathcal{A}_{1k}(u_1) := (2, \frac{1}{k}), \mathcal{B}_{1k}(u_1) := 0, \mathcal{A}_{2k}(u_2) := (-2, -\frac{1}{k}), \mathcal{B}_{2k}(u_2) := 1 + \frac{1}{k}$ for $u_1, u_2 \in \mathbb{R}$. We see that $C(p_k) = \{x := (x_1, x_2) \in \mathbb{R}^2 \mid -1 - \frac{1}{k} \le 2x_1 + \frac{1}{k} x_2 \le 0\}$, and that $p_k \to \overline{p}$ as $k \to \infty$.

We claim that $S(p_k) = \emptyset$ for all $k \ge 1$. To see this, for each $k \ge 1$, take $x_l := (l, -1 - k - 2lk), l \in \mathbb{R}$. Then, it holds that $x_l \in C(p_k)$ and $c_k^\top x_l = 2l \to -\infty$ as $l \to -\infty$.

4. Semicontinuity and Lipschitz continuity of the optimal value function. In this section, we explore the upper/lower semicontinuity and order-Lipschitz continuity of the optimal value function v of (PRLP) defined by (1.3).

Proposition 4.1. **(Upper semicontinuity of the optimal value function)** *Let the Slater condition hold for the problem (PRLP) at $p := (c, \mathcal{A}, \mathcal{B}, 0) \in Z$. Then, the optimal value function $v : Z \to \overline{\mathbb{R}}$ is upper semicontinuous at p.*

Proof. Note that the Slater condition ensures that $C(p) \neq \emptyset$, and thus, $v(p) < +\infty$. Let us first consider the case, where $v(p) = -\infty$. Take an arbitrary sequence $p_k := (c_k, \mathcal{A}_k, \mathcal{B}_k, \alpha_k) \in Z, k \in \mathbb{N}$, such that $p_k \to p$ as $k \to \infty$. We need to show that $v(p_k) \to -\infty$ as $k \to \infty$.

Since $v(p) = -\infty$, it holds that for any $\lambda > 0$, there exists $x_\lambda \in C(p)$ such that $c^\top x_\lambda < -\lambda$. We can find a neighborhood U of c and a neighborhood V of x_λ such that

$$\widetilde{c}^\top x < -\lambda \text{ for all } \widetilde{c} \in U \text{ and all } x \in V. \tag{4.1}$$

Since $p_k \to p$ as $k \to \infty$, we may assume without loss of generality that $c_k \in U$ for all $k \in \mathbb{N}$.

The Slater condition being satisfied at p ensures that there exists $\widehat{x} \in \mathbb{R}^n$ such that

$$\mathcal{A}_j(u_j)^\top \widehat{x} < \mathcal{B}_j(u_j),\ \forall u_j \in U_j^0,\ j = 1, \ldots, q, \tag{4.2}$$

We will show that there exists $k_0 \in \mathbb{N}$ such that $x_r := \frac{1}{r}\widehat{x} + (1 - \frac{1}{r})x_\lambda \in C(p_k)$ for all $k \geq k_0$, where $r \in (1, +\infty)$ is chosen sufficiently large such that $x_r \in V$.

To see this, let us observe by (4.2) and the relation $x_\lambda \in C(p)$ that

$$\mathcal{A}_j(u_j)^\top x_r = \frac{1}{r}\mathcal{A}_j(u_j)^\top \widehat{x} + \left(1 - \frac{1}{r}\right)\mathcal{A}_j(u_j)^\top x_\lambda < \mathcal{B}_j(u_j),\ \forall u_j \in U_j^0,\ j = 1, \ldots, q.$$

By virtue of Lemma 2.1, we find $\epsilon > 0$ such that

$$\widetilde{\mathcal{A}}_j(u_j)^\top x_r < \widetilde{\mathcal{B}}_j(u_j),\ \forall u_j \in U_j^{\widetilde{\alpha}},\ j = 1, \ldots, q$$

for all $\widetilde{p} := (\widetilde{c}, \widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\alpha}) \in Z$ satisfying $||\widetilde{p} - p|| < \epsilon$. Then, let $k_0 \in \mathbb{N}$ be such that $||p_k - p|| < \epsilon$ for all $k \geq k_0$. It holds that

$$\mathcal{A}_{jk}(u_j)^\top x_r < \mathcal{B}_{jk}(u_j),\ \forall u_j \in U_j^{\alpha_k},\ j = 1, \ldots, q,\ k \geq k_0,$$

showing that $x_r \in C(p_k)$ for all $k \geq k_0$.

So, we use (4.1) to assert that

$$v(p_k) := \inf\{c_k^\top x \mid x \in C(p_k)\} \leq c_k^\top x_r < -\lambda \tag{4.3}$$

for all $k \geq k_0$. Since $\lambda > 0$ was taken arbitrarily, (4.3) proves that $v(p_k) \to -\infty$ as $k \to \infty$.

We now consider the case, where $-\infty < v(p) < +\infty$. Let $\varepsilon > 0$. By definition, there exists $x_\varepsilon \in C(p)$ such that

$$c^\top x_\varepsilon < v(p) + \frac{\varepsilon}{2}. \tag{4.4}$$

This entails that there exist a neighborhood $\overline{U}$ of c and a neighborhood $\overline{V}$ of x_ε such that

$$\widetilde{c}^\top x < c^\top x_\varepsilon + \frac{\varepsilon}{2} \text{ for all } \widetilde{c} \in \overline{U} \text{ and all } x \in \overline{V}. \tag{4.5}$$

Note that the Slater condition at p guarantees that $\overline{x}_r := \frac{1}{r}\widehat{x} + (1 - \frac{1}{r})x_\varepsilon \in \overline{V}$ by choosing large enough $r \in (1, +\infty)$, where $\widehat{x}$ is given as in (4.2). Similarly as above, we can find $\overline{\epsilon} > 0$ such that $\overline{x}_r \in C(\widetilde{p})$ for all $\widetilde{p} := (\widetilde{c}, \widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\alpha}) \in Z$ satisfying $||\widetilde{p} - p|| < \overline{\epsilon}$. Note further that $\overline{\epsilon}$ can be chosen smaller such that $\widetilde{c} \in \overline{U}$ whenever $||\widetilde{p} - p|| < \overline{\epsilon}$. Now, we deduce from (4.4) and (4.5) that

$$v(\widetilde{p}) := \inf\{\widetilde{c}^\top x \mid x \in C(\widetilde{p})\} \leq \widetilde{c}^\top \overline{x}_r < v(p) + \varepsilon$$

for all $\widetilde{p} \in Z$ satisfying $||\widetilde{p} - p|| < \overline{\epsilon}$. In conclusion, the optimal value function $v : Z \to \overline{\mathbb{R}}$ is upper semicontinuous at p. □

THEOREM 4.2. **(Lower semicontinuity of the optimal value function)** *Let $p := (c, \mathcal{A}, \mathcal{B}, 0) \in Z$ be such that $-\infty < v(p) < +\infty$. Then, the optimal value function $v : Z \to \overline{\mathbb{R}}$ is lower semicontinuous at p if and only if $S(p)$ is nonempty and bounded.*

Proof. [$\Longrightarrow$] Assume that $v : Z \to \overline{\mathbb{R}}$ is lower semicontinuous at p. Take $\lambda > v(p)$, and let $0 < \epsilon_0 < \lambda - v(p)$. By the definition of v, there is $x_{\epsilon_0} \in C(p)$ such that

$$c^\top x_{\epsilon_0} < v(p) + \epsilon_0 < \lambda, \tag{4.6}$$

and hence, the sublevel set $\Omega := \{x \in C(p) \mid c^\top x \le \lambda\}$ is nonempty. The closedness of Ω is easy to see. We will show that Ω is bounded. If this is not the case, then we may assume that there exists a sequence $x_k \in \Omega, k \in \mathbb{N}$, such that $||x_k|| \to +\infty$ as $k \to \infty$. Then, $\{\frac{x_k}{||x_k||}\}$ is a bounded sequence and thus has a convergent subsequence; without loss of generality, we assume that $\frac{x_k}{||x_k||} \to w \in \mathbb{R}^n$ with $||w|| = 1$ as $k \to \infty$. On the one hand, the relation $x_k \in \Omega$ ensures that

$$c^\top \left(\frac{x_k}{||x_k||}\right) \le \frac{\lambda}{||x_k||}$$

for k large enough, and thus, we arrive at

$$c^\top w \le 0 \tag{4.7}$$

by letting $k \to \infty$.

On the other hand, the relation $x_k \in \Omega$ entails that $x_k \in C(p)$ for all $k \in \mathbb{N}$. Thus, it holds that $\mathcal{A}_j(u_j)^\top x_k \le \mathcal{B}_j(u_j),\ \forall u_j \in U_j^0,\ j = 1, \dots, q$, which in turn imply that

$$\mathcal{A}_j(u_j)^\top \left(\frac{x_k}{||x_k||}\right) \le \frac{1}{||x_k||}\mathcal{B}_j(u_j),\ \forall u_j \in U_j^0,\ j = 1, \dots, q$$

for k large enough. By letting $k \to \infty$, we arrive at

$$\mathcal{A}_j(u_j)^\top w \le 0,\ \forall u_j \in U_j^0,\ j = 1, \dots, q. \tag{4.8}$$

Denote $x_r := x_{\epsilon_0} + rw$ for $r > 0$. The relation $x_{\epsilon_0} \in C(p)$ means that $\mathcal{A}_j(u_j)^\top x_{\epsilon_0} \le \mathcal{B}_j(u_j),\ \forall u_j \in U_j^0,\ j = 1, \dots, q$. It together with (4.8) ensures that $\mathcal{A}_j(u_j)^\top x_r = \mathcal{A}_j(u_j)^\top x_{\epsilon_0} + r\mathcal{A}_j(u_j)^\top w \le \mathcal{B}_j(u_j),\ \forall u_j \in U_j^0,\ j = 1, \dots, q$ for every $r > 0$. So, $x_r \in C(p)$ for every $r > 0$.

Now, put $p_k := (c_k, \mathcal{A}, \mathcal{B}, 0)$, where $c_k := c - \frac{1}{k}w$ for each $k \in \mathbb{N}$. It is easy to see that $C(p_k) = C(p)$ for all $k \in \mathbb{N}$ and that $p_k \to p$ as $k \to \infty$.

For each $k \in \mathbb{N}$ and $r > 0$, it is true that

$$\begin{aligned} v(p_k) := \inf\{c_k^\top x \mid x \in C(p_k)\} \le c_k^\top x_r &= c^\top x_{\epsilon_0} - \frac{1}{k}w^\top x_{\epsilon_0} + rc^\top w - \frac{1}{k}r||w||^2 \\ &< v(p) + \epsilon_0 - \frac{1}{k}w^\top x_{\epsilon_0} - \frac{1}{k}r, \end{aligned} \tag{4.9}$$

where the inequality holds by virtue of (4.6) and (4.7). It contradicts the lower semicontinuity of $v : Z \to \overline{\mathbb{R}}$ at p with $-\infty < v(p) < +\infty$ since for a given $k \in \mathbb{N}$, $(\epsilon_0 - \frac{1}{k}w^\top x_{\epsilon_0} - \frac{1}{k}r) \to -\infty$ as $r \to +\infty$.

Consequently, the set Ω must be bounded. We conclude that the function $c^\top x$ attains its minimum over the compact set Ω; i.e., there exists $\overline{x} \in \Omega$ such that $c^\top \overline{x} \le c^\top x$ for all $x \in \Omega$. It follows that $c^\top \overline{x} \le c^\top x$ for all $x \in C(p)$. So, $\overline{x} \in S(p)$, or $S(p) \neq \emptyset$. Furthermore, it is obvious that the boundedness of Ω yields the boundedness of $S(p)$ due to $S(p) \subset \Omega$.

[$\Longleftarrow$] Conversely, let $S(p) \neq \emptyset$, and let $M > 0$ be such that

$$[x \in S(p)] \Longrightarrow [||x|| \le M]. \tag{4.10}$$

We need to prove that $v : Z \to \overline{\mathbb{R}}$ is lower semicontinuous at p. Assume on the contrary that there exist a real number $\varepsilon_0 > 0$ and a sequence $p_k := (c_k, \mathcal{A}_k, \mathcal{B}_k, \alpha_k), k \in \mathbb{N}$, such that $p_k \to p$ as $k \to \infty$, and

$$v(p_k) < v(p) - \varepsilon_0 \ \text{ for all } \ k \in \mathbb{N}. \tag{4.11}$$

Let W be an open set such that $S(p) \subset W$. By (4.10), one can assume further that W is bounded. Taking $\overline{x} \in S(p)$, we will prove that there exists a neighborhood V of p in Z such that

$$\Omega(\widetilde{p}) := \{x \in C(\widetilde{p}) \mid \widetilde{c}^\top x \leq c^\top \overline{x}\} \subset W \text{ for all } \widetilde{p} := (\widetilde{c}, \widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\alpha}) \in V. \tag{4.12}$$

Suppose on the contrary that there exists a sequence $\widetilde{p}_k := (\widetilde{c}_k, \widetilde{\mathcal{A}}_k, \widetilde{\mathcal{B}}_k, \widetilde{\alpha}_k), k \in \mathbb{N}$, such that $\widetilde{p}_k \to p$ as $k \to \infty$ and $\Omega(\widetilde{p}_k) \nsubseteq W$. In this circumstance, we can find a sequence $x_k \in \Omega(\widetilde{p}_k)$ such that $x_k \in \mathbb{R}^n \setminus W$ for all $k \in \mathbb{N}$.

We assert that the sequence $x_k, k \in \mathbb{N}$, is bounded. If this is not the case, then we may assume $||x_k|| \to +\infty$ as $k \to \infty$. Then, $\{\frac{x_k}{||x_k||}\}$ is a bounded sequence and thus has a convergent subsequence; without loss of generality, we assume that $\frac{x_k}{||x_k||} \to w \in \mathbb{R}^n$ with $||w|| = 1$ as $k \to \infty$. On the one hand, the relation $x_k \in \Omega(\widetilde{p}_k)$ ensures that

$$\widetilde{c}_k^\top \left(\frac{x_k}{||x_k||}\right) \leq \frac{c^\top \overline{x}}{||x_k||} \tag{4.13}$$

for k large enough. Letting $k \to \infty$ in (4.13), we obtain that

$$c^\top w \leq 0. \tag{4.14}$$

On the other hand, the relation $x_k \in \Omega(\widetilde{p}_k)$ entails that $x_k \in C(\widetilde{p}_k)$ for all $k \in \mathbb{N}$. Thus, it holds that $\widetilde{\mathcal{A}}_{jk}(u_j)^\top x_k \leq \widetilde{\mathcal{B}}_{jk}(u_j),\ \forall u_j \in U_j^{\widetilde{\alpha}_k},\ j = 1, \ldots, q$, which in turn imply that

$$\widetilde{\mathcal{A}}_{jk}(u_j)^\top \left(\frac{x_k}{||x_k||}\right) \leq \frac{1}{||x_k||}\widetilde{\mathcal{B}}_{jk}(u_j),\ \forall u_j \in U_j^{\widetilde{\alpha}_k},\ j = 1, \ldots, q$$

for k large enough. By letting $k \to \infty$, we arrive at

$$\mathcal{A}_j(u_j)^\top w \leq 0,\ \forall u_j \in U_j^0,\ j = 1, \ldots, q, \tag{4.15}$$

where we should bear in mind that $U_j^0 \subset U_j^{\widetilde{\alpha}_k},\ j = 1, \ldots, q$ for all $k \in \mathbb{N}$.

The relation $\overline{x} \in C(p)$ means that $\mathcal{A}_j(u_j)^\top \overline{x} \leq \mathcal{B}_j(u_j),\ \forall u_j \in U_j^0,\ j = 1, \ldots, q$. This together with (4.15) ensures that $\mathcal{A}_j(u_j)^\top(\overline{x} + \lambda w) = \mathcal{A}_j(u_j)^\top \overline{x} + \lambda \mathcal{A}_j(u_j)^\top w \leq \mathcal{B}_j(u_j),\ \forall u_j \in U_j^0,\ j = 1, \ldots, q$ for every $\lambda > 0$. So, $\overline{x} + \lambda w \in C(p)$ for all $\lambda > 0$. Moreover, by (4.14), $c^\top(\overline{x} + \lambda w) = c^\top \overline{x} + \lambda c^\top w \leq c^\top \overline{x}$ for all $\lambda > 0$. It confirms that $\overline{x} + \lambda w \in S(p)$ for all $\lambda > 0$. Hence, due to (4.10), $||\overline{x} + \lambda w|| \leq M$ for all $\lambda > 0$, which is absurd. So, the sequence $x_k, k \in \mathbb{N}$, must be bounded.

On the one side, we may assume that

$$x_k \to x_0 \in \mathbb{R}^n \setminus W \text{ as } k \to \infty. \tag{4.16}$$

On the other side, the relation $x_k \in \Omega(\widetilde{p}_k)$ entails that $x_k \in C(\widetilde{p}_k)$ and that $\widetilde{c}_k^\top x_k \leq c^\top \overline{x}$ for all $k \in \mathbb{N}$. Passing to the limit as $k \to \infty$, we arrive at $x_0 \in C(p)$ and $c^\top x_0 \leq c^\top \overline{x}$, which guarantee that $x_0 \in S(p)$ due to $\overline{x} \in S(p)$. Furthermore, by $S(p) \subset W$, we conclude that $x_0 \in W$, which clearly contradicts (4.16). So, our assertion in (4.12) is valid.

Now, for each $k \in \mathbb{N}$, we assert by the definition of $v(p_k)$ that there exists $\widetilde{x}_k \in C(p_k)$ such that $c_k^\top \widetilde{x}_k < v(p_k) + \frac{1}{k}$. This together with (4.11) gives

$$c_k^\top \widetilde{x}_k < v(p) + \frac{1}{k} - \varepsilon_0, \tag{4.17}$$

and thus, $c_k^\top \widetilde{x}_k < v(p) = c^\top \overline{x}$ for all $k > \frac{1}{\varepsilon_0}$, where $\overline{x} \in S(p)$ as chosen above. Hence, $\widetilde{x}_k \in \Omega(p_k)$ for all $k > \frac{1}{\varepsilon_0}$. This fact together with (4.12) entails that there exists $k_0 \in \mathbb{N}$ such that $\widetilde{x}_k \in W$ for all $k \geq k_0$. It means that the sequence $\widetilde{x}_k, k \geq k_0$ is bounded due to the boundedness of W. There is no loss of generality in assuming that $\widetilde{x}_k \to \widetilde{x}_0 \in C(p)$ as $k \to \infty$. With this in hand, we get by (4.17) that $c^\top \widetilde{x}_0 \leq v(p) - \varepsilon_0$, which is a contradiction. It says that the assertion in (4.11) is not true, and so, $v : Z \to \overline{\mathbb{R}}$ is lower semicontinuous at p. □

THEOREM 4.3. **(Order-Lipschitz continuity of the optimal value function)** *Let $p := (c, \mathcal{A}, \mathcal{B}, 0) \in Z$ be such that $-\infty < v(p) < +\infty$. Let the Slater condition hold for the problem (PRLP) at p. Then, the following statements are equivalent.*
(i) *The optimal value function $v : Z \to \overline{\mathbb{R}}$ is locally order-Lipschitz continuous at p, i.e., there exist a real number $\kappa > 0$ and a neighborhood V of p in Z such that*

$$v(\widetilde{p}) - v(\check{p}) \leq \kappa ||\widetilde{p} - \check{p}|| \tag{4.18}$$

for $\widetilde{p} := (\widetilde{c}, \widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\alpha}) \in V, \check{p} := (\check{c}, \check{\mathcal{A}}, \check{\mathcal{B}}, \check{\alpha}) \in V$ satisfying $\widetilde{\alpha} \leq \check{\alpha}$.
(ii) *$S(p)$ is nonempty and bounded.*

Proof. [**Proving** (i) $\Longrightarrow$ (ii)] Assume the assertion in (4.18) holds. Let $\epsilon > 0$. We choose a neighborhood V_1 of p in Z such that $V_1 \subset V$ and that $||\widetilde{p} - p|| \leq \frac{\epsilon}{\kappa}$ for all $\widetilde{p} \in V_1$. Then, for each $\widetilde{p} := (\widetilde{c}, \widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\alpha}) \in V_1$, it is obvious that $0 \leq \widetilde{\alpha}$, and thus, we get by (4.18) that

$$v(\widetilde{p}) \geq v(p) - \kappa ||\widetilde{p} - p|| \geq v(p) - \epsilon,$$

which shows that the function $v : Z \to \overline{\mathbb{R}}$ is lower semicontinuous at p. Thanks to Theorem 4.2, we conclude that (ii) is valid.
[**Proving** (ii) $\Longrightarrow$ (i)] Let $S(p) \neq \emptyset$, and let $M > 0$ be such that

$$[x \in S(p)] \Longrightarrow [||x|| \leq M]. \tag{4.19}$$

In this circumstance, we apply Theorem 3.5(i) to assert that there exists a neighborhood V_1 of p in Z such that $S(\widetilde{p}) \neq \emptyset$ for all $\widetilde{p} \in V_1$. We may assume further that the Slater condition holds for the problem (PRLP) at every $\widetilde{p} \in V_1$.

Let W be an open set such that $S(p) \subset W$. By (4.19), one can assume further that W is bounded. By Theorem 3.1(ii), the solution map $S : Z \rightrightarrows \mathbb{R}^n$ is upper semicontinuous at p. Hence, we can find a neighborhood V_2 of p in Z (assuming without loss of generality that $V_2 \subset V_1$) such that $S(\widetilde{p}) \subset W$ for all $\widetilde{p} \in V_2$. The boundedness of W guarantees that there exists a real number $M_1 > 0$ such that

$$||x|| \leq M_1 \text{ for all } x \in S(\widetilde{p}) \text{ whenver } \widetilde{p} \in V_2. \tag{4.20}$$

Next, we will prove that there exist a real number $M_2 > 0$ and a neighborhood V of p in Z (assuming that $V \subset V_2$) such that for $\widetilde{p} := (\widetilde{c}, \widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\alpha}) \in V$ and $\widetilde{\lambda}_j^0 \geq 0, \widetilde{\lambda}_j^i \in \mathbb{R}, j = 1, \ldots, q, i = 1, \ldots, s$ satisfying

$$\begin{cases} \widetilde{c} + \sum\limits_{j=1}^{q} (\widetilde{\lambda}_j^0 \widetilde{a}_j^0 + \sum_{i=1}^s \widetilde{\lambda}_j^i \widetilde{a}_j^i) = 0, \quad \widetilde{c}^\top \widetilde{x} + \sum\limits_{j=1}^{q} (\widetilde{\lambda}_j^0 \widetilde{b}_j^0 + \sum_{i=1}^s \widetilde{\lambda}_j^i \widetilde{b}_j^i) = 0, \\ \widetilde{\lambda}_j^0 (A_j^0 + \widetilde{\alpha} I_j) + \sum_{i=1}^s \widetilde{\lambda}_j^i A_j^i \succeq 0, j = 1, \ldots, q, \end{cases} \tag{4.21}$$

where $\widetilde{x} \in S(\widetilde{p})$, one has

$$\sum_{j=1}^{q} (\widetilde{\lambda}_j^0)^2 + \sum_{j=1}^{q} \sum_{i=1}^{s} (\widetilde{\lambda}_j^i)^2 \leq M_2. \tag{4.22}$$

Assume on the contrary that the assertion is not true. Then, there exist a sequence

$$p_k := (c_k, \mathcal{A}_k, \mathcal{B}_k, \alpha_k) \in Z, \ k \in \mathbb{N},$$

such that $p_k \to p$ as $k \to \infty$, and sequences $\lambda_{jk}^0 \geq 0, \lambda_{jk}^i \in \mathbb{R}, j = 1, \ldots, q, i = 1, \ldots, s, k \in \mathbb{N}$, such that

$$\begin{cases} c_k + \sum\limits_{j=1}^{q} (\lambda_{jk}^0 a_{jk}^0 + \sum\limits_{i=1}^{s} \lambda_{jk}^i a_{jk}^i) = 0, \quad c_k^\top x_k + \sum\limits_{j=1}^{q} (\lambda_{jk}^0 b_{jk}^0 + \sum\limits_{i=1}^{s} \lambda_{jk}^i b_{jk}^i) = 0, \\ \lambda_{jk}^0 (A_j^0 + \alpha_k I_j) + \sum\limits_{i=1}^{s} \lambda_{jk}^i A_j^i \succeq 0, j = 1, \ldots, q, \end{cases} \tag{4.23}$$

with some $x_k \in S(p_k), k \in \mathbb{N}$, and

$$\sum_{j=1}^{q}(\lambda_{jk}^0)^2 + \sum_{j=1}^{q}\sum_{i=1}^{s}(\lambda_{jk}^i)^2 \to +\infty \ \text{ as } \ k \to \infty.$$

Since $p_k \to p$ as $k \to \infty$, there is no loss of generality in assuming that $p_k \in V_2$ for all $k \in \mathbb{N}$. Then, taking (4.20) into account, we may assume that $x_k \to \overline{x}$ as $k \to \infty$. For $k \in \mathbb{N}$ large enough, by putting $\widetilde{\lambda}_{jk}^0 := \frac{\lambda_{jk}^0}{\sqrt{\sum_{j=1}^{q}(\lambda_{jk}^0)^2 + \sum_{j=1}^{q}\sum_{i=1}^{s}(\lambda_{jk}^i)^2}}, \widetilde{\lambda}_{jk}^i := \frac{\lambda_{jk}^i}{\sqrt{\sum_{j=1}^{q}(\lambda_{jk})^2 + \sum_{j=1}^{q}\sum_{i=1}^{s}(\lambda_{jk}^i)^2}}, j = 1, \ldots, q, i = 1, \ldots, s,$ it holds that $\sum_{j=1}^{q}(\widetilde{\lambda}_{jk}^0)^2 + \sum_{j=1}^{q}\sum_{i=1}^{s}(\widetilde{\lambda}_{jk}^i)^2 = 1$, and hence, we may assume that $\widetilde{\lambda}_{jk}^0 \to \widetilde{\lambda}_j^0 \geq 0$ and $\widetilde{\lambda}_{jk}^i \to \widetilde{\lambda}_j^i \in \mathbb{R}, j = 1, \ldots, q, i = 1, \ldots, s$ as $k \to \infty$ with $\sum_{j=1}^{q}(\widetilde{\lambda}_j^0)^2 + \sum_{j=1}^{q}\sum_{i=1}^{s}(\widetilde{\lambda}_j^i)^2 = 1$. By letting $\widetilde{\lambda}_k := \frac{1}{\sqrt{\sum_{j=1}^{q}(\lambda_{jk}^0)^2 + \sum_{j=1}^{q}\sum_{i=1}^{s}(\lambda_{jk}^i)^2}}$, we get by (4.23) that

$$\begin{cases} \widetilde{\lambda}_k c_k + \sum_{j=1}^{q}(\widetilde{\lambda}_{jk}^0 a_{jk}^0 + \sum_{i=1}^{s}\widetilde{\lambda}_{jk}^i a_{jk}^i) = 0, \quad \widetilde{\lambda}_k c_k^\top x_k + \sum_{j=1}^{q}(\widetilde{\lambda}_{jk}^0 b_{jk}^0 + \sum_{i=1}^{s}\widetilde{\lambda}_{jk}^i b_{jk}^i) = 0, \\ \widetilde{\lambda}_{jk}^0(A_j^0 + \alpha_k I_j) + \sum_{i=1}^{s}\widetilde{\lambda}_{jk}^i A_j^i \succeq 0, j = 1, \ldots, q. \end{cases} \tag{4.24}$$

Letting $k \to \infty$ in (4.24), we obtain that

$$\sum_{j=1}^{q}(\widetilde{\lambda}_j^0 a_j^0 + \sum_{i=1}^{s}\widetilde{\lambda}_j^i a_j^i) = 0, \quad \sum_{j=1}^{q}\widetilde{\lambda}_j^0 b_j^0 + \sum_{j=1}^{q}\sum_{i=1}^{s}\widetilde{\lambda}_j^i b_j^i = 0, \tag{4.25}$$

$$\widetilde{\lambda}_j^0 A_j^0 + \sum_{i=1}^{s}\widetilde{\lambda}_j^i A_j^i \succeq 0, j = 1, \ldots, q. \tag{4.26}$$

Similar to the proof of Theorem 3.1(i), for a given $j \in \{1, \ldots, q\}$, we assert by the compactness of U_j^0 and (4.26) that if $\widetilde{\lambda}_j^0 = 0$, then $\widetilde{\lambda}_j^i = 0$ for all $i = 1, \ldots, s$. This fact together with the equality $\sum_{j=1}^{q}(\widetilde{\lambda}_j^0)^2 + \sum_{j=1}^{q}\sum_{i=1}^{s}(\widetilde{\lambda}_j^i)^2 = 1$ ensures that there exists $j \in \{1, \ldots, q\}$ such that $\widetilde{\lambda}_j^0 \neq 0$, i.e., $\sum_{j=1}^{q}\widetilde{\lambda}_j^0 \neq 0$. As shown in the proof of Theorem 3.1(i), the relation (4.26) entails the existence of $\widetilde{u}_j \in U_j^0, j = 1, \ldots, q$ such that

(4.27)
$$\widetilde{\lambda}_j^0\big(a_j^0 + \sum_{i=1}^{s}\widetilde{u}_j^i a_j^i\big)^\top x = \big(\widetilde{\lambda}_j^0 a_j^0 + \sum_{i=1}^{s}\widetilde{\lambda}_j^i a_j^i\big)^\top x, \ \widetilde{\lambda}_j^0\big(b_j^0 + \sum_{i=1}^{s}\widetilde{u}_j^i b_j^i\big) = \widetilde{\lambda}_j^0 b_j^0 + \sum_{i=1}^{s}\widetilde{\lambda}_j^i b_j^i, \ j = 1, \ldots, q$$

for each $x \in \mathbb{R}^n$.

Since the Slater condition holds for the problem (PRLP) at p, we find $\widehat{x} \in \mathbb{R}^n$ such that

$$\big(a_j^0 + \sum_{i=1}^{s}\widetilde{u}_j^i a_j^i\big)^\top \widehat{x} < b_j^0 + \sum_{i=1}^{s}\widetilde{u}_j^i b_j^i, \ j = 1, \ldots, q.$$

This follows that

$$\sum_{j=1}^{q}\widetilde{\lambda}_j^0\big(a_j^0 + \sum_{i=1}^{s}\widetilde{u}_j^i a_j^i\big)^\top \widehat{x} < \sum_{j=1}^{q}\widetilde{\lambda}_j^0\big(b_j^0 + \sum_{i=1}^{s}\widetilde{u}_j^i b_j^i\big),$$

due to $\sum_{j=1}^{q} \widetilde{\lambda}_j^0 \neq 0$. Taking (4.27) into account, we arrive at

$$\Big(\sum_{j=1}^{q} \widetilde{\lambda}_j^0 a_j^0 + \sum_{j=1}^{q}\sum_{i=1}^{s} \widetilde{\lambda}_j^i a_j^i\Big)^\top \widehat{x} - \sum_{j=1}^{q} \widetilde{\lambda}_j^0 b_j^0 - \sum_{j=1}^{q}\sum_{i=1}^{s} \widetilde{\lambda}_j^i b_j^i < 0,$$

which contradicts (4.25). Therefore, we conclude that the assertion in (4.22) is valid.

Now, take any $\widetilde{p} := (\widetilde{c}, \widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\alpha}) \in V$ and $\check{p} := (\check{c}, \check{\mathcal{A}}, \check{\mathcal{B}}, \check{\alpha}) \in V$ such that $\widetilde{\alpha} \leq \check{\alpha}$. Letting $\widetilde{x} \in S(\widetilde{p})$ and $\check{x} \in S(\check{p})$, we get by Lemma 2.2 that

$$\begin{cases} \exists \widetilde{\lambda}_j^0 \geq 0, \widetilde{\lambda}_j^i \in \mathbb{R}, j = 1, \ldots, q, i = 1, \ldots, s \text{ such that} \\ \widetilde{c} + \sum\limits_{j=1}^{q} (\widetilde{\lambda}_j^0 \widetilde{a}_j^0 + \sum_{i=1}^{s} \widetilde{\lambda}_j^i \widetilde{a}_j^i) = 0,\ \widetilde{c}^\top \widetilde{x} + \sum\limits_{j=1}^{q} (\widetilde{\lambda}_j^0 \widetilde{b}_j^0 + \sum_{i=1}^{s} \widetilde{\lambda}_j^i \widetilde{b}_j^i) = 0, \\ \text{and } \widetilde{\lambda}_j^0 (A_j^0 + \widetilde{\alpha} I_j) + \sum_{i=1}^{s} \widetilde{\lambda}_j^i A_j^i \succeq 0, j = 1, \ldots, q, \end{cases} \tag{4.28}$$

and

$$\begin{cases} \exists \check{\lambda}_j^0 \geq 0, \check{\lambda}_j^i \in \mathbb{R}, j = 1, \ldots, q, i = 1, \ldots, s \text{ such that} \\ \check{c} + \sum\limits_{j=1}^{q} (\check{\lambda}_j^0 \check{a}_j^0 + \sum_{i=1}^{s} \check{\lambda}_j^i \check{a}_j^i) = 0,\ \check{c}^\top \check{x} + \sum\limits_{j=1}^{q} (\check{\lambda}_j^0 \check{b}_j^0 + \sum_{i=1}^{s} \check{\lambda}_j^i \check{b}_j^i) = 0, \\ \text{and } \check{\lambda}_j^0 (A_j^0 + \check{\alpha} I_j) + \sum_{i=1}^{s} \check{\lambda}_j^i A_j^i \succeq 0, j = 1, \ldots, q. \end{cases} \tag{4.29}$$

Note as above that the linear matrix inequalities in (4.28) ensure the existence of $\widetilde{u}_j \in U_j^{\widetilde{\alpha}}, j = 1, \ldots, q$ such that

$$\widetilde{\lambda}_j^0 \big(\widetilde{a}_j^0 + \sum_{i=1}^{s} \widetilde{u}_j^i \widetilde{a}_j^i\big)^\top x = \big(\widetilde{\lambda}_j^0 \widetilde{a}_j^0 + \sum_{i=1}^{s} \widetilde{\lambda}_j^i \widetilde{a}_j^i\big)^\top x,\ \widetilde{\lambda}_j^0 \big(\widetilde{b}_j^0 + \sum_{i=1}^{s} \widetilde{u}_j^i \widetilde{b}_j^i\big) = \widetilde{\lambda}_j^0 \widetilde{b}_j^0 + \sum_{i=1}^{s} \widetilde{\lambda}_j^i \widetilde{b}_j^i,\ j = 1, \ldots, q$$

for each $x \in \mathbb{R}^n$. Hence, by (4.28),

$$\widetilde{c}^\top x = -\sum_{j=1}^{q} (\widetilde{\lambda}_j^0 \widetilde{a}_j^0 + \sum_{i=1}^{s} \widetilde{\lambda}_j^i \widetilde{a}_j^i)^\top x = -\sum_{j=1}^{q} \widetilde{\lambda}_j^0 \big(\widetilde{a}_j^0 + \sum_{i=1}^{s} \widetilde{u}_j^i \widetilde{a}_j^i\big)^\top x = -\sum_{j=1}^{q} \widetilde{\lambda}_j^0 \widetilde{\mathcal{A}}_j (\widetilde{u}_j)^\top x \tag{4.30}$$

for any $x \in \mathbb{R}^n$, and

$$\widetilde{c}^\top \widetilde{x} = -\sum_{j=1}^{q} (\widetilde{\lambda}_j^0 \widetilde{b}_j^0 + \sum_{i=1}^{s} \widetilde{\lambda}_j^i \widetilde{b}_j^i) = -\sum_{j=1}^{q} \widetilde{\lambda}_j^0 \big(\widetilde{b}_j^0 + \sum_{i=1}^{s} \widetilde{u}_j^i \widetilde{b}_j^i\big) = -\sum_{j=1}^{q} \widetilde{\lambda}_j^0 \widetilde{\mathcal{B}}_j (\widetilde{u}_j). \tag{4.31}$$

Then, we get by (4.30) and (4.31) that

$$\begin{aligned} \widetilde{c}^\top (\widetilde{x} - \check{x}) &= \sum_{j=1}^{q} \widetilde{\lambda}_j^0 \widetilde{\mathcal{A}}_j (\widetilde{u}_j)^\top \check{x} - \sum_{j=1}^{q} \widetilde{\lambda}_j^0 \widetilde{\mathcal{B}}_j (\widetilde{u}_j) \\ &= \sum_{j=1}^{q} \widetilde{\lambda}_j^0 \check{\mathcal{A}}_j (\widetilde{u}_j)^\top \check{x} - \sum_{j=1}^{q} \widetilde{\lambda}_j^0 \widetilde{\mathcal{B}}_j (\widetilde{u}_j) + \sum_{j=1}^{q} \widetilde{\lambda}_j^0 \big[\widetilde{\mathcal{A}}_j (\widetilde{u}_j) - \check{\mathcal{A}}_j (\widetilde{u}_j)\big]^\top \check{x} \\ &\leq \sum_{j=1}^{q} \widetilde{\lambda}_j^0 \big[\check{\mathcal{B}}_j (\widetilde{u}_j) - \widetilde{\mathcal{B}}_j (\widetilde{u}_j)\big] + \sum_{j=1}^{q} \widetilde{\lambda}_j^0 \big[\widetilde{\mathcal{A}}_j (\widetilde{u}_j) - \check{\mathcal{A}}_j (\widetilde{u}_j)\big]^\top \check{x}, \end{aligned} \tag{4.32}$$

where the inequality in (4.32) holds as $\check{x} \in C(\check{p})$ and $\widetilde{u}_j \in U_j^{\widetilde{\alpha}} \subset U_j^{\check{\alpha}}, j = 1, \ldots, q$.

Now, taking (4.20), (4.22) and (4.32) into account, we obtain

$$\begin{aligned} v(\widetilde{p}) - v(\check{p}) = \widetilde{c}^\top \widetilde{x} - \check{c}^\top \check{x} &= (\widetilde{c} - \check{c})^\top \check{x} + \widetilde{c}^\top (\widetilde{x} - \check{x}) \\ &\leq M_1 ||\widetilde{p} - \check{p}|| + q\sqrt{M_2} ||\widetilde{p} - \check{p}|| + q M_1 \sqrt{M_2} ||\widetilde{p} - \check{p}|| \\ &= \kappa ||\widetilde{p} - \check{p}||, \end{aligned}$$

where $\kappa := M_1 + q\sqrt{M_2} + qM_1\sqrt{M_2} > 0$. So, the assertion of (i) holds and thus, the proof is complete. □

The following corollary recovers the nonemptiness and boundedness of the solution map for the problem (PRLP) at a given parameter and a characterization for the Lipschitz continuity of the optimal value function in the case of *fixed* index sets.

COROLLARY 4.4. **(Lipschitz continuity of the optimal value function)** *Let $\Lambda := \{0\}$, and let $p := (c, \mathcal{A}, \mathcal{B}, 0) \in Z$ be such that $-\infty < v(p) < +\infty$. Let the Slater condition hold for the problem (PRLP) at p. Then, the following statements are equivalent.*
(i) *The optimal value function $v : Z \to \overline{\mathbb{R}}$ is locally Lipschitz continuous at p, i.e., there exist a real number $\kappa > 0$ and a neighborhood V of p in Z such that*

$$|v(\widetilde{p}) - v(\check{p})| \leq \kappa ||\widetilde{p} - \check{p}|| \tag{4.33}$$

for $\widetilde{p} := (\widetilde{c}, \widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, 0) \in V, \check{p} := (\check{c}, \check{\mathcal{A}}, \check{\mathcal{B}}, 0) \in V$.
(ii) *$S(p)$ is nonempty and bounded.*

Proof. The proof is followed directly by Theorem 4.3. □

Remark 4.5. The implication of (ii) $\Longrightarrow$ (i) in Corollary 4.4 was given in [1, Theorem 3.5] and in [2, Theorem 4.2(iii)]. Note further that the proof of [2, Theorem 4.2(iii)] was presented in a primal approach with an arbitrary index set.

5. Conclusions. In this paper, we have studied a parametric robust linear problem (PRLP) that includes data perturbations on the objective and constraints as well as the uncertainty sets. Employing a dual approach, we have examined the stability and sensitivity properties of (PRLP) by exploring how the behaviors of the optimal value function and the solution map of the underlying problem change with respect to the changes of the parameters. It has been shown that, under the Slater condition and the nonemptiness of the solution map of (PRLP) at a reference point, the solution map is always closed, and if assume in addition that the solution map at this point is bounded then the solution map is upper semicontinuous. A characterization in terms of the solution existence has been established for the lower semicontinuity of the solution map. In this way, we have been able to examine the lower and upper semicontinuities and order-Lipschitz property of the optimal value function of (PRLP).

By considering the index set parameters fixed, we have derived the nonemptiness and boundedness of the solution sets and a characterization for the Lipschitz continuity of the optimal value function for semi-infinite linear programs. It would be interesting to see if we can analogously examine the stability properties of the feasible set mapping and tailor their behaviours to the obtained stability properties of the solution map or optimal value function. It is also worth extending these approaches to explore other related properties including calmness [15] or more general classes of parametric optimization problems. Furthermore, we would like to deploy appropriate applications to examine the solution stability and sensitivity in practical scenarios such as the demand and supply relationships in the electric power market uplifting as in [20], or the heat and mass transport principles in the process engineering modelling as in [28], where the problem data frequently involve uncertainty factors.

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