On the supersymmetric partition function in QCD-inspired random matrix models

Yan V Fyodorov[1,2] **and Gernot Akemann**[3]

[1]Department of Mathematical Sciences, Brunel University
Uxbridge, UB8 3PH, United Kingdom
[2]Petersburg Nuclear Physics Institute, Gatchina 188350, Russia
[3] Service de Physique Théorique, CEA/DSM/SPhT
Unité de recherche associée au CNRS
F-91191 Gif-sur-Yvette Cedex, France

Abstract

We show that the expression for the supersymmetric partition function of the chiral Unitary (Laguerre) Ensemble conjectured recently by Splittorff and Verbaarschot [13] follows from the general expression derived recently by Fyodorov and Strahov [16].

A class of random matrices that has attracted a considerable attention recently [1, 2, 3, 4, 5, 6, 8, 7, 9, 10] is the so-called *chiral* (Gaussian) Unitary Ensemble (chGUE), also known as the Laguerre ensemble. The corresponding matrices are of the form $\hat{D} = \begin{pmatrix} \mathbf{0} & \hat{W} \\ \hat{W}^\dagger & \mathbf{0} \end{pmatrix}$, where $\hat{W}$ stands for a complex matrix, with $\hat{W}^\dagger$ being its Hermitian conjugate. The off-diagonal block structure is characteristic for systems with chiral symmetry. The chiral ensemble was introduced to provide a background for calculating the universal part of the microscopic level density for the Euclidian QCD Dirac operator, see [11] and references therein. Independently and simultaneously it was realised that the same chiral ensemble is describing a new group structure associated with scattering in disordered mesoscopic wires [2]. One of the main objects of interest in QCD is the so-called Euclidean partition function used to describe a system of quarks characterized by n_f flavors and quark masses m_f interacting with the Yang-Mills gauge fields. At the level of Random Matrix Theory the true partition function is replaced by the matrix integral:

$$Z_{n_f}(\hat{M}_f) = \int \mathcal{D}\hat{W} \prod_{k=1}^{n_f} \det\{i\hat{D} + m_f^{(k)} \mathbf{1}_{2N}\} e^{-N \mathrm{Tr} V(\hat{W}^\dagger \hat{W})} \tag{1}$$

where $\hat{M}_f = \mathrm{diag}\left(m_f^{(1)}, ..., m_f^{(n_f)}\right)$ and $V(z)$ is a suitable potential. Here the integration over complex $\hat{W}$ replaces the functional integral over gauge field configurations

[11]. Then the calculation of the partition function amounts to performing the ensemble average of the product of characteristic polynomials of $i\hat{D}$ over the probability density $P(W) \propto e^{-N\mathrm{Tr}V(\hat{W}^\dagger\hat{W})}$. In the general case of non-zero topological charge $\nu > 0$ the matrices $\hat{W}$ have to be chosen rectangular of size $N \times (N+\nu)$ [11]. For simplicity one may choose the probability distribution to be Gaussian as defined by the formula: $d\mathcal{P}(W) \propto d\hat{W} d\hat{W}^\dagger \exp - \left[N \mathrm{Tr} \hat{W}^\dagger \hat{W}\right]$.

The characteristic feature of the chiral ensemble is the presence of a particular point $\lambda = 0$ in the spectrum, also called the "hard edge" [3]. The eigenvalues of chiral matrices appear in pairs $\pm\lambda_k$, $k = 1, ..., N$. Far from the hard edge the statistics of eigenvalues is practically the same as for usual GUE matrices without chiral structure, but in the vicinity of the edge eigenvalues behave very differently.

Let $\mathcal{Z}_N[m]$ be the following spectral determinant (characteristic polynomial of $i\hat{D}$):

$$\mathcal{Z}_N[m] = \det\left(m^2 \mathbf{1}_N + \hat{W}^\dagger \hat{W}\right) . \tag{2}$$

and let us consider a more general (supersymmetric) partition function for the chGUE defined as

$$\mathcal{K}(\hat{M}_f, \hat{M}_b) = \left\langle \frac{\prod_{j=1}^{L} \mathcal{Z}_N\left[m_f^{(j)}\right]}{\prod_{j=1}^{M} \mathcal{Z}_N\left[m_b^{(j)}\right]} \right\rangle_W \tag{3}$$

where $\hat{M}_f = \mathrm{diag}\left(m_f^{(1)}, ..., m_f^{(L)}\right)$, $\hat{M}_b = \mathrm{diag}\left(m_b^{(1)}, ..., m_b^{(M)}\right)$. This correlation function contains much more information on spectra of chiral matrices than the partition function (1) since it involves both product and ratios of the characteristic polynomials. Many efforts were spent on developing methods allowing one to calculate particular cases of such a general supersymmetric partition (or correlation) function [9, 10]. In particular, the case $\nu = 0$ was completely solved in [15] by a variant of the supermatrix method [12, 10] augmented with a generalization of an Itzykson-Zuber type integrals [4, 5] to integration over non-compact group manifolds. In the recent paper [13] Splittorff and Verbaarschot conjectured the result for arbitrary integer $\nu > 0$ in the microscopic (sometimes also called "chiral") large-N limit: $N \to \infty$ such that $\hat{X}_{b,f} = 2N\hat{M}_{b,f}$ is finite. The authors of [13] used the advanced version of the replica method suggested recently by Kanzieper [14]. The final result is given in terms of a determinant containing modified Bessel functions $I_l(z)$ ("compact integrals") and their noncompact partners - Macdonald functions $K_l(z)$. The goal of the present Letter is to show that the case $\nu \neq 0$ considered in [13] in fact follows from a very general expression derived in the recent paper [16]. The demonstration of this fact also provides a natural explanation why both compact and non-compact integrals must appear on equal basis.

The eigenvalues $x_1, ..., x_N$ of the $N \times N$ positive definite matrix $H = \hat{W}^\dagger \hat{W}$ are known to be distributed according to the Laguerre ensemble density function

$$\mathcal{P}(x_1, ..., x_N) \propto \Delta^2(\hat{X}) \prod_{i=1}^{N} w_\nu(x_i) \tag{4}$$

where $w_\nu(x) = x^\nu e^{-Nx}$ and $\Delta(\hat{X}) = \prod_{i>j}(x_i - x_j)$. Note that the spectral determinant $\mathcal{Z}_N(m) = (-1)^N \prod_{i=1}^{N} [(-m^2) - x_i]$ is just the characteristic polynomial of matrices H from the Laguerre ensemble taken at negative real values of the spectral parameter. As is proved in the paper [16] one can express the general correlation function of the characteristic polynomials for an arbitrary unitary invariant ensemble of $\beta = 2$ symmetry class in terms of a $(M+L)$-sized determinant. The main building blocks of that determinant are (monic) orthogonal polynomials $\pi_n(x) = x^n + ...$ satisfying

$$\int_D dx\, w(x) \pi_n(x) \pi_m(x) = \delta_{nm} c_n^2 \tag{5}$$

where $w(x)$ is a general weight function, c_n are normalization constants and D is the corresponding interval of orthogonality. A novel feature revealed in [16] is that for $M > 0$ such a determinant structure contains the *Cauchy transforms* of the orthogonal polynomials

$$h_n(\epsilon) = \frac{1}{2\pi i} \int_D dx \frac{w(x)}{x - \epsilon} \pi_n(x) \tag{6}$$

alongside with the orthogonal polynomials themselves. For them to be well defined we need to have $Im(\epsilon) \neq 0$.

Actually, the partition function Eq.(3) is given by [16]:

$$\mathcal{K}(\hat{M}_f, \hat{M}_b) \propto \frac{1}{\triangle(\hat{M}_b^2)\triangle(\hat{M}_f^2)} \tag{7}$$

$$\times \det \begin{vmatrix} h_{N-M}\left(-[m_b^{(1)}]^2\right) & h_{N-M+1}\left(-[m_b^{(1)}]^2\right) & \dots & h_{N+L-1}\left(-[m_b^{(1)}]^2\right) \\ \vdots & \vdots & & \vdots \\ h_{N-M}\left(-[m_b^{(M)}]^2\right) & h_{N-M+1}\left(-[m_b^{(M)}]^2\right) & \dots & h_{N+L-1}\left(-[m_b^{(M)}]^2\right) \\ \pi_{N-M}\left(-[m_f^{(1)}]^2\right) & \pi_{N-M+1}\left(-[m_f^{(1)}]^2\right) & \dots & \pi_{N+L-1}\left(-[m_f^{(1)}]^2\right) \\ \vdots & \vdots & & \vdots \\ \pi_{N-M}\left(-[m_f^{(L)}]^2\right) & \pi_{N-M+1}\left(-[m_f^{(L)}]^2\right) & \dots & \pi_{N+L-1}\left(-[m_f^{(L)}]^2\right) \end{vmatrix}.$$

For the Laguerre ensemble of matrices H with positive eigenvalues $\hat{X} = \mathrm{diag}(x_1, ..., x_N)$, the weight function is just $w_\nu(x) = x^\nu e^{-Nx}$, the domain is $D = [0 \leq x < \infty]$ and the monic polynomials are $\pi_n(x) = \frac{(-1)^n}{N^n} n!\, L_n^\nu(xN)$, with $L_n^\nu(xN)$ being the standard Laguerre polynomials. Here ν can be taken real valued with $\nu > -1$. To calculate the Cauchy transform Eq.(6) we exploit a well-known integral representation for the Laguerre polynomials containing the Bessel function $J_\nu(x)$:

$$\pi_n(x) = \frac{(-1)^n}{N^{n+\nu/2}}\, e^{Nx} x^{-\nu/2} \int_0^\infty dt\, e^{-t} t^{n+\nu/2} J_\nu\left(2\sqrt{Nxt}\right) . \tag{8}$$

Let us consider, for definiteness, $Im(\epsilon) > 0$ and further employ the integral representation

$$\frac{1}{x-\epsilon} = i \int_0^\infty d\tau e^{-i\tau(x-\epsilon)} \tag{9}$$

Then replacing $\pi_n(x)$ in (6) by (8) and $1/(x-\epsilon)$ by (9) we easily perform the integration over x first, then integrate over τ and arrive at the following representation (cf.Eq.(8)):

$$h_n(\epsilon) = \frac{(-1)^n}{2N^{n+\nu/2}} \epsilon^{\nu/2} \int_0^\infty dt\, e^{-t} t^{n+\nu/2} H_\nu^{(1)}\left(2\sqrt{N\epsilon t}\right) . \tag{10}$$

Here $H_\nu^{(1)}(z)$ is the Hankel function of the first order. Being actually interested in analytically continued values of $\pi_n(x), h_n(x)$ for the region $x = -m^2 < 0$ we introduce the modified Bessel and Macdonald functions according to $I_\nu(z) = e^{-i\pi\nu/2} J_\nu(iz)$ and $K_\nu(z) = \frac{i\pi}{2} e^{i\pi\nu/2} H_\nu^{(1)}(iz)$. We then have

$$\pi_n(-m^2) = \frac{(-1)^n}{N^{n+\nu/2}}\, e^{-N m^2} m^{-\nu} \int_0^\infty dt\, e^{-t} t^{n+\nu/2} I_\nu\left(2m\sqrt{Nt}\right) \tag{11}$$

$$h_n(-m^2) = \frac{(-1)^n}{N^{n+\nu/2}} \frac{m^\nu}{i\pi} \int_0^\infty dt\, e^{-t} t^{n+\nu/2} K_\nu\left(2m\sqrt{Nt}\right) \tag{12}$$

Substituting such representations into the expression Eq.(7) it is easy to satisfy oneself that the right-hand side can be rewritten as $(M+L)$-fold integral

$$\begin{aligned} \mathcal{K}(\hat{M}_f, \hat{M}_b) \propto & \frac{1}{\triangle(\hat{M}_b^2)\triangle(\hat{M}_f^2)} e^{-N\sum_{j=1}^{L}[m_f^{(j)}]^2} \left[\frac{\det M_b}{\det M_f}\right]^\nu \\ \times & \int_{t_i>0} d\hat{t} \det(\hat{t})^{N-M} \Delta(\hat{t}) e^{-N \mathrm{Tr}\hat{t}} \prod_{l=1}^{L} \left[t_l^{\nu/2} I_\nu(2m_f^{(l)} N\sqrt{t_l})\right] \prod_{l=L+1}^{L+M} \left[t_l^{\nu/2} K_\nu(2m_b^{(l-L)} N\sqrt{t_l})\right] . \end{aligned} \tag{13}$$

Here $\hat{t} > 0$ is a diagonal matrix of the size $M+L$ with entries $t_1, ..., t_{M+L}$, and we rescaled $t \to Nt$. Such an equation generalizes the integral representation Eqs.(28)-(29) from [15] to nonzero values of ν. It is valid for any integer N, L, M. The chiral limit $N \to \infty$ can be performed exactly along the same lines as in [15] and the result emerging is the one conjectured by Splittorff and Verbaarschot [13], [17]:

$$\mathcal{K}(X_f, X_b) \propto \left[\frac{\det X_b}{\det X_f}\right]^{\nu} \frac{1}{\triangle(X_b^2)\triangle(X_f^2)} \det\left[X_i^{j-1}\mathcal{J}_{\nu+j-1}(X_i)\right]_{i,j=1,\dots,M+L}. \tag{14}$$

Here $X_f = X_{\{i=1,\dots,L\}}$ and $X_b = X_{\{i=L+1,\dots,L+M\}}$ denote the rescaled fermionic and bosonic masses respectively as well as $\mathcal{J}_i = I_i$ for $i = 1,\dots,L$ and $\mathcal{J}_i = K_i$ for $i = L+1,\dots,L+M$. Note that the presence of "compact" (modified Bessel) and "non-compact" (Macdonald) functions in the final expressions is a direct consequence of the presence of both orthogonal polynomials and their non-polynomial partners (Cauchy transforms) in the determinantal representation. One may also wish to consider a more general type of potentials V in the probability density, see e.g [7]. The related universality questions will be addressed elsewhere [18].

YVF is grateful to Jac Verbaarschot for stimulating his interest in the problem and useful communications. This work was supported by EPSRC grant GR/R13838/01 "Random Matrices close to unitary or Hermitian" (YVF) and by a Heisenberg fellowship of the Deutsche Forschungsgemeinschaft (GA).

References

[1] E.V. Shuryak and J.J.M. Verbaarschot, *Nucl. Phys.* **A560** (1993) 306; J.J.M. Verbaarschot and I. Zahed, *Phys.Rev.Lett.* **70** (1993) 3852

[2] K. Slevin and T. Nagao, *Phys.Rev.B* **50** (1994) 2380 ; AV Andreev et al., *Nucl.Phys.B*[FS] **432** (1994) 487

[3] P.J. Forrester, *Nucl.Phys.B*[FS] **402** (1993) 709 ; C. Tracy and H. Widom, *Commun.Math.Phys.* **161** (1994) 289;

[4] A.D. Jackson et al., *Phys.Lett.*B **387** (1996) 355; *Nucl.Phys.B*[FS] **479** (1996) 707; *ibid* **506** (1996) 612

[5] T. Guhr and T. Wettig, *J.Math.Phys.* **37** (1996) 6395; and *Nucl.Phys.B* **506** (1997) 589

[6] J. Jurkiewicz et al., *Nucl.Phys.B* **478** (1996) 605; Erratum-ibid. **513** (1998) 759; T. Wilke et al., *Phys.Rev.D* **57** (1998) 6486 **392** (1997) 155

[7] G. Akemann et al., *Nucl.Phys.B* **487**, (1997) 721; E. Kanzieper and V. Freilikher, *Phil.Mag.B* **77** 1161 (1998); P.H. Damgaard and S.M. Nishigaki, *Nucl.Phys.B* **518**, (1998) 495

[8] E. Brezin and S. Hikami, *Comm.Math.Phys.* **214** (2000) 111

[9] D. Dalmazi and J.J.M. Verbaarschot, *Nucl.Phys.B* **592** (2001) 419

[10] Y.V. Fyodorov, *Nucl.Phys.B*[PM] **621** (2002) 643

[11] J.J.M. Verbaarschot and T. Wettig, *Annu.Rev.Nucl.Part.Sci.* **50** (2000) 343

[12] K.B. Efetov, "Supersymmetry in Disorder and Chaos" (Cambridge University Press, Cambridge 1997)

[13] K. Splittorff and J.J.M. Verbaarschot, *Phys.Rev.Lett.* **90** (2003) 041601

[14] E. Kanzieper, *Phys.Rev.Lett.* **89** (2002) 250201

[15] Y.V. Fyodorov and E. Strahov, *Nucl.Phys.B* **647** (2002) 581

[16] Y.V. Fyodorov and E. Strahov, *J. Phys. A: Math.Gen.* **36** (2003) 3203

[17] Note that the factor $\left[\frac{\det X_b}{\det X_f}\right]^\nu$ is absent in the corresponding expression in [13]. Instead, it is (implicitly) included in the normalisation of the probability measure of the chiral random matrix ensemble.

[18] G. Akemann and Y.V. Fyodorov, hep-th/0304095