Published in: *Engineering Analysis with Boundary Elements*, Vol. 26, 2002, 681-690

Localized Boundary-Domain Integral Formulations for Problems with Variable Coefficients

S.E. Mikhailov

Dept. of Mathematics, Glasgow Caledonian Univ., Cowcaddens Road, Glasgow, G4 0BA, UK

Abstract

Specially constructed localized parametrixes are used in this paper instead of a fundamental solution to reduce a Boundary Value Problem with variable coefficients to a Localized Boundary-Domain Integral or Integro-Differential Equation (LBDIE or LBDIDE). After discretization, this results in a sparsely populated system of linear algebraic equations, which can be solved by well-known efficient methods. This make the method competitive with the Finite Element Method for such problems. Some methods of the parametrix localization are discussed and the corresponding LBDIEs and LBDIDEs are introduced. Both mesh-based and meshless algorithms for the localized equations discretization are described.

Key words: Parametrix, Boundary-Domain Integral Equation, Boundary-Domain Integro-Differential Equation, Boundary Element Method, Localization, Mesh-based Algorithm, Meshless Algorithm, Domain Decomposition
1991 MSC: 65N, 35J25, 35C15, 35Q, 45A05

Email address: s.mikhailov@gcal.ac.uk (S.E. Mikhailov).
URL: www.maths.gcal.ac.uk/people/contactssmik.html (S.E. Mikhailov).

1 Introduction

The Boundary Integral Equation (BIE) method (boundary element method, elastic potential method) has been intensively developed over recent decades both in theory and in engineering applications. Its popularity is due to the possibility (at least for some problems with constant coefficients) of reducing a Boundary Value Problem (BVP) for a linear partial differential equation in a domain to an integral equation on the domain boundary, that is, to diminish the problem dimensionality by one. It leads to a diminution of the linear algebraic equations system, which results from discretization, and allows to obtain numerical solutions using small computer resources. The main thing necessary for the reduction of a BVP to a BIE is a fundamental solution to the original partial differential equation, available in an analytical form and/or cheaply calculated. After the fundamental solution is used in the corresponding Green formulae, one can reduce the problem to a boundary integral equation.

However, such a fundamental solution is generally not available if the coefficients of the original BVP are not constant. The BVPs of heat transfer with variable heat conductivity coefficients and the BVPs of elastic shells particularly belong to this category. One can use, in this case, a parametrix (Levi function), which is usually available, instead of the fundamental solution in the Green formulae. Parametrix correctly describes the main part of the fundamental solution but is not required to satisfy the original differential equations apart from the singular point. This allows a reduction of the problem not to boundary but to Boundary-Domain Integral or Integro-Differential Equation (BDIE or BDIDE) [1–3], see also [4–7]. For numerical solving the BDIE or BDIDE, one should discretize not only the domain boundary but also the domain itself and arrives after discretization at a system of linear algebraic equations, as in the Finite Element Method (FEM), without any dimension diminution. Unfortunately, this system unlike FEM is fully populated which prevents the use of economical methods developed for sparsely populated system solution.

To prevent this difficulty and to make the BDIE/BDIDE method competitive with the FEM, some *localized* parametrixes are constructed and used in this paper to reduce BVPs with variable coefficients to *localized* BDIEs or BDIDEs, developing the approach of [8]. This results, after discretization, in sparsely populated systems of linear algebraic equations, which can be solved by well known efficient methods. The local boundary integral equation method of [9,10] can be considered as a particular realization of the localized BDIE/BDIDE method described here.

2 BVP, fundamental solution, parametrix, integral and integro-differential equations

2.1 Stationary heat transfer problem in an inhomogeneous body

The presented approach is sufficiently general but for illustration of the idea we consider a stationary heat transfer boundary value problem in an isotropic inhomogeneous 2D or 3D body Ω, with a prescribed temperature $\bar{u}(x)$ on a closed part $\partial_D\Omega$ of the boundary $\partial\Omega$ and prescribed heat flux $\bar{t}(x)$ on the remaining part $\partial_N\Omega$ of $\partial\Omega$. That is, we consider an equation

$$(Lu)(x) := \frac{\partial}{\partial x_i}\left[a(x)\frac{\partial u(x)}{\partial x_i}\right] = f(x), \quad x \in \Omega \tag{1}$$

with the mixed boundary conditions

$$u(x) = \bar{u}(x),\ x \in \partial_D\Omega \tag{2}$$
$$Tu(x) = \bar{t}(x),\ x \in \partial_N\Omega \tag{3}$$

where Ω is an open (without boundaries) domain, $u(x)$ is an unknown temperature, $a(x)$ is a known variable thermo-conductivity coefficient, $f(x)$ is a known distributed heat source, T is a surface flux operator, $(Tu)(x) := a(x)\partial u(x)/\partial n(x)$, $n(x)$ is an external normal vector to the boundary $\partial\Omega$, $\bar{u}(x)$ and $\bar{t}(x)$ are known functions. Summation in repeated indices is supposed from 1 to 2 in the 2D and from 1 to 3 in the 3D case unless stated otherwise.

The Green formula for the differential operator L has the form

$$\int_{\Omega} [uLv - vLu]\, d\Omega = \int_{\partial\Omega} [uTv - vTu]\, d\Gamma \tag{4}$$

2.2 Fundamental solution, integral and integro-differential equations

Suppose $F(x, y)$ is a fundamental solution for the operator L, that is, a solution to the equation

$$L_x F(x, y) = \delta(x - y),$$

where $\delta(x - y)$ is the Dirac delta-function. Then taking $u(x)$ in (4) as an unknown solution of (1) and $v(x)$ as the fundamental solution $F(x, y)$, one obtains, by the usual way (see e.g. [11]), the integral equality:

$$c(y)u(y) - \int_{\partial\Omega} [u(x)T_x F(x,y) - F(x,y)Tu(x)]\, d\Gamma(x) = \int_{\Omega} F(x,y)f(x)d\Omega(x), \quad (5)$$

$$c(y) = c(y;\Omega) = \begin{cases} 1 \text{ if } y \in \Omega, \\ 0 \text{ if } y \notin \bar{\Omega} \\ \alpha(y)/(2\pi) \text{ if } y \in \partial\Omega \text{ and } \Omega \subset \mathbb{R}^2 \\ \alpha(y)/(4\pi) \text{ if } y \in \partial\Omega \text{ and } \Omega \subset \mathbb{R}^3 \end{cases} \quad (6)$$

where $\alpha(y)$ is an interior space angle at a corner point y of the boundary $\partial\Omega$, particularly, $c(y) = 1/2$ if y is a smooth point of the boundary. (Note that the signs in front of the surface integrals in (5) throughout this paper differ from those in [8], where they correspond in fact to the use of the *internal* normal vector n in the flux operator T definition.)

Different combinations of representation (5) with boundary conditions (2) and (3) lead to different integral or integro-differential systems, which, in turn, can lead to different numerical realizations. We will in particular consider two of them in this paper.

One way is to substitute boundary conditions (2) and (3) into (5), to introduce a new variable $t(x) = Tu(x)$ for the unknown flux on $\partial_D\Omega$, and to use (5) at $y \in \Omega \cup \partial\Omega$ to reduce BVP (1), (2), (3) to a *Boundary-Domain Integral Equation* (BDIE) for $u(x)$ at $x \in \Omega \cup \partial_N\Omega$ and $t(x)$ at $x \in \partial_D\Omega$,

$$c^0(y)u(y) - \int_{\partial_N\Omega} u(x)T_x F(x,y)d\Gamma(x) + \int_{\partial_D\Omega} F(x,y)t(x)d\Gamma(x) = \mathcal{F}^0(y), \quad (7)$$

$$\mathcal{F}^0(y) := [c^0(y) - c(y)]\bar{u}(y) + \mathcal{F}(y), \qquad y \in \Omega \cup \partial\Omega,$$

$$\mathcal{F}(y) := \int_{\partial_D\Omega} \bar{u}(x)T_x F(x,y)d\Gamma(x) - \int_{\partial_N\Omega} F(x,y)\bar{t}(x)d\Gamma(x) + \int_{\Omega} F(x,y)f(x)d\Omega(x), \quad (8)$$

$$c^0(y) = 0 \text{ if } y \in \partial_D\Omega, \ c^0(y) = c(y) \text{ if } y \in \Omega \cup \partial_N\Omega. \quad (9)$$

Since the left hand side of integral equation (7) at the boundary points $y \in \partial\Omega$ is expressed in terms of only boundary values $u(x)$ ($x \in \partial_N\Omega$) and $t(x)$ ($x \in \partial_D\Omega$), one can split (7) and first solve the equation at $y \in \partial\Omega$ for $u(x)$ ($x \in \partial_N\Omega$) and $t(x)$ ($x \in \partial_D\Omega$), and then use equation (7) at the remaining points $y \in \Omega$ for calculating $u(y)$ at the internal points. By this way one reduces the original BVP (1), (2), (3) to a direct *Boundary Integral Equation*

(BIE) with well known advantages (diminishing of the problem dimensionality by one) and disadvantages (system of linear algebraic equations with a fully populated matrix after discretization).

Another approach includes substituting boundary conditions (2) and (3) into integral equality (5) but leaving T as a differential flux operator acting on u at the Dirichlet boundary part $\partial_D\Omega$ and using the resulting *Boundary-Domain Integro-Differential Equation* only on $\Omega \cup \partial\Omega_N$,

$$c(y)u(y) - \int\limits_{\partial_N\Omega} u(x)T_xF(x,y)d\Gamma(x) + \int\limits_{\partial_D\Omega} F(x,y)Tu(x)d\Gamma(x) = \mathcal{F}(y), \quad (10)$$

$$y \in \Omega \cup \partial_N\Omega,$$

where $\mathcal{F}(y)$ is given by (8). Complementing the BDIDE with the Dirichlet boundary condition (2) at $y \in \partial_D\Omega$ reduces BVP (1), (2), (3) to a *Boundary-Domain Integro-Differential Problem*, BDIDP (10), (2) for $u(x)$, $x \in \Omega \cup \partial_N\Omega$.

2.3 Parametrix, integral and integro-differential equations

For the partial differential operators with variable coefficients, like L in (1), a fundamental solution is usually not available in an explicit form or the form is too expensive to use for numerical solution of the BIE. However, a *parametrix* is often available instead, which is a function $P(x,y)$ satisfying

$$L_xP(x,y) = \delta(x-y) + R(x,y),$$

where the remainder term $R(x,y)$ as function of $x \in \Omega$ has not more than a weak (integrable) singularity at $x = y$.

One can check that a parametrix for (1) is given by the fundamental solution to the same equation but with the "frozen" coefficient $a(x) = a(y)$. The equation then becomes the Laplace equation up to the constant multiplier and the parametrix for (1) is $P(x,y) = F_\Delta(x,y)/a(y)$, where $F_\Delta(x,y)$ is a fundamental solution for the Laplace operator Δ,

$$F_\Delta(x,y) = \frac{\ln|x-y|}{2\pi}, \quad x,y \in \mathbb{R}^2; \quad F_\Delta(x,y) = \frac{-1}{4\pi|x-y|}, \quad x,y \in \mathbb{R}^3,$$

where $|x-y| = \sqrt{(x_i-y_i)(x_i-y_i)}$. Then

$$P(x,y) = \frac{\ln|x-y|}{2\pi a(y)}, \quad R(x,y) = \frac{x_i-y_i}{2\pi a(y)|x-y|^2}\frac{\partial a(x)}{\partial x_i}, \quad x,y \in \mathbb{R}^2, \; (11)$$

$$P(x,y) = \frac{-1}{4\pi a(y)|x-y|}, \quad R(x,y) = \frac{x_i - y_i}{4\pi a(y)|x-y|^3} \frac{\partial a(x)}{\partial x_i}, \quad x, y \in \mathbb{R}^3. \tag{12}$$

Substituting in (4) $P(x,y)$ for $v(x)$ and taking $u(x)$ as a solution to (1), we arrive at an integral equality,

$$c(y)u(y) - \int_{\partial\Omega} [u(x)T_x P(x,y) - P(x,y)Tu(x)]\, d\Gamma(x) + \int_{\Omega} R(x,y)u(x)d\Omega(x) = \int_{\Omega} P(x,y)f(x)d\Omega(x) \tag{13}$$

where $c(y)$ is given by (6).

As in the previous subsection, substituting boundary conditions (2) and (3) into (13), introducing a new variable $t(x) = Tu(x)$ for the unknown flux on $\partial_D\Omega$, and using (13) at $y \in \Omega \cup \partial\Omega$ reduces BVP (1), (2), (3) to the following BDIE for $u(x)$ at $x \in \Omega \cup \partial_N\Omega$ and $t(x)$ at $x \in \partial_D\Omega$,

$$c^0(y)u(y) - \int_{\partial_N\Omega} u(x)T_x P(x,y)d\Gamma(x) + \int_{\partial_D\Omega} P(x,y)t(x)d\Gamma(x) + \int_{\Omega} R(x,y)u(x)d\Omega(x) = \mathcal{F}^0(y), \quad y \in \Omega \cup \partial\Omega. \tag{14}$$

$$\mathcal{F}^0(y) := [c^0(y) - c(y)]\bar{u}(y) + \mathcal{F}(y),$$
$$\mathcal{F}(y) := \int_{\partial_D\Omega} \bar{u}(x)T_x P(x,y)d\Gamma(x) - \int_{\partial_N\Omega} P(x,y)\bar{t}(x)d\Gamma(x) + \int_{\Omega} P(x,y)f(x)d\Omega(x), \tag{15}$$

where $c^0(y)$ is given by (9).

Since even for boundary points y, the last term in the left hand side of (14) includes the unknown values of u over the whole domain Ω, this BDIE does not lead to a *Boundary* Integral Equation as in the case when the parametrix is a fundamental solution, described in the previous subsection.

Using another approach, one can substitute boundary conditions (2) and (3) into integral equality (13) but leave T as a differential flux operator acting on u on the Dirichlet boundary part $\partial_D\Omega$ and use the following BDIDE only at $y \in \Omega \cup \partial\Omega_N$,

$$c(y)u(y) - \int_{\partial_N\Omega} u(x)T_xP(x,y)d\Gamma(x) + \int_{\partial_D\Omega} P(x,y)Tu(x)d\Gamma(x)$$
$$+\int_{\Omega} R(x,y)u(x)d\Omega(x) = \mathcal{F}(y), \quad y \in \Omega \cup \partial\Omega_N, \tag{16}$$

where $\mathcal{F}(y)$ is given by (15). Complementing the BDIDE with the Dirichlet boundary condition (2) at $y \in \partial_D\Omega$ reduces BVP (1), (2), (3) to a BDIDP (16), (2) for $u(x)$, $x \in \Omega \cup \partial_N\Omega$. As we will see below, this approach can lead, after discretization, to a system with a diminished number of linear algebraic equations.

3 Localized parametrix and BDIE/BDIDP

BDIE (14) as well as BDIDP (16), (2) can be reduced after some discretization to a system of linear algebraic equations and solved numerically. However, the system will include unknowns not only at the boundary but also at internal points (similar to the Finite Element Methods). Moreover, since the commonly used parametrixes, see e.g. (11), (12), are highly non-local, i.e. do not vanish for virtually all x, the matrix of the system will be fully populated (unlike the Finite Element Methods) - this prevents the use of well elaborated methods for sparsely populated systems.

To avoid this difficulty, we present some ideas on how to construct *localized* parametrixes and consequently *localized* BDIE/BDIDP (LBDIE/LBDIDP). This is based on the fact that a parametrix is not unique and is defined up to any function $\phi(x,y)$ such that $L_x\phi(x,y)$ has no more than weak singularities. In other words, all parametrixes $P(x,y)$ for a differential operator L have the same singularity at $x = y$ but can differ at other points. Thus we can perturb an available (not localized) parametrix $P^0(x,y)$ additively or multiplicatively by a proper function so as to localize it.

Particularly, we can consider a function

$$P(x,y) = \chi(x,y)P^0(x,y) \tag{17}$$

where $\chi(x,y)$ is a cut-off function, such that $\chi(y,y) = 1$ and $\chi(x,y) = 0$ at x not belonging to a localization domain $\omega(y)$ (a vicinity of y), see Fig. 1. Then $P(x,y)$ has the same singularity as $P^0(x,y)$ at $x = y$ but is localized (non zero) only on $\omega(y)$. Further we have,

$$L_x(P) = L_x(\chi P^0) = L_x(P^0) + L_x((1-\chi)P^0) = \delta(x-y) + R(x,y),$$
$$R(x,y) = R^0(x,y) + L_x((1-\chi)P^0).$$

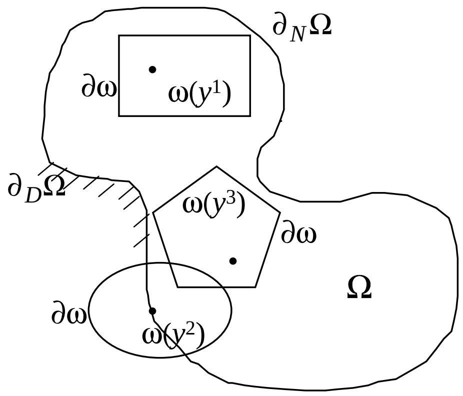

Fig. 1. A body Ω with localization domains $\omega(y^i)$

Consequently, if χ is smooth enough, then R will have the necessary properties of the remainder, that is, $P(x, y)$ given by (17) is also a parametrix. Some non-smooth cut-off functions are considered below as well.

3.1 Discontinuous localization

The simplest cut-off function is piecewise constant. Let a domain $\omega(y) \ni y$ be an arbitrary open neighborhood of a point y. Then the piecewise constant function $\chi(\omega)$ which is the characteristic function of the set ω is given by

$$\chi(\omega) = \left\{ \begin{array}{l} 1, \;\; x \in \bar{\omega} \\ 0, \;\; x \notin \bar{\omega} \end{array} \right\}.$$

If one uses this function in (17), one arrives at a discontinuous localized parametrix

$$P(x, y) = \left\{ \begin{array}{ll} P^0(x, y), & x \in \bar{\omega}(y) \\ 0, & x \notin \bar{\omega}(y) \end{array} \right\}.$$

Instead of further substituting this discontinuous parametrix into the Green formula (4), which would demand a careful analysis of the $L_x P(x, y)$ behavior in the distribution sense, we apply the Green formula not to the domain Ω but to its intersection with $\omega(y)$ and arrive at the integral equality localized on the intersection $\omega(y) \cap \Omega$ and on its boundary $\partial[\omega(y) \cap \Omega]$,

$$c(y)u(y) - \int_{\bar{\omega}(y)\cap\partial\Omega} u(x) T_x P(x, y) d\Gamma(x) + \int_{\bar{\omega}(y)\cap\partial\Omega} P(x, y) T u(x) d\Gamma(x)$$
$$- \int_{\Omega\cap\partial\omega(y)} u(x) T_x P(x, y) d\Gamma(x) + \int_{\Omega\cap\partial\omega(y)} P(x, y) T u(x) d\Gamma(x)$$

$$+\int\limits_{\omega(y)\cap\Omega} R(x,y)u(x)d\Omega(x) = \int\limits_{\omega(y)\cap\Omega} P(x,y)f(x)d\Omega(x) \tag{18}$$

Thus, the simplest localization by the piecewise constant cut-off function leads to the localized integral equality. It includes the volume integral along the part of $\omega(y)$ belonging to Ω with unknown function u and integrals along the part of the boundary $\partial\omega(y)$ of the localization domain belonging to Ω with unknown function u and its flux Tu, in addition to integrals along the part of the global boundary $\partial\Omega$ intersecting with the closure $\bar{\omega}(y)$ of the localization domain $\omega(y)$ with unknown function u or flux Tu.

As for its non-localized counterpart (13), we can use equality (18) to arrive either to BDIE or to BDIDP.

Substitution of boundary conditions (2) and (3) in (18) and introduction of a new variable $t(x) = Tu(x)$ at $x \in \partial\Omega_D$ reduces BVP (1), (2), (3) to the following BDIDE for $u(x)$, $x \in \Omega \cup \partial_N\Omega$ and $t(x) = Tu(x)$, $x \in \partial_D\Omega$,

$$c^0(y)u(y) - \int\limits_{\bar{\omega}(y)\cap\partial_N\Omega} u(x)T_xP(x,y)d\Gamma(x) + \int\limits_{\bar{\omega}(y)\cap\partial_D\Omega} P(x,y)t(x)d\Gamma(x) \tag{19a}$$

$$- \int\limits_{\Omega\cap\partial\omega(y)} u(x)T_xP(x,y)d\Gamma(x) + \int\limits_{\Omega\cap\partial\omega(y)} P(x,y)Tu(x)d\Gamma(x) \tag{19b}$$

$$+ \int\limits_{\omega(y)\cap\Omega} R(x,y)u(x)d\Omega(x) = \mathcal{F}^0(y), \quad y \in \Omega \cup \partial\Omega, \tag{19c}$$

$$\mathcal{F}^0(y) := [c^0(y) - c(y)]\bar{u}(y) + \mathcal{F}(y), \tag{20}$$

$$\mathcal{F}(y) := \int\limits_{\bar{\omega}(y)\cap\partial_D\Omega} \bar{u}(x)T_xP(x,y)d\Gamma(x) - \int\limits_{\bar{\omega}(y)\cap\partial_N\Omega} P(x,y)\bar{t}(x)d\Gamma(x) + \int\limits_{\omega(y)\cap\Omega} P(x,y)f(x)d\Omega(x). \tag{21}$$

Unlike its non-localized counterpart (16), the above equation is not integral but integro-differential, since it includes an unknown flux $Tu(x)$ on $\Omega\cap\partial\omega(y)$.

We can also substitute boundary conditions (2) and (3) into integral equality (18) but leave T as a differential flux operator, acting on u at the Dirichlet boundary part $\partial_D\Omega$, and use the following BDIDE only at $y \in \Omega \cup \partial\Omega_N$,

$$c(y)u(y) - \int\limits_{\bar{\omega}(y)\cap\partial_N\Omega} u(x)T_xP(x,y)d\Gamma(x) + \int\limits_{\bar{\omega}(y)\cap\partial_D\Omega} P(x,y)Tu(x)d\Gamma(x) \tag{22a}$$

$$- \int_{\Omega \cap \partial\omega(y)} u(x) T_x P(x,y) d\Gamma(x) + \int_{\Omega \cap \partial\omega(y)} P(x,y) T u(x) d\Gamma(x) \tag{22b}$$

$$+ \int_{\omega(y) \cap \Omega} R(x,y) u(x) d\Omega(x) = \mathcal{F}(y), \quad y \in \Omega \cup \partial_N \Omega, \tag{22c}$$

where $\mathcal{F}(y)$ is given by (21). We arrive then to BDIDP (22), (2) for $u(x)$, $x \in \Omega \cup \partial_N \Omega$.

Note that if $P(x,y)$ is a fundamental solution at $x \in \omega(y)$, then the last (volume) integrals with $R(x,y)$ disappear in the left hand sides of (18), (19) and (22).

3.2 Relation with the subdomain (domain decomposition) method

Let us cover the domain Ω by a subdomain closure set $\bar{\Omega}_i \subset \bar{\Omega}$ and take $\omega(y) = \Omega_i$ if $y \in \Omega_i$. Suppose parametrixes $P_i(x,y)$ are used in each subdomain Ω_i. Then localized integral representation (18) generates a *subdomain* (*domain decomposition*) version of the BDIE if we replace $c(y) = c(y;\Omega)$ by the coefficient $c(y;\Omega_i)$ corresponding to Ω_i. Particularly, for the case when the Ω_i do not intersect but can have interfaces with each other, one can introduce a new variable $t(x) = Tu(x)$ for the unknown flux also on $\partial\Omega_i \backslash \partial_N \Omega$ and arrive at the BDIE on the subdomains and their boundaries

$$\begin{aligned}
c^0(y)u(y) &- \int_{\partial\Omega_i \cap \partial_N \Omega} u(x) T_x P_i(x,y) d\Gamma(x) + \int_{\partial\Omega_i \cap \partial_D \Omega} P_i(x,y) t(x) d\Gamma(x) \\
&- \int_{\Omega \cap \partial\Omega_i} u(x) T_x P_i(x,y) d\Gamma(x) + \int_{\Omega \cap \partial\Omega_i} P_i(x,y) t(x) d\Gamma(x) \\
&+ \int_{\Omega_i \cap \Omega} R(x,y) u(x) d\Omega(x) = \mathcal{F}^0(y), \quad y \in \bar{\Omega}_i,
\end{aligned} \tag{23}$$

$$\begin{aligned}
\mathcal{F}^0(y) := [c^0(y) - c(y;\Omega_i)]\bar{u}(y) &+ \int_{\bar{\Omega}_i \cap \partial_D \Omega} \bar{u}(x) T_x P_i(x,y) d\Gamma(x) \\
- \int_{\bar{\Omega}_i \cap \partial_N \Omega} P_i(x,y) \bar{t}(x) d\Gamma(x) &+ \int_{\Omega_i \cap \Omega} P_i(x,y) f(x) d\Omega(x),
\end{aligned}$$

$$c^0(y) = 0 \text{ if } y \in \partial\Omega_i \cap \partial_D \Omega, \quad c^0(y) = c(y;\Omega_i) \text{ if } y \in \bar{\Omega}_i \backslash \partial_D \Omega.$$

which should be complemented by the interface conditions for u and t in neighboring subdomains Ω_i.

Let some fundamental solutions $F_i(x,y)$ be used as parametrixes $P_i(x,y)$ on each subdomain. Then the last volume integral disappears in BDIE (23) and it can be considered at $y \in \partial\Omega_i$ along with the interface conditions as the well-known *subdomain* (*domain decomposition*) version of the BIE for $u(x),\ x \in \partial\Omega_i \backslash \partial_D\Omega$ and $t(x),\ x \in \partial\Omega_i \backslash \partial_N\Omega$, and as an integral representation for $u(y)$ at $y \in \Omega_i$ after the BIE is solved.

3.3 Continuous non-smooth localization

To get rid of integrals including Tu on $\partial\omega(y)$ in (18), (19) and (22), one can construct a localized parametrix $P(x,y)$ vanishing on the boundary $\partial\omega(y)$ but not necessarily with vanishing parametrix flux $T_x P(x,y)$.

Internally smooth localization

If the parametrix $P(x,y)$ is continuous in $x \in \bar{\Omega}$ and smooth in $x \in \omega(y)$ except at $x = y$ and vanishes on $\partial\omega(y)$, this reduces integral equality (18) to the following one

$$\begin{aligned} c(y)u(y) - \int\limits_{\bar{\omega}(y)\cap\partial\Omega} u(x)T_x P(x,y)d\Gamma(x) + \int\limits_{\omega(y)\cap\partial\Omega} P(x,y)Tu(x)d\Gamma(x) \\ - \int\limits_{\Omega\cap\partial\omega(y)} u(x)T_x P(x,y)d\Gamma(x) \\ + \int\limits_{\omega(y)\cap\Omega} R(x,y)u(x)d\Omega(x) = \int\limits_{\omega(y)\cap\Omega} P(x,y)f(x)d\Omega(x) \end{aligned} \tag{24}$$

As a consequence, the second integral disappears in (19b) and BDIDE (19) becomes BDIE. Similarly, BDIDP (22), (2) also simplifies since the second integral disappears in (22b).

Different methods can be used to obtain a parametrix $P(x,y)$ vanishing on $\partial\omega(y)$. Particularly, the Green function on $\omega(y)$ (the difference between a fundamental solution and a function called therein a "companion solution") for a corresponding BVP with "frozen" coefficients and without junior derivative terms in the differential operator L, was employed as a parametrix $P^0(x,y)$ in [9,10]. However, the Green function is available in an analytical form only for sufficiently simple shapes of the localization domain $\omega(y)$, e.g. for a ball.

It seems to be simpler and more universal to construct a proper localized parametrix using formula (17), where $\chi(x,y)$ is a continuous in $x \in \Omega$ cut-off function, which is smooth in $\omega(y)$ and equal to zero both on the boundary and outside of $\omega(y)$, whereas P^0 is an available parametrix (e.g. a fundamental solution for a corresponding differential operator with "frozen" coefficients and without junior derivative terms).

Some examples of such cut-off functions localized on a ball $\omega_B(y;\rho) := \{x : |x-y| < \rho\}$ of a radius ρ around a point y or on a cube $\omega_C(y;\rho) = \{x : |x_i - y_i| < \rho,\ i = 1,...,n\}$ with an edge 2ρ around a point y in $\mathbb{R}^n$ are

$$\chi(x,y) = \left(1 - \frac{|x-y|^2}{\rho^2}\right), \quad x \in \bar{\omega}_B(y;\rho);$$
$$\chi(x,y) = \prod_{i=1}^{n}\left(1 - \frac{(x_i-y_i)^2}{\rho^2}\right), \quad x \in \bar{\omega}_C(y;\rho),$$

while $\chi(x,y) = 0$ at x outside the corresponding localization domain $\omega(y)$. Here $\prod_{i=1}^n a_i := a_1 a_2 ... a_n$.

Internally piecewise smooth localization

Let us consider a piecewise smooth cut-off function $\chi(x,y)$ that is continuous in $x \in \bar{\Omega}$, equal to zero at $x \notin \omega$, is continuously differentiable with respect to x in open subdomains $\omega_k(y)$, which constitute (together with their interfaces) $\omega(y)$, and has some jumps in normal derivatives with respect to x on interfaces γ between $\omega_k(y)$. Such cut-off functions naturally appear if we consider a mesh on Ω, take the mesh elements adjacent to a node point y as $\omega_k(y)$ and shape function corresponding to the node y as values of $\chi(x,y)$ in $\omega_k(y)$.

For example, if $\omega(y) \subset \mathbb{R}^2$ consists of triangles $\omega_k(y)$ with y being an apex of all them, Fig. 2, then $\chi(x,y)$ can be taken to be piecewise linear,

$$\chi(x,y) = 1 - \frac{|(x^{(k1)}-y)\times(x-y) + (x-y)\times(x^{(k2)}-y)|}{|(x^{(k1)}-y)\times(x^{(k2)}-y)|}, \quad x \in \omega_k(y)$$

Here $x^{(k1)}, x^{(k2)}$ are corners of $\omega_k(y)$ supplementary to y and $a\times b$ denotes the vector product of vectors a and b.

If $\omega(y) \subset \mathbb{R}^3$ consists of tetrahedrons $\omega_k(y)$ with y being an apex of all them, then $\chi(x,y)$ also can be taken piecewise linear,

$$\begin{aligned}\chi(x,y) = 1 - \{&(x^{(k1)}-y)\cdot[(x^{(k2)}-y)\times(x-y)] \\ &+(x^{(k1)}-y)\cdot[(x-y)\times(x^{(k3)}-y)] \\ &+(x-y)\cdot[(x^{(k2)}-y)\times(x^{(k3)}-y)]\} \\ /\ \{&(x^{(k1)}-y)\cdot[(x^{(k2)}-y)\times(x^{(k3)}-y)]\}\end{aligned}$$

for $x \in \bar{\omega}_k(y)$, where $x^{(k1)}, x^{(k2)}, x^{(k3)}$ are corners of $\omega_k(y)$ supplementary to y.

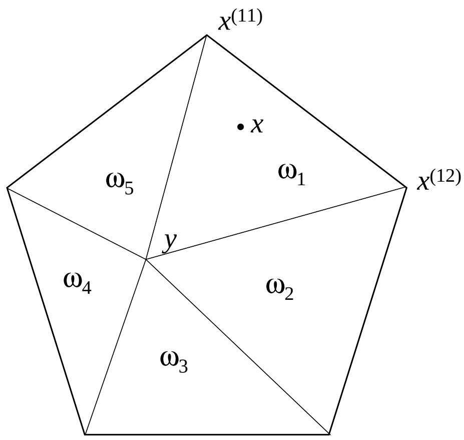

Fig. 2. Localization domain $\omega(y)$ consisting of triangles $\omega_k(y)$

Here $a \cdot b$ denotes the inner product and $a \cdot [b \times c]$ the mixed product of vectors a, b, and c.

We can substitute this piecewise smooth (except at point $x = y$) parametrix $P(x, y) = \chi(x, y)P^0(x, y)$ into the Green formula on the intersection of $\omega_k(y)$ with Ω and arrive at integral equalities similar to (18) but localized on the intersection $\omega_k(y) \cap \Omega$ and on its boundary $\partial[\omega_k(y) \cap \Omega]$. Summing up the integral equalities generated by each subdomain $\omega_k(y)$, we arrive at the BDIE localized on $\omega(y) \cap \Omega$, its boundary $\partial[\omega(y) \cap \Omega]$ and on the net interface γ between the smoothness subdomains $\omega_k(y)$,

$$c(y)u(y) - \int\limits_{\bar{\omega}(y)\cap\partial\Omega} u(x)T_xP(x,y)d\Gamma(x) + \int\limits_{\omega(y)\cap\partial\Omega} P(x,y)Tu(x)d\Gamma(x) \tag{25a}$$

$$- \int\limits_{\Omega\cap\partial\omega(y)} u(x)T_xP(x,y)d\Gamma(x) \tag{25b}$$

$$- \int\limits_{\gamma\cap\Omega} u(x) \left[T_xP(x,y)\Big|_{\gamma-} + T_xP(x,y)\Big|_{\gamma+} \right] d\Gamma(x) \tag{25c}$$

$$+ \int\limits_{[\omega(y)\setminus\gamma]\cap\Omega} R(x,y)u(x)d\Omega(x) = \int\limits_{\omega(y)\cap\Omega} P(x,y)f(x)d\Omega(x) \tag{25d}$$

We account for continuity of $\chi(x, y)$ and consequently $P(x, y)$ on γ and understand under the fluxes $T_xP(x, y)\Big|_{\gamma\mp}$ the limiting values of the fluxes on γ in the directions of the outer normal vectors $n(x)|_{\gamma\mp}$ for the adjacent subdomains $\omega_k(y)$, where $n(x)|_{\gamma-} = -n(x)|_{\gamma+}$. Obtaining (25), it was also taken into account that if $y \in \omega(y)$ belongs to an interface between some subdomains $\omega_k(y)$, $k = 1, ..., K$, and is an internal point of Ω, then $\sum_1^K c(y; \omega_k)(y) = 1$ owing to (6). However, if $y \in \omega(y)$ belongs to an interface between some subdomains $\omega_k(y)$, $k = 1, ..., K$, and is a boundary point of Ω, then $\sum_1^K c(y; \omega_k \cap \Omega)(y) =$

$c(y;\Omega)$. This means, the coefficients $c(y)$ in (25) are given by the same expression (6), that is, are determined by the position of y in Ω and does not depend on $\chi(x,y)$ and $\omega(y)$.

As a consequence of the jumps of the parametrix flux inside $\omega(y)$ and vanishing the parametrix on $\partial\omega(y)$, BDIDE (19) changes for this case since the second integral in (19b) is replaced by (25c) and the equation remains integro-differential. Similarly, BDIDP (22), (2) also changes for this case since the second integral in (22b) is replaced by (25c).

3.4 Globally smooth localization

To simplify the integral representation even further by getting rid of the remaining integral along $\partial\omega(y)$, one can employ a smooth in $x\in\bar\Omega$ cut-off function $\chi(x,y)$, which vanishes on $\partial\omega(y)$ together with its normal derivative in x. Then the same holds true also for the parametrix $P(x,y)=\chi(x,y)P^0(x,y)$.

Some examples of smooth cut-off functions localized on a ball $\omega_B(y;\rho)$ of a radius ρ around a point y or on a cube $\omega_C(y;\rho)$ with an edge 2ρ around a point y in $\mathbb{R}^n$, are

$$\chi(x,y)=\left(1-\frac{|x-y|^2}{\rho^2}\right)^2,\ x\in\bar\omega_B(y;\rho)$$
$$\chi(x,y)=\prod_{i=1}^{n}\left(1-\frac{(x_i-y_i)^2}{\rho^2}\right)^2,\ x\in\bar\omega_C(y;\rho)$$
$$\chi(x,y)=\exp\left(1-\frac{\rho^2}{\rho^2-|x-y|^2}\right),\ x\in\bar\omega_B(y;\rho)$$
$$\chi(x,y)=\prod_{i=1}^{n}\exp\left(1-\frac{\rho^2}{\rho^2-(x_i-y_i)^2}\right),\ x\in\bar\omega_C(y;\rho)$$

where $\chi(x,y)=0,\ x\notin\bar\omega(y)$. The first two functions χ are continuous and have continuous first derivatives and the last two are infinitely smooth in $\mathbb{R}^n$.

Integral representation (18) is reduced for such a parametrix to the following one,

$$c(y)u(y)-\int_{\omega(y)\cap\partial\Omega}u(x)T_xP(x,y)d\Gamma(x)+\int_{\omega(y)\cap\partial\Omega}P(x,y)Tu(x)d\Gamma(x)$$
$$+\int_{\omega(y)\cap\Omega}R(x,y)u(x)d\Omega(x)=\int_{\omega(y)\cap\Omega}P(x,y)f(x)d\Omega(x)\qquad(26)$$

If y is an internal point sufficiently far from the boundary $\partial\Omega$, then the boundary integrals along $\omega(y) \cap \partial\Omega$ vanish in the left hand side of (26) while only the volume integral along $\omega(y)$ remains. If y is a boundary point of Ω or an internal point near the boundary of Ω, then the volume integral along the intersection of $\omega(y)$ with Ω and the both boundary integrals along the intersection of $\partial\Omega$ with $\omega(y)$ remain.

As a consequence, both integrals along $\Omega(y) \cap \partial\omega(y)$ disappear in (19b) and BDIDE (19) becomes BDIE for this case. Similarly, BDIDP (22), (2) also simplifies since the both integrals along $\Omega(y) \cap \partial\omega(y)$ disappear in (22b).

4 Discretization of LBDIE/LBDIDP

To reduce LBDIDE (19) or BDIDP (22), (2) to a system of linear algebraic equations, one can first employ an interpolation or approximation formula to express the unknown function u at any point of integration or source point in terms of the values of the same or an auxiliary function at some node points. To make sure the system will be sparsely populated, the interpolation/approximation formula should be also local. After substitution of the interpolation/approximation formula into a LBDIDE/LBDIDP, either the collocation or the Petrov-Galerkin method can be applied. Only the first method is discussed below.

As demonstrated above, there is a lot of flexibility in constructing appropriate cut-off functions for rather arbitrary shapes and combinations of volume elements. It seems to be preferable to use continuous cut-off functions to eliminate the unknown flux Tu from the formulation except on the global boundary $\partial\Omega$ (or even smooth cut-off functions to get rid of all the surface integrals along $\partial\omega(y)$ and γ) and work with the simplified LBDIEs or LBDIDPs. We will consider below the sufficiently general case of the discontinuous localization described in subsection 3.1 and show the simplifications for more smooth localizations.

4.1 Mesh-based discretization

Suppose the domain Ω is covered by a mesh of closed volume elements e_k with nodes set up at the corners, edges, faces, and/or inside the elements. Let J be the total number of nodes x^i ($i = 1, 2, ..., J$), from which there are J_D nodes on $\partial_D\Omega$. One can use each node x^i as a collocation point for an LBDIE with a localization domain $\omega(x^i)$. Let the union of closures of the volume elements that intersect with $\omega(x^i)$ be called the *total* localization domain $\tilde{\omega}(x^i)$, Fig.

3. Then the closure $\bar{\omega}(x^i) \cap \bar{\Omega}$ belongs to $\tilde{\omega}(x^i)$. If $\omega(x^i)$ is sufficiently small, then $\tilde{\omega}(x^i)$ consists only of the elements adjacent to the collocation point x^i. If $\omega(x^i)$ is from the very beginning chosen as consisting only of the elements adjacent to the collocation point x^i, which seems to be reasonable in practical calculations, then $\tilde{\omega}(x^i) = \bar{\omega}(x^i)$. Let $J_{\tilde{\omega}}(x^i)$ be the number of nodes belonging to $\tilde{\omega}(x^i)$.

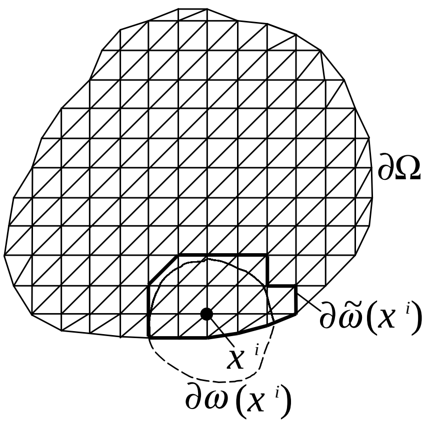

Fig. 3. A localization domain $\omega(x^i)$ and a total localization domain $\tilde{\omega}(x^i)$ associated with a collocation point x^i at a mesh-based discretization of a body Ω

Let us consider a continuous interpolation of $u(x)$ at any point $x \in \Omega$ in terms of the values of $u(x^j)$ at the node points x^j belonging to the same element $e_k \subset \Omega$ as x,

$$u(x) = \sum_j u(x^j)\phi_{kj}(x), \quad x, x^j \in e_k,$$

where the shape functions $\phi_{kj}(x)$ are localized on e_k. Collecting the interpolation formulae for all $x \in \tilde{\omega}(x^i)$, we have

$$u(x) = \sum_{x^j \in \tilde{\omega}(x^i)} u(x^j)\Phi_{ij}(x), \ x \in \tilde{\omega}(x^i), \tag{27}$$

$$\Phi_{ij}(x) = \begin{cases} \phi_{kj}(x) \text{ if } x, x^j \in e_k \subset \tilde{\omega}(x^i) \\ 0 \qquad \text{otherwise} \end{cases} \tag{28}$$

Consequently, $\Phi_{ij}(x) = 0$ if $x^j \notin \tilde{\omega}(x^i)$. We can also use a local interpolation of $t(x) = (Tu)(x^j)$ along only boundary nodes belonging to $\tilde{\omega}(x^i) \cap \partial_D\Omega$,

$$t(x) = \sum_{x^j \in \tilde{\omega}(x^i) \cap \partial_D\Omega} t(x^j)\Phi'_{ij}(x), \ x \in \tilde{\omega}(x^i) \cap \partial_D\Omega. \tag{29}$$

Here $\Phi'_{ij}(x)$ are the boundary shape functions obtained similar to $\Phi_{ij}(x)$ in (28) and such that $\Phi'_{ij}(x) = 0$ if $x^j \notin \tilde{\omega}(x^i) \cap \partial_D\Omega$.

After substitution of interpolations (27), (29), e.g., in LBDIDE (19), we arrive at the following system of J linear algebraic equations for J unknowns $u(x^j)$, $x^j \in \Omega \cup \partial_N \Omega$ and $t(x^j) = (Tu)(x^j)$, $x^j \in \partial_D \Omega$,

$$c^0(x^i)u(x^i) + \sum_{x^j \in \Omega \cup \partial_N \Omega} K^0_{ij} u(x^j) + \sum_{x^j \in \partial \Omega} Q_{ij} t(x^j) \\ = \mathcal{F}^0(x^i) - \sum_{x^j \in \partial_D \Omega} K^0_{ij} \bar{u}(x^j), \quad i = 1, ...J, \quad \text{no sum in } i, \quad (30)$$

where $\mathcal{F}^0(x^i)$ is calculated from (20),

$$K^0_{ij} = - \int\limits_{\bar{\omega}(x^i) \cap \partial_N \Omega} \Phi_{ij}(x) T_x P(x, x^i) d\Gamma(x) + \int\limits_{\omega(x^i) \cap \Omega} R(x, x^i) \Phi_{ij}(x) d\Omega(x) \quad (31a)$$

$$- \int\limits_{\Omega \cap \partial \omega(x^i)} \Phi_{ij}(x) T_x P(x, x^i) d\Gamma(x) + \int\limits_{\Omega \cap \partial \omega(x^i)} P(x, x^i) T \Phi_{ij}(x) d\Gamma(x), \quad (31b)$$

$$Q_{ij} = \int\limits_{\bar{\omega}(x^i) \cap \partial_D \Omega} P(x, x^i) \Phi'_{ij}(x) d\Gamma(x) \quad (32)$$

Instead, one can arrive at the system of only $J - J_D$ algebraic equations for $J - J_D$ unknowns $u(x^j)$, $x^j \in \Omega \cup \partial_N \Omega$, if one substitutes interpolation formulae (27) in BDIDP (22), (2),

$$c(x^i)u(x^i) + \sum_{x^j \in \Omega \cup \partial_N \Omega} K_{ij} u(x^j) = \mathcal{F}(x^i) - \sum_{x^j \in \partial_D \Omega} K_{ij} \bar{u}(x^j), \quad (33)$$

$$x^i \in \Omega \cup \partial_N \Omega, \quad (\text{no sum in } i),$$

where $\mathcal{F}(x^i)$ is calculated from (21),

$$K_{ij} = - \int\limits_{\bar{\omega}(x^i) \cap \partial_N \Omega} \Phi_{ij}(x) T_x P(x, x^i) d\Gamma(x) + \int\limits_{\bar{\omega}(x^i) \cap \partial_D \Omega} P(x, x^i) T \Phi_{ij}(x) d\Gamma(x) \quad (34a)$$

$$- \int\limits_{\Omega \cap \partial \omega(x^i)} \Phi_{ij}(x) T_x P(x, x^i) d\Gamma(x) + \int\limits_{\Omega \cap \partial \omega(x^i)} P(x, x^i) T \Phi_{ij}(x) d\Gamma(x) \quad (34b)$$

$$+ \int\limits_{\omega(x^i) \cap \Omega} R(x, x^i) \Phi_{ij}(x) d\Omega(x). \quad (34c)$$

4.2 Meshless discretization

For a meshless discretization, one needs a method of local interpolation or approximation of a function along randomly distributed nodes x^i, for example, the moving least squares (MLS) approximation method, see e.g. [9,10] and the references therein. This leads to an approximation of a function $u(x)$ along values of an auxiliary function $\hat{u}(x)$ in the nodes x^i belonging to a localization domain $\omega_0(x)$ of the approximation method,

$$u(x) = \sum_j \hat{u}(x^j)\phi_j(x), \quad x^j \in \omega_0(x), \tag{35}$$

where $\phi_j(x)$ are known shape functions.

We will suppose all the approximation nodes x^i belong to $\bar{\Omega}$ and will use them also as collocation points for the LBDIDE/LBDIDP discretization. Let, as before, J be the total number of nodes x^j $(i = 1, 2, ..., J)$, J_D from which be the number of the nodes on $\partial_D\Omega$. Formula (35) can be used to approximate $u(x)$ both at a collocation point x^i and at integration points x from a localization domain $\omega(x^i)$. Then (35) implies a total approximation of $u(x)$, for all $x \in \bar{\omega}(x^i)$,

$$u(x) = \sum_{x^j \in \tilde{\omega}(x^i)} \hat{u}(x^j)\Phi_{ij}(x), \; x \in \bar{\omega}(x^i); \tag{36}$$

$$\Phi_{ij}(x) = \begin{cases} \phi_j(x) \text{ if } x^j \in \omega_0(x) \subset \tilde{\omega}(x^i) \\ 0 \qquad \text{otherwise} \end{cases} \tag{37}$$

where $\tilde{\omega}(x^i) := \cup_{x \in \bar{\omega}(x^i) \cap \bar{\Omega}} \omega_0(x)$ is a total localization domain, Fig. 4. Consequently, $\Phi_{ij}(x) = 0$ if $x^j \notin \tilde{\omega}(x^i)$. Let $J_{\tilde{\omega}}(x^i)$ be the number of nodes $x^j \in \tilde{\omega}(x^i)$.

Let $\Phi'_{ij}(x)$ be the shape functions obtained similar to $\Phi_{ij}(x)$ in (37) for a local approximation of $t(x) = (Tu)(x^j)$ along only boundary nodes belonging to $\tilde{\omega}(x^i) \cap \partial_D\Omega$, such that $\Phi'_{ij}(x) = 0$ if $x^j \notin \tilde{\omega}(x^i) \cap \partial_D\Omega$. Then

$$t(x) = \sum_{x^j \in \tilde{\omega}(x^i)} \hat{t}(x^j)\Phi'_{ij}(x), \; x \in \bar{\omega}(x^i) \cap \partial_D\Omega. \tag{38}$$

After substitution of approximations (36), (38), e.g., in LBDIDE (19) and boundary conditions (2), we arrive at the following system of $J + J_D$ linear algebraic equations with respect to the J unknowns $\hat{u}(x^j)$, $x^j \in \bar{\Omega}$ and J_D unknowns $\hat{t}(x^j)$, $x^j \in \partial_D\Omega$,

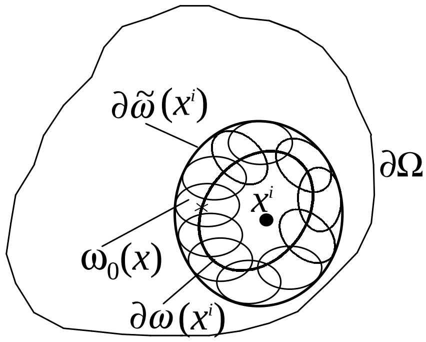

Fig. 4. A localization domain $\omega(x^i)$ and a total localization domain $\tilde{\omega}(x^i)$ associated with a collocation point x^i at a meshless discretization of a body Ω

$$c^0(x^i) \sum_{x^j \in \tilde{\omega}(x^i)} \hat{u}(x^j)\Phi_{ij}(x^i) + \sum_{j=1}^{J} K^0_{ij}\hat{u}(x^j) + \sum_{x_j \in \partial\Omega} Q_{ij}\hat{t}(x^j) = \mathcal{F}^0(x^i), \; i = 1, ...J \quad (39)$$

$$\sum_{x^j \in \tilde{\omega}(x^i)} \hat{u}(x^j)\Phi_{ij}(x^i) = \bar{u}(x^i), \quad x^i \in \partial_D\Omega, \qquad \text{no sum in } i, \quad (40)$$

where $\mathcal{F}^0(x^i)$ is calculated from (20), K^0_{ij} and Q_{ij} are expressed by (31), (32) with the shape functions Φ_{ij}, $\Phi'_{ij}(x)$ from (37) and (38).

Alternatively, one can arrive at another system of J algebraic equations with respect to J unknowns $\hat{u}(x^j)$, $x^j \in \bar{\Omega}$ if one substitutes approximation formulae (36) in BDIDP (22), (2),

$$c(x^i) \sum_{x^j \in \tilde{\omega}(x^i)} \hat{u}(x^j)\Phi_{ij}(x^i) + \sum_{j=1}^{J} K_{ij}\hat{u}(x^j) = \mathcal{F}(x^i), \quad x^i \in \Omega \cup \partial_N\Omega, \quad (41)$$

$$\sum_{x^j \in \tilde{\omega}(x^i)} \hat{u}(x^j)\Phi_{ij}(x^i) = \bar{u}(x^i), \quad x^i \in \partial_D\Omega, \qquad \text{no sum in } i, \quad (42)$$

where $\mathcal{F}(x^i)$ is calculated from (21) and K_{ij} is expressed by (34) with the shape functions Φ_{ij} from (37).

4.3 Remarks on the discretizations

Application of the discretization algorithms to the both LBDIE (19) and LBDIDP (22), (2) needs differentiation of the shape functions $\phi_{kj}(x)$ to calculate $T\Phi_{ij}(x)$ in the second integrals in (31b), (34a), and (34b). However, if the parametrix is continuously localized, then the second integrals in (31b) and (34b) disappear and the need to differentiate remains only for (34a) in the discrete versions of LBDIDP (22), (2). If the parametrix is localized and globally smooth (except its singular point $x = y$), then all the integrals in (31b) and (34b) disappear simplifying the matrix calculations for algebraic systems (30), (33), (39)-(40), and (41)-(42) even further.

From the definitions in both mesh based and meshless methods, $\Phi_{ij}(x) = T\Phi_{ij}(x) = \Phi'_{ij}(x) = 0$ and consequently $K^0_{ij} = Q_{ij} = K_{ij} = 0$ if $x^j \notin \tilde{\omega}(x^i)$. This means, each of equations in systems (30), (33), (39)-(40), and (41)-(42) has not more than $J_{\tilde{\omega}}(x^i) \ll J$ non-zero entries, what manifests the systems are sparsely populated.

5 Concluding Remarks

Partial differential equations with variable coefficients generally do not possess explicit and cheaply calculated fundamental solutions and this prevents reduction of BVPs for such equations to BIEs. Fortunately, such equations often possess simple and cheaply calculated parametrixes. Localization of a parametrix by multiplication by a cut-off function with a local support allows the reduction of a BVP to a localized boundary-domain integral or integro-differential equation/problem, which ends up, after a discretization, in a system of linear algebraic equations with a sparsely populated matrix. This makes the method competitive with the finite element method. Examples of different cut-off functions with different smoothness leading to different LBDIEs/LBDIDEs/LBDIDPs demonstrate the method high flexibility. Localized algorithms for both mesh-based and meshless discretization are presented showing the great potential of the LBDIE/ LBDIDE/LBDIDP method for numerical applications to different BVPs in science and engineering.

Even in some situations when a cheap fundamental solution is available, it seems to be profitable to treat it as a parametrix and obtain a localized BDIE/LBDIDE/LBDIDP based on it. (For a particular localization, such an approach was in fact employed in [9].) This make sense especially for elongated or flattened bodies, where the nonlocal connection between remote points in the *traditional* BIEs looks artificial from the physical point of view and leads to ill-posed algebraic systems after discretization. The localized approach allows to obtain a discrete algebraic system of higher dimension but with a very sparsely populated and well-posed matrix.

As was mentioned at the end of subsection 3.1, the localized parametrix approach leads also to the domain decomposition (subdomains) method. Following this approach further, one can use BIE in some of the subdomains (provided that there is a fundamental solution available there) and LBDIE/ LBDIDE/LBDIDP in the remaining subdomains. This seems particularly promising for infinite or semi-infinite domains, where BIE is employed in an infinite (semi-infinite) subdomain with constant BVP coefficients and LBDIE/LBDIDE/LBDIDP in the remaining finite subdomain with variable coefficients. Speaking more generally, in many problems, where the BIE/FEM combination proved to be efficient, the subdomain combination of BIE with LBDIE/LBDIDE/LBDIDP should be not less efficient and more natural.

So as not to obscure the main idea of the localization approach, we deliberately avoided specification of appropriate function classes where the considered integral and integro-differential operators act. However this should be done to consider some important features of the LBDIE/LBDIDE/LBDIDP such as existence and uniqueness of solution, spectral properties, equivalence to the original BVP, convergence of approximate solutions. This should be the subject of special investigations leading to an optimal choice of the cut-off functions, localization domains, node points, and hopefully to effective iteration methods based on this information [12].

References

1. Hilbert D. Jahrbericht der deutschen Mathematiker–Vereinigung 1907; 16:77–78.

2. Levi EE. I problemi dei valori al contorno per le equazioni lineari totalmente ellittiche alle derivate parziali. Met. Soc. It. dei Sc. XL 1909;16.

3. Hilbert D. Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen. Leipzig: Teubner, 1912.

4. Antes H. Über singuläre Lastfälle in einer linearen Schalentheorie. Ingenieur–Archiv 1976;45:99–114.

5. Beskos DE. Dynamic analysis of plates and shallow shells by the D/BEM. In: Voyiadis GZ, Karamanlidis D, editors. Advances in the theory of plates and shells, 1990, p.177–196, Amsterdam.

6. Beskos DE. Static and dynamic analysis of shells. In: Bescos DE, editor. Boundary Element Analysis of Plates and Shells, 1991, p.93–140.

7. Pomp A. Levi functions for linear elliptic systems with variable coefficients including shell equations, Computational Mechanics 1998;22:93–99.

8. Mikhailov SE. About localized boundary–domain integral equations for problems with variable coefficients. In: Harris PJ, editor. Third UK Conference on Boundary Integral Methods, University of Brighton Press, ISBN 1–901177–32–7, 2001, p.67–76.

9. Zhu T, Zhang J–D, Atluri SN. A local boundary integral equation (LBIE) method in computational mechanics, and a meshless discretization approach. Computational Mechanics 1998;21:223–235.

10. Zhu T, Zhang J–D, Atluri SN A meshless numerical method based on the local boundary integral equation (LBIE) to solve linear and non–linear boundary value problems. Engineering Analysis with Boundary Elements 1999;**23**:375–389.

11. Brebbia CA, Telles JCF, Wrobel LC. Boundary Element Techniques. Springer–Verlag, 1984.

12. Mikhailov SE. Finite–dimensional perturbations of linear operators and some applications to boundary integral equations, Engineering Analysis with Boundary Elements 1999;23:805–813.