Observer-Based H_∞ Control for Systems with Repeated Scalar Nonlinearities and Multiple Packet Losses

Hongli Dong*, Zidong Wang and Huijun Gao

Abstract

This paper is concerned with the H_∞ control problem for a class of systems with repeated scalar nonlinearities and multiple missing measurements. The nonlinear system is described by a discrete-time state equation involving a repeated scalar nonlinearity which typically appears in recurrent neural networks. The measurement missing phenomenon is assumed to occur, simultaneously, in the communication channels from the sensor to the controller and from the controller to the actuator, where the missing probability for each sensor/actuator is governed by an individual random variable satisfying a certain probabilistic distribution in the interval [0 1]. Attention is focused on the analysis and design of an observer-based feedback controller such that the closed-loop control system is stochastically stable and preserves a guaranteed H_∞ performance. Sufficient conditions are obtained for the existence of admissible controllers. It is shown that the controller design problem under consideration is solvable if certain linear matrix inequalities (LMIs) are feasible. Three examples are provided to illustrate the effectiveness of the developed theoretical results.

Keywords

Observer-based H_∞ control; repeated scalar nonlinearity; stochastic stability; multiple missing measurements; linear matrix inequalities.

I. Introduction

H_∞ control problem has long been an important research topic for its theoretical and practical significance in systems control. The main idea of H_∞ control is to reduce the effect of the disturbance input on the regulated output according to a prescribed level. Much effort has been made to H_∞ controller design in order to guarantee desired robust stability, see for example [1, 9, 16, 26, 32, 33, 36]. This control is often realized with the assumption that the entire state is available. In many systems, such an assumption does not hold, and it is necessary to design observers which produce an estimate of the system state. Therefore, it is not surprising that, the dynamic output feedback H_∞ control problem has recently gained particular research attention (see, e.g., [11, 18, 19, 22, 24, 31] and the references therein).

In the past decade, an increasing number of control applications, which have the control-loops closed via a shared communication network, have been investigated, see [7,12,13,29,34,35,37] for some recent publications. These control systems are known as networked control systems (NCS) that differ from traditional control

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systems using direct point-to-point links because the network introduces additional dynamics in the closed-loop system. There are different ways of modeling the dynamics introduced in the closed-loop system by the networks, such as packet losses, time-varying delays and data quantization, etc. The attention of this paper is focused on observer-based feedback control of nonlinear systems subject to packet losses.

It should be pointed out that, in most of the existing literature, it has been implicitly assumed that the packet loss problem occurs only in the channel from the sensor to the controller. Another typical kind of packet losses, which happen in the channel from the controller to the actuator, has not yet been fully investigated [29]. Also, in reported results concerning packet losses, the probability 0 is usually used to stand for an entire signal missing and the probability 1 denotes the intactness (i.e., there is no signal missing at all), and all the sensors or actuators are assumed to have the same missing probability (see e.g., the aforementioned literature and [10, 28]). Such a description, however, does have its limitations since it cannot cover some practical cases, for example, the case when only *partial* information is missing and the case when the *individual* sensor or actuator has different missing probability [30]. Furthermore, the networked control problem for nonlinear systems has not been paid adequate research efforts due primarily to the mathematical difficulty. Therefore, in this paper, we are motivated to develop a more reasonable model for a class of nonlinear systems in which the missing probability for each sensor/actuator is specified by an *individual* random variable satisfying a certain probabilistic distribution in the interval [0 1] that could be any commonly used discrete distribution.

In this paper, we aim to deal with the H_∞ control problem for a class of nonlinear NCSs with random packet losses in sensor-to-controller and controller-to-actuator channels. The packet loss phenomenon is assumed to be random and could be different for individual sensor/actuator, which is modeled by an individual random variable satisfying a certain probabilistic distribution on the interval [0 1]. Such a probabilistic distribution can be any commonly used discrete distributions. The repeated scalar nonlinearity [4,6,8], which typically appears in recurrent neural networks, is employed to describe the networked systems. The objective is to analyze and design an observer-based controller such that the closed-loop system is stochastically stable with guaranteed H_∞ performance. Both the stability analysis and controller design problems are thoroughly investigated. It is shown that the addressed controller design problem is solvable if certain linear matrix inequalities (LMIs) are feasible. Three examples are exploited to illustrate the effectiveness of the proposed design method.

The rest of this paper is organized as follows. Section II formulates the problem under consideration. The stability condition and H_∞ performance of the closed-loop observer-based feedback control system are given in Section III. The H_∞ controller design problem is solved in Section IV. The validity of this approach is demonstrated by three illustrative examples in Section V. Finally, in Section VI, the conclusion is given.

Notation. The notation used in the paper is fairly standard. The superscript "T" stands for matrix transposition, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}^{m\times n}$ is the set of all real matrices of dimension $m \times n$; I and 0 represent the identity matrix and zero matrix, respectively. The notation $P > 0$ means that P is real symmetric and positive definite; the notation $\|A\|$ refers to the norm of a matrix A defined by $\|A\| = \sqrt{\mathrm{tr(A^TA)}}$ and $\|\cdot\|_2$ stands for the usual l_2 norm. In symmetric block matrices or complex matrix expressions, we use an asterisk $(*)$ to represent a term that is induced by symmetry, and diag $\{\dots\}$ stands for a block-diagonal matrix. In addition, $\mathbb{E}\{x\}$ and $\mathbb{E}\{x|\, y\}$ will, respectively, mean expectation of x and expectation of x conditional on y. The set of all nonnegative integers is denoted by $\mathbb{I}^+$ and the set of all nonnegative real numbers is represented by $\mathbb{R}^+$. OL denotes the class of all continuous nondecreasing convex functions $\Gamma : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Gamma(0) = 0$ and $\Gamma(r) > 0$ for $r > 0$. If A is a matrix, $\lambda_{\max}(A)$ (respectively, $\lambda_{\min}(A)$) means the largest (respectively, smallest) eigenvalue of A. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

II. Problem Formulation

A. The Physical Plant

In this paper, we consider the discrete-time system with repeated scalar nonlinearities described as following:

$$\begin{cases} x_{k+1} = Af(x_k) + B_2u_k + B_1w_k \\ z_k = C_1f(x_k) + D_1w_k \\ y_{ck} = C_2x_k + D_2w_k \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^n$ represents the state vector; $u_k \in \mathbb{R}^m$ is the control input; $z_k \in \mathbb{R}^r$ is the controlled output; $y_{ck} \in \mathbb{R}^p$ is the process output; $w_k \in \mathbb{R}^q$ is the disturbance input which belongs to $l_2[0,\infty)$; $A, B_1, B_2, C_1, C_2, D_1$ and D_2 are known real matrices with appropriate dimensions. f is a nonlinear function satisfying the following assumption as in [5].

Assumption 1: The nonlinear function $f : \mathbb{R} \to \mathbb{R}$ in system (1) satisfies

$$\forall a, b \in \mathbb{R} \quad |f(a) + f(b)| \leq |a + b|\,. \tag{2}$$

In the sequel, for the vector $x = [x_1\ x_2 \cdots x_n]^T$, we denote

$$f(x) \triangleq [f(x_1)\ f(x_2)\ \ \cdots\ \ f(x_n)]^T.$$

Remark 1: The model (1) is called a system with repeated scalar nonlinearity [4,6,8]. Note that f is odd (by putting $b = -a$) and 1-Lipschitz (by putting $b = -b$). Therefore, f encapsulates some typical classes of nonlinearities, such as i) the semilinear function (i.e., the standard saturation $sat(s) := s$ if $|s| \leq 1$ and $sat(s) := sgn(s)$ if $|s| > 1$); ii) the hyperbolic tangent function that has been extensively used for activation function in neural networks; and iii) the sine function.

Remark 2: The plant model structure (1) can be used to describe a broad class of real-time dynamical systems, such as marketing and production control problem [14, 20, 23], digital control systems having saturation type nonlinearities on the state or on the controller [2, 15], recurrent artificial neural networks (see e.g. [3] and the references therein), n-stand cold rolling mills [20], neural networks defined on hypercubes [21], fixed-point state-space digital filters using saturation overflow arithmetic [5, 17], manufacturing systems for decision-making [27] and so on.

B. The Controller

The dynamic observer-based control scheme for the system (1) is described by

$$\begin{cases} \hat{x}_{k+1} = Af(\hat{x}_k) + B_2u_k + L(y_k - \hat{y}_k) \\ \hat{y}_k = C_2\hat{x}_k \\ \hat{u}_k = K\hat{x}_k \end{cases} \tag{3}$$

where $\hat{x}_k \in \mathbb{R}^n$ is the state estimate of the system (1), $y_k \in \mathbb{R}^p$ is the measured output, $\hat{u}_k \in \mathbb{R}^m$ is the control input without transmission missing, and $L \in \mathbb{R}^{n\times p}$ and $K \in \mathbb{R}^{m\times n}$ are the observer and controller gains, respectively.

C. The Communication Links

Quality-of-Service (QoS) is an important performance index for NCS. As in [29], in this paper, we consider the following two QoS measures: 1) the point-to-point network allowable data-dropout rate that is used to indicate the probability of data packet dropout in data transmission and 2) the point-to-point network throughput that is used to indicate how fast the signal can be sampled and sent as a packet through the network. Obviously, the sampling period h and the data-dropout rate ρ determine the control performance.

Different from [29], in this paper, we assume that the data are multiple-packet transmitted, and the dropout rate varies as the packet varies. It is clear that the network allowable data-dropout rate is related with the packet scheduler, backlog controller, and algorithm complex of loss dropper policy. In this paper, we are concerned with the multiple random packet losses in both the sensor-to-controller channel and the controller-to-actuator channel.

As discussed in the introduction, due to the existence of the communication links, the phenomenon of data packet dropout will inevitably induce missing observations. That is, the process output is probably not equivalent to the measured output (i.e. $y_{ck} \neq y_k$). In this paper, the measurement with multiple communication packet loss is described by

$$y_k = \Xi y_{ck} = \sum_{i=1}^{p} \alpha_i (C_{2i} x_k + D_{2i} w_k), \tag{4}$$

where $\Xi := \mathrm{diag}\{\alpha_1, ..., \alpha_{\mathrm{p}}\}$ with α_i $(i = 1, ..., p)$ being p unrelated random variables which are also unrelated with w_k. It is assumed that α_i has the probabilistic density function $q_i(s)$ $(i = 1, ..., p)$ on the interval [0 1] with mathematical expectation μ_i and variance σ_i^2. C_{2i} and D_{2i} are defined by

$$C_{2i} := \mathrm{diag}\{\underbrace{0, \cdots, 0}_{\mathrm{i-1}}, 1, \underbrace{0, \cdots, 0}_{\mathrm{p-i}}\}\mathrm{C}_2, \quad \mathrm{D}_{2\mathrm{i}} := \mathrm{diag}\{\underbrace{0, \cdots, 0}_{\mathrm{i-1}}, 1, \underbrace{0, \cdots, 0}_{\mathrm{p-i}}\}\mathrm{D}_2.$$

α_i could satisfy any discrete probabilistic distribution on the interval [0 1], which includes the widely used Bernoulli distribution as a special case. In the sequel, we denote $\bar{\Xi} = \mathbb{E}\{\Xi\}$.

Similarly, the control input with multiple communication packet loss is described by

$$u_k = \Omega \hat{u}_k = \sum_{j=1}^{m} \beta_j K_j \hat{x}_k, \tag{5}$$

where $\Omega = \mathrm{diag}\{\beta_1, ..., \beta_{\mathrm{m}}\}$ with β_j $(j = 1, ..., m)$ being m unrelated random variables and

$$K_j = \mathrm{diag}\{\underbrace{0, \cdots, 0}_{\mathrm{j-1}}, 1, \underbrace{0, \cdots, 0}_{\mathrm{m-j}}\}\mathrm{K}.$$

It is assumed that β_j has the probabilistic density function $m_j(s)$ on the interval [0 1] with mathematical expectation ϑ_j and variance ξ_j^2. We define $\bar{\Omega} = \mathbb{E}\{\Omega\}$.

Remark 3: It can be noticed from (4) and (5) that the diagonal matrices Ξ and Ω, which consist of random variables, are introduced to reflect the random multiple packet losses in, respectively, the sensor-to-controller and controller-to-actuator channels. The random packet-loss mode in the sensor output has been recently studied in many NCS papers, most of which were concerned with the linear system with single packet-loss in the sensor-to-controller channel only. To the best of the authors' knowledge, so far, there has been little research on the control problem for nonlinear systems in the presence of *multiple* packet losses in *both* sensor-to-controller *and* controller-to-actuator channels, and the purpose of this paper is therefore to shorten such a gap. More specifically, the description in (4)-(5) represents one of the first few attempts to reflect, in a probabilistic way, the multiple packet loss phenomenon in both sensor-to-controller and controller-to-actuator channels.

Remark 4: In real systems, the measurement data may be transferred through multiple sensors and actuators. For different sensor or actuator, the data missing probability may be different. In this sense, it would be more reasonable to assume that the data missing law for each individual sensor/actuator satisfies individual probabilistic distribution. In equation (4), the diagonal matrix Ξ represents the whole missing status where the random variable α_i corresponds to the ith sensor. We notice that the data loss (also called packet dropout or measurement missing) phenomenon has been extensively studied and several models have been introduced. The Bernoulli distributed model is arguably the most popular one in which 0 is used to stand for an entire missing of signals and 1 denotes the intactness. However, due to various reasons such as sensor aging and sensor temporal failure, the data missing at one moment might be partial, and therefore the missing probability cannot be simply described by 0 or 1. In (4), α_i can take value on the interval [0 1] and the probability for α_i to take different values may differ from each other. It is easy to see that the Bernoulli distribution is included as a special case. Similar discussion can be applied to the diagonal matrix Ω.

D. The Closed-loop System

Letting the estimation error be

$$e_k := x_k - \hat{x}_k, \tag{6}$$

the closed-loop system can be obtained as follows by substituting (3), (4) and (5) into (1) and (6)

$$\begin{cases} x_{k+1} = Af(x_k) + B_2\bar{\Omega}Kx_k + B_2(\Omega-\bar{\Omega})Kx_k - B_2\bar{\Omega}Ke_k - B_2(\Omega-\bar{\Omega})Ke_k + B_1w_k \\ e_{k+1} = A[f(x_k) - f(\hat{x}_k)] + (LC_2 - L\bar{\Xi}C_2)x_k - L(\Xi-\bar{\Xi})C_2x_k - LC_2e_k \\ \qquad\quad + \left[(B_1 - L\bar{\Xi}D_2) - L(\Xi-\bar{\Xi})D_2\right]w_k, \end{cases} \tag{7}$$

or, in a compact form,

$$\varsigma_{k+1} = \breve{A}\eta_k + \bar{A}\varsigma_k + \psi_k\hat{A}\varsigma_k + \bar{B}w_k \tag{8}$$

where

$$\varsigma_k = \begin{bmatrix} x_k^T & e_k^T \end{bmatrix}^T, \quad \eta_k = \begin{bmatrix} f^T(x_k) & f^T(x_k) - f^T(\hat{x}_k) \end{bmatrix}^T, \quad \breve{A} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} B_2\bar{\Omega}K & -B_2\bar{\Omega}K \\ LC_2 - L\bar{\Xi}C_2 & -LC_2 \end{bmatrix},$$

$$\psi_k = \begin{bmatrix} B_2(\Omega-\bar{\Omega}) & 0 \\ 0 & L(\Xi-\bar{\Xi}) \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} K & -K \\ -C_2 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B_1 \\ (B_1 - L\bar{\Xi}D_2) - L(\Xi-\bar{\Xi})D_2 \end{bmatrix}.$$

It should be pointed out that, in the closed-loop system (8), the stochastic matrices Ξ and Ω appear, which make the difference from 1) the traditional deterministic system without random packet losses and 2) the system with single random packet loss. Before proceeding further, we introduce the following definition, assumption and lemmas, which will be needed for the derivation of our main results.

Definition 1: [25] The solution $\varsigma_k = 0$ of the closed-loop system in (8) with $w_k \equiv 0$ is said to be stochastically stable if, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $\mathbb{E}\{||\varsigma_k||\} < \varepsilon$ whenever $k \in \mathbb{I}^+$ and $||\varsigma_0|| < \delta$.

Assumption 2: [29] The matrix B_2 is of full column rank, i.e., $\mathrm{rank}(B_2) = m$.

Remark 5: For the matrix B_2 of full column rank, there always exist two orthogonal matrices $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{m\times m}$ such that

$$\tilde{B}_2 = UB_2V = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} B_2V = \begin{bmatrix} \Sigma \\ 0 \end{bmatrix}, \tag{9}$$

where $U_1 \in \mathbb{R}^{m\times n}$ and $U_2 \in \mathbb{R}^{(n-m)\times n}$, and $\Sigma = \mathrm{diag}\{\tau_1, \tau_2, \cdots, \tau_{\mathrm{m}}\}$, where τ_i $(i = 1, 2, \cdots, m)$ are nonzero singular values of B_2.

Lemma 1: [29] For the matrix $B_2 \in \mathbb{R}^{n\times m}$ with full column rank, if matrix P_1 is of the following structure:

$$P_1 = U^T \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix} U = U_1^T P_{11} U_1 + U_2^T P_{22} U_2, \tag{10}$$

where $P_{11} \in \mathbb{R}^{m\times m} > 0$ and $P_{22} \in \mathbb{R}^{(n-m)\times(n-m)} > 0$, and U_1 and U_2 are defined in (9), then there exists a non-singular matrix $P \in \mathbb{R}^{m\times m}$ such that $B_2P = P_1B_2$.

Remark 6: [29] The purpose of Lemma 1 is to find a solution P to $B_2P = P_1B_2$, which will later facilitate our development of the LMI approach to the controller design. The assumption of B_2 being full column rank is just for presentation convenience, which does not lose any generality, as we can always conduct the congruence transformation on B_2. If the condition (10) holds, then P exists but it may not be unique unless B_2 is square and non-singular.

Lemma 2: [25] If there exist a Lyapunov function $V(\varsigma_k)$ and a function $\Gamma(r) \in OL$ satisfying the following conditions

$$V(0) = 0, \tag{11}$$

$$\Gamma(||\varsigma||) \leq V(\varsigma), \tag{12}$$

$$\mathbb{E}\left\{V(\varsigma_{k+1})\right\} - \mathbb{E}\left\{V(\varsigma_k)\right\} < 0, \qquad k \in \mathbb{I}^+, \tag{13}$$

then the solution $\varsigma_k = 0$ of the closed-loop system in (8) with $w_k \equiv 0$ is stochastically stable.

In this paper, we aim to design the controller (3) for the system (1) such that, in the presence of multiple random packet losses, the closed-loop system (8) is stochastically stable and the H_∞ performance constraint is satisfied. To be more specific, we describe the problem as follows.

Problem HCMDL (H_∞ control with multiple data losses):
For given the communication link parameters $\bar{\Xi}$ and $\bar{\Omega}$ and the scalar $\gamma > 0$, design the controller (3) for the system (1) such that the closed-loop system satisfies the following two performance requirements:
i) (stochastic stability) the closed-loop system in (8) is stochastically stable in the sense of Definition 1;
ii) (H_∞ performance) under zero initial condition, the controlled output z_k satisfies $||z||_{\mathbb{E}} \leq \gamma||w||_2$, where

$$||\bar{z}||_{\mathbb{E}} \triangleq \mathbb{E}\left\{\sqrt{\sum_{k=0}^{\infty} z_k^T z_k}\right\}, \tag{14}$$

and $\|\cdot\|_2$ stands for the usual l_2 norm.

If the above two conditions are satisfied, the closed-loop system is said to be stochastically stable with a guaranteed H_∞ performance γ, and the problem *HCMDL* is solved.

III. H_∞ Control Performance Analysis

In this section, the problem *HCMDL* formulated in the previous section will be tackled via a quadratic matrix inequality approach described in the following theorem.

Theorem 1: Suppose that both the controller gain matrix K and the observer gain matrix L are given. The closed-loop system in (8) is stochastically stable with a guaranteed H_∞ performance γ if there exist positive definite matrices P_1, P_2 and two scalars $\rho_1 > 0$, $\rho_2 > 0$ satisfying

$$\begin{bmatrix} \Lambda + \Lambda_1 & \Lambda_2 \\ \Lambda_2^T & \Lambda_3 \end{bmatrix} < 0, \tag{15}$$

$$P_1 \leq \rho_1 I, \quad P_2 \leq \rho_2 I \tag{16}$$

where

$$
\begin{aligned}
\Lambda &= 2\begin{bmatrix} B_2\bar{\Omega}K & -B_2\bar{\Omega}K \\ LC_2 - L\bar{\Xi}C_2 & -LC_2 \end{bmatrix}^T \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} B_2\bar{\Omega}K & -B_2\bar{\Omega}K \\ LC_2 - L\bar{\Xi}C_2 & -LC_2 \end{bmatrix} \\
&\quad + \begin{bmatrix} 2\rho_1\lambda_1 I - P_1 & 0 \\ 0 & 2\rho_2\lambda_1 I - P_2 \end{bmatrix} + \sum_{j=1}^{m}\xi_j^2 \bar{B}_j^T P_1 \bar{B}_j + \sum_{i=1}^{p}\sigma_i^2 \bar{C}_i^T P_2 \bar{C}_i, \\
\bar{B}_j &= \begin{bmatrix} B_2K_j & -B_2K_j \end{bmatrix}, \;\; \bar{C}_i = \begin{bmatrix} -LC_{2i} & 0 \end{bmatrix}, \\
\Lambda_1 &= \begin{bmatrix} \rho_1\lambda_1 I + 2\lambda_2 I & 0 \\ 0 & \rho_2\lambda_1 I \end{bmatrix}, \\
\Lambda_2 &= \begin{bmatrix} (B_2\bar{\Omega}K)^T P_1 B_1 + (LC_2 - L\bar{\Xi}C_2)^T P_2 (B_1 - L\bar{\Xi}D_2) + \sum_{i=1}^{p}\sigma_i^2 (LC_{2i})^T P_2 (LD_{2i}) \\ -(B_2\bar{\Omega}K)^T P_1 B_1 - (LC_2)^T P_2 (B_1 - L\bar{\Xi}D_2) \end{bmatrix}, \\
\Lambda_3 &= 2B_1^T P_1 B_1 + 2(B_1 - L\bar{\Xi}D_2)^T P_2 (B_1 - L\bar{\Xi}D_2) + \sum_{i=1}^{p}\sigma_i^2 (LD_{2i})^T P_2 (LD_{2i}) + 2D_1^T D_1 - \gamma^2 I, \\
\lambda_1 &= \lambda_{\max}(A^T A), \;\; \lambda_2 = \lambda_{\max}(C_1^T C_1). \qquad (17)
\end{aligned}
$$

Proof: We first prove the stochastic stability of the closed-loop system. For this purpose, assume $w_k \equiv 0$ and define the following Lyapunov function candidate as:

$$
V_k = x_k^T P_1 x_k + e_k^T P_2 e_k, \qquad (18)
$$

where P_1 and P_2 are solutions to (15)-(16). Then, along the trajectory of (7), we have

$$
\begin{aligned}
E\{\Delta V_k\} &= \mathbb{E}\{V_{k+1} | x_k, ..., x_0,\, e_k, ..., e_0\} - V_k \\
&= \mathbb{E}\left\{x_{k+1}^T P_1 x_{k+1} + e_{k+1}^T P_2 e_{k+1}\right\} - x_k^T P_1 x_k - e_k^T P_2 e_k \\
&= \mathbb{E}\Big\{[Af(x_k) + B_2\bar{\Omega}Kx_k - B_2\bar{\Omega}Ke_k + B_2(\Omega - \bar{\Omega})Kx_k - B_2(\Omega - \bar{\Omega})Ke_k]^T P_1 \\
&\quad \times [Af(x_k) + B_2\bar{\Omega}Kx_k - B_2\bar{\Omega}Ke_k + B_2(\Omega - \bar{\Omega})Kx_k - B_2(\Omega - \bar{\Omega})Ke_k] \\
&\quad + [A(f(x_k) - f(\hat{x}_k)) + (LC_2 - L\bar{\Xi}C_2)x_k - L(\Xi - \bar{\Xi})C_2x_k - LC_2e_k]^T P_2 \\
&\quad \times [A(f(x_k) - f(\hat{x}_k)) + (LC_2 - L\bar{\Xi}C_2)x_k - L(\Xi - \bar{\Xi})C_2x_k - LC_2e_k] - x_k^T P_1 x_k - e_k^T P_2 e_k\Big\} \\
&= [Af(x_k) + B_2\bar{\Omega}Kx_k - B_2\bar{\Omega}Ke_k]^T P_1 [Af(x_k) + B_2\bar{\Omega}Kx_k - B_2\bar{\Omega}Ke_k] \\
&\quad + [A(f(x_k) - f(\hat{x}_k)) + (LC_2 - L\bar{\Xi}C_2)x_k - LC_2e_k]^T P_2 [A(f(x_k) - f(\hat{x}_k)) \\
&\quad + (LC_2 - L\bar{\Xi}C_2)x_k - LC_2e_k] - x_k^T P_1 x_k - e_k^T P_2 e_k \\
&\quad + \mathbb{E}\{[B_2(\Omega - \bar{\Omega})Kx_k - B_2(\Omega - \bar{\Omega})Ke_k]^T P_1 [B_2(\Omega - \bar{\Omega})Kx_k - B_2(\Omega - \bar{\Omega})Ke_k]\} \\
&\quad + \mathbb{E}\{[-L(\Xi - \bar{\Xi})C_2x_k]^T P_2 [-L(\Xi - \bar{\Xi})C_2x_k]\}.
\end{aligned}
$$

It follows from the definition (4) and (5) that

$$
\begin{aligned}
&\mathbb{E}\{\alpha_i - \mu_i\}\{\alpha_l - \mu_l\} = \begin{cases} \sigma_i^2 & i = l \\ 0 & i \neq l, \end{cases} \qquad (i, l = 1, \ldots, p), \\
&\mathbb{E}\{\beta_j - \vartheta_j\}\{\beta_q - \vartheta_q\} = \begin{cases} \xi_j^2 & j = q \\ 0 & j \neq q, \end{cases} \qquad (j, q = 1, \ldots, m),
\end{aligned} \qquad (19)
$$

and then we can obtain that

$$\begin{aligned}
& \mathbb{E}\left\{[B_2(\Omega-\bar{\Omega})Kx_k - B_2(\Omega-\bar{\Omega})Ke_k]^T P_1 [B_2(\Omega-\bar{\Omega})Kx_k - B_2(\Omega-\bar{\Omega})Ke_k]\right\} \qquad (20)\\
= & \ \mathbb{E}\left\{\left[B_2\sum_{j=1}^{m}(\beta_j-\vartheta_j)K_j\,(x_k-e_k)\right]^T P_1\left[B_2\sum_{j=1}^{m}(\beta_j-\vartheta_j)K_j\,(x_k-e_k)\right]\right\}\\
= & \ \sum_{j=1}^{m}\sum_{q=1}^{m}\mathbb{E}\{\beta_j-\vartheta_j\}\{\beta_q-\vartheta_q\}[B_2K_j\,(x_k-e_k)]^T P_1[B_2K_j\,(x_k-e_k)]\\
= & \ \sum_{j=1}^{m}\xi_j^2[B_2K_j\,(x_k-e_k)]^T P_1[B_2K_j\,(x_k-e_k)],\\
& \mathbb{E}\{[-L(\Xi-\bar{\Xi})C_2x_k]^T P_2[-L(\Xi-\bar{\Xi})C_2x_k]\} = \sum_{i=1}^{p}\sigma_i^2[LC_{2i}x_k]^T P_2[LC_{2i}x_k]. \qquad (21)
\end{aligned}$$

From the elementary inequality $2a^Tb \le a^Ta + b^Tb$, it follows that

$$2f^T(x_k)A^TP_1B_2\bar{\Omega}K(x_k-e_k) \le f^T(x_k)A^TP_1Af(x_k) + (x_k-e_k)^T(B_2\bar{\Omega}K)^TP_1(B_2\bar{\Omega}K)(x_k-e_k) \qquad (22)$$

and, in the same way, we have

$$\begin{aligned}
& 2[f(x_k)-f(\hat{x}_k)]^TA^TP_2[(LC_2-L\bar{\Xi}C_2)x_k - LC_2e_k]\\
\le & \ [f(x_k)-f(\hat{x}_k)]^TA^TP_2A[f(x_k)-f(\hat{x}_k)]\\
& +[(LC_2-L\bar{\Xi}C_2)x_k - LC_2e_k]^TP_2[(LC_2-L\bar{\Xi}C_2)x_k - LC_2e_k]. \qquad (23)
\end{aligned}$$

Next, we obtain from (2) and (16) that

$$f^T(x_k)A^TP_1Af(x_k) \le \rho_1\lambda_{\max}(A^TA)x_k^Tx_k, \qquad (24)$$

$$[f(x_k)-f(\hat{x}_k)]^TA^TP_2A[f(x_k)-f(\hat{x}_k)] \le \rho_2\lambda_{\max}(A^TA)e_k^Te_k, \qquad (25)$$

then from (18)-(25), we have

$$\begin{aligned}
E\{\Delta V_k\} \le & \ 2\rho_1\lambda_{\max}(A^TA)x_k^Tx_k + 2(B_2\bar{\Omega}Kx_k - B_2\bar{\Omega}Ke_k)^TP_1(B_2\bar{\Omega}Kx_k - B_2\bar{\Omega}Ke_k) + 2\rho_2\lambda_{\max}(A^TA)e_k^Te_k\\
& +2[(LC_2-L\bar{\Xi}C_2)x_k - LC_2e_k]^TP_2[(LC_2-L\bar{\Xi}C_2)x_k - LC_2e_k] - x_k^TP_1x_k - e_k^TP_2e_k\\
& +\sum_{j=1}^{m}\xi_j^2[B_2K_j\,(x_k-e_k)]^TP_1[B_2K_j\,(x_k-e_k)] + \sum_{i=1}^{p}\sigma_i^2(LC_{2i}x_k)^TP_2(LC_{2i}x_k)\\
= & \ \varsigma_k^T\Lambda\varsigma_k.
\end{aligned}$$

By Schur complement, (15) implies that $\Lambda < 0$, hence

$$E\{\Delta V_k\} = \varsigma_k^T\Lambda\varsigma_k < 0$$

which satisfies (13). Taking $\Gamma(x_{k,}\, e_k) = \lambda_{\min}(P_1)x_k^2 + \lambda_{\min}(P_2)e_k^2$ such that $\Gamma(x_{k,}\, e_k) \in OL$, we obtain

$$\Gamma(||x_k, e_k||) = \lambda_{\min}(P_1)||x_k||^2 + \lambda_{\min}(P_2)||e_k||^2 \le x_k^TP_1x_k + e_k^TP_2e_k = V(x_k, e_k),$$

which satisfies (12). Considering $V(0) = 0$, it follows readily from Lemma 2 that the closed-loop system in (8) with $w_k \equiv 0$ is stochastically stable.

Next, the H_∞ performance criterion for the closed-loop system in (8) will be established. For presentation convenience, we denote

$$\hat{\Xi}_1 := \quad Af(x_k)+B_2\bar{\Omega}Kx_k+B_2(\Omega-\bar{\Omega})Kx_k-B_2\bar{\Omega}Ke_k-B_2(\Omega-\bar{\Omega})Ke_k+B_1w_k \tag{26}$$

$$\begin{aligned}\hat{\Xi}_2 := \quad & A(f(x_k)-f(\hat{x}_k))+(LC_2-L\bar{\Xi}C_2)x_k-L(\Xi-\bar{\Xi})C_2x_k-LC_2e_k \\ & +((B_1-L\bar{\Xi}D_2)-L(\Xi-\bar{\Xi})D_2)w_k]^T\end{aligned} \tag{27}$$

Now, Assuming zero initial conditions and in view of the above notations, an index is introduced and calculated as follows:

$$\begin{aligned}J \;=\; & \mathbb{E}\left\{V_{k+1}\right\}-\mathbb{E}\left\{V_k\right\}+\mathbb{E}\left\{z_k^Tz_k\right\}-\gamma^2\mathbb{E}\left\{w_k^Tw_k\right\} \\ =\; & \mathbb{E}\{\hat{\Xi}_1^TP_1\hat{\Xi}_1+\hat{\Xi}_2^TP_2\hat{\Xi}_2-x_k^TP_1x_k-e_k^TP_2e_k\}+[C_1f(x_k)+D_1w_k]^T[C_1f(x_k)+D_1w_k]-\gamma^2w_k^Tw_k \\ \leq\; & \eta_k^T\Lambda\eta_k+(B_1w_k)^TP_1[Af(x_k)+B_2\bar{\Omega}Kx_k-B_2\bar{\Omega}Ke_k+B_1w_k]+[Af(x_k)+B_2\bar{\Omega}Kx_k-B_2\bar{\Omega}Ke_k]^TP_1B_1w_k \\ & +\mathbb{E}\{[(B_1-L\bar{\Xi}D_2)w_k-L(\Xi-\bar{\Xi})D_2w_k]^TP_2[A(f(x_k)-f(\hat{x}_k)) \\ & +(LC_2-L\bar{\Xi}C_2)x_k-L(\Xi-\bar{\Xi})C_2x_k-LC_2e_k]\} \\ & +\mathbb{E}\{[A(f(x_k)-f(\hat{x}_k))+(LC_2-L\bar{\Xi}C_2)x_k-L(\Xi-\bar{\Xi})C_2x_k-LC_2e_k]^TP_2 \\ & \times[(B_1-L\bar{\Xi}D_2)w_k-L(\Xi-\bar{\Xi})D_2w_k]\} \\ & +\mathbb{E}\{[(B_1-L\bar{\Xi}D_2)w_k-L(\Xi-\bar{\Xi})D_2w_k]^TP_2[(B_1-L\bar{\Xi}D_2)w_k-L(\Xi-\bar{\Xi})D_2w_k]\} \\ & +[C_1f(x_k)+D_1w_k]^T[C_1f(x_k)+D_1w_k]-\gamma^2w_k^Tw_k \\ =\; & \begin{bmatrix}\varsigma_k \\ w_k\end{bmatrix}^T\begin{bmatrix}\Lambda+\Lambda_1 & \Lambda_2 \\ \Lambda_2^T & \Lambda_3\end{bmatrix}\begin{bmatrix}\varsigma_k \\ w_k\end{bmatrix},\end{aligned}$$

where Λ, Λ_1, Λ_2, Λ_3 come from (17). It follows from (15) that $J<0$, that is,

$$\mathbb{E}\left\{V_{k+1}\right\}-\mathbb{E}\left\{V_k\right\}+\mathbb{E}\left\{z_k^Tz_k\right\}-\gamma^2\mathbb{E}\left\{w_k^Tw_k\right\}<0. \tag{28}$$

Summing up (28) from zero to ∞ with respect to k yields

$$\sum_{k=0}^{\infty}\mathbb{E}\{\|z_k\|^2\}<\gamma^2\sum_{k=0}^{\infty}\mathbb{E}\{\|w_k\|^2\}+\mathbb{E}\left\{V_0\right\}-\mathbb{E}\left\{V_\infty\right\}.$$

Since $\varsigma_0=0$ and the system (8) is stochastically stable, it is easy to conclude that

$$\sum_{k=0}^{\infty}\mathbb{E}\{\|z_k\|^2\}<\gamma^2\sum_{k=0}^{\infty}\mathbb{E}\{\|w_k\|^2\},$$

which is equivalent to (14). The proof is complete. ■

IV. Controller Design

In this section, we will deal with the controller design problem and derive the explicit expression of the controller parameters, that is, to determine the controller parameters in (3) such that the closed-loop system in (8) is stochastically stable and the controlled output z_k satisfies (14).

Theorem 2: Consider the system (1). There exists a dynamic observer-based controller in the form of (3) such that the closed-loop system in (8) is stochastically stable with a guaranteed H_∞ performance γ, if there exist positive-definite matrices $P_{11}\in\mathbb{R}^{m\times m}$, $P_{22}\in\mathbb{R}^{(n-m)\times(n-m)}$, $P_2\in\mathbb{R}^{n\times n}$, real matrices $M_j\in\mathbb{R}^{m\times n}$ (j=1,...,m), $N\in\mathbb{R}^{n\times p}$ and two scales $\rho_1>0$, $\rho_2>0$ satisfying

$$\begin{bmatrix}\Pi_1 & \Pi_2^T \\ \Pi_2 & \Pi_3\end{bmatrix}<0, \tag{29}$$

$$P_1 \leq \rho_1 I, \tag{30}$$

$$P_2 \leq \rho_2 I. \tag{31}$$

Where

$$\Pi_1 = diag\left\{-P_1 + 3\rho_1\lambda_1 I + 2\lambda_2 I, -P_2 + 3\rho_2\lambda_1 I, -\gamma^2 I + 2D_1^T D_1\right\},$$

$$\Pi_3 = diag\left\{-P_1, -P_2, -P_1, -P_2, -P_1, -P_2, -\hat{P}_1, -\hat{P}_2\right\},$$

$$\Pi_2 = \begin{bmatrix} \sum_{j=1}^{m}\vartheta_j\ B_2 M_j & -\sum_{j=1}^{m}\vartheta_j\ B_2 M_j & P_1 B_1 \\ NC_2 - N\bar{\Xi}C_2 & -NC_2 & P_2 B_1 - N\bar{\Xi}D_2 \\ \sum_{j=1}^{m}\vartheta_j\ B_2 M_j & -\sum_{j=1}^{m}\vartheta_j\ B_2 M_j & 0 \\ NC_2 - N\bar{\Xi}C_2 & -NC_2 & 0 \\ 0 & 0 & P_1 B_1 \\ 0 & 0 & P_2 B_1 - N\bar{\Xi}D_2 \\ \hat{B} & -\hat{B} & 0 \\ \hat{C} & 0 & \hat{D} \end{bmatrix}, \tag{32}$$

$$\hat{B} = [\xi_1 M_1^T B_2^T, \cdots, \xi_m M_m^T B_2^T]^T,\ \hat{C} = [-\sigma_1 {C_{21}}^T N^T, \cdots, -\sigma_p {C_{2p}}^T N^T]^T,$$

$$\hat{P}_1 = \mathrm{diag}\{\underbrace{P_1, \cdots, P_1}_{\mathrm{m}}\},\ \hat{\mathrm{P}}_2 = \mathrm{diag}\{\underbrace{P_2, \cdots, P_2}_{\mathrm{p}}\},$$

$$\hat{D} = [-\sigma_1 {D_{21}}^T N^T, \cdots, -\sigma_p {D_{2p}}^T N^T]^T,\ P_1 := \mathrm{U}_1^{\mathrm{T}}\mathrm{P}_{11}\mathrm{U}_1 + \mathrm{U}_2^{\mathrm{T}}\mathrm{P}_{22}\mathrm{U}_2.$$

Furthermore, the controller parameters are given by

$$K = \sum_{j=1}^{m} V\Sigma^{-1}P_{11}^{-1}\Sigma V^T M_j, \quad L = P_2^{-1}N. \tag{33}$$

Proof: From Theorem 1, we know that there exists a dynamic observer-based controller such that (8) is stochastically stable with a guaranteed H_∞ performance γ if there exist positive definite matrices P_1 and P_2 satisfying (15). Noticing

$$P_1 B_2 \bar{\Omega} K = \sum_{j=1}^{m}\vartheta_j\ P_1 B_2 K_j,$$

and applying the Schur complement to (15), we have

$$\begin{bmatrix} \Pi_1 & \hat{\Pi}_2^T \\ \hat{\Pi}_2 & \Pi_3 \end{bmatrix} < 0,$$

where

$$\hat{\Pi}_2 = \begin{bmatrix} \sum_{j=1}^{m}\vartheta_j\ P_1B_2K_j & -\sum_{j=1}^{m}\vartheta_j\ P_1B_2K_j & P_1B_1 \\ P_2LC_2 - P_2L\bar{\Xi}C_2 & -P_2LC_2 & P_2B_1 - P_2L\bar{\Xi}D_2 \\ \sum_{j=1}^{m}\vartheta_j\ P_1B_2K_j & -\sum_{j=1}^{m}\vartheta_j\ P_1B_2K_j & 0 \\ P_2LC_2 - P_2L\bar{\Xi}C_2 & -P_2LC_2 & 0 \\ 0 & 0 & P_1B_1 \\ 0 & 0 & P_2B_1 - P_2L\bar{\Xi}D_2 \\ \check{B} & -\check{B} & 0 \\ \check{C} & 0 & \check{D} \end{bmatrix}, \tag{34}$$

$$\check{B} = [\xi_1(P_1B_2K_1)^T, \cdots, \xi_m(P_1B_2K_m)^T]^T,\ \ \hat{C} = [-\sigma_1(P_2LC_{21})^T, \cdots, -\sigma_p(P_2LC_{2p})^T]^T,$$

$$\check{D} = [-\sigma_1(P_2LD_{21})^T, \cdots, -\sigma_p(P_2LD_{2p})^T]^T.$$

Since there exist $P_{11} > 0$ and $P_{22} > 0$ such that $P_1 = U_1^T P_{11} U_1 + U_2^T P_{22} U_2$, where U_1 and U_2 are defined in (9), it follows from Lemma 1 that there exists a non-singular matrix $P \in \mathbb{R}^{m\times m}$ such that $B_2P = P_1B_2$. Now let us calculate such a matrix P from the relation $B_2P = P_1B_2$ as follows:

$$P_1U^T \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^T = U^T \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^T P,$$

i.e.

$$U^T \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix} \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^T = U^T \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^T P,$$

which implies that

$$P = (V^T)^{-1}\Sigma^{-1}P_{11}\Sigma V^T. \tag{35}$$

Since $B_2P = P_1B_2$, we define

$$M_j = PK_j, \quad N = P_2L, \tag{36}$$

and we can obtain (29) and (33) readily. The proof is now complete. ■

Remark 7: As we can see from Theorem *2*, in the presence of multiple random packet losses, the H_∞ control problem is solved for systems with repeated scalar nonlinearities, and an observer-based feedback controller is designed to stochastically stabilize the networked system and also achieve the prescribed H_∞ disturbance rejection attenuation level. The possible future research directions include real-time applications of the proposed filtering theory in telecommunications, and further extensions of the present results to more complex systems with unreliable communication links, such as sampled-data systems, bilinear systems, time-delay systems and more general nonlinear systems.

Remark 8: The packet dropout or missing measurement problems have recently attracted considerable research interest [7, 12, 13, 29, 34, 35, 37]. In particular, in [25, 29], the filtering and control problems have been investigated for a general class of nonlinear discrete-time stochastic systems with *single packet loss* where the missing probability is assumed to be either 0 (complete loss) or 1 (no loss). In [30], the filtering problem has been considered for a class of discrete-time uncertain stochastic nonlinear time-delay systems with probabilistic missing measurements, which are assumed to occur only in the channel from the sensor to the controller, and the control problem has not been taken into account. Different from existing literature, this paper exhibits the following distinctive features: 1) the packet loss phenomenon is assumed to be random and could be different for individual sensor/actuator; 2) the packet loss is modeled by an individual random variable satisfying a certain probabilistic distribution on the interval [0 1], which is not restricted to be 0 or 1; 3) the random packet losses in both sensor-to-controller and controller-to-actuator channels are considered; and 4) the observer-based H_∞ control problem is dealt with that facilitates the investigation on the packet loss from controller to actuator.

V. Illustrative Examples

In this section, three simulation examples are presented to illustrate the usefulness and flexibility of the observer-based controller design method developed in this paper.

Example 1: In this example, we are interested in designing the observer-based controller for systems with repeated scalar nonlinearities and multiple missing measurements. The system data of (7) are given as follows:

$$A=\begin{bmatrix}-1&0&0\\0&0.8&0.5\\0.5&0&0\end{bmatrix},\ B_1=\begin{bmatrix}0.1&1&0.4\\0.2&-0.2&1\\0.2&-0.1&0.1\end{bmatrix},\ B_2=\begin{bmatrix}1&0&0.6\\0&0.3&0.5\\1&1.2&-0.5\end{bmatrix},\ C_1=\begin{bmatrix}0.5&0&0\\0.4&0.5&0.5\\1&0.2&0.2\end{bmatrix},$$
$$C_2=\begin{bmatrix}1&1&5\\0.2&-0.4&0.4\\0&0&0.3\end{bmatrix},\ D_1=\begin{bmatrix}0.1&-1&0.3\\0.2&-0.3&0.3\\1&0.1&-0.1\end{bmatrix},\ D_2=\begin{bmatrix}-0.2&0.1&0.2\\0&0.2&1\\1&0.4&0\end{bmatrix}. \tag{37}$$

Assuming that the probabilistic density functions of α_1, α_2 and α_3 in [0 1] are described by

$$q_1(s_1)=\begin{cases}0 & s_1=0\\0.1 & s_1=0.5\\0.9 & s_1=1\end{cases},\qquad q_2(s_2)=\begin{cases}0.1 & s_2=0\\0.1 & s_2=0.5\\0.8 & s_2=1\end{cases},\qquad q_3(s_3)=\begin{cases}0 & s_3=0\\0.2 & s_3=0.5\\0.8 & s_3=1\end{cases}, \tag{38}$$

from which the expectations and variances can be easily calculated as $\mu_1 = 0.95$, $\mu_2 = 0.85$, $\mu_3 = 0.9$, $\sigma_1 = 0.15$, $\sigma_2 = 0.32$ and $\sigma_3 = 0.2$. In the same way, we assume the probabilistic density functions of β_1, β_2 and β_3 in [0 1] to be

$$m_1(s_1)=\begin{cases}0 & s_1=0\\0.4 & s_1=0.5\\0.6 & s_1=1\end{cases},\quad m_2(s_2)=\begin{cases}0.05 & s_2=0\\0.15 & s_2=0.5\\0.8 & s_2=1\end{cases},\quad m_3(s_3)=\begin{cases}0 & s_3=0\\0.2 & s_3=0.5\\0.8 & s_3=1\end{cases}, \tag{39}$$

from which we can calculate that $\vartheta_1 = 0.8$, $\vartheta_2 = 0.875$, $\vartheta_3 = 0.9$, $\xi_1 = 0.245$, $\xi_2 = 0.268$ and $\xi_3 = 0.2$. Here, the nonlinear function $f(x_k) = \sin(x_k)$ satisfies (2). By applying Theorem 2, we can obtain an admissible solution as follows:

$$K=\begin{bmatrix}-0.0364&0.0708&-0.3856\\-0.0242&0.0550&-0.3159\\0.0255&-0.0579&0.3356\end{bmatrix},\qquad L=\begin{bmatrix}0.0509&0.3040&0.8950\\0.1242&0.4020&0.3154\\0.0040&0.0247&0.0488\end{bmatrix}.$$

For the purpose of simulation, we let the initial conditions be $x_0 = [1\ 0\ 0]^T$, $\hat{x}_0 = [0\ 0\ 0]^T$, and the disturbance input be $w_k = [k^{-2}\ k^{-2}\ k^{-2}]^T$. Fig. 1 displays the state responses of the uncontrolled system, which are apparently unstable. Fig. 2 shows the state simulation results of the closed-loop system, from which we can see that the desired objective is achieved.

Example 2: In this example, we aim to illustrate the effectiveness of our results for different measurement missing cases. Here

$$A=\begin{bmatrix}-1&0&-0.9\\2&0.8&0.5\\0.5&0&1.2\end{bmatrix},$$

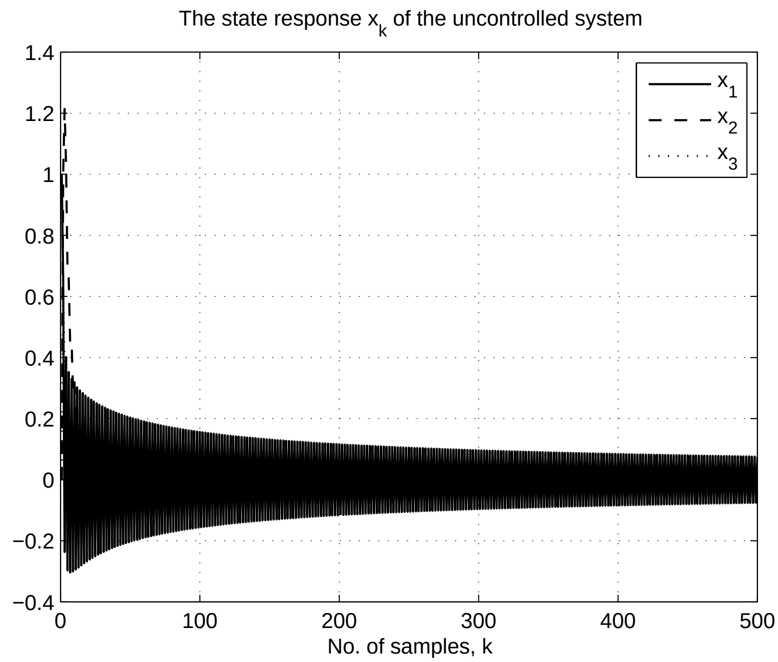

Fig. 1. The state evolution x_k of uncontrolled system in Example 1

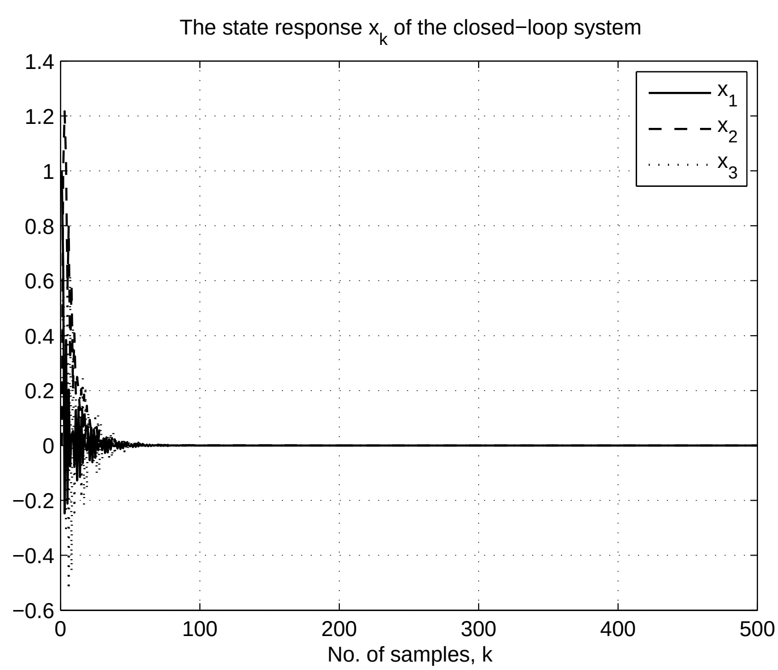

Fig. 2. The state evolution x_k of controlled system in Example 1

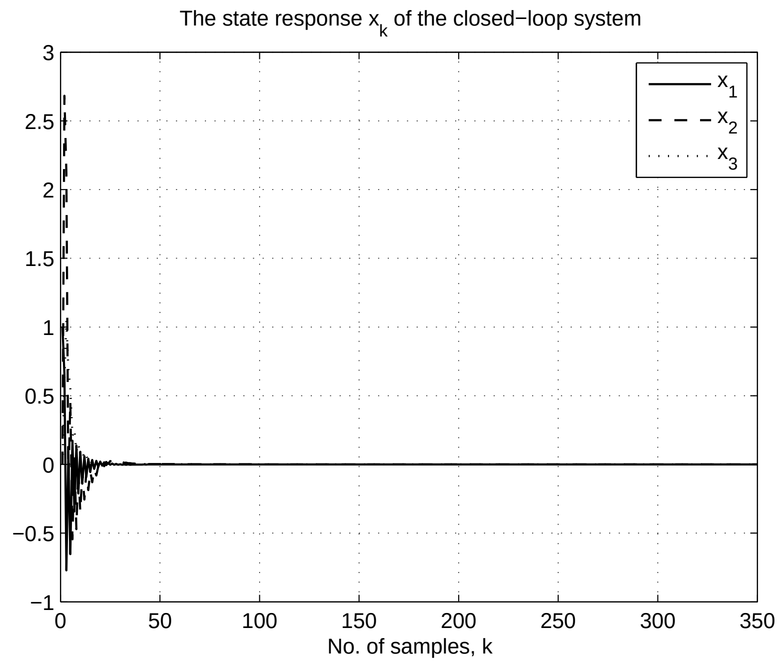

Fig. 3. The state evolution x_k when the packet-loss probability is relatively lower in Example 2

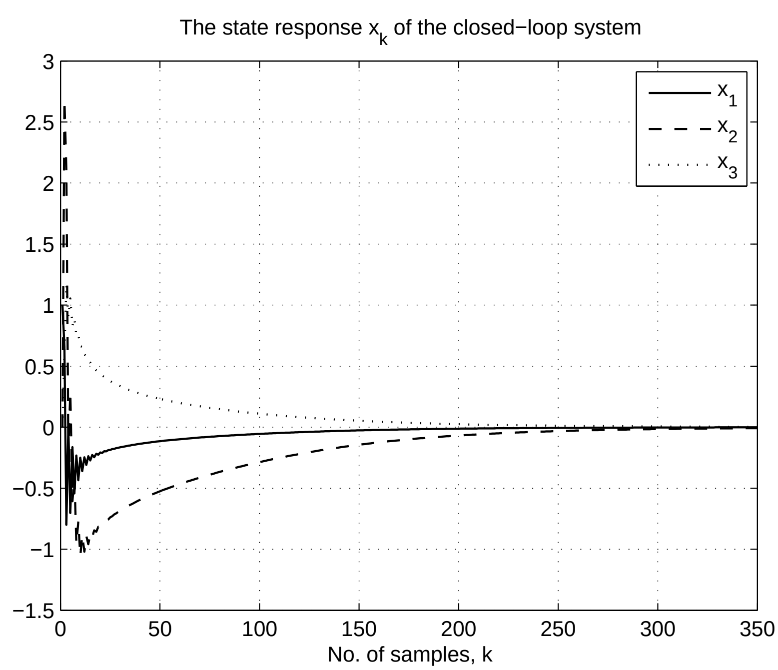

Fig. 4. The state evolution x_k when the packet-loss probability is relatively higher in Example 2

and the other system data of (7) is the same as in Example 1. First, we assume the probabilistic density functions of α_1, α_2, α_3 and β_1, β_2, β_3 are the same as (38) and (39) respectively, and obtain an admissible solution as follows:

$$K = \begin{bmatrix} -0.0276 & 0.0551 & -0.3031 \\ -0.0087 & 0.0295 & -0.1799 \\ 0.0095 & -0.0318 & 0.1967 \end{bmatrix}, \quad L = \begin{bmatrix} 0.0514 & 0.3017 & 0.8770 \\ 0.1217 & 0.3971 & 0.3288 \\ 0.0042 & 0.0236 & 0.0414 \end{bmatrix},$$

for which the simulation result of the state responses is given in Fig. 3 that confirms the realization of our design goal.

Next, let us consider the case when the multiple packet-loss probability becomes higher. Take the probabilistic density functions of α_1, α_2 and α_3 in [0 1] as

$$q_1(s_1) = \begin{cases} 0 & s_1 = 0 \\ 0.3 & s_1 = 0.5 \\ 0.7 & s_1 = 1 \end{cases}, \quad q_2(s_2) = \begin{cases} 0 & s_2 = 0 \\ 0.3 & s_2 = 0.5 \\ 0.7 & s_2 = 1 \end{cases}, \quad q_3(s_3) = \begin{cases} 0 & s_3 = 0 \\ 0.5 & s_3 = 0.5 \\ 0.5 & s_3 = 1 \end{cases},$$

and the probabilistic density functions of β_1, β_2 and β_3 in [0 1] as

$$m_1(s_1) = \begin{cases} 0.8 & s_1 = 0 \\ 0.1 & s_1 = 0.5 \\ 0.1 & s_1 = 1 \end{cases}, \quad m_2(s_2) = \begin{cases} 0.2 & s_2 = 0 \\ 0.1 & s_2 = 0.5 \\ 0.7 & s_2 = 1 \end{cases}, \quad m_3(s_3) = \begin{cases} 0.2 & s_3 = 0 \\ 0.1 & s_3 = 0.5 \\ 0.7 & s_3 = 1 \end{cases}.$$

By calculating the expectations and variances of the random variables, we have arrived at the following solution:

$$K = \begin{bmatrix} -0.0135 & 0.0676 & -0.2887 \\ 0.0424 & 0.0102 & 0.1361 \\ -0.0465 & -0.0465 & -0.1507 \end{bmatrix}, \quad L = \begin{bmatrix} 0.0570 & 0.3216 & 1.0009 \\ 0.1433 & 0.4253 & 0.3136 \\ 0.0071 & 0.0317 & 0.0495 \end{bmatrix},$$

Again, the simulation result of the state responses are depicted in Fig. 4. As we can see from Figs. 3-4, when the packet losses are severer, the dynamical behavior of the NCS takes longer to converge and, furthermore, the robustness of the closed-loop system is rather degraded.

Example 3: Following [14, 20, 23], we consider a factory that produces two kinds of products ($j = 1, 2$) sharing common resources and raw materials, like colour TV and black/white TV, PC and laptop computer, etc. Fig. 5 shows the schematic diagram of the system under consideration. The information transmissions are conducted through networks which are subject to possible missing measurements. Due to the unknown disturbance input and probabilistic packet losses, it would be of practical significance to design an observer-based controller to stabilize the systems while maintaining certain system performances, for which the developed theory in this paper can be ideally applied.

During the kth period (quarter or season), we define

s_{jk} : amount of sales of product j

a_{jk} : advertisement cost spent for product j

i_{jk} : amount of inventory of product j

p_{jk} : production of product j

Let

$$x_k = \begin{bmatrix} p_{1,k+1} \\ p_{2,k+1} \\ i_{1k} \\ i_{2k} \end{bmatrix}, \quad u_k = \begin{bmatrix} s_{1k} \\ s_{2k} \\ a_{1k} \\ a_{2k} \end{bmatrix}.$$

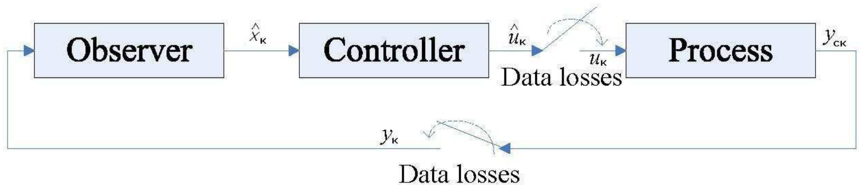

Fig. 5. Closed-loop system with data losses

The effect of advertisements on sales in the marketing process and the interlink between production and the amount of sales in the production process (assuming one-period gestation lag) may be expressed dynamically by the following form:

$$x_{k+1} = Af(x_k) + B_1 w_k + B_2 u_k + \triangle B_2 u_{k-d_1}$$

where $f(x_k)$ is a saturation nonlinearity function, $\triangle B_2 u_{k-1}$ describes the uncertain changes in advertisements cost, d_1 denotes the time-delay and y_k represents the measured production of product j. The purpose of this example is to design an observer-based H_∞ control that renders the closed-loop system stochastically stable and guarantees an appropriate level of performance. It is readily seen that the above model is a special case of (1) when $\triangle B_2 = 0$.

Now, let's consider a specific example for the above combined marketing and production control problem, where

$$A = \begin{bmatrix} 0.7 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ -0.7 & 0 & 0.9 & 0 \\ 0 & -0.5 & 0 & 0.9 \end{bmatrix}, B_1 = \begin{bmatrix} 0 & 0.2 & 0 & 0 \\ 0.1 & 0 & 0 & 0 \\ 0 & -0.2 & 0 & 0 \\ -0.1 & 0 & 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & 0 & 0.8 & 0 \\ 0.1 & 0 & 0 & 0 \\ 0 & -0.2 & 0 & 0 \\ -0.1 & 0 & 0 & 0.4 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0.5 & 0 & 0.7 & 0 \\ 0.4 & 0.5 & 0 & 0.5 \\ 1 & 0 & 0.5 & 0 \\ 0.2 & -0.5 & 0 & 0.9 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0.4 & 0.5 & 0 & 0.5 \\ 1 & 0 & 1 & 0 \\ 0 & -0.5 & 0.2 & 1 \end{bmatrix}, D_1 = \begin{bmatrix} 0.1 & 0 & -1 & 0.3 \\ 0.4 & 0.5 & 0 & 0.5 \\ 1 & 0 & 0.5 & 0 \\ 0.2 & -0.5 & 0 & 1 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} -0.5 & 0.2 & 0.7 & 0 \\ 0.4 & 0.5 & 0 & 0.5 \\ -1 & 0 & 0.5 & 0 \\ 0.2 & -0.5 & 0 & 0.9 \end{bmatrix}$$

We assume the probabilistic density functions of α_1, α_2, α_3 and β_1, β_2, β_3 are the same as (38) and (39), respectively, and α_4 and β_4 are described by

$$q_4(s_4) = \begin{cases} 0.05 & s_4 = 0 \\ 0.15 & s_4 = 0.5 \\ 0.8 & s_4 = 1 \end{cases}, \quad m_4(s_4) = \begin{cases} 0.1 & s_4 = 0 \\ 0.1 & s_4 = 0.5 \\ 0.8 & s_4 = 1 \end{cases}.$$

Here, the saturation nonlinear function $f(x_k)$ satisfies

$$\begin{cases} f(x_k) = x_k & |x_k| \leq 1 \\ f(x_k) = 1 & x_k > 1 \\ f(x_k) = -1 & x_k < -1 \end{cases}$$

The state response of the closed-loop system is shown in Fig. 6, which illustrates the effectiveness of the H_∞ controller design.

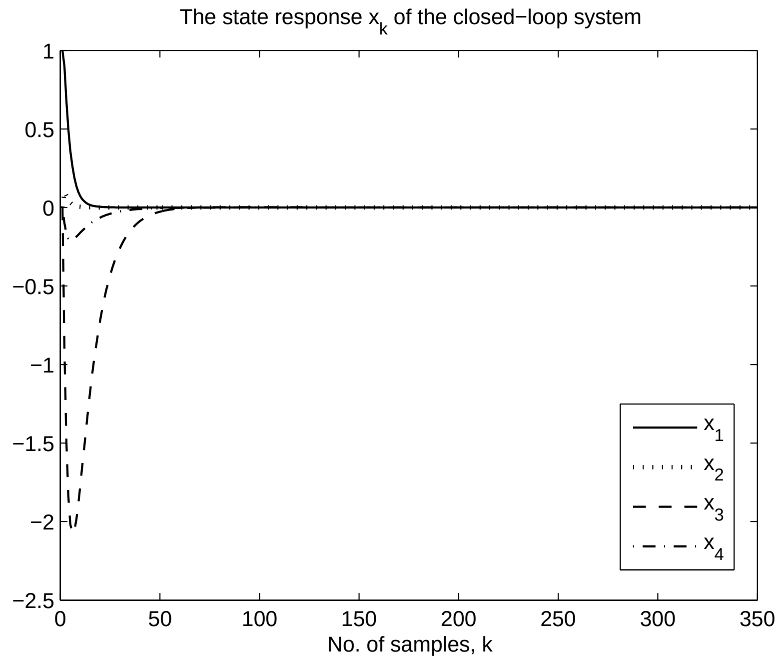

Fig. 6. The state evolution x_k of Example 3

As a summary of this section, all the simulation results have further confirmed our theoretical analysis for the problem of observer-based H_∞ control for systems with repeated scalar nonlinearities and multiple missing measurements.

VI. Conclusions

In this paper, A novel H_∞ control problem has been considered for systems with repeated scalar nonlinearities under multiple missing measurements. The random communication packet losses have been allowed to occur, simultaneously, in the communication channels from the sensor to the controller and from the controller to the actuator, and the missing probability for each sensor is governed by an individual random variable satisfying a certain probabilistic distribution in the interval [0 1]. In the presence of random packet losses, an observer-based feedback controller has been designed to stochastically stabilize the networked system. Both the stability analysis and controller synthesis problems have been investigated in detail. Simulation results have demonstrated the usefulness and feasibility of the addressed control scheme.

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