Skew-orthogonal Laguerre polynomials for chiral real asymmetric random matrices

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Abstract. We apply the method of skew-orthogonal polynomials (SOP) in the complex plane to asymmetric random matrices with real elements, belonging to two different classes. Explicit integral representations valid for arbitrary weight functions are derived for the SOP and for their Cauchy transforms, given as expectation values of traces and determinants or their inverses, respectively. Our proof uses the fact that the joint probability distribution function for all combinations of real eigenvalues and complex conjugate eigenvalue pairs can be written as a product. Examples for the SOP are given in terms of Laguerre polynomials for the chiral ensemble (also called the non-Hermitian real Wishart-Laguerre ensemble), both without and with the insertion of characteristic polynomials. Such characteristic polynomials play the role of mass terms in applications to complex Dirac spectra in field theory. In addition, for the elliptic real Ginibre ensemble we recover the SOP of Forrester and Nagao in terms of Hermite polynomials.

Keywords: skew-orthogonal Laguerre polynomials, real asymmetric random matrices, characteristic polynomials, Cauchy transform

1. Introduction

Classical orthogonal polynomials (OP) are one of the principal standard tools used to solve problems in Random Matrix Theory (RMT). The three classical Wigner-Dyson ensembles with Gaussian elements can be solved in terms of Hermite polynomials, whereas their chiral counterparts require the use of Laguerre polynomials. Whilst for the symmetry classes with unitary invariance ($\beta = 2$) the corresponding scalar product is symmetric with the standard weight, for the classes with orthogonal ($\beta = 1$) or symplectic ($\beta = 4$) symmetry the scalar product becomes skew-symmetric, with the details – including the weight – dependent on the symmetry class. The corresponding polynomials are then called skew-orthogonal polynomials (SOP). For details of all these cases we refer to [1], as well as to [2] and [3] for reviews on SOP. What all these ensembles have in common is that each of their solutions can be expressed using the kernel of the corresponding (S)OP as a building block.

The solution of RMT in terms of (S)OP is exact for finite matrix size N. Moreover, when taking one of the possible large-N limits in the bulk of the spectrum, or at the soft or hard edge, the standard Plancherel-Rotach asymptotics can be used. Much work has been done on the question of universality, i.e. the extent to which the asymptotics also hold for non-Gaussian weights, see e.g. [4] for a review.

This setup of (S)OP has been generalised to the complex plane in order to solve non-Hermitian RMT. However, for the standard Ginibre ensembles the (S)OP are simply given in terms of monic powers (which holds for all weights that are rotationally invariant in $\mathbb{C}$). Only when considering so-called elliptic deformations of the Gaussian Ginibre weight do Hermite polynomials on $\mathbb{C}$ appear, as was first observed in [5]. The corresponding quaternionic ($\beta = 4$) elliptic Ginibre ensemble was solved in terms of some Hermite SOP in [6], and only very recently was the real elliptic Ginibre case ($\beta = 1$) solved in terms of another set of Hermite SOP [7].

The chiral counterparts of two of these ensembles were introduced in [8] ($\beta = 2$) and [9] ($\beta = 4$), where they were solved in terms of Laguerre OP on $\mathbb{C}$ and Laguerre SOP respectively. The obvious question then arises about the existence of Laguerre SOP for the chiral $\beta = 1$ ensemble [10, 11] which we answer affirmatively in this article, thereby completing the set of classical Hermite and Laguerre (S)OP in the complex plane for these non-Hermitian RMT.

One complication arises for the ensembles with Laguerre (S)OP in the complex plane: due to the integration over angular variables the elliptic ensembles that are Gaussian in terms of the matrix elements lead to non-Gaussian weight functions for the eigenvalues (for $\beta = 1$ elliptic Ginibre we have also a complementary error function). The Bessel function of the second kind appearing here for all three $\beta = 1, 2, 4$ makes the orthogonality question much more involved.

For this reason we first provide a new integral representation for the $\beta = 1$ SOP, valid in both the elliptic Ginibre and chiral symmetry classes for arbitrary weight functions. In view of earlier results for SOP on $\mathbb{R}$ for both $\beta = 1, 4$ [12, 13], as well as for complex SOP for $\beta = 4$ [6, 9], this representation comes very naturally. Moreover, it was shown very recently in [14] that the $\beta = 1$ and $\beta = 4$ Ginibre ensembles can be treated on an equal footing. We will rederive this relation amongst these symmetry classes from a different angle. We will also derive a new integral representation for the Cauchy transforms of the SOP on $\mathbb{C}$ valid for both $\beta = 1$ and 4. This extends the expression for Cauchy transforms on $\mathbb{C}$ for $\beta = 2$ in [15].

One important ingredient necessary in order to derive these results is the factorisation of the joint probability distribution function (jpdf) for $\beta = 1$, which is originally given by a sum over all possible combinations of real and complex conjugate eigenvalues [16, 17]. Such a factorisation, which uses the symmetrisation over all eigenvalues, might have been expected from the fact that the partition function can be written as a single Pfaffian over double integrals [18].

Our integral representation allows us to derive the $\beta = 1$ Laguerre SOP on $\mathbb{C}$ in a straightforward fashion. The known Hermite SOP on $\mathbb{C}$ of [7] also follow easily. As a third and important example for our general formalism we explicitly compute the SOP for the chiral $\beta = 1$ ensemble with the insertion of mass terms. Such insertions play a crucial role in the application of RMT to the complex Dirac operator spectrum in Quantum Chromodynamics (QCD) and related field theories at small quark chemical potential in the low density phase, see e.g. [19] for reviews, as well as very recently in the high density phase corresponding to maximal non-Hermiticity [20].

For this third example we exploit the fact that averages (and ratios) of the required characteristic polynomials have been computed very recently for the non-Hermitian $\beta = 1$ symmetry classes in [21]. Together with our interpretation of the building blocks there as the kernel, SOP and their Cauchy transforms this completes the analogy to earlier computations of such averages and ratios for $\beta = 2$ in [22, 15, 23] and $\beta = 4$ in [9, 24].

This article is organised as follows. In section 2 we recall the definition of the two RMT with real asymmetric matrices including mass terms, their respective weight functions and eigenvalue representations. Section 3 is devoted to a factorisation proof of the jpdf of real and complex eigenvalues, where we give two different arguments. The new integral representations for the SOP and their Cauchy transforms are then shown in subsections 4.1 and 4.2 respectively. In section 5 we provide three explicit examples for SOP including Hermite in subsection 5.1, Laguerre SOP in subsection 5.2 and SOP including masses in subsection 5.3. In the Appendices we collect together short proofs for some mathematical identities used in the text.

2. The Matrix Models

We will show how to solve the following two matrix models of real asymmetric matrices in terms of skew-orthogonal polynomials (SOP) in the complex plane.

The first model is given by the chiral extension of the elliptic real Ginibre ensemble

$$\mathcal{Z}_{ch}^{(N_f)}(\{m\}) \sim \int_{\mathbb{R}^{2N(N+\nu)}} dA\, dB \prod_{f=1}^{N_f} \det \begin{pmatrix} m_f \mathbb{1}_{N\times N} & A \\ B^T & m_f \mathbb{1}_{(N+\nu)\times(N+\nu)} \end{pmatrix} \tag{2.1}$$

$$\times \exp\left[-\frac{1}{2}\eta_+ \mathrm{Tr}(AA^T + BB^T) + \eta_- \mathrm{Tr}(AB^T)\right] ,$$

$$\text{with}\ \ \eta_\pm \equiv \frac{1 \pm \mu^2}{4\mu^2} . \tag{2.2}$$

Here A and B are real asymmetric matrices of size $N \times (N + \nu)$, and $\mu \in (0, 1]$ is a non-Hermiticity parameter. The integration runs over all independent real matrix elements of A and B with a flat measure. The product of determinants or characteristic polynomials is motivated by the addition of N_f quark flavours in applications to QCD at finite density [19, 20]. The model can be written as a Gaussian two-matrix model with two independent real asymmetric matrices P and Q, with $A = P + \mu Q$ and $B = P - \mu Q$

(see [11]). In the limit $\mu \to 0$ the model reduces to the chiral Gaussian Orthogonal Ensemble (chGOE).

The second matrix model is also a generalisation of the elliptic real Ginibre ensemble and is given by

$$\mathcal{Z}_{Gin}^{(N_f)}(\{m\}) \sim \int_{\mathbb{R}^{N^2}} dJ \prod_{f=1}^{N_f} |\det[J - im_f]|^2 \exp\left[-\frac{1}{1-\tau^2}\mathrm{Tr}(JJ^T - \tau J^2)\right] . \tag{2.3}$$

Here J is a real asymmetric matrix of size N^2, and $\tau \in [0,1)$ is the non-Hermiticity parameter. We again integrate over all the independent matrix elements of J. The model can alternatively be written as a two-matrix model with symmetric and anti-symmetric matrices S and A with Gaussian elements, where $J = S + A\sqrt{(1-\tau)/(1+\tau)}$, see also [25] for $N_f = 0$. In the limit $\tau \to 1$ the model reduces to the GOE when $N_f = 0$. The extra determinants correspond to the imaginary mass terms coming in pairs of opposite sign. These are also motivated from applications to QCD with a chemical potential, but this time in three dimensions, see [26].

Once we switch to an eigenvalue basis for the Dirac matrix $\mathcal{D}$ for the first model, where

$$\mathcal{D} \equiv \left(\begin{array}{cc} \mathbf{0}_{N\times N} & A \\ B^T & \mathbf{0}_{(N+\nu)\times(N+\nu)} \end{array} \right) , \tag{2.4}$$

and to the eigenvalues of the matrix J for the second model, both models can be treated along the same lines. Because the characteristic equation for both $\mathcal{D}$ and J is real its solutions are either real or come in complex conjugate pairs. However, because of the chirality of $\mathcal{D}$ there is a peculiarity here:

$$0 = \det[\Lambda \mathbb{1}_{2N+\nu} - \mathcal{D}] = \Lambda^\nu \det[\Lambda^2 \mathbb{1}_N - AB^T] = \Lambda^\nu \prod_{j=1}^{N} (\Lambda^2 - \Lambda_j^2) . \tag{2.5}$$

Whilst the Λ_j^2 are indeed either real or come in complex conjugate pairs, the Dirac eigenvalues Λ_j are consequently real ($\Lambda_j^2 > 0$), purely imaginary ($\Lambda_j^2 < 0$), or come in quadruplets ($\pm\Lambda_j, \pm\Lambda_j^*$); there are also ν generic zero-eigenvalues. For simplicity and to keep the presentation of the two models parallel we will mainly consider changed variables $z_j \equiv \Lambda_j^2$ in the following.

Following [27, 10] the partition functions in eqs. (2.1) and (2.3) above can be written as follows, where the normalisation is to be determined later, see eq. (4.16),

$$\mathcal{Z}_{2N+\chi} = N! \sum_{n=0}^{N} \prod_{k=1}^{2n+\chi} \int_{\mathbb{R}} dx_k \prod_{m=1}^{N-n} \int_{\mathbb{C}} d^2 z_m \, P_{2n+\chi, N-n}(x, z, z^*) . \tag{2.6}$$

Here we sum over all the possible ways of splitting the total number $(2N + \chi)$ of eigenvalues into $K \equiv 2n + \chi$ real eigenvalues $\{x_k\}$ and $M \equiv N - n$ complex conjugate eigenvalue pairs $\{z_m, z_m^*\}$. A product with an upper limit less than its lower limit is defined as unity. Note that in this expression we have only one complex integration for each complex conjugate eigenvalue pair. The differentials of the complex eigenvalues are defined over the real and imaginary part, i.e. $d^2z_m = d\Re\mathrm{e}z_m d\Im\mathrm{m}z_m$. In the following we treat the cases of an even ($\chi = 0$) and odd ($\chi = 1$) total number of eigenvalues on the same footing. The jpdf for a fixed number K of real eigenvalues and M complex eigenvalue pairs is defined as

$$P_{K,M}(x,z,z^*) \equiv \prod_{k=1}^{K} h(x_k) \prod_{m=1}^{M} \Big(g(z_m, z_m^*)\, 2i\Theta(\Im\mathrm{m}\, z_m) \Big)\ \Delta_{K+2M}(\{x\},\{z,z^*\})$$
$$\times \prod_{k=2}^{K} \Theta(x_k - x_{k-1}) \prod_{m=2}^{M} \Theta(\Re\mathrm{e}\, z_m - \Re\mathrm{e}\, z_{m-1}) \tag{2.7}$$

with the weight specified in eqs. (2.9) and (2.11) below ‡. Here Θ is the Heaviside distribution, and the Vandermonde determinant is defined as

$$\Delta_N(\{z\}) = \prod_{k>l}^{N} (z_k - z_l) = \det_{1\le a,b \le N} \left[z_a^{b-1} \right] . \tag{2.8}$$

In eq. (2.7) we explicitly specify the number K of real eigenvalues and M complex eigenvalue pairs, with the set of arguments labelled as $x_1, \ldots, x_K, z_1, z_1^*, \ldots, z_M, z_M^*$ in $\Delta_{K+2M}(\{x\},\{z,z^*\})$. The factors $2i\Theta(\Im\mathrm{m}\, z_m)$ and the ordering of the real eigenvalues $\Theta(x_k - x_{k-1})$ in eq. (2.7) allow us to omit the modulus sign around the Vandermonde determinant. The ordering of the real parts $\Theta(\Re\mathrm{e}\, z_m - \Re\mathrm{e}\, z_{m-1})$ is needed to make the transformation to upper triangular 2×2 block form of the matrices A and B^T (or J) unique when computing the Jacobian [11] (see also [28]). The latter and part of the former can be dropped later due to the symmetrising integration as will be shown in the next section.

We also mention that the partition function can be written as a single Pfaffian [18, 27], eq. (3.8) below, and we come back to the consequences for factorisation in the next section.

We now give the weight functions for our two models. Looking at eq. (2.5) for the chiral model we are only interested in the eigenvalues of the matrix $C = AB^T$. Because the N_f extra mass terms compared with [11] depend only on C their addition to the jpdf in [11] is trivial and so we only give the result. The corresponding weight functions in eq. (2.7) for the real eigenvalues x and complex eigenvalues $z = x + iy$ read

$$h_{ch}(x) \qquad \equiv 2|x|^{\nu/2} K_{\nu/2}(\eta_+|x|) \exp[\eta_- x] \prod_{f=1}^{N_f} (x + m_f^2)\ , \tag{2.9}$$

$$g_{ch}(z_1,z_2) \equiv 2|z_1 z_2|^{\nu/2} \exp[\eta_-(z_1+z_2)] \prod_{f=1}^{N_f} (z_1 + m_f^2)(z_2 + m_f^2)$$
$$\times \int_0^\infty \frac{dt}{t} \exp\Big[-\eta_+^2 t (z_1^2 + z_2^2) - \frac{1}{4t} \Big] K_{\nu/2}\left(2\eta_+^2 t z_1 z_2 \right) \mathrm{erfc}\left(\eta_+ \sqrt{t} |z_2 - z_1| \right) .$$

‡ In contrast to [11] we distinguish the weights for real and complex eigenvalues by different symbols (h and g respectively).

In addition we have a trivial overall factor arising from the generic zero eigenvalues in the mass terms:

$$P^{ch}_{K,M}(x,z,z^*) \;= \prod_{f=1}^{N_f} m_f^{\nu}\; P_{K,M}(x,z,z^*) \tag{2.10}$$

For the second model eq. (2.3) we have instead

$$h_{Gin}(x) \qquad \equiv \exp[-x^2]\prod_{f=1}^{N_f}(x^2+m_f^2)\;, \tag{2.11}$$

$$g_{Gin}(z_1,z_2) \equiv \exp[-z_1^2-z_2^2]\;\mathrm{erfc}\left(\frac{|z_1-z_2|}{\sqrt{1-\tau}}\right)\prod_{f=1}^{N_f}(z_1^2+m_f^2)(z_2^2+m_f^2)\;.$$

For the two Gaussian examples above the following relation is satisfied

$$\lim_{\Im m\, z\to 0} g(z,z^*) = h(\Re e\, z)^2\;, \tag{2.12}$$

relating the two weights.

As a general remark here and in the following we can allow for more general weight functions $h(x)$ and $g(z_1,z_2)$ in eq. (2.7) that do not necessarily follow from a matrix representation. For example, we could generalise the weights in eqs. (2.9) and (2.11) by multiplying by a factor $\exp[-V(z_1,z_2)]$ where V is a polynomial in z_1 and z_2 §. Moreover one can independently choose h and g instead fulfilling the relation (2.12).

3. Factorisation of the joint probability distribution

In this section we prove that the jpdf inside the partition function can be written in a factorised form. For this to be possible, it is essential that we integrate over all the eigenvalues, leading to a symmetrisation. Hence this applies equally to the expectation value of any operator symmetric in all variables. Examples for this are the computation of the gap probability or expectation values leading to integral representations of the SOP in the next section. However, such a factorisation can also be found for the k-point density correlation functions when summing over all possibilities of splitting k into real and complex eigenvalue pairs.

Let us first state the result for the partition function eq. (2.6) in terms of a single product for the weights

$$\mathcal{Z}_{2N+\chi} \;=\; \int_{\mathbb{R}} dy^{\chi}\; h^{\chi}(y)\prod_{k=1}^{2N}\int_{\mathbb{C}} d^2z_k\prod_{j=1}^{N}F(z_{2j-1},z_{2j})\;\Delta_{\chi+2N}(y,\{z\})\;, \tag{3.1}$$

where we define the anti-symmetric function

$$F(z_1,z_2) \equiv ig(z_1,z_2)(\Theta(\Im\mathrm{m}\, z_1)-\Theta(\Im\mathrm{m}\, z_2))\,\delta^2(z_2-z_1^*) + \frac{1}{2}h(z_1)h(z_2)\delta(\Im\mathrm{m}\, z_1)\delta(\Im\mathrm{m}\, z_2)\mathrm{sgn}(\Re\mathrm{e}\, z_2-\Re\mathrm{e}\, z_1)\,. \tag{3.2}$$

§ A so-called harmonic potential could be realised as a matrix model, by multiplying eqs. (2.1) and (2.3) with $\exp[-\mathrm{Tr}V(AB^T)]$ or $\exp[-\mathrm{Tr}V(J^2)]$. Although at finite N these are formal expressions due to lack of convergence this can be dealt with in the large-N limit.

Note that for an even number of variables ($\chi = 0$ in eq. (3.1)) the integration and weight for the real variable y have to be dropped, as well as the argument y inside the Vandermonde determinant. In eq. (3.1) we integrate over $2N$ independent complex variables in contrast to the ordered integration in complex conjugated pairs. The standard two-dimensional delta function in eq. (3.2) reads $\delta^2(z) = \delta(x)\delta(y)$ for $z = x+iy$.

We will prove this factorisation in two different ways. One is by explicitly summing up all the terms in eq. (2.6) to make a single product. The results for expectation values of characteristic polynomials in [21] are built up on this idea, although a proof of this was not given. The second way is by starting from a single Pfaffian representation of the partition function derived in [18, 27] and using (in reverse) a proof of a slightly generalised version of the de Bruijn integral formula. Because both derivations are short and illustrate different aspects we decided to present both.

The first derivation of the factorisation of the jpdf goes as follows. From eqs. (2.7) and (2.6) we obtain

$$\mathcal{Z}_{2N+\chi} = N! \sum_{n=0}^{N} \frac{1}{(N-n)!} \prod_{l=1}^{2n+\chi} \int_{\mathbb{R}} dx_l\, h(x_l) \prod_{j=2}^{2n+\chi} \Theta(x_j - x_{j-1})$$
$$\times \prod_{m=1}^{N-n} \int_{\mathbb{C}} (d^2 z_m\, g(z_m, z_m^*)\, 2i\Theta(\Im\mathrm{m}\, z_m))\ \Delta_{(2n+\chi)+2(N-n)}(\{x\},\{z,z^*\})\ , \tag{3.3}$$

where we use that the integrand is totally symmetric under a permutation of two pairs of complex conjugated eigenvalues. Dropping the ordering of the real parts leads to a factor $1/(N-n)!$.

The ordering of the real variables can be simplified by applying the method of integration over alternating variables [1] twice. Pulling the integrations over odd variables in, dropping the symmetrisation giving a factor $1/n!$ and then pulling them back out leads to the following result:

$$\mathcal{Z}_{2N+\chi} = \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} \prod_{l=1}^{2n+\chi} \int_{\mathbb{R}} dx_l\, h(x_l) \prod_{j=1}^{n} \Theta(x_{2j} - x_{2j-1})$$
$$\times \prod_{m=1}^{N-n} \int_{\mathbb{C}} (d^2 z_m\, g(z_m, z_m^*)\, 2i\Theta(\Im\mathrm{m}\, z_m))\ \Delta_{(2n+\chi)+2(N-n)}(\{x\},\{z,z^*\})\ . \tag{3.4}$$

Isolating the integration dx_χ we can do the sum over multiple integrations, using the binomial formula and the permutation invariance of the integrand under exchanging pairs of complex numbers:

$$a \equiv \int_{\mathbb{R}} dx_1\, h(x_1) \int_{\mathbb{R}} dx_2\, h(x_2)\, \Theta(x_2 - x_1)$$
$$= \int_{\mathbb{C}} d^2 z_1\, h(z_1) \int_{\mathbb{C}} d^2 z_2\, h(z_2)\, \Theta(\Re\mathrm{e}\, z_2 - \Re\mathrm{e}\, z_1)\, \delta(\Im\mathrm{m}\, z_1)\, \delta(\Im\mathrm{m}\, z_2) \tag{3.5}$$

$$b \equiv \int_{\mathbb{C}} d^2 z\, g(z,z^*)\, 2i\Theta(\Im\mathrm{m}\, z) = \int_{\mathbb{C}} d^2 z_1 \int_{\mathbb{C}} d^2 z_2\, g(z_1,z_2)\, 2i\Theta(\Im\mathrm{m}\, z_1)\, \delta^2(z_2 - z_1^*)\ , \tag{3.6}$$

with

$$\sum_{n=0}^{N} \frac{N!}{n!(N-n)!} a^n b^{N-n} \ = \ (a+b)^N \ . \tag{3.7}$$

Pulling out all $2N$ independent complex integrations of the N-fold product this leads to eq. (3.1) as claimed, after making the function $F(z_1, z_2)$ manifestly anti-symmetric.

We now come to our second argument, starting from the result derived in [18, 27]. This states that, including normalisation,

$$\mathcal{Z}_{2N} \ = \ N! \operatorname*{Pf}_{1\leq a,b\leq 2N} \left[\int_{\mathbb{C}^2} d^2z_1 d^2z_2 F(z_1, z_2)[z_1^{a-1} z_2^{b-1} - z_2^{a-1} z_1^{b-1}] \right], \tag{3.8}$$

where for simplicity we only state the even case, i.e. with $2N$ eigenvalues. The sign of the Pfaffian is defined as in [1] such that the Pfaffian of the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes \mathbb{1}_N$ is unity. Here $F(z_1, z_2)$ is the function from eq. (3.2). By using a slight generalisation of the proof of the de Bruijn integral formula, in reverse,

$$\prod_{k=1}^{2N} \int_{\mathbb{C}} d^2z_k \prod_{j=1}^{N} F(z_{2j-1}, z_{2j}) \det_{1\leq a,b\leq N} [\{f_a(z_{2b-1}), g_a(z_{2b})\}]$$
$$= N! \operatorname*{Pf}_{1\leq a,b\leq 2N} \left[\int_{\mathbb{C}^2} d^2u\, d^2v F(u,v)[f_a(u)g_b(v) - f_b(u)g_a(v)] \right] \tag{3.9}$$

where we refer to Appendix A for a derivation (cf. Appendix C.2 in [29]), we obtain

$$\mathcal{Z}_{2N} \ = \prod_{k=1}^{2N} \int_{\mathbb{C}} d^2z_k \prod_{j=1}^{N} F(z_{2j-1}, z_{2j}) \det_{1\leq a\leq 2N;\, 1\leq b\leq N} \left[\{z_{2b-1}^{a-1}, z_{2b}^{a-1}\}\right]. \tag{3.10}$$

Here the last determinant is simply the Vandermonde determinant, and thus we have arrived again at eq. (3.1).

All of the above arguments can be generalised, by including an arbitrary observable that is symmetric under the exchange of all the eigenvalues. Perhaps the simplest example, which we will also encounter in the next section, is a factorising operator,

$$f(x, z, z^*) = f^\chi(x) \prod_{j=1}^{2N} f(z_j)\,. \tag{3.11}$$

The individual factors do not affect the symmetry arguments above and we obtain

$$\langle f(x,z,z^*)\rangle_{2N+\chi} = \frac{1}{\mathcal{Z}_{2N+\chi} N!} \int_{\mathbb{R}} dy^\chi (h(y)f(y))^\chi \prod_{k=1}^{2N} \int_{\mathbb{C}} d^2z_k f(z_k) \prod_{j=1}^{N} F(z_{2j-1}, z_{2j})$$
$$\times\, \Delta_{\chi+2N}(y, \{z\}) \tag{3.12}$$

for general expectation values. An explicit example for such an operator is the characteristic polynomial. This result can be generalised to non-factorising observables symmetric in the eigenvalues. Since the monomials in the traces of a matrix can be traced back to products over characteristic polynomials eq. (3.12) is true for all symmetric polynomials in the eigenvalues. With Weierstraß' approximation theorem all symmetric functions are built of polynomials in the traces, and as a limiting case one can also construct distributions like the Dirac distribution. This means that all observables symmetric in the eigenvalues fulfil a similar equation as (3.12) in a weak sense.

As a further remark the symplectic non-Hermitian ensembles with $\beta = 4$ [9, 6] are already of the factorised form eq. (3.1) from the onset, with

$$F^{(\beta=4)}(z_1, z_2) = i(z_2 - z_1)w(z_1, z_2)\delta^2(z_2 - z_1^*) \tag{3.13}$$

and we always have $\chi = 0$ due to symmetry. The symmetric real weight $w(z_1, z_2)$ can be found in [6] and [9] for the Gaussian Ginibre and chiral classes respectively. Therefore all statements we derive from the form of eq. (3.1) automatically hold true for these symmetry classes as well. The factorisation thus unifies the non-Hermitian ensembles for $\beta = 1$ and 4; in fact, this was already pointed out in [14, 21]. In [14], this was found in a different way without using factorisation.

4. Integral representation of skew-orthogonal polynomials and their Cauchy transforms

In this section we will derive integral representations for the SOP for general weight functions, using the results from the previous section. For this we will only need the result for an even total number of eigenvalues (i.e. $\chi = 0$).

All the matrix or complex eigenvalue models introduced previously can be solved for all eigenvalue density correlation functions in terms of the following skew-symmetric kernel

$$\mathcal{K}_{2N}(z_1, z_2) = \sum_{k,l=0}^{2N-1} \mathcal{A}_{kl}^{-1}\, p_k(z_1)\, p_l(z_2) \tag{4.1}$$

where

$$\mathcal{A}_{kl} \equiv 2 \int_{\mathbb{C}^2} d^2z_1\, d^2z_2\, F(z_1, z_2)\, p_k(z_1)\, p_l(z_2)\ . \tag{4.2}$$

In fact, the kernel is only a property of the measure $F(z_{2j-1}, z_{2j})$ and not of the particular choice of the polynomials $\{p_k(z)\}$; in [27] these were chosen to be monic. For an odd total number of eigenvalues a similar representation to eq. (4.1) holds, but with a modification to the last row and column of the matrix $\mathcal{A}$; see [28] and [30]. Here we will choose the polynomials $p_k(z)$ to be skew-orthogonal with respect to the following anti-symmetric scalar product

$$\langle f|g\rangle = -\langle g|f\rangle \equiv \int_{\mathbb{C}^2} d^2z_1\, d^2z_2\, F(z_1, z_2) \det \begin{bmatrix} f(z_1) & g(z_1) \\ f(z_2) & g(z_2) \end{bmatrix}, \tag{4.3}$$

defined for two functions $f(z)$ and $g(z)$ that are integrable with respect to the weight functions contained in $F(z_1, z_2)$. This includes the particular function $g(z_1, z_2)$ from eq. (2.7).

Our skew-orthogonal polynomials $q_k(z)$ are defined to satisfy

$$\begin{aligned} \langle q_{2k}|q_{2l+1}\rangle &= h_k\delta_{kl} \ , \\ \langle q_{2k}|q_{2l}\rangle \quad &= 0 \ = \ \langle q_{2k+1}|q_{2l+1}\rangle \ \forall k,l \geq 0 \ , \end{aligned} \tag{4.4}$$

where the $h_k > 0$ are their positive (squared skew) norms, see eq. (4.16). This leads to a block diagonal matrix $\mathcal{A} = \text{diag}\,(h_0\epsilon, \ldots, h_{N-1}\epsilon)$ that can be easily inverted, where ϵ is the anti-symmetric 2×2 matrix with elements $\epsilon_{12} = 1 = -\epsilon_{21}$. The kernel can be written as a single sum in terms of the SOP:

$$\mathcal{K}_{2N}(z_1, z_2) = \sum_{k=0}^{N-1} \frac{1}{h_k}\,(q_{2k+1}(z_1)q_{2k}(z_2) - q_{2k+1}(z_2)q_{2k}(z_1)). \tag{4.5}$$

The kernel for $2N + 1$ contains the same SOP plus a correction term, see also [30] for the Ginibre ensemble and [1] for the GOE.

4.1. Skew-orthogonal polynomials

After all this preparation we come to our second result, an explicit representation of the SOP. They are given in terms of the following expectation values

$$q_{2n}(z) = \Big\langle \det(z - J)\Big\rangle_{2n} = \Big\langle \prod_{j=1}^{2n}(z - z_j)\Big\rangle_{2n} \tag{4.6}$$

for the even polynomials, and

$$\begin{aligned} q_{2n+1}(z) &= \Big\langle \det(z - J)[\text{Tr}J + z + c]\Big\rangle_{2n} \ = \ \Big\langle \det(z - J)\text{Tr}J\Big\rangle_{2n} \ + \ (z + c)\, q_{2n}(z) \\ &= \Big\langle \prod_{j=1}^{2n}(z - z_j)\Big[\sum_{i=1}^{2n} z_i \ + \ z \ + \ c\Big]\Big\rangle_{2n} \ , \end{aligned} \tag{4.7}$$

for the odd polynomials, which are both expectation values over an even number of eigenvalues $2n$, $n \geq 1$. For $n = 0$ we simply have $q_0(z) = 1$ and $q_1(z) = z + c$, by definition. It is easy to see by taking large arguments that these representations are in monic normalisation, *viz* $q_n(z) = z^n + \mathcal{O}(z^{n-1})$. Similar expressions hold in terms of the matrix $\mathcal{D}$ from eq. (2.4) for the chiral model (see subsection 5.2 for more details), whilst the representations given in terms of squared eigenvalues $\Lambda_j^2 = z_j$ are identical. Eqs. (4.6) and (4.7) were also shown for particular non-Hermitian ensembles in refs. [12, 13].

The set of odd polynomials is not unique because of the anti-symmetry of the skew product (4.3), as the arbitrary constant c times the even polynomial drops out. In most of the following we will keep the constant $c \neq 0$ though.

The proof of the first integral representation eq. (4.6) goes as follows. We can write the product times the Vandermonde determinant of dimension $2n$ as a Vandermonde determinant of dimension $2n + 1$, and so

$$q_{2n}(z) = \frac{1}{\mathcal{Z}_{2n}} \prod_{k=1}^{2n} \int_{\mathbb{C}} d^2 z_k \prod_{j=1}^{n} F(z_{2j-1}, z_{2j}) \det_{1\leq a\leq 2n+1;1\leq b\leq 2n} \left[\{z_b^{a-1}\}|z^{a-1}\right]. \tag{4.8}$$

We can now apply a slight modification of the generalisation of the de Bruijn integral formula proved in Appendix C.2 of [29],

$$\prod_{k=1}^{2n}\int_{\mathbb{C}} d^2z_k \prod_{j=1}^{n} F(z_{2j-1},z_{2j}) \det_{1\leq a\leq 2n+m;\,1\leq j\leq n;\,1\leq i\leq m}\left[\{f_a(z_{2j-1}),g_a(z_{2j})\}|\ \alpha_{ai}\ \right]$$
$$= (-)^{m(m-1)/2} n! \operatorname*{Pf}_{1\leq a,b\leq 2n+m;\,1\leq i\leq m}\left[\begin{array}{cc}\left\{\int_{\mathbb{C}^2} d^2u\, d^2v\, F(u,v)[f_a(u)g_b(v)-f_b(u)g_a(v)]\right\} & \alpha_{ai}\\ -\alpha^T_{ib} & 0\end{array}\right] \tag{4.9}$$

The overall sign can be seen by choosing $[\alpha_{ai}] = [\ \mathbf{0}_{m,\,2n}\ \mathbb{1}_m\ ]^T$. Here α is a constant matrix, which in our case in eq. (4.8) is a simple vector with $m=1$. In contrast to the usual de Bruijn formula we integrate over $2n$ variables here instead of n, as is shown to hold in Appendix A (see also Appendix C.2 in [29]).

Denoting the basis functions of monic powers by $e_a(z)\equiv z^a$ and using the fact that we have equal functions $f_a(z)=g_a(z)=e_{a-1}(z)$ above we arrive at

$$q_{2n}(z) = \frac{n!}{\mathcal{Z}_{2n}} \operatorname*{Pf}_{1\leq a,b\leq 2n+1}\left[\begin{array}{cc}\{\langle e_{a-1}|e_{b-1}\rangle\} & e_{a-1}(z)\\ -e_{b-1}(z) & 0\end{array}\right]. \tag{4.10}$$

It is easy to see that using the definition of the skew product in eq. (4.3) and performing one more integration we have

$$\langle q_{2n}|e_c\rangle \ =\ 0\ \ \forall c=0,\dots 2n\ , \tag{4.11}$$

because the corresponding Pfaffian vanishes. Using the linearity of the skew product we can deduce that the even polynomials $q_{2n}(z)$ in eq. (4.6) are skew-orthogonal to all polynomials of lower and equal degree.

To prove the second integral representation eq. (4.7) we need a further identity for manipulating Vandermonde determinants,

$$\sum_{a=1}^{N} z_a\ \Delta_N(\{z\})\ =\ \det\left[\begin{array}{ccc}1 & \dots & 1\\ z_1 & \dots & z_N\\ \vdots & & \vdots\\ z_1^{N-2} & \dots & z_N^{N-2}\\ z_1^{N} & \dots & z_N^{N}\end{array}\right] \equiv \widetilde{\Delta}_N(\{z\})\ , \tag{4.12}$$

which is proved in Appendix B. We can now proceed as in eq. (4.8), by first incorporating the product in eq. (4.7) into a larger Vandermonde determinant, and then applying the identity (4.12) for $2n+1$:

$$q_{2n+1}(z) = \frac{1}{\mathcal{Z}_{2n}}\prod_{k=1}^{2n}\int_{\mathbb{C}} d^2z_k \prod_{j=1}^{n} F(z_{2j-1},z_{2j}) \det_{1\leq a,b\leq 2n}\left[\begin{array}{cc}\{z_b^{a-1}\} & z^{a-1}\\ z_b^{2n+1} & z^{2n+1}\end{array}\right]. \tag{4.13}$$

For simplicity we have set $c=0$ here as it does not affect the proof. Again applying the integral formula eq. (4.9), with a slightly modified range of indices compared with the even polynomial case, we obtain

$$q_{2n+1}(z) = \frac{n!}{\mathcal{Z}_{2n}} \underset{1\leq a,b\leq 2n+1}{\mathrm{Pf}} \left[\begin{array}{ccc} \{\langle e_{a-1}|e_{b-1}\rangle\} & \langle e_{a-1}|e_{2n+1}\rangle & e_{a-1}(z) \\ \langle e_{2n+1}|e_{b-1}\rangle & 0 & e_{2n+1}(z) \\ -e_{b-1}(z) & -e_{2n+1}(z) & 0 \end{array} \right] . \tag{4.14}$$

On taking the skew product (4.3) of this result it obviously follows that

$$\langle q_{2n+1}|e_c\rangle \ = \ 0 \ \ \forall c = 0, \ldots 2n-1 \ , \tag{4.15}$$

this being the skew-orthogonality of the odd polynomials $q_{2n+1}(z)$ given by eq. (4.7) to all polynomials of degree less than or equal to $2n-1$.

As a last step we will verify the coefficient of the only non-vanishing skew product which due to linearity and eq. (4.11) equals $\langle q_{2n}|e_{2n+1}\rangle = \langle q_{2n}|q_{2n+1}\rangle$. To do so we will first determine the partition function in terms of the norms. It follows along the lines of eq. (4.8). Inside the Vandermonde determinant there we could choose any set of polynomials in monic normalisation, after applying the invariance properties of the determinant. We thus have

$$\begin{aligned} \mathcal{Z}_{2n} &= \prod_{k=1}^{2n} \int_{\mathbb{C}} d^2 z_k \prod_{j=1}^{n} F(z_{2j-1}, z_{2j}) \underset{1\leq a,b\leq 2n}{\det} [q_{a-1}(z_b)] \ = \ n! \underset{1\leq a,b\leq 2n}{\mathrm{Pf}} [\langle q_{a-1}|q_{b-1}\rangle] \\ &= n! \prod_{a=0}^{n-1} h_a \ . \end{aligned} \tag{4.16}$$

In the second step we applied once more the integral identity (4.9), with $m=0$ and the matrix α absent. Due to the skew-orthogonality, the matrix inside the Pfaffian becomes block diagonal, with the norms h_k times ϵ down the diagonal, which finally leads to the product of the norms.

Using eq. (4.10) after replacing the monic powers with polynomials q_k we have

$$\begin{aligned} \langle q_{2n}|q_{2n+1}\rangle &= \frac{n!}{\mathcal{Z}_{2n}} \underset{1\leq a,b\leq 2n+1}{\mathrm{Pf}} \left[\begin{array}{cc} \{\langle q_{a-1}|q_{b-1}\rangle\} & \langle q_{a-1}|q_{2n+1}\rangle \\ -\langle q_{2n+1}|q_{b-1}\rangle & 0 \end{array} \right] \\ &= \frac{n!}{\mathcal{Z}_{2n}} \prod_{a=0}^{n} h_a \ = \ h_n \ , \end{aligned} \tag{4.17}$$

and thus the consistency of the normalisation of our integral representations (4.6) and (4.7) with respect to eq. (4.4). This concludes our proof of the integral representations of the SOP satisfying eq. (4.4). An entirely different derivation of the same results can be made by a mapping to the $\beta = 4$ symplectic case. When mapping our $F(z_1, z_2)$ as in eq. (3.13) we could in principle copy the orthogonality proof from [6] where the representations eq. (4.6) and (4.7) were derived for $\beta = 4$.

It is worth mentioning that the same representation for SOP holds for Hermitian RMT at $\beta = 1$ and 4 with real eigenvalues as was shown earlier in [12, 13]. However, for all four cases – two Hermitian and two non-Hermitian – the jpdf and corresponding skew products are different. In contrast for $\beta = 2$ all OP are obtained from a single relation as in eq. (4.6), in both Hermitian and non-Hermitian RMT [22]. The fact that the same integral representation for SOP holds both in non-chiral [12, 13] and chiral ensembles is straightforward in the Hermitian case. However, for non-Hermitian ensembles this becomes nontrivial comparing [6] vs. [9] for $\beta = 4$, and [10] for $\beta = 1$. This is due to the two-matrix model structure of the chiral ensembles, where the change to an eigenvalue basis requires detailed calculations.

Let us finish this subsection with some remarks. For an even number of eigenvalues $\chi = 0$ the anti-symmetric kernel eq. (4.1) (or (4.5)) can itself be expressed as the expectation value of two characteristic polynomials for $\beta = 1$ [10]‖ (and $\beta = 4$ [24])

$$\Big\langle \det(\lambda - J)\det(\gamma - J)\Big\rangle_{2N} \;=\; h_N \frac{\mathcal{K}_{2N+2}(\lambda,\gamma)}{\lambda-\gamma} \quad \text{with} \quad \lambda \neq \gamma \;, \tag{4.18}$$

and similarly for the chiral ensemble. This equation is valid for arbitrary weight functions. In fact we will partly use this relation to determine the set of odd polynomials eq. (4.7) in section 5 below. So why are eqs. (4.6) and (4.7) interesting if the kernel itself can be independently determined as a building block? It is because the integral representations we just derived, and the explicit determination of the SOP in some examples in the next section, complete the list of classical polynomials in the complex plane for the three elliptic Ginibre ensembles and their chiral extensions.

The determination of the SOP through an ansatz, and subsequently the direct verification of the relations (4.4) for skew-orthogonal Hermite polynomials, was already a formidable task for the elliptic real Ginibre ensemble as can be seen from [7]. Because of the non-Gaussian form of the chiral weight eq. (2.9) this is even more so true for skew-orthogonal Laguerre polynomials. The integral representations derived here provide an alternative, constructive approach, leading to a new result for skew-orthogonal Laguerre polynomials.

4.2. Cauchy transforms

We now come to the definition and integral representation of the Cauchy transforms $t_k(z)$. It is very natural to define them with respect to the scalar product eq. (4.3) as follows:

$$t_n(\kappa) \equiv \int_{\mathbb{C}^2} d^2z_1\, d^2z_2\, F(z_1,z_2) \det\begin{bmatrix} q_n(z_1) & \dfrac{1}{\kappa - z_1} \\ q_n(z_2) & \dfrac{1}{\kappa - z_2}\end{bmatrix} = \Big\langle q_n \Big| \frac{1}{\kappa - z}\Big\rangle \;. \tag{4.19}$$

We will now show that the following integral representations hold:

$$t_{2n}(\kappa) = h_n \Big\langle \frac{1}{\det(\kappa - J)}\Big\rangle_{2n+2} = h_n \Big\langle \prod_{j=1}^{2n+2} \frac{1}{(\kappa - z_j)}\Big\rangle_{2n+2} \tag{4.20}$$

for the Cauchy transforms of the even polynomials, and

$$t_{2n+1}(\kappa) = h_n \Big\langle \frac{\mathrm{Tr}J - (\kappa + c)}{\det(\kappa - J)}\Big\rangle_{2n+2} \;=\; h_n \Big\langle \frac{\mathrm{Tr}J}{\det(\kappa - J)}\Big\rangle_{2n+2} \;-\; (\kappa + c)\, t_{2n}(\kappa)$$

‖ Note that the overall constant in front of the kernel has been chosen here to be consistent with the standard choice in eq. (4.5).

$$= h_n \left\langle \frac{\sum_{a=1}^{2n+2} z_a - (\kappa + c)}{\prod_{j=1}^{2n+2} (\kappa - z_j)} \right\rangle_{2n+2} \tag{4.21}$$

for the odd polynomials. Note that the averages for $t_{2n}(\kappa)$ and $t_{2n+1}(\kappa)$ run over $2n+2$ variables, instead of $2n$ as for the polynomials $q_{2n}(z)$ and $q_{2n+1}(z)$. This implies in particular that $t_0(\kappa) \neq$ constant, see also eq. (4.22) below.

The correct overall prefactors can also easily be seen. From expanding the geometric series in the definition eq. (4.19) for large arguments, and using eqs. (4.11) and (4.15) as well as the anti-symmetry of the first non-vanishing skew product, it follows that the Cauchy transforms are indeed Laurent series with the following coefficients

$$\begin{aligned} t_{2n}(\kappa) &= + \frac{h_n}{\kappa^{2n+2}} + \mathcal{O}\left(\frac{1}{\kappa^{2n+3}}\right), \\ t_{2n+1}(\kappa) &= - \frac{h_n}{\kappa^{2n+1}} + \mathcal{O}\left(\frac{1}{\kappa^{2n+2}}\right). \end{aligned} \tag{4.22}$$

The form given in eqs. (4.20) and (4.21) is completely analogous to eqs. (4.6) and (4.7), as well as to the corresponding result for $\beta = 2$. Let us also remark that such a representation was not known before in the non-Hermitian $\beta = 4$ symmetry class, and that both translate into new representations for $\beta = 1, 4$ in the Hermitian limit.

We begin by proving the representation for the Cauchy transforms of the even polynomials. Inserting the result eq. (4.6) into the definition (4.19) we have

$$\begin{aligned} t_{2n}(\kappa) &= \int_{\mathbb{C}^2} d^2 z_1\, d^2 z_2\, F(z_1, z_2) \left[\frac{\langle \det(z_1 - J) \rangle_{2n}}{\kappa - z_2} - \frac{\langle \det(z_2 - J) \rangle_{2n}}{\kappa - z_1} \right] \\ &= \frac{1}{\mathcal{Z}_{2n}} \int_{\mathbb{C}^2} d^2 z_{2n+1}\, d^2 z_{2n+2}\, F(z_{2n+1}, z_{2n+2}) \prod_{k=1}^{2n} \int_{\mathbb{C}} d^2 z_k \prod_{j=1}^{n} F(z_{2j-1}, z_{2j})\ \Delta_{2n}(\{z\}) \\ &\quad \times \left[\frac{\prod_{j=1}^{2n} (z_{2n+1} - z_j)}{\kappa - z_{2n+2}} - \frac{\prod_{j=1}^{2n} (z_{2n+2} - z_j)}{\kappa - z_{2n+1}} \right] \\ &= \frac{1}{(n+1) \mathcal{Z}_{2n}} \prod_{k=1}^{2n+2} \int_{\mathbb{C}} d^2 z_k \prod_{j=1}^{n+1} F(z_{2j-1}, z_{2j}) \\ &\quad \times \sum_{k=1}^{n+1} \left[\frac{\Delta_{2n+1}(\{z\}_{l \neq 2k})}{\kappa - z_{2k}} - \frac{\Delta_{2n+1}(\{z\}_{l \neq 2k-1})}{\kappa - z_{2k-1}} \right] \\ &= \frac{1}{(n+1) \mathcal{Z}_{2n}} \prod_{k=1}^{2n+2} \int_{\mathbb{C}} d^2 z_k \prod_{j=1}^{n+1} F(z_{2j-1}, z_{2j}) \det_{1 \leq a \leq 2n+2;\, 1 \leq b \leq 2n+1} \left[\{z_a^{b-1}\} \middle| \frac{1}{\kappa - z_a} \right] \\ &= \frac{1}{(n+1) \mathcal{Z}_{2n}} \prod_{k=1}^{2n+2} \int_{\mathbb{C}} d^2 z_k \prod_{j=1}^{n+1} F(z_{2j-1}, z_{2j})\ \frac{\prod_{i>j}^{2n+2} (z_i - z_j)}{\prod_{l=1}^{2n+2} (\kappa - z_l)} \\ &= h_n \left\langle \frac{1}{\det(\kappa - J)} \right\rangle_{2n+2}. \end{aligned} \tag{4.23}$$

In the first step we have simply written out the expectation value and renamed the additional two integration variables. The products in the numerator can be incorporated into a larger Vandermonde determinant. Next we can symmetrise the integrand with respect to an exchange of any pair of variables z_{2j}, z_{2j+1} leading to a prefactor $1/(n+1)$. The resulting expression can be seen to be the expansion of a Vandermonde determinant plus an extra column. In the last step we use an identity that was proved in [29], see eqs. (3.3) vs (3.7) there¶, deriving different representations for Berezinians

$$\det_{1\leq a\leq 2n;\, 1\leq b\leq 2n-1}\left[\{z_a^{b-1}\}\middle|\frac{1}{\kappa-z_a}\right] = \frac{\prod_{a>b}^{2n}(z_a-z_b)}{\prod_{l=1}^{2n}(\kappa-z_l)}\ . \tag{4.24}$$

This can be used to express the Cauchy transform as an expectation value, after providing the correct normalisation factor from eq. (4.16) in the last step.

The proof for the odd Cauchy transforms follows along the same lines. For simplicity we set $c=0$ here, which can easily be reinstated at the end:

$$\begin{aligned}
t_{2n+1}(\kappa) &= \frac{1}{\mathcal{Z}_{2n}}\prod_{k=1}^{2n+2}\int_{\mathbb{C}} d^2z_k \prod_{j=1}^{n+1}F(z_{2j-1},z_{2j})\left[\frac{\left(\sum_{l=1}^{2n} z_l\ + z_{2n+1}\right)\prod_{j=1}^{2n}(z_{2n+1}-z_j)}{\kappa-z_{2n+2}}\right.\\
&\quad \left. -\ \frac{\left(\sum_{l=1}^{2n} z_l\ + z_{2n+2}\right)\prod_{j=1}^{2n}(z_{2n+2}-z_j)}{z_{2n+1}-\kappa}\right]\Delta_{2n}(\{z\})\\
&= \frac{1}{(n+1)\mathcal{Z}_{2n}}\prod_{k=1}^{2n+2}\int_{\mathbb{C}} d^2z_k \prod_{j=1}^{n+1}F(z_{2j-1},z_{2j})\\
&\quad \times\sum_{k=1}^{n+1}\left[\frac{\widetilde{\Delta}_{2n+1}(\{z\}_{l\neq 2k})}{\kappa-z_{2k}}-\frac{\widetilde{\Delta}_{2n+1}(\{z\}_{l\neq 2k-1})}{\kappa-z_{2k-1}}\right]\\
&= \frac{1}{(n+1)\mathcal{Z}_{2n}}\prod_{k=1}^{2n+2}\int_{\mathbb{C}} d^2z_k \prod_{j=1}^{n+1}F(z_{2j-1},z_{2j})\det_{1\leq a\leq 2n+2;\, 1\leq b\leq 2n}\left[\{z_a^{b-1}\}\middle| z_a^{2n+1}\middle|\frac{1}{\kappa-z_a}\right]\\
&= \frac{1}{(n+1)\mathcal{Z}_{2n}}\prod_{k=1}^{2n+2}\int_{\mathbb{C}} d^2z_k \prod_{j=1}^{n+1}F(z_{2j-1},z_{2j})\ \frac{\left(\sum_{l=1}^{2n+2} z_l\ -\kappa\right)\prod_{i>j}^{2n+2}(z_i-z_j)}{\prod_{l=1}^{2n+2}(\kappa-z_l)}\\
&= h_n\left\langle\frac{\mathrm{Tr}J\ -\kappa}{\det(\kappa-J)}\right\rangle_{2n+2}\ .
\end{aligned}\tag{4.25}$$

Here we included the product into the Vandermonde determinant as before, as well as the additional sum leading to the modified Vandermonde determinant $\widetilde{\Delta}$ defined in the right-hand side of eq. (4.12). In the last step we simply need a slightly modified version of the identity eq. (4.24) which is derived in Appendix C. The inclusion of the arbitrary constant $c\neq 0$ follows simply by shifting $\kappa\to\kappa+c$ in the numerator but not in the denominator. This concludes the derivation of all the integral representations of the SOP and their Cauchy transforms. In principle the simple expectation values in

¶ Note that we order products here so that there is no sign in eq. (2.8) for the Vandermonde determinant.

eqs. (4.6) and (4.7) as well as eqs. (4.20) and (4.21) could be computed explicitly using supersymmetric vectors depending on ordinary variables and Grassmannians. In the explicit examples given in the next section we shall give the resulting SOP only.

5. Examples for skew-orthogonal polynomials

In this section we will give three examples of skew-orthogonal polynomials in the complex plane: Hermite, Laguerre and Laguerre with mass terms. Although the first of these were already known, our derivation is new. The second and third are new examples.

In principle there are two different methods. In the first of these we directly use the integral representations; for the even polynomials these are the expectations of a single determinant, eq. (4.6), and for the odd polynomials the expectations of a determinant multiplied by a trace, eq. (4.7). Both can be calculated in one step by computing the expectation of the product of two determinants (which is proportional to the kernel) and either taking limits or differentiating, and using the fact that the determinant is the generating functional of all independent matrix invariants. The expectations can then be computed using Grassmannians, and because this was already explicitly done in [10] we can be very brief here.

The second method follows the general setup outlined at the start of section 4. Given the kernel in terms of general polynomials eq. (4.1), and choosing them to be skew-orthogonal eq. (4.3), the individual polynomials can be "read off" from the kernel in eq. (4.5) by differentiation (or taking limits):

$$\begin{aligned} q_{2n}(z) \quad &= h_n \frac{1}{(2n+1)!}\frac{\partial^{2n+1}}{\partial u^{2n+1}}\mathcal{K}_{2n+2}(u,z) \ = \ h_n \lim_{u\to\infty}\frac{\mathcal{K}_{2n+2}(u,z)}{u^{2n+1}} \\ q_{2n+1}(z) &= -\,h_n \frac{1}{(2n)!}\frac{\partial^{2n}}{\partial u^{2n}}\mathcal{K}_{2n+2}(u,z)\Big|_{u=0} \ + \ c\, q_{2n}(z)\ . \end{aligned} \tag{5.1}$$

This is possible whenever the kernel has already been independently determined, e.g. by the above procedure detailed in [10] (see also [27] for another method). In addition the norms h_k can be read off from the kernel as the leading coefficients.

Of course both methods lead to the same answer. In the third example the kernel including the masses as well as the partition function itself have not previously been computed explicitly and so this also constitutes a new result.

5.1. Example I: skew-orthogonal Hermite polynomials

In general the calculation of the expectation of a single determinant (or the product of two determinants) is straightforward, even without switching to an eigenvalue basis: we express the determinant as an integral over anti-commuting (Grassmann) variables, and then the Gaussian random matrices can be integrated out. After a Hubbard-Stratonovich transformation the anti-commuting variables can also be integrated out. Because the procedure was carried out and explained in detail for two determinants with $N_f = 0$ in this model in [10] we only quote here the result for our first example, the expectation with respect to model eq. (2.3)+:

$$\Big\langle \det(z-J)\,\det(u-J)\Big\rangle_N = N! \sum_{l=0}^{N} \tau^l \sum_{k=0}^{l} \frac{1}{k!\,2^k}\, H_k\left(\frac{z}{\sqrt{2\tau}}\right)\, H_k\left(\frac{u}{\sqrt{2\tau}}\right) \tag{5.2}$$

where τ is the non-Hermiticity parameter, and the $H_k(z)$ are the standard Hermite polynomials. Hence, for the even polynomials we can simply project out the second determinant to give

$$q_{2k}(z) = \lim_{u\to\infty} \frac{\Big\langle \det(z-J)\,\det(u-J)\Big\rangle_{2k}}{u^{2k}} = \left(\frac{\tau}{2}\right)^k H_{2k}\left(\frac{z}{\sqrt{2\tau}}\right) . \tag{5.3}$$

Here we used the following result to calculate the single term in the double sum that survives the limiting process:

$$\lim_{u\to\infty} \frac{1}{u^N}\, H_N\left(\frac{u}{\alpha}\right) = \left(\frac{2}{\alpha}\right)^N . \tag{5.4}$$

This equation also implies that the even polynomials eq. (5.3) are in monic normalisation as they should be, starting with $q_0(z) = 1$. Alternatively we could of course have differentiated eq. (5.2) N times with respect to u.

For the odd polynomials, we use the fact that the determinant is the generating functional for symmetric functions, and in particular for the trace:

$$\frac{1}{(N-1)!}\, \frac{\partial^{N-1}}{\partial u^{N-1}} \det(u\mathbb{1}_N - J)\Big|_{u=0} = -\mathrm{Tr} J \ , \tag{5.5}$$

where J is an $N \times N$ matrix, and $N \geq 1$. Applying this to eq. (5.2) on the left-hand side allows us to obtain $q_{2k+1}(z)$ from eq. (4.7), for $k \geq 1$:

$$\begin{aligned} q_{2k+1}(z) &= -\frac{1}{(2k-1)!}\, \frac{\partial^{2k-1}}{\partial u^{2k-1}} \Big\langle \det(z-J)\,\det(u-J)\Big\rangle_{2k}\Big|_{u=0} + (z+c)q_{2k}(z) \\ &= -(2k)\left(\sqrt{\frac{\tau}{2}}\right)^{2k-1} (\tau+1) H_{2k-1}\left(\frac{z}{\sqrt{2\tau}}\right) + (z+c)\left(\frac{\tau}{2}\right)^k H_{2k}\left(\frac{z}{\sqrt{2\tau}}\right) \\ &= \frac{\tau^{k+\frac{1}{2}}}{2^{k+\frac{1}{2}}} H_{2k+1}\left(\frac{z}{\sqrt{2\tau}}\right) + 2k\, \frac{\tau^{k-\frac{1}{2}}}{2^{k-\frac{1}{2}}} H_{2k-1}\left(\frac{z}{\sqrt{2\tau}}\right) + c\,\frac{\tau^k}{2^k} H_{2k}\left(\frac{z}{\sqrt{2\tau}}\right) . \end{aligned} \tag{5.6}$$

In the first step only two terms survive the differentiation after setting $u = 0$; we used the following properties of Hermite polynomials in addition to eq. (5.4)

$$\frac{d^{n-1}}{dz^{n-1}} H_n(z) = 2^n n!\, z \tag{5.7}$$

$$H_{n+1}(z) = 2zH_n(z) - 2nH_{n-1}(z) \ , \quad \text{for} \quad n \geq 1 \ , \tag{5.8}$$

as well as the recurrence relation to simplify eq. (5.6) in the last line. This form makes it more transparent that the arbitrary addition of $cq_{2k}(z)$ is the only even Hermite

+ The double sum can be simplified by using the Christoffel-Darboux identity [10].

polynomial appearing in this example. From eq. (5.4) it also follows that $q_{2k+1}(z)$ is in monic normalisation, and for $k=0$ we have $q_1(z)=z+c$ by definition. Defining

$$C_k(z) \equiv \left(\frac{\tau}{2}\right)^{\frac{k}{2}} H_k\left(\frac{z}{\sqrt{2\tau}}\right) \tag{5.9}$$

we reobtain the following final simple result from [7]

$$\begin{aligned} q_{2k}(z) \quad &= C_{2k}(z), \\ q_{2k+1}(z) &= C_{2k+1}(z) - 2kC_{2k-1}(z) - cC_{2k}(z)\ . \end{aligned} \tag{5.10}$$

The norms $h_k = 2(\tau+1)\sqrt{2\pi}\,(2k)!$ can be determined either by direct calculation of the scalar product eq. (4.4) of the SOP which we just obtained, as was done in [7], or by computing the partition function*.

Let us emphasise that in our derivation the skew-orthogonality of the polynomials is automatically satisfied due to their general integral representation eqs. (4.6) and (4.7). This is in contrast to [7], where the skew-orthogonality was explicitly verified for the weights eq. (2.11) in the complex plane.

5.2. Example II: skew-orthogonal Laguerre polynomials

In this subsection we turn to entirely new expressions for skew-orthogonal Laguerre polynomials for our chiral model eq. (2.1). We will start with the so-called quenched case ($N_f=0$) and then add mass terms in the next subsection.

Expressed in terms of the $2n$ eigenvalues z_j of AB^T these are the same as before; however, the matrix $\mathcal{D}$ in eq. (2.4) also has ν generic zero eigenvalues, and so we repeat the integral representations eqs. (4.6) and (4.7) for completeness, and also to make contact with [10]. For the even polynomials, we have

$$q_{2n}(z) = \frac{1}{z^{\nu/2}}\Big\langle \det(\sqrt{z}\mathbb{1}_{4n+\nu} - \mathcal{D})\Big\rangle_{4n+\nu} = \Big\langle \det(z\mathbb{1}_{2n} - AB^T)\Big\rangle_{2n} \tag{5.11}$$

and for the odd polynomials

$$\begin{aligned} q_{2n+1}(z) &= \frac{1}{z^{\nu/2}}\left\langle \det(\sqrt{z}\mathbb{1}_{4n+\nu} - \mathcal{D})\left[\frac{1}{2}\mathrm{Tr}\mathcal{D}^2 + z + c\right]\right\rangle_{4n+\nu} \\ &= \Big\langle \det(z\mathbb{1}_{2n} - AB^T)\,\mathrm{Tr}AB^T\Big\rangle_{2n} \ +\ (z+c)\,q_{2n}(z)\,. \end{aligned} \tag{5.12}$$

The starting point for what we need for our calculations, namely the expectation of two determinants, was again given in detail in [10] and thus we merely state the result:

$$\begin{aligned} &\Big\langle \det(z\mathbb{1}_{2n} - AB^T)\det(u\mathbb{1}_{2n} - AB^T)\Big\rangle_{2n} \\ &= (2n)!\,(2n+\nu)!(4\mu^2\eta_+)^{4n}\sum_{l=0}^{2n}\left(\frac{\eta_-}{\eta_+}\right)^{2l}\sum_{k=0}^{l}\frac{k!}{(k+\nu)!}L_k^\nu\left(\frac{z}{4\mu^2\eta_-}\right)L_k^\nu\left(\frac{u}{4\mu^2\eta_-}\right), \end{aligned} \tag{5.13}$$

where we recall the notation (2.2).

* The lower order terms from combining eqs. (4.5) and (4.18) will provide ratios of norms h_N/h_k and thus the k-dependence only.

We thus obtain the even polynomials by simply projecting out the second determinant

$$q_{2k}(z) = \lim_{u\to\infty} \frac{1}{u^{2k}} \Big\langle \det(z\mathbb{1}_{2k} - AB^T)\det(u\mathbb{1}_{2k} - AB^T) \Big\rangle_{2k}$$
$$= (4\mu^2\eta_-)^{2k}\,(2k)!\,L_{2k}^{\nu}\left(\frac{z}{4\mu^2\eta_-}\right). \tag{5.14}$$

Here we have used the following relation for the Laguerre polynomials

$$\lim_{u\to\infty} \frac{1}{u^N} L_N^{\nu}\left(\frac{u}{\alpha}\right) \;=\; \frac{(-1)^N}{N!\,\alpha^N}\ , \tag{5.15}$$

which also confirms that the even polynomial is properly normalised.

For the odd polynomials we again need to take derivatives as in eq. (5.5) to obtain for $k \geq 1$

$$q_{2k+1}(z) = -\frac{1}{(2k-1)!}\frac{\partial^{2k-1}}{\partial u^{2k-1}}\Big\langle \det(z\mathbb{1}_{2k} - AB^T)\det(u\mathbb{1}_{2k} - AB^T)\Big\rangle_{2k}\Big|_{u=0}$$
$$+ (z+c)q_{2k}(z)$$
$$= (4\mu^2\eta_-)^{2k+1}(2k)!\,(2k+\nu)\left(2kL_{2k}^{\nu}\left(\frac{z}{4\mu^2\eta_-}\right) + \left(1+\frac{\eta_+^2}{\eta_-^2}\right)L_{2k-1}^{\nu}\left(\frac{z}{4\mu^2\eta_-}\right)\right)$$
$$+ (z+c)(4\mu^2\eta_-)^{2k}\,(2k)!\,L_{2k}^{\nu}\left(\frac{z}{4\mu^2\eta_-}\right)$$
$$= -(4\mu^2\eta_-)^{2k+1}\,(2k+1)!\,L_{2k+1}^{\nu}\left(\frac{z}{4\mu^2\eta_-}\right) + c'(4\mu^2\eta_-)^{2k}\,(2k)!\,L_{2k}^{\nu}\left(\frac{z}{4\mu^2\eta_-}\right)$$
$$+ (2k+\nu)(4\mu^2\eta_+)^2(4\mu^2\eta_-)^{2k-1}\,(2k)!L_{2k-1}^{\nu}\left(\frac{z}{4\mu^2\eta_-}\right), \tag{5.16}$$

where the new arbitrary constant

$$c' \;\equiv\; c + (4\mu^2\eta_-)(4k^2+4k+1+(2k+1)\nu) \tag{5.17}$$

now depends on k, ν and μ. The above result was obtained after using the corresponding relations for Laguerre polynomials:

$$\frac{d^{n-1}}{dz^{n-1}}L_n^{\nu}(z) \quad = (-1)^n(z-(n+\nu)) \tag{5.18}$$

$$(n+1)L_{n+1}^{\nu}(z) = (2n+\nu+1-z)L_n^{\nu}(z) - (n+\nu)L_{n-1}^{\nu}(z)\ ,\text{for}\ \ n \geq 1\ . \tag{5.19}$$

It is easy to see that the polynomials are again monic, due to eq. (5.15). This once more fixes $q_1(z) = z + c$. We can now define

$$C_k^{\nu}(z) \equiv (4\mu^2\eta_-)^k\,k!\,L_k^{\nu}\left(\frac{z}{4\mu^2\eta_-}\right) \tag{5.20}$$

allowing us to write

$$q_{2k}(z) \;\;= +C_{2k}^{\nu}(z),$$
$$q_{2k+1}(z) = -C_{2k+1}^{\nu}(z) + (1+\mu^2)^2\,(2k)(2k+\nu)C_{2k-1}^{\nu}(z) + c'C_{2k}^{\nu}(z)\ , \tag{5.21}$$

giving the new skew-orthogonal Laguerre polynomials up to an arbitrary constant. The final result compares with the similar form of eq. (5.10).

For the norms we find $h_k = 8\pi(4\mu^2)(2k)!\,(2k+\nu)!\,(4\mu^2\eta_+)^{4k+\nu+1}$ where the k-dependence again follows from the ratio of the norms h_N/h_k, see eqs. (4.18) and (4.5), whereas the overall constant factor can be deduced from the partition function, see eq. (3.46) in [11], and taking the ratio for consecutive values of N.

5.3. Example III: inclusion of mass terms in the chiral model

Our third example gives the SOP again for weights including $N_f > 0$ mass terms, which is also called the unquenched case. We will exemplify this using the chiral model eq. (2.1) where such terms are more common due to applications in QCD. However, the same insertion of mass terms can be done in the non-chiral model eq. (2.3) following the same lines.

Our main point here will be to express the SOP for $N_f > 0$ in terms of the SOP for $N_f = 0$ (and the corresponding kernel), which we have already calculated. To indicate which polynomials we are referring to we will use a superscript, as in $q_k^{(N_f)}(z)$, and correspondingly for the kernel and expectations.

Our derivation relies heavily on [21] where all the expectation values of products and ratios of characteristic polynomials (or determinants) have been expressed in terms of Pfaffian expressions of matrices containing a small number of building blocks; in our case these building blocks will be the quenched ($N_f = 0$) SOP and kernel.

To begin we first express the unquenched integral representations eqs. (4.6) and (4.7) in terms of ratios of quenched expectations. For the even polynomials we have

$$\begin{aligned} q_{2n}^{(N_f)}(z) &= \left\langle \det(z\mathbb{1}_{2n} - AB^T) \right\rangle_{2n}^{(N_f)} \\ &= \frac{\left\langle \det(z\mathbb{1}_{2n} - AB^T) \prod_{f=1}^{N_f} \det(m_f^2 \mathbb{1}_{2n} - AB^T) \right\rangle_{2n}^{(0)}}{\left\langle \prod_{f=1}^{N_f} \det(m_f^2 \mathbb{1}_{2n} - AB^T) \right\rangle_{2n}^{(0)}} \,, \end{aligned} \tag{5.22}$$

and similarly for the odd polynomials we have

$$q_{2n+1}^{(N_f)}(z) = \frac{\left\langle \det(z\mathbb{1}_{2n} - AB^T)\,\mathrm{Tr}AB^T \prod_{f=1}^{N_f} \det(m_f^2 \mathbb{1}_{2n} - AB^T) \right\rangle_{2n}^{(0)}}{\left\langle \prod_{f=1}^{N_f} \det(m_f^2 \mathbb{1}_{2n} - AB^T) \right\rangle_{2n}^{(0)}} + (z+c)q_{2n}^{(N_f)}(z). \tag{5.23}$$

We will now give all the building blocks for these expressions. The first building block in the denominator, the expectation value of the mass term, simultaneously provides us with the massive partition function itself:

$$\frac{\mathcal{Z}_{ch\,2N}^{(N_f)}(\{m\})}{\mathcal{Z}_{ch\,2N}^{(0)}} = \prod_{f=1}^{N_f} m_f^{\nu} \left\langle \prod_{f=1}^{N_f} \det(m_f^2 \mathbb{1}_{2N} - AB^T) \right\rangle_{2N}^{(0)} . \tag{5.24}$$

The masses to the power of ν, the number of generic zero eigenvalues, of course cancel in the ratios for the $q_k^{(N_f)}(z)$ above. Using the results from [21] and expressing the expectation values there in terms of our quenched kernel and even SOP we obtain:

$$\frac{\mathcal{Z}^{(N_f)}_{ch\,2N}(\{m\})}{\mathcal{Z}^{(0)}_{ch\,2N}} = \frac{\prod_{f=1}^{N_f} m_f^\nu}{\Delta_{N_f}(\{m^2\})}(-)^{N_f/2}\prod_{j=N}^{N+(N_f-2)/2} h_j^{(0)} \underset{1\le f,g\le N_f}{\mathrm{Pf}}\left[\mathcal{K}^{(0)}_{2N+N_f}(m_f^2,m_g^2)\right] \qquad N_f \text{ even}$$

$$\frac{\mathcal{Z}^{(N_f)}_{ch\,2N}(\{m\})}{\mathcal{Z}^{(0)}_{ch\,2N}} = \frac{\prod_{f=1}^{N_f} m_f^\nu}{\Delta_{N_f}(\{m^2\})}(-)^{(N_f-1)/2}\prod_{j=N}^{N+(N_f-3)/2} h_j^{(0)}$$
$$\times \underset{1\le f,g\le N_f}{\mathrm{Pf}}\begin{bmatrix} 0 & q^{(0)}_{2N+N_f-1}(m_g^2) \\ -q^{(0)}_{2N+N_f-1}(m_f^2) & \mathcal{K}^{(0)}_{2N+N_f-1}(m_f^2,m_g^2)\end{bmatrix}, \quad N_f \text{ odd} \tag{5.25}$$

where we have to distinguish even and odd numbers of flavours N_f. The product over the norms in the prefactor is equal to unity when the upper limit is $N-1$. Compared to [21] we have used the following identity

$$\underset{1\le f,g\le N_f}{\mathrm{Pf}}\begin{bmatrix} 0 & q^{(0)}_{2M}(m_g^2) \\ -q^{(0)}_{2M}(m_f^2) & \mathcal{K}^{(0)}_{2M+2}(m_f^2,m_g^2)\end{bmatrix} = \underset{1\le f,g\le N_f}{\mathrm{Pf}}\begin{bmatrix} 0 & q^{(0)}_{2M}(m_g^2) \\ -q^{(0)}_{2M}(m_f^2) & \mathcal{K}^{(0)}_{2M}(m_f^2,m_g^2)\end{bmatrix} \tag{5.26}$$

which can be easily seen by adding multiples of the first row and column to the remaining rows and columns, in order to eliminate the leading SOP in the kernels and hence shifting their index down by two. This result for the partition function (or expectation values of characteristic polynomials) precisely equals the corresponding result for $\beta = 4$ in [24]♯.

The even polynomials now easily follow from eq. (5.25), by choosing one of the masses to be the argument. We obtain

$$q^{(N_f)}_{2N}(z) = \frac{\mathrm{Pf}\begin{bmatrix} 0 & q^{(0)}_{2N+N_f}(z) & q^{(0)}_{2N+N_f}(m_g^2) \\ -q^{(0)}_{2N+N_f}(z) & 0 & \mathcal{K}^{(0)}_{2N+N_f}(z,m_g^2) \\ -q^{(0)}_{2N+N_f}(m_f^2) & \mathcal{K}^{(0)}_{2N+N_f}(m_f^2,z) & \mathcal{K}^{(0)}_{2N+N_f}(m_f^2,m_g^2)\end{bmatrix}}{\prod_{f=1}^{N_f}(z-m_f^2)\ \mathrm{Pf}\left[\mathcal{K}^{(0)}_{2N+N_f}(m_f^2,m_g^2)\right]} \tag{5.27}$$

for N_f even. Here and in the following we suppress the indices of the Pfaffian which run from 1 to N_f in both the even and odd cases. For N_f odd we obtain

$$q^{(N_f)}_{2N}(z) = -\frac{h^{(0)}_{N+(N_f-1)/2}\mathrm{Pf}\begin{bmatrix} 0 & \mathcal{K}^{(0)}_{2N+N_f+1}(z,m_g^2) \\ \mathcal{K}^{(0)}_{2N+N_f+1}(m_f^2,z) & \mathcal{K}^{(0)}_{2N+N_f+1}(m_f^2,m_g^2)\end{bmatrix}}{\prod_{f=1}^{N_f}(z-m_f^2)\ \mathrm{Pf}\begin{bmatrix} 0 & q^{(0)}_{2N+N_f-1}(m_g^2) \\ -q^{(0)}_{2N+N_f-1}(m_f^2) & \mathcal{K}^{(0)}_{2N+N_f-1}(m_f^2,m_g^2)\end{bmatrix}}. \tag{5.28}$$

Next we determine the massive kernel using eq. (4.18)

$$\mathcal{K}^{(N_f)}_{2N}(z,u) = \frac{(z-u)}{h^{(N_f)}_{N-1}}\Big\langle \det(z\mathbb{1}_{2N}-AB^T)\det(u\mathbb{1}_{2N}-AB^T)\Big\rangle^{(N_f)}_{2N-2} \tag{5.29}$$
$$= \frac{(z-u)}{h^{(N_f)}_{N-1}}\frac{\Big\langle \det(z\mathbb{1}_{2n}-AB^T)\det(u\mathbb{1}_{2N}-AB^T)\prod_{f=1}^{N_f}\det(m_f^2\mathbb{1}_{2n}-AB^T)\Big\rangle^{(0)}_{2N-2}}{\Big\langle \prod_{f=1}^{N_f}\det(m_f^2\mathbb{1}_{2n}-AB^T)\Big\rangle^{(0)}_{2N-2}}.$$

♯ Notice a typo in [24] in eq. (2.8) compared to the correct Theorem 1 in eq. (3.1) there.

Using eq. (5.25) with two extra masses we obtain for N_f even

$$\mathcal{K}_{2N}^{(N_f)}(z,u) = -\ \frac{\mathrm{Pf}\begin{bmatrix} 0 & \mathcal{K}^{(0)}_{2N+N_f}(u,z) & \mathcal{K}^{(0)}_{2N+N_f}(u,m_g^2) \\ \mathcal{K}^{(0)}_{2N+N_f}(z,u) & 0 & \mathcal{K}^{(0)}_{2N+N_f}(z,m_g^2) \\ \mathcal{K}^{(0)}_{2N+N_f}(m_f^2,u) & \mathcal{K}^{(0)}_{2N+N_f}(m_f^2,z) & \mathcal{K}^{(0)}_{2N+N_f}(m_f^2,m_g^2) \end{bmatrix}}{\prod_{f=1}^{N_f}(z-m_f^2)(u-m_f^2)\ \mathrm{Pf}\left[\mathcal{K}^{(0)}_{2N+N_f}(m_f^2,m_g^2)\right]} . \tag{5.30}$$

Here the mass dependent inverse norm $1/h_{N-1}^{(N_f)}$ has been eliminated using the following identity, leading to a shift in the index of the kernels in the denominator by $+2$. Following eq. (4.16) we can write

$$\frac{h_{N-1}^{(N_f)}}{h_{N-1}^{(0)}} = \frac{\mathcal{Z}_{ch\,2N}^{(N_f)}(\{m\})}{\mathcal{Z}_{ch\,2N}^{(0)}}\ \frac{\mathcal{Z}_{ch\,2N-2}^{(0)}}{\mathcal{Z}_{ch\,2N-2}^{(N_f)}(\{m\})} \tag{5.31}$$

$$= \begin{cases} \dfrac{h^{(0)}_{N+(N_f-2)/2}}{h^{(0)}_{N-1}} \dfrac{\mathrm{Pf}\left[\mathcal{K}^{(0)}_{2N+N_f}(m_f^2,m_g^2)\right]}{\mathrm{Pf}\left[\mathcal{K}^{(0)}_{2N-2+N_f}(m_f^2,m_g^2)\right]} & ,\ N_f \text{ even} \\[3em] \dfrac{h^{(0)}_{N+(N_f-3)/2}}{h^{(0)}_{N-1}} \dfrac{\mathrm{Pf}\begin{bmatrix} 0 & q^{(0)}_{2N+N_f-1}(m_g^2) \\ -q^{(0)}_{2N+N_f-1}(m_f^2) & \mathcal{K}^{(0)}_{2N+N_f-1}(m_f^2,m_g^2)\end{bmatrix}}{\mathrm{Pf}\begin{bmatrix} 0 & q^{(0)}_{2N+N_f-3}(m_g^2) \\ -q^{(0)}_{2N+N_f-3}(m_f^2) & \mathcal{K}^{(0)}_{2N+N_f-3}(m_f^2,m_g^2)\end{bmatrix}} & ,\ N_f \text{ odd.} \end{cases}$$

Likewise for N_f odd we have

$$\mathcal{K}_{2N}^{(N_f)}(z,u) = \frac{-1}{\prod_{f=1}^{N_f}(z-m_f^2)(u-m_f^2)\mathrm{Pf}\begin{bmatrix} 0 & q^{(0)}_{2N+N_f-1}(m_g^2) \\ -q^{(0)}_{2N+N_f-1}(m_f^2) & \mathcal{K}^{(0)}_{2N+N_f-1}(m_f^2,m_g^2)\end{bmatrix}}$$
$$\times\,\mathrm{Pf}\begin{bmatrix} 0 & q^{(0)}_{2N+N_f-1}(z) & q^{(0)}_{2N+N_f-1}(u) & q^{(0)}_{2N+N_f-1}(m_g^2) \\ -q^{(0)}_{2N+N_f-1}(z) & 0 & \mathcal{K}^{(0)}_{2N+N_f-1}(z,u) & \mathcal{K}^{(0)}_{2N+N_f-1}(z,m_g^2) \\ -q^{(0)}_{2N+N_f-1}(u) & \mathcal{K}^{(0)}_{2N+N_f-1}(u,z) & 0 & \mathcal{K}^{(0)}_{2N+N_f-1}(u,m_g^2) \\ -q^{(0)}_{2N+N_f-1}(m_f^2) & \mathcal{K}^{(0)}_{2N+N_f-1}(m_f^2,z) & \mathcal{K}^{(0)}_{2N+N_f-1}(m_f^2,u) & \mathcal{K}^{(0)}_{2N+N_f-1}(m_f^2,m_g^2) \end{bmatrix}. \tag{5.32}$$

Let us pause with a few remarks. Following [11] these expressions for the massive kernel determine all massive eigenvalue correlation functions for $2N$, in terms of the known quenched kernel and the quenched even SOP that were given in Example II above. In particular it is transparent that even for finite N the unquenched kernel (when properly normalised by the massive weight) is given by the quenched kernel plus some correction terms. The same structure thus prevails for all unquenched eigenvalue correlation functions.

For $2N+1$, correction terms to the massive kernel will also include the massive SOP $q_{2N}^{(N_f)}(z)$, when following e.g. [30]. Thus all massive eigenvalue correlation functions follow in this case as well.

As the final step we will give the massive odd SOP $q_{2N+1}^{(N_f)}(z)$. Here we will follow eq. (5.1) and determine them from the kernel, rather than eq. (5.5). As an aside, above we could have alternatively determined the kernel first and then the even SOP from eq. (5.1) as well. A slight generalisation of eq. (5.1) reads

$$q_{2n+1}^{(N_f)}(z) = -h_n^{(N_f)} \frac{1}{(2n+k)!} \frac{\partial^{2n+k}}{\partial u^{2n+k}} \left(\prod_{l=1}^{k} (u-a_l) \mathcal{K}_{2n+2}^{(N_f)}(u,z) \right) \Bigg|_{u=0} + \ c' \, q_{2n}^{(N_f)}(z) \quad (5.33)$$

where $k \geq 0$, and the a_l are some arbitrary constants. It is easy to see that we only get a non-vanishing result when $2n$ or $2n+1$ of the derivatives act on the kernel and not the prefactor. This is true because the function in the brackets is a polynomial of order $2n+k+1$ in the variable u. Hence, the differentiation yields the coefficients of the monomials of order $2n$ and $2n+1$ in the variable u of the kernel $\mathcal{K}_{2n+2}^{(N_f)}$.

In choosing $k = N_f$ and the $a_l = m_l^2$ we can use this relation to cancel the factor $\prod_{f=1}^{N_f}(u - m_f^2)$ in the denominator of eq. (5.30) that would otherwise have to be differentiated as well. We thus obtain from eq. (5.33) and (5.30) that

$$\begin{aligned} q_{2N+1}^{(N_f)}(z) = & - h_N^{(N_f)} \frac{1}{(2N+N_f)!} \frac{\partial^{2N+N_f}}{\partial u^{2N+N_f}} \left(\prod_{f=1}^{N_f} (u-m_f^2) \mathcal{K}_{2N+2}^{(N_f)}(u,z) \right) \Bigg|_{u=0} + \ c' \, q_{2N}^{(N_f)}(z) \\ = & \frac{\mathrm{Pf} \begin{bmatrix} 0 & q_{2N+N_f+1}^{(0)}(z) & q_{2N+N_f+1}^{(0)}(m_g^2) \\ -q_{2N+N_f+1}^{(0)}(z) & 0 & \mathcal{K}_{2N+N_f}^{(0)}(z,m_g^2) \\ -q_{2N+N_f+1}^{(0)}(m_f^2) & \mathcal{K}_{2N+N_f}^{(0)}(m_f^2,z) & \mathcal{K}_{2N+N_f}^{(0)}(m_f^2,m_g^2) \end{bmatrix}}{\prod_{f=1}^{N_f}(z-m_f^2) \ \mathrm{Pf}\left[\mathcal{K}_{2N+N_f}^{(0)}(m_f^2,m_g^2) \right]} + c' q_{2N}^{(N_f)}(z) \\ & \text{for } \ N_f \text{ even.} \end{aligned} \quad (5.34)$$

Here we have pulled the derivatives inside the Pfaffian leading to the quenched SOP of shifted odd index, and used eq. (5.31) and the identity corresponding to eq. (5.26) for the odd polynomials. For odd N_f we obtain

$$\begin{aligned} q_{2N+1}^{(N_f)}(z) = & \frac{1}{\prod_{f=1}^{N_f}(z-m_f^2) \ \mathrm{Pf} \begin{bmatrix} 0 & q_{2N+N_f-1}^{(0)}(m_g^2) \\ -q_{2N+N_f-1}^{(0)}(m_f^2) & \mathcal{K}_{2N+N_f-1}^{(0)}(m_f^2,m_g^2) \end{bmatrix}} \\ & \times \mathrm{Pf} \begin{bmatrix} 0 & q_{2N+N_f+1}^{(0)}(z) & \tilde{c} & q_{2N+N_f+1}^{(0)}(m_g^2) \\ -q_{2N+N_f+1}^{(0)}(z) & 0 & q_{2N+N_f-1}^{(0)}(z) & \mathcal{K}_{2N+N_f+1}^{(0)}(z,m_g^2) \\ -\tilde{c} & -q_{2N+N_f-1}^{(0)}(z) & 0 & q_{2N+N_f-1}^{(0)}(m_g^2) \\ -q_{2N+N_f+1}^{(0)}(m_f^2) & \mathcal{K}_{2N+N_f+1}^{(0)}(m_f^2,z) & -q_{2N+N_f-1}^{(0)}(m_f^2) & \mathcal{K}_{2N+N_f+1}^{(0)}(m_f^2,m_g^2) \end{bmatrix} \\ & + c' q_{2N}^{(N_f)}(z) \ \text{ for } \ N_f \text{ odd.} \end{aligned} \quad (5.35)$$

The constants $\tilde{c}$ in the Pfaffian in the numerator can be absorbed into the even polynomial $c' q_{2N}^{(N_f)}(z)$, which can be seen as follows. Just as the determinants of two matrices that only differ by a single row (or column) can be added, a similar statement holds for Pfaffians: due to linearity the Pfaffians of two anti-symmetric matrices that only differ by a single row and its transposed column can be added. We can thus split off the $\tilde{c}$-part from the Pfaffian above to obtain

$$\mathrm{Pf}\begin{bmatrix} 0 & q^{(0)}_{2N+N_f+1}(z) & \tilde{c} & q^{(0)}_{2N+N_f+1}(m_g^2) \\ -q^{(0)}_{2N+N_f+1}(z) & 0 & 0 & \mathcal{K}^{(0)}_{2N+N_f+1}(z,m_g^2) \\ -\tilde{c} & 0 & 0 & 0 \\ -q^{(0)}_{2N+N_f+1}(m_f^2) & \mathcal{K}^{(0)}_{2N+N_f+1}(m_f^2,z) & 0 & \mathcal{K}^{(0)}_{2N+N_f+1}(m_f^2,m_g^2) \end{bmatrix}$$
$$= -\,\mathrm{Pf}\begin{bmatrix} 0 & \tilde{c} & 0 & 0 \\ -\tilde{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{K}^{(0)}_{2N+N_f+1}(z,m_g^2) \\ 0 & 0 & \mathcal{K}^{(0)}_{2N+N_f+1}(m_f^2,z) & \mathcal{K}^{(0)}_{2N+N_f+1}(m_f^2,m_g^2) \end{bmatrix} \qquad (5.36)$$

which is proportional to the numerator of the even polynomials with odd N_f in eq. (5.28). Thus the final result for the odd polynomial with odd N_f is eq. (5.35) with $\tilde{c}=0$ and $c' \to c''$. This ends our third example for the SOP and kernel including masses.

Similar expressions could be given for the non-chiral model eq. (2.3), as well as for the Cauchy transforms of the unquenched SOP.

6. Conclusions

In this paper we have completed the analysis of the set of (skew-) orthogonal polynomials in the complex plane that apply to the chiral extensions of the three elliptic Ginibre ensembles. By constructing an explicit integral representation we found a new set of skew-orthogonal Laguerre polynomials in the complex plane which provide an alternative method of solving the chiral ensemble with real asymmetric elements. Our integral representation is also valid for the real elliptic Ginibre ensemble; in fact, it holds for arbitrary weight functions g and h in these two classes. Furthermore we also gave a new integral representation of the Cauchy transforms of these polynomials which holds not only for the two symmetry classes with real matrix elements ($\beta=1$) but also for quaternion real matrix elements ($\beta=4$).

An important ingredient for our results was a proof that the probability distribution in the partition function factorises for $\beta=1$. This offers another unifying view of the non-Hermitian $\beta=1$ and $\beta=4$ symmetry classes, both chiral and non-chiral.

There are many more non-Hermitian ensembles, in addition to the three Ginibre classes and their chiral counterparts, all six of which have now been solved. It thus remains an open question whether corresponding sets of orthogonal or skew-orthogonal polynomials exist for the other ensembles. It is possible that, just as in the real case, the known polynomials also apply to some of these other non-Hermitian symmetry classes, once a complex eigenvalue representation has been found for them.

As an application of our results we have shown how to construct the skew-orthogonal polynomials and the kernel when including N_f characteristic polynomials or mass terms into our models. These constitute the building blocks for the massive partition function and eigenvalue density correlation functions. Consequently this will allow us to study the complex Dirac operator spectrum for Quantum Chromodynamics with two colours and non-vanishing quark chemical potential, both in the low and high density phases. The study of the large-N limit needed for this partly follows from the known quenched $N_f = 0$ results and is left for future investigations.

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Appendix A. Generalisation of the de Bruijn integral formula

In this appendix we slightly generalise the standard de Bruijn integral formula that reads

$$\prod_{j=1}^{n}\int_{\mathbb{C}} d^2z_j\, w(z_j) \det_{1\leq a\leq 2n;\,1\leq l\leq n}[\{f_a(z_l), g_a(z_l)\}]$$
$$= n! \operatorname*{Pf}_{1\leq a,b\leq 2n}\left[\int_{\mathbb{C}} d^2u\, w(u)[f_a(u)g_b(u) - f_b(u)g_a(u)]\right] . \qquad \text{(A.1)}$$

Here $w(z)$ is a weight function in the complex plane and f and g are functions such that the integrals exist. The proof is usually done by a Laplace expansion into 2×2 blocks that each depend on a single variable z_l.

If we start out with $2n$ instead of n integrations over a product of an anti-symmetric weight $F(u, v)$ and let f and g depend on different variables, we have on the left-hand side

$$\prod_{k=1}^{2n}\int_{\mathbb{C}} d^2z_k \prod_{j=1}^{n} F(z_{2j-1}, z_{2j}) \det_{1\leq a\leq 2n;\,1\leq b\leq n}[\{f_a(z_{2b-1}), g_a(z_{2b})\}]$$
$$= \prod_{j=1}^{n}\int_{\mathbb{C}^2} d^2z_{2j-1}d^2z_{2j}F(z_{2j-1}, z_{2j}) \sum_{\sigma}(-)^{\sigma}\prod_{j=1}^{n}\det\left[\begin{array}{cc} f_{\sigma(2j-1)}(z_{2j-1}) & g_{\sigma(2j-1)}(z_{2j}) \\ f_{\sigma(2j)}(z_{2j-1}) & g_{\sigma(2j)}(z_{2j}) \end{array}\right]$$
$$= n! \operatorname*{Pf}_{1\leq a,b\leq 2n}\left[\int_{\mathbb{C}^2} d^2u\, d^2vF(u, v)[f_a(u)g_b(v) - f_b(u)g_a(v)]\right] . \qquad \text{(A.2)}$$

Here $(-)^{\sigma}$ is the sign of the permutation of the $2n$ variables, and the sum is over all $(2n)!$ permutations which satisfy the restriction $\sigma(1) < \sigma(2) < \ldots < \sigma(2n)$. Note that each pair $\{z_{2j-1}, z_{2j}\}$ only appears in one subdeterminant. This gives the Pfaffian as a result (see e.g. [1] for a definition).

Appendix B. Relation to a modified Vandermonde determinant

In this appendix we prove an identity related to Vandermonde determinants, which is needed to derive the integral representation eq. (4.7) for the odd polynomials $q_{2n+1}(z)$. For completeness we repeat the relation eq. (4.12) which is to be shown here,

$$\sum_{i=1}^{N} z_i\ \Delta_N(\{z\}) \ =\ \det\begin{bmatrix} 1 & \dots & 1 \\ z_1 & \dots & z_N \\ \vdots & & \vdots \\ z_1^{N-2} & \dots & z_N^{N-2} \\ z_1^{N} & \dots & z_N^{N} \end{bmatrix} \equiv \widetilde{\Delta}_N(\{z\})\ . \tag{B.1}$$

Using the second representation from eq. (2.8), $\Delta_N(\{z\}) = \det_{1\leq a,b\leq N}[z_a^{b-1}]$, one can see that the modified Vandermonde determinant in eq. (B.1) has a mismatch of 1 in the powers in the last row compared to the Vandermonde determinant.

Our proof uses that $\widetilde{\Delta}_N(\{z\})$ is simply the coefficient of power u^{N-1} in a Leibniz expansion of the Vandermonde determinant $\Delta_{N+1}(\{z\}, u)$ of size $N+1$ with respect to the last column in the extra variable u. This term can be singled out by a differentiation,

$$\frac{1}{(N-1)!}\ \frac{\partial^{N-1}}{\partial u^{N-1}}\bigg|_{u=0}\ \Delta_{N+1}(\{z\}, u) = -\widetilde{\Delta}_N(\{z\})\ . \tag{B.2}$$

On the other hand, using again eq. (2.8) that $\Delta_N(\{z\}) = \prod_{j>k}^{N}(z_j - z_k)$, one can write

$$\Delta_{N+1}(\{z\}, u) = \prod_{a=1}^{N}(u - z_a)\Delta_N(\{z\}). \tag{B.3}$$

Combining eqs. (B.2) and (B.3) we obtain the result (B.1).

Appendix C. Cauchy-type identity for the modified Vandermonde determinant

In this appendix we prove the following identity,

$$\det_{1\leq a\leq 2n+2;\, 1\leq b\leq 2n}\left[\left\{z_a^{b-1}\right\}\Big| z_a^{2n+1}\Big|\frac{1}{\kappa - z_a}\right] \ =\ \frac{\left(\sum_{k=1}^{2n+2} z_k\ - \kappa\right)\prod_{i>j}^{2n+2}(z_i - z_j)}{\prod_{l=1}^{2n+2}(\kappa - z_l)}\ , \tag{C.1}$$

which is a slight modification of identity (4.24) with a mismatch by one power in the last but one column. In fact we will use the identity (4.24) to prove the above. Expanding the left-hand side with respect to the last but one column we have

$$\begin{aligned}
&\sum_{k=1}^{2n+2}(-)^{2n+2-k-1} z_k^{2n+1} \det_{1\leq a\neq k\leq 2n+2;\, 1\leq b\leq 2n}\left[\left\{z_a^{b-1}\right\}\Big|\frac{1}{\kappa - z_a}\right] \\
&= (-)\sum_{k=1}^{2n+2}(-)^{2n+2-k} z_k^{2n+1}\frac{\prod_{i>j;\, i,j\neq k}^{2n+2}(z_i - z_j)}{\prod_{l\neq k}^{2n+2}(\kappa - z_l)}\ \frac{(\kappa - z_k)}{(\kappa - z_k)} \\
&= (-)\frac{\kappa\Delta_{2n+2}(\{z\}) - \widetilde{\Delta}_{2n+2}(\{z\})}{\prod_{l=1}^{2n+2}(\kappa - z_l)}
\end{aligned}$$

$$= \frac{\left(\sum_{k=1}^{2n+2} z_k \ - \kappa\right) \prod_{i>j}^{2n+2}(z_i - z_j)}{\prod_{l=1}^{2n+2}(\kappa - z_l)} \ . \tag{C.2}$$

In the second step we used the identity (4.24) and multiplied by unity to complete the product in the denominator. For the numerator we obtain the modified Vandermonde determinant $\widetilde{\Delta}$ from eq. (4.12) and a proper Vandermonde determinant, both of size $2n+2$, after resumming the expansion. Writing out explicitly $\widetilde{\Delta}_{2n+2}$ from the left-hand side of eq. (B.1) yields the right-hand side of our identity (C.1).

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