Bounded H_∞ Synchronization and State Estimation for Discrete Time-Varying Stochastic Complex Networks Over a Finite Horizon

Bo Shen, Zidong Wang, *Senior Member, IEEE*, and Xiaohui Liu

***Abstract*— In this paper, new synchronization and state estimation problems are considered for an array of coupled discrete time-varying stochastic complex networks over a finite horizon. A novel concept of bounded H_∞ synchronization is proposed to handle the time-varying nature of the complex networks. Such a concept captures the transient behavior of the time-varying complex network over a finite horizon, where the degree of bounded synchronization is quantified in terms of the H_∞-norm. A general sector-like nonlinear function is employed to describe the nonlinearities existing in the network. By utilizing a time-varying real-valued function and the Kronecker product, criteria are established that ensure the bounded H_∞ synchronization in terms of a set of recursive linear matrix inequalities (RLMIs), where the RLMIs can be computed recursively by employing available MATLAB toolboxes. The bounded H_∞ state estimation problem is then studied for the same complex network, where the purpose is to design a state estimator to estimate the network states through available output measurements such that, over a finite horizon, the dynamics of the estimation error is guaranteed to be bounded with a given disturbance attenuation level. Again, an RLMI approach is developed for the state estimation problem. Finally, two simulation examples are exploited to show the effectiveness of the results derived in this paper.**

***Index Terms*— Bounded H_∞ synchronization, complex networks, discrete-time networks, finite horizon, recursive linear matrix inequalities, stochastic networks, time-varying networks, transient behavior.**

I. Introduction

COMPLEX networks are made up of interconnected nodes and are used to describe various systems of the real world. Many real-world systems can be described by complex networks, such as the World Wide Web, telephone call graphs, neural networks, scientific citation web, etc. Since the discoveries of the "small-world" and "scale-free" properties of complex networks [1], [2], complex networks have become a focus of research and have attracted increasing attention from various fields of science and engineering. In particular, special attention has been paid to the synchronization problem for dynamical complex networks, in which each node is regarded as a dynamical element. It has been shown that the synchronization is ubiquitous in many system models of the natural world, for example, the large-scale and complex networks of chaotic oscillators [3]–[10], the coupled systems exhibiting spatiotemporal chaos and autowaves [11], [12], and the array of coupled neural networks [13]–[21].

Recently, the synchronization problem for discrete-time stochastic complex networks has drawn much research attention since it is rather challenging to understand the interaction topology of complex networks because of the *discrete and random* nature of network topology [22]. On one hand, discrete-time networks could be more suitable to model digitally transmitted signals in many application areas such as image processing, time-series analysis, quadratic optimization problems, and system identification. On the other hand, the stochastic disturbances over a real complex network may result from the release of probabilistic causes such as neurotransmitters [23], random phase-coupled oscillators [24], and packet dropouts [25]. A great number of results are in the recent literature on the general topic of stochastic synchronization problem for discrete-time complex networks. For example, in [26], the problem of stochastic synchronization analysis has been investigated for a new array of coupled discrete-time stochastic complex networks with randomly occurred nonlinearities and time delays. The synchronization stability problem has been studied in [27] for a class of complex dynamical networks with Markovian jumping parameters and mixed time delays. In [28], the delay-distribution-dependent stability has been discussed for stochastic discrete-time neural networks with randomly mixed time-varying delays.

Although the synchronization problem for discrete-time stochastic complex networks is now attracting increasing research attention, there are still several open problems deserving further investigation. In a real world, virtually all complex networks are time-varying, that is, all the network parameters are explicitly dependent on time. For example, a major challenge in biological networks is to understand and model, quantitatively, the dynamic topological and functional properties of biological networks. Such time- or condition-

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B. Shen is with the School of Information Science and Technology, Donghua University, Shanghai 200051, China (e-mail: shenbodh@gmail.com).

Z. Wang is with the Department of Information Systems and Computing, Brunel University, Uxbridge, Middlesex UB8 3PH, U.K. He is also with the School of Information Science and Technology, Donghua University, Shanghai 200051, China (e-mail: zidong.wang@brunel.ac.uk).

X. Liu is with the Department of Information Systems and Computing, Brunel University, Uxbridge, Middlesex UB8 3PH, U.K. (e-mail: xiaohui.liu@brunel.ac.uk).

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specific biological circuitries are referred to as time-varying networks or structural nonstationary networks, which are common in biological systems. The synchronization problem for time-varying complex networks has received some scattered research interest, where most literature has focused on time-varying coupling or time-varying delay terms. For example, in [29], a time-varying complex dynamical network model has been introduced, and it has been revealed that the synchronization of such a model is completely determined by the inner-coupling matrix, the eigenvalues, and the corresponding eigenvectors of the coupling configuration matrix of the network. Very recently, in [30], a class of controlled time-varying complex dynamical networks with similarity has been investigated, and a decentralized holographic structure controller is designed to stabilize the network asymptotically at its equilibrium states. It should be pointed out that, up to now, the general synchronization results for complex networks with time-varying network parameters have been very few, especially when the networks exhibit both discrete-time and stochastic natures.

In fact, for a truly time-varying discrete stochastic complex network, it is often theoretically difficult and practically unnecessary to establish easy-to-verify criteria for ensuring the global or asymptotical synchronization (steady-state behavior). Instead, we would be more interested in the transient behaviors *over a finite time interval*, e.g., the boundedness of the synchronization errors in the mean square and the disturbance rejection attenuation level of the error evolutions. For example, in biological networks, gene promoters can be in various epigenetic states and undergo interactions with many molecules in a highly transient, probabilistic, and combinatorial way, and therefore the resulting complex dynamics can only be analyzed within a finite period [31]. Despite its clear engineering insight, the synchronization problem for time-varying discrete stochastic complex networks poses some fundamental difficulties. 1) How can we define the synchronization concept over a finite horizon? 2) How can we quantify the attenuation level of the synchronization against exogenous disturbances? 3) How can we develop an effective technique to derive mathematically verifiable synchronization criteria? These questions may well explain why the synchronization problem for time-varying complex networks with or without stochastic disturbances is still open, and such a situation is the first motivation of our current investigation.

Closely associated with the synchronization problem is the so-called state estimation problem for complex networks. For large-scale complex networks, it is quite common that only partial information about the network nodes (states) is accessible from the network outputs. Therefore, in order to make use of key network nodes in practice, it becomes necessary to *estimate* the network nodes through available measurements. Note that the state estimation problem for neural networks (a special class of complex networks) was first addressed in [32] and has then drawn particular research interests (see [33], [34]) where the networks are deterministic and continuous-time. Recently, the state estimation problem for complex networks has also gained much attention, see [35]. When it comes to the transient behaviors of time-varying complex networks, similar to the synchronization problem, two natural questions are, how to define the estimator error over a finite horizon in a quantitative way and how to establish the existence conditions for the desired estimators. It is, therefore, the second motivation in our paper to offer satisfactory answers to the two questions.

In this paper, we aim to deal with the synchronization and state estimation problems for an array of coupled discrete time-varying stochastic complex networks over a finite horizon. The contribution of this paper is mainly twofold: 1) a novel concept of bounded H_∞ synchronization is proposed to reflect the time-varying nature of the complex networks and quantify the attenuation level of the disturbance rejection via the H_∞-norm, and 2) both synchronization and state estimation problems are solved by utilizing a time-varying real-valued function, the Kronecker product, as well as the recursive linear matrix inequalities (RLMIs). Rather than the commonly used Lipschitz-type function, a more general sector-like nonlinear function is employed to describe the nonlinearities existing in the network. We first define the concept of bounded H_∞ synchronization for the stochastic complex networks in the discrete-time domain. By utilizing a time-varying real-valued function and the Kronecker product, we show that the addressed synchronization problem can be converted into the feasibility problem of a set of RLMIs. We then turn to the state estimation problem for the same complex networks. Through available output measurements, we aim to design a state estimator to estimate the network states such that the dynamics of the estimation error is bounded in an H_∞ sense. Again, an RLMI approach is used, with the main proof omitted, for the state estimation case. Two simulation examples are provided to show the usefulness of the proposed synchronization and state estimation schemes.

Notation: The notation used here is fairly standard except where otherwise stated. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\|A\|$ refers to the norm of a matrix A defined by $\|A\| = \sqrt{\text{trace}(A^T A)}$. The notation $X \geq Y$ (respectively, $X > Y$), where X and Y are real symmetric matrices, means that $X - Y$ is positive semidefinite (respectively, positive definite). M^T represents the transpose of the matrix M. I denotes the identity matrix of compatible dimension. $\text{diag}\{\cdots\}$ stands for a block-diagonal matrix and the notation $\text{diag}_n\{*\}$ is employed to stand for $\text{diag}\{\overbrace{*, \ldots, *}^{n}\}$. Moreover, we may fix a probability space $(\Omega, \mathscr{F}, \text{Prob})$, where Prob, the probability measure, has a total mass 1. $\mathbb{E}\{x\}$ stands for the expectation of the stochastic variable x with respect to the given probability measure Prob. The asterisk $*$ in a matrix is used to denote a term induced by symmetry. Matrices, if they are not explicitly specified, are assumed to have compatible dimensions.

II. Problem Formulation and Preliminaries

Let a finite discrete time horizon be given as $[0 \quad N] := \{0, 1, 2, \ldots, N\}$. Consider the following array of stochastic discrete time-varying complex networks consisting of M

coupled nodes of the form

$$x_i(k+1) = f(k, x_i(k)) + \sum_{j=1}^{M} w_{ij}\Gamma x_j(k) + B_i(k)v(k) + g_i(k, x_i(k))\omega(k), \qquad i = 1, 2, \ldots, M \tag{1}$$

with output

$$z_i(k) = E(k)x_i(k), \qquad i = 1, 2, \ldots, M \tag{2}$$

where $x_i(k) \in \mathbb{R}^n$ is the state vector of the ith node, $z_i(k) \in \mathbb{R}^m$ is the controlled output of the ith node, $\Gamma = \operatorname{diag}\{r_1, r_2, \ldots, r_n\}$ is a matrix linking the jth state variable if $r_j \neq 0$, and $W = (w_{ij})_{M\times M}$ is the coupled configuration matrix of the network with $w_{ij} \geq 0$ $(i \neq j)$ but not all zero. As usual, the coupling configuration matrix $W = (w_{ij})_{M\times M}$ is symmetric (i.e., $W = W^T$) and satisfies

$$\sum_{j=1}^{M} w_{ij} = \sum_{j=1}^{M} w_{ji} = 0, \qquad i = 1, 2, \ldots, M. \tag{3}$$

$\omega(k)$ is a 1-D, zero-mean Gaussian white noise sequence on a probability space $(\Omega, \mathscr{F}, \text{Prob})$ with $\mathbb{E}\{\omega^2(k)\} = 1$. Let $(\Omega, \mathscr{F}, \{\mathscr{F}_k\}_{k\in[0\ \ N]}, \text{Prob})$ be a filtered probability space where $\{\mathscr{F}_k\}_{k\in[0\ \ N]}$ is the family of sub σ-algebras of $\mathscr{F}$ generated by $\{\omega(k)\}_{k\in[0\ \ N]}$. In fact, each $\mathscr{F}_k$ is assumed to be the minimal σ-algebras generated by $\{\omega(i)\}_{0\leq i\leq k-1}$, while $\mathscr{F}_0$ is assumed to be some given sub σ-algebras of $\mathscr{F}$, independent of $\mathscr{F}_k$ for all $1 \leq k \leq N$ [36], and the initial value $x_i(0)$ $(i = 1, 2, \ldots, M)$ belongs to $\mathscr{F}_0$.

For the exogenous disturbance input $v(k) \in \mathbb{R}^q$, it is assumed that $v = \{v(k)\}_{k\in[0\ \ N]} \in l_2([0\ \ N], \mathbb{R}^q)$, where $l_2([0\ \ N], \mathbb{R}^q)$ is the space of nonanticipatory square-summable stochastic process $v = \{v(k)\}_{k\in[0\ \ N]}$ with respect to $\{\mathscr{F}_k\}_{k\in[0\ \ N]}$ with the norm

$$\|v\|^2_{[0\ \ N]} = \mathbb{E}\left\{\sum_{k=0}^{N} \|v(k)\|^2\right\} = \sum_{k=0}^{N} \mathbb{E}\left\{\|v(k)\|^2\right\}.$$

The nonlinear vector-valued function $f : [0\ \ N] \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be continuous and satisfies the following sector-bounded condition [26], [35]:

$$[f(k,x) - f(k,y) - U_1(k)(x-y)]^T[f(k,x) - f(k,y) - U_2(k)(x-y)] \leq 0, \quad \forall x, y \in \mathbb{R}^n \tag{4}$$

for all $k \in [0\ \ N]$, where $U_1(k)$ and $U_2(k)$ are real matrices of appropriate dimensions.

The noise intensity function vector $g_i : [0\ \ N] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies the following conditions:

$$\begin{aligned} &g_i(k, 0) = 0 \\ &\|g_i(k,x) - g_j(k,y)\|^2 \leq \|V(k)(x-y)\|^2, \quad \forall x, y \in \mathbb{R}^n \end{aligned} \tag{5}$$

for all $k \in [0\ \ N]$ and $i, j = 1, 2, \ldots, M$, where $V(k)$ is a constant matrix.

For the purpose of simplicity, we introduce the following notations:

$$\begin{aligned} x(k) &= \left[x_1^T(k)\, x_2^T(k)\, \cdots\, x_M^T(k)\right]^T \\ B(k) &= \left[B_1^T(k)\, B_2^T(k)\, \cdots\, B_M^T(k)\right]^T \\ \mathcal{F}(k, x(k)) &= \left[f^T(k, x_1(k))\, f^T(k, x_2(k))\, \cdots\, f^T(k, x_M(k))\right]^T \\ \mathcal{G}(k, x(k)) &= \left[g_1^T(k, x_1(k))\, g_2^T(k, x_2(k))\, \cdots\, g_M^T(k, x_M(k))\right]^T. \end{aligned} \tag{6}$$

By using the Kronecker product, the complex networks (1) can be rewritten in the following compact form:

$$x(k+1) = \mathcal{F}(k, x(k)) + (W \otimes \Gamma)x(k) + B(k)v(k) + \mathcal{G}(k, x(k))\omega(k). \tag{7}$$

To proceed, we introduce the following definition for the bounded H_∞ synchronization.

Definition 1: The stochastic discrete time-varying complex network (1) or (7) is said to be boundedly H_∞-synchronized with a disturbance attenuation γ over a finite horizon $[0\ \ N]$ if the following holds:

$$\sum_{1\leq i<j\leq M} \|z_i - z_j\|^2_{[0\ \ N]} \leq \gamma^2 \left\{\|v\|^2_{[0\ \ N]} + \mathbb{E}\{x^T(0)Sx(0)\}\right\} \tag{8}$$

for the given positive scalar $\gamma > 0$ and positive definite matrix $S = S^T > 0$.

Remark 1: In the past few years, the synchronization problems of complex networks have been well studied over the infinite time horizon, see [35], where all synchronization errors between the subsystems of a complex network are required to asymptotically approach zero. However, for the inherently time-varying complex networks addressed in this paper, we are more interested in the transient behavior of the synchronization over a specified time interval. In other words, we like to examine the transient behavior over a finite horizon rather than the steady-state property over an infinite horizon. For this purpose, we make one of the first few attempts to define the notion of bounded H_∞-synchronization with a disturbance attenuation level so as to characterize the performance requirement of the synchronization over a finite horizon. It is noticed that, if the constraint (8) is met, then the synchronization error between any pair of subsystems of the complex network is guaranteed to be bounded. Furthermore, the H_∞ performance index $\gamma > 0$ is used to quantify the attenuation level of the synchronization error dynamics against exogenous disturbances.

In this paper, our aim is to investigate the bounded H_∞-synchronization problem and establish easy-to-verify criteria for the stochastic discrete time-varying complex network (1) over a finite time horizon. Later, we shall address the finite-horizon H_∞ state estimation problem by designing the finite-horizon H_∞ estimators for the stochastic discrete time-varying complex network (1).

III. Bounded H_∞-Synchronization of Discrete Time-Varying Complex Networks

In this section, we deal with the bounded H_∞-synchronization problem for the stochastic discrete time-varying complex network (1) with a given disturbance attenuation level over a finite time horizon. The following lemma is important and will be used in the sequel.

Lemma 1 [35]: Let $\mathcal{U} = (\alpha_{ij})_{M\times M}$, $P \in \mathbb{R}^{n\times n}$, $x = \begin{bmatrix} x_1^T & x_2^T & \cdots & x_M^T \end{bmatrix}^T$, and $y = \begin{bmatrix} y_1^T & y_2^T & \cdots & y_M^T \end{bmatrix}^T$ with $x_i, y_i \in \mathbb{R}^n$ $(i = 1, 2, \ldots, M)$. If $\mathcal{U}^T = \mathcal{U}$ and each row sum of $\mathcal{U}$ is zero, then

$$x^T(\mathcal{U} \otimes P)y = -\sum_{1\le i<j\le M} \alpha_{ij}(x_i - x_j)^T P(y_i - y_j). \tag{9}$$

The following theorem provides a sufficient condition under which the complex network (1) is boundedly H_∞-synchronized with the given disturbance attenuation level over a finite time horizon.

Theorem 1: Let the positive scalar $\gamma > 0$ and the initial positive definite matrix $S^T = S > 0$ be given. The stochastic discrete time-varying complex network (1) or (7) is boundedly H_∞-synchronized with the disturbance attenuation γ over a finite horizon $[0 \;\; N]$ if there exist a family of positive definite matrices $\{P(k)\}_{0\le k\le N+1}$ and two families of positive scalars $\{\lambda_1(k)\}_{0\le k\le N}$, $\{\lambda_2(k)\}_{0\le k\le N}$ satisfying the initial condition

$$\sum_{1\le i<j\le M} \mathbb{E}\left\{(x_i(0) - x_j(0))^T P(0)(x_i(0) - x_j(0))\right\} \le \gamma^2 \mathbb{E}\left\{x^T(0)Sx(0)\right\} \tag{10}$$

and the RLMIs (11) shown at the bottom of the page, for all $0 \le k \le N$ and $1 \le i < j \le M$, where

$$\begin{aligned}
\Theta_{ij}^{(1)}(k) &= -Mw_{ij}^{(2)}\Gamma^T P(k+1)\Gamma - P(k) + E^T(k)E(k) \\
&\quad -\lambda_1(k)\tilde{U}_1(k) + \lambda_2(k)V^T(k)V(k), \\
\Theta_{ij}^{(2)}(k) &= -Mw_{ij}\Gamma^T P(k+1) - \lambda_1(k)\tilde{U}_2(k), \\
\Theta_{ij}^{(3)}(k) &= -Mw_{ij}\Gamma^T P(k+1)\mathbf{B}_{ij}(k), \\
\Theta_{ij}^{(4)}(k) &= P(k+1)\mathbf{B}_{ij}(k), \\
\Theta_{ij}^{(5)}(k) &= -\frac{2\gamma^2 I}{M(M-1)} + \mathbf{B}_{ij}^T(k)P(k+1)\mathbf{B}_{ij}(k), \\
\tilde{U}_1(k) &= \frac{U_1^T(k)U_2(k) + U_2^T(k)U_1(k)}{2}, \\
\tilde{U}_2(k) &= -\frac{U_1^T(k) + U_2^T(k)}{2}, \\
\mathbf{B}_{ij}(k) &= B_i(k) - B_j(k), \quad w_{ij}^{(2)} = \sum_{k=1}^{M} w_{ik}w_{kj}.
\end{aligned} \tag{12}$$

Proof: See Appendix I. ■

Remark 2: It should be pointed out that the RLMI technique [37], [38] serves as an effective approach to investigating the problems of H_∞ filtering and control in a finite time horizon. In Theorem 1, the RLMI approach has been applied, for the first time, to deal with the synchronization problem for the discrete time-varying stochastic complex network and derive a criterion for testing the bounded H_∞-synchronization in terms of a set of RLMIs.

Remark 3: Different from the infinite time horizon case, the asymptotical behavior of synchronization error is not required to be analyzed for a time-varying complex network over a finite time horizon and, therefore, the synchronization criterion given in Theorem 1 takes care of the boundedness of the synchronization error but does not actually guarantee its convergence. In case the considered complex network is time-invariant and its steady-state property over an infinite horizon is a concern, an LMIs-based asymptotical synchronization criterion can be easily deduced from the RLMIs (11) as long as the variables $P(k)$, $\lambda_1(k)$, and $\lambda_2(k)$ are taken as constant variables P, λ_1, and λ_2, respectively.

IV. Finite-Horizon H_∞ State Estimation for Discrete Time-Varying Complex Networks

In this section, the finite-horizon H_∞ state estimation problem is first formulated for the stochastic discrete time-varying complex network (1), and then an array of time-varying H_∞ estimators is designed by using the RLMI approach.

Suppose that the measurement of the complex network (1) is of the form

$$y_i(k) = C_i(k)x_i(k) + D_i(k)v(k), \quad i = 1, 2, \ldots, M \tag{13}$$

where $y_i(k) \in \mathbb{R}^r$ is the measured output vector from the ith node of the complex network.

Based on the measurements $y_i(k)$ $(i = 1, 2, \ldots, M)$, we construct the following state estimator:

$$\begin{cases}
\hat{x}_i(k+1) = f(k, \hat{x}_i(k)) + \sum_{j=1}^{M} w_{ij}\Gamma\hat{x}_j(k) \\
\qquad\qquad\quad +K_i(k)(y_i(k) - C_i(k)\hat{x}_i(k)) \\
\hat{z}_i(k) = E(k)\hat{x}_i(k), \quad i = 1, 2, \ldots, M
\end{cases} \tag{14}$$

where $\hat{x}_i(k) \in \mathbb{R}^n$ is the estimate of network state $x_i(k)$, $\hat{z}_i(k) \in \mathbb{R}^m$ is the estimate of output $z_i(k)$, and $K_i(k) \in \mathbb{R}^{n\times r}$ is the estimator parameter to be designed. The initial values of estimators are assumed to be zeros, i.e., $\hat{x}_i(0) = 0$ for all $i = 1, 2, \ldots, M$.

By setting the estimation error $e_i = x_i - \hat{x}_i$ and the filtering error $\tilde{z}_i = z_i - \hat{z}_i$, the error dynamics of complex network can

$$\Phi_{ij}(k) = \begin{bmatrix}
\Theta_{ij}^{(1)}(k) & \Theta_{ij}^{(2)}(k) & 0 & \Theta_{ij}^{(3)}(k) \\
* & P(k+1) - \lambda_1(k)I & 0 & \Theta_{ij}^{(4)}(k) \\
* & * & P(k+1) - \lambda_2(k)I & 0 \\
* & * & * & \Theta_{ij}^{(5)}(k)
\end{bmatrix} \le 0 \tag{11}$$

be obtained from (1), (13), and (14) as follows:

$$\begin{cases} e_i(k+1) = -K_i(k)C_i(k)e_i(k) + \tilde{f}(k, e_i(k)) \\ \qquad + \sum_{j=1}^{M} w_{ij}\Gamma e_j(k) + (B_i(k) - K_i(k)D_i(k))v(k) \\ \qquad + g_i(k, e_i(k) + \hat{x}_i(k))\omega(k) \\ \tilde{z}_i(k) = E(k)e_i(k) \end{cases} \tag{15}$$

where $\tilde{f}(k, e_i(k)) = f(k, x_i(k)) - f(k, \hat{x}_i(k))$.

Introducing the notations

$$\begin{aligned} \hat{x}(k) &= \begin{bmatrix} \hat{x}_1^T(k) & \hat{x}_2^T(k) & \cdots & \hat{x}_M^T(k) \end{bmatrix}^T, \\ e(k) &= \begin{bmatrix} e_1^T(k) & e_2^T(k) & \cdots & e_M^T(k) \end{bmatrix}^T, \\ \tilde{z}(k) &= \begin{bmatrix} \tilde{z}_1^T(k) & \tilde{z}_2^T(k) & \cdots & \tilde{z}_M^T(k) \end{bmatrix}^T, \\ K(k) &= \text{diag}\{K_1(k), K_2(k), \ldots, K_M(k)\}, \\ C(k) &= \text{diag}\{C_1(k), C_2(k), \ldots, C_M(k)\}, \\ D(k) &= \begin{bmatrix} D_1^T(k) & D_2^T(k) & \cdots & D_M^T(k) \end{bmatrix}^T, \\ E_\Lambda(k) &= \text{diag}_M\{E(k)\}, \\ \tilde{\mathcal{F}}(k, e(k)) &= \begin{bmatrix} \tilde{f}^T(k, e_1(k)) & \tilde{f}^T(k, e_2(k)) & \cdots & \tilde{f}^T(k, e_M(k)) \end{bmatrix}^T \end{aligned} \tag{16}$$

we can rewrite the error dynamics of complex networks (15) in the following compact form:

$$\begin{cases} e(k+1) = (-K(k)C(k) + W \otimes \Gamma)e(k) + \tilde{\mathcal{F}}(k, e(k)) \\ \qquad + (B(k) - K(k)D(k))v(k) \\ \qquad + \mathcal{G}(k, e(k) + \hat{x}(k))\omega(k) \\ \tilde{z}(k) = E_\Lambda(k)e(k) \end{cases} \tag{17}$$

where $B(k)$ and $\mathcal{G}(k, x(k))$ are defined in (6).

In this section, we aim to design the time-varying estimators (14) for the stochastic discrete time-varying complex network (1) such that the filtering error $\tilde{z}(k)$ satisfies the following H_∞ performance constraint:

$$\|\tilde{z}\|^2_{[0\ \ N]} \le \gamma^2 \left\{ \|v\|^2_{[0\ \ N]} + \mathbb{E}\{e^T(0)Se(0)\} \right\} \tag{18}$$

for the given disturbance attenuation level $\gamma > 0$ and positive definite matrix $S^T = S > 0$.

In the following theorem, a sufficient condition is given to guarantee that the filtering error satisfies the H_∞ performance constraint (18).

Theorem 2: Let the scalar $\gamma > 0$, initial positive definite matrix $S^T = S > 0$, and estimator parameters $K_i(k)$ $(i = 1, 2, \ldots, M)$ be given. The filtering error $\tilde{z}(k)$ satisfies the H_∞ performance constraint (18) if there exist a family of positive definite matrices $\{P(k)\}_{0 \le k \le N+1}$ and three families of positive scalars $\{\varepsilon_1(k)\}_{0 \le k \le N}$, $\{\varepsilon_2(k)\}_{0 \le k \le N}$, $\{\mu(k)\}_{0 \le k \le N+1}$ satisfying the initial condition

$$\mathbb{E}\left\{e^T(0)P(0)e(0)\right\} + \mu(0) \le \gamma^2 \mathbb{E}\left\{e^T(0)Se(0)\right\} \tag{19}$$

and the RLMIs (20) shown at the bottom of page, for all $0 \le k \le N$, where

$$\begin{aligned} \Xi_1(k) &= -P(k) + E_\Lambda^T(k)E_\Lambda(k) - \varepsilon_1(k)\tilde{U}_{1\Lambda}(k) \\ &\quad + \varepsilon_2(k)V_\Lambda^T(k)V_\Lambda(k), \\ \Xi_2(k) &= -C^T(k)K^T(k)P(k+1) + (W \otimes \Gamma)^T P(k+1), \\ \Xi_3(k) &= B^T(k)P(k+1) - D^T(k)K^T(k)P(k+1), \\ \Xi_4(k) &= \mu(k+1) - \mu(k) + \varepsilon_2(k)\hat{x}^T(k)V_\Lambda^T(k)V_\Lambda(k)\hat{x}(k), \\ \tilde{U}_{1\Lambda}(k) &= \frac{U_{1\Lambda}^T(k)U_{2\Lambda}(k) + U_{2\Lambda}^T(k)U_{1\Lambda}(k)}{2}, \\ \tilde{U}_{2\Lambda}(k) &= -\frac{U_{1\Lambda}^T(k) + U_{2\Lambda}^T(k)}{2}, \\ U_{1\Lambda}(k) &= \text{diag}_M\{U_1(k)\}, \quad U_{2\Lambda}(k) = \text{diag}_M\{U_2(k)\}, \\ V_\Lambda(k) &= \text{diag}_M\{V(k)\}. \end{aligned} \tag{21}$$

Proof: See Appendix II. ■

After establishing the analysis results, we are now ready to deal with the design problem of the finite-horizon H_∞ estimators for the stochastic network (1). The following result can be readily derived from Theorem 2, and therefore its proof is omitted for saving space.

Theorem 3: Let the scalar $\gamma > 0$ and initial positive definite matrix $S^T = S > 0$ be given. The finite-horizon H_∞ estimation problem is solvable for the time-varying stochastic complex network (1) if there exist a family of positive definite diagonal block matrices $\{P(k) = \text{diag}\{P_1(k), P_2(k), \ldots, P_M(k)\}\}_{0 \le k \le N+1}$, a family of diagonal block matrices $\{X(k) = \text{diag}\{X_1(k), X_2(k), \ldots, X_M(k)\}\}_{0 \le k \le N}$, and three families of positive scalars $\{\varepsilon_1(k)\}_{0 \le k \le N}$, $\{\varepsilon_2(k)\}_{0 \le k \le N}$, $\{\mu(k)\}_{0 \le k \le N+1}$ satisfying the initial condition (19) and the RLMIs (22) shown at the bottom of the page,

$$\begin{bmatrix} \Xi_1(k) & -\varepsilon_1(k)\tilde{U}_{2\Lambda}(k) & 0 & 0 & \varepsilon_2(k)V_\Lambda^T(k)V_\Lambda(k)\hat{x}(k) & \Xi_2(k) & 0 \\ * & -\varepsilon_1(k)I & 0 & 0 & 0 & P(k+1) & 0 \\ * & * & -\gamma^2 I & 0 & 0 & \Xi_3(k) & 0 \\ * & * & * & -\varepsilon_2(k)I & 0 & 0 & P(k+1) \\ * & * & * & * & \Xi_4(k) & 0 & 0 \\ * & * & * & * & * & -P(k+1) & 0 \\ * & * & * & * & * & * & -P(k+1) \end{bmatrix} \le 0 \tag{20}$$

$$\begin{bmatrix} \Xi_1(k) & -\varepsilon_1(k)\tilde{U}_{2\Lambda}(k) & 0 & 0 & \varepsilon_2(k)V_\Lambda^T(k)V_\Lambda(k)\hat{x}(k) & \bar{\Xi}_2(k) & 0 \\ * & -\varepsilon_1(k)I & 0 & 0 & 0 & P(k+1) & 0 \\ * & * & -\gamma^2 I & 0 & 0 & \bar{\Xi}_3(k) & 0 \\ * & * & * & -\varepsilon_2(k)I & 0 & 0 & P(k+1) \\ * & * & * & * & \Xi_4(k) & 0 & 0 \\ * & * & * & * & * & -P(k+1) & 0 \\ * & * & * & * & * & * & -P(k+1) \end{bmatrix} \le 0 \tag{22}$$

where

$$\begin{aligned}\bar{\Xi}_2(k) &= -C^T(k)X^T(k) + (W \otimes \Gamma)^T P(k+1) \\ \bar{\Xi}_3(k) &= B^T(k)P(k+1) - D^T(k)X^T(k),\end{aligned} \tag{23}$$

$\Xi_1(k)$, $\Xi_4(k)$, $\tilde{U}_{2\Lambda}(k)$, and $V_\Lambda(k)$ are defined in Theorem 2. Furthermore, if (19) and (22) are true, the desired estimators are given by (14) with the following parameters:

$$K_i(k) = P_i^{-1}(k+1)X_i(k), \quad i = 1, 2, \ldots, M \tag{24}$$

for all $0 \le k \le N$.

Remark 4: In Theorem 3, a criterion is established to ensure the existence of the desired estimator gains, and the explicit expression of such estimator gains is characterized in terms of the solution to a set of RLMIs. Note that such RLMIs can be effectively solved and checked by the algorithms such as the interior-point method. The state estimate at current time is involved in RLMIs (22), which means that more current information is used to estimate the state the next time. In this sense, the estimator design scheme in terms of RLMIs (22) can potentially improve the accuracies of the state estimation.

V. Illustrative Examples

In this section, two simulation examples are presented to demonstrate the effectiveness of the established criteria on the bounded H_∞-synchronization as well as the finite-horizon H_∞ state estimation problems for the complex network (1).

Consider a stochastic time-varying complex network (1) with four nodes in a given finite time horizon $k \in [0 \quad 25]$. The coupling configuration matrix are assumed to be $W = (w_{ij})_{M\times M}$ with

$$w_{ij} = \begin{cases} -0.3, & i = j \\ 0.1, & i \ne j \end{cases}$$

and the inner-coupling matrix is given as $\Gamma = \text{diag}_4\{0.1\}$.

The nonlinear time-varying function $f(k, x_i(k))$ is chosen as

$$f(k, x_i(k)) = \begin{cases} \begin{bmatrix} -0.15x_{i1}(k) + 0.1x_{i2}(k) + \tanh(0.1x_{i1}(k)) \\ 0.25x_{i2}(k) - \tanh(0.1x_{i2}(k)) \end{bmatrix}, & 0 \le k < 10 \\ \begin{bmatrix} 0.25x_{i1}(k) - \tanh(0.15x_{i1}(k)) \\ 0.1x_{i2}(k) \end{bmatrix}, & 10 \le k \le 25 \end{cases}$$

and the disturbance matrices are taken as

$$B_1(k) = \begin{bmatrix} 0.14 + 0.1\sin(6(k-1)) \\ 0.12 \end{bmatrix}, \quad B_2(k) = \begin{bmatrix} -0.13 \\ 0.1 \end{bmatrix},$$

$$B_3(k) = \begin{bmatrix} 0 \\ -0.15 \end{bmatrix}, \quad B_4(k) = \begin{bmatrix} 0.15 \\ 0.1 \end{bmatrix}.$$

The noise intensity function is simplified to $g_i(k, x_i(k)) = V_i(k)x_i(k)$, with

$$V_i(k) = \begin{bmatrix} 0.09 & -0.117 \\ -0.045 & 0.135 \end{bmatrix}, \quad i = 1, 2, 3, 4.$$

Then, it is easily verified that

$$U_1(k) = \begin{cases} \begin{bmatrix} -0.15 & 0.1 \\ 0 & 0.25 \end{bmatrix}, & 0 \le k < 10 \\ \begin{bmatrix} 0.25 & 0 \\ 0 & 0.1 \end{bmatrix}, & 10 \le k \le 25 \end{cases}$$

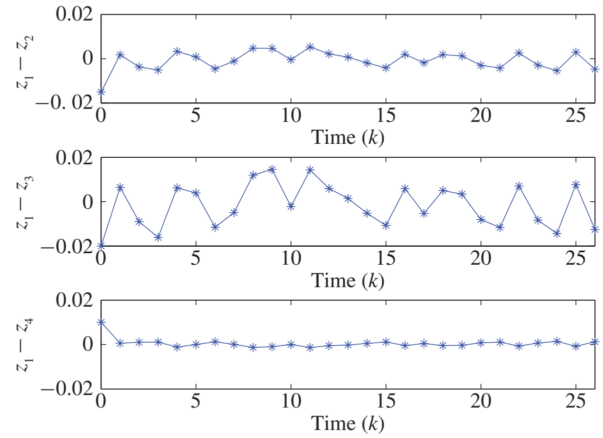

Fig. 1. Synchronization errors between $z_1(k)$ and $z_i(k)$ $(i = 2, 3, 4)$.

$$U_2(k) = \begin{cases} \begin{bmatrix} -0.05 & 0.1 \\ 0 & 0.15 \end{bmatrix}, & 0 \le k < 10 \\ \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, & 10 \le k \le 25 \end{cases}$$

and $V(k) = \begin{bmatrix} 0.09 & -0.117 \\ -0.045 & 0.135 \end{bmatrix}$.

We are now ready to deal with the bounded H_∞-synchronization problem as well as the finite-horizon H_∞ state estimation problem over the given finite horizon for the complex network (1) with above parameters.

Example 1: In this example, let us test the bounded H_∞-synchronization of the complex network based on our established criterion. Set the initial values of the complex network as

$$x_1(0) = \begin{bmatrix} 0.1 & -0.15 \end{bmatrix}^T \quad x_2(0) = \begin{bmatrix} 0.15 & -0.1 \end{bmatrix}^T$$
$$x_3(0) = \begin{bmatrix} 0.2 & -0.1 \end{bmatrix}^T \quad x_4(0) = \begin{bmatrix} 0.1 & -0.2 \end{bmatrix}^T.$$

Let the disturbance attenuation level and the positive definite matrix be $\gamma = 0.7071$ and $S = \text{diag}_8\{1\}$, respectively. In order to check whether the complex network mentioned above is boundedly H_∞-synchronized with the given disturbance attenuation level γ, we first choose the initial positive definite matrices $P(0) = \text{diag}_2\{1\}$ to satisfy the initial condition (10). Then the set of RLMIs (11) in Theorem 1 can be solved recursively by using MATLAB (with the YALMIP 3.0), and a set of feasible solutions is obtained as shown in Table I. According to Theorem 1, the array of stochastic discrete time-varying complex networks can reach the bounded H_∞-synchronization with the given disturbance attenuation level γ.

In the simulation, the exogenous disturbance input $v(k)$ is selected as a random variable that obeys uniform distribution over $[-0.25 \quad 0.25]$. The simulation results are presented in Fig. 1, which plots the synchronization error between the output $z_1(k)$ and outputs $z_i(k)$ $(i = 2, 3, 4)$. It can be seen from Fig. 1 that all synchronization errors are indeed bounded, which verifies the effectiveness of the synchronization criteria proposed in Theorem 1.

Remark 5: Recently, considerable research efforts have been made on the synchronization problems of complex networks, and various synchronization concepts have been

TABLE I

VARIABLES $P(k)$, $\lambda_1(k)$, $\lambda_2(k)$

k	$P(k)$	$\lambda_1(k)$	$\lambda_2(k)$
0	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	1.0879	1.1482
1	$\begin{bmatrix} 0.1922 & -0.0019 \\ -0.0019 & 0.2030 \end{bmatrix}$	0.7973	0.8180
2	$\begin{bmatrix} 0.1410 & -0.0008 \\ -0.0008 & 0.1480 \end{bmatrix}$	0.8296	0.8484
3	$\begin{bmatrix} 0.1477 & -0.0003 \\ -0.0003 & 0.1544 \end{bmatrix}$	0.8351	0.8544
4	$\begin{bmatrix} 0.1496 & 0.0000 \\ 0.0000 & 0.1560 \end{bmatrix}$	0.8367	0.8561
5	$\begin{bmatrix} 0.1503 & 0.0002 \\ 0.0002 & 0.1565 \end{bmatrix}$	0.8372	0.8565
$\vdots$	$\vdots$	$\vdots$	$\vdots$
21	$\begin{bmatrix} 0.1790 & -0.0045 \\ -0.0045 & 0.1797 \end{bmatrix}$	0.8217	0.8318
22	$\begin{bmatrix} 0.1721 & -0.0036 \\ -0.0036 & 0.1709 \end{bmatrix}$	0.8926	0.9029
23	$\begin{bmatrix} 0.1904 & -0.0032 \\ -0.0032 & 0.1874 \end{bmatrix}$	0.8364	0.8471
24	$\begin{bmatrix} 0.1825 & -0.0024 \\ -0.0024 & 0.1786 \end{bmatrix}$	0.8314	0.8418
25	$\begin{bmatrix} 0.1832 & -0.0019 \\ -0.0019 & 0.1786 \end{bmatrix}$	0.8328	0.8432

TABLE II

ESTIMATOR PARAMETERS $K_i(k)$ $(i = 1, 2, 3, 4)$

k	$K_1(k)$	$K_2(k)$	$K_3(k)$	$K_4(k)$
0	$\begin{bmatrix} 0.1208 \\ 0.1461 \end{bmatrix}$	$\begin{bmatrix} -0.0697 \\ 0.1414 \end{bmatrix}$	$\begin{bmatrix} -0.0604 \\ -0.2137 \end{bmatrix}$	$\begin{bmatrix} 0.1615 \\ 0.1459 \end{bmatrix}$
1	$\begin{bmatrix} 0.0847 \\ 0.1299 \end{bmatrix}$	$\begin{bmatrix} -0.1207 \\ 0.1047 \end{bmatrix}$	$\begin{bmatrix} -0.0077 \\ -0.1569 \end{bmatrix}$	$\begin{bmatrix} 0.1452 \\ 0.1127 \end{bmatrix}$
2	$\begin{bmatrix} 0.0634 \\ 0.1300 \end{bmatrix}$	$\begin{bmatrix} -0.1134 \\ 0.1090 \end{bmatrix}$	$\begin{bmatrix} -0.0462 \\ -0.1892 \end{bmatrix}$	$\begin{bmatrix} 0.1673 \\ 0.1344 \end{bmatrix}$
3	$\begin{bmatrix} 0.0469 \\ 0.1191 \end{bmatrix}$	$\begin{bmatrix} -0.1230 \\ 0.1046 \end{bmatrix}$	$\begin{bmatrix} -0.0051 \\ -0.1542 \end{bmatrix}$	$\begin{bmatrix} 0.1485 \\ 0.1019 \end{bmatrix}$
4	$\begin{bmatrix} 0.0411 \\ 0.1209 \end{bmatrix}$	$\begin{bmatrix} -0.1282 \\ 0.1009 \end{bmatrix}$	$\begin{bmatrix} -0.0016 \\ -0.1512 \end{bmatrix}$	$\begin{bmatrix} 0.1496 \\ 0.1013 \end{bmatrix}$
5	$\begin{bmatrix} 0.0398 \\ 0.1253 \end{bmatrix}$	$\begin{bmatrix} -0.1093 \\ 0.1137 \end{bmatrix}$	$\begin{bmatrix} -0.0193 \\ -0.1640 \end{bmatrix}$	$\begin{bmatrix} 0.1473 \\ 0.1089 \end{bmatrix}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
21	$\begin{bmatrix} 0.1473 \\ 0.1198 \end{bmatrix}$	$\begin{bmatrix} -0.1364 \\ 0.1018 \end{bmatrix}$	$\begin{bmatrix} 0.0063 \\ -0.1510 \end{bmatrix}$	$\begin{bmatrix} 0.1522 \\ 0.1001 \end{bmatrix}$
22	$\begin{bmatrix} 0.1174 \\ 0.1201 \end{bmatrix}$	$\begin{bmatrix} -0.1302 \\ 0.1001 \end{bmatrix}$	$\begin{bmatrix} 0.0002 \\ -0.1501 \end{bmatrix}$	$\begin{bmatrix} 0.1502 \\ 0.1001 \end{bmatrix}$
23	$\begin{bmatrix} 0.0933 \\ 0.1198 \end{bmatrix}$	$\begin{bmatrix} -0.1370 \\ 0.1024 \end{bmatrix}$	$\begin{bmatrix} 0.0062 \\ -0.1514 \end{bmatrix}$	$\begin{bmatrix} 0.1512 \\ 0.1007 \end{bmatrix}$
24	$\begin{bmatrix} 0.0689 \\ 0.1200 \end{bmatrix}$	$\begin{bmatrix} -0.1308 \\ 0.1001 \end{bmatrix}$	$\begin{bmatrix} 0.0008 \\ -0.1500 \end{bmatrix}$	$\begin{bmatrix} 0.1504 \\ 0.1000 \end{bmatrix}$
25	$\begin{bmatrix} 0.0528 \\ 0.1201 \end{bmatrix}$	$\begin{bmatrix} -0.1317 \\ 0.1001 \end{bmatrix}$	$\begin{bmatrix} 0.0015 \\ -0.1501 \end{bmatrix}$	$\begin{bmatrix} 0.1512 \\ 0.1001 \end{bmatrix}$

proposed, such as asymptotical synchronization [14], [29], exponential synchronization [4], [35], and exponential H_∞ synchronization [16]. However, as far as we know, all the synchronization concepts in the existing literature are concerned with the case of infinite time horizon and only the asymptotical behavior of the synchronization has been analyzed. As a distinguishing feature, the notion of bounded H_∞-synchronization proposed in this paper can be used to characterize the transient behavior of the synchronization over a specified time interval. In other words, the derived bounded H_∞-synchronization criterion can guarantee that: 1) the synchronization error over a given time interval is bounded, and 2) the influence from the external disturbances and the initial states to the synchronization error is attenuated with a given H_∞-norm γ. This has been well verified by the simulation results of Example 1.

Example 2: In this example, we deal with the finite-horizon H_∞ state estimation problem. The initial values of complex network are set as

$$x_1(0) = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}^T \quad x_2(0) = \begin{bmatrix} -0.2 & 0.1 \end{bmatrix}^T$$
$$x_3(0) = \begin{bmatrix} -0.1 & -0.15 \end{bmatrix}^T \quad x_4(0) = \begin{bmatrix} -0.15 & -0.1 \end{bmatrix}^T$$

the disturbance attenuation level is given as $\gamma = 1$, and the positive definite matrix is taken as $S = \mathrm{diag}_8\{5\}$. We choose the initial positive definite matrices $P_1(0) = P_2(0) = P_3(0) = P_4(0) = \mathrm{diag}_2\{1\}$ and positive scalar $\mu(0) = 0.5$ to satisfy the initial condition (19). By using MATLAB (with the YALMIP 3.0) again, the set of RLMIs (22) in Theorem 3 can be solved recursively, and all desired estimator parameters can be derived. Table II lists all estimator parameters $K_i(k)$ $(i = 1, 2, 3, 4)$ and the variables $P_i(k)$ $(i = 1, 2, 3, 4)$ and $\mu(k)$ are shown in Table III.

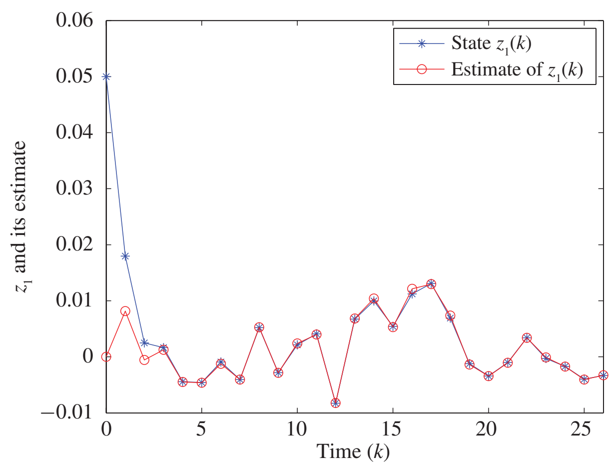

Fig. 2. Output $z_1(k)$ and its estimate $\hat{z}_1(k)$.

In the simulation, the exogenous disturbance input $v(k)$ is the same as that used in Example 1. Simulation results are presented in Figs. 2–5 which show the output $z_i(k)$ and its estimate $\hat{z}_i(k)$ $(i = 1, 2, 3, 4)$. The simulation has confirmed that the designed estimators perform very well.

Remark 6: From above simulation examples, it can be seen that the developed RLMI-based algorithms are implemented where the initial variable matrices are chosen beforehand to satisfy the conditions (10) and (19). For the synchronization algorithm, the selection of initial matrices is independent of the initial values of the complex network, which can be seen from the condition (10). In other words, the H_∞-synchronization of the complex network depends only on the given attenuation

TABLE III

VARIABLES $P_i(k)$ $(i = 1, 2, 3, 4)$ AND $\mu(k)$

k	$P_1(k)$	$P_2(k)$	$P_3(k)$	$P_4(k)$	$\mu(k)$
0	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	0.5000
1	$\begin{bmatrix} 6.4331 & -1.7597 \\ -1.7597 & 6.3939 \end{bmatrix}$	$\begin{bmatrix} 5.9195 & -1.8174 \\ -1.8174 & 6.5502 \end{bmatrix}$	$\begin{bmatrix} 6.4184 & -1.4750 \\ -1.4750 & 6.8707 \end{bmatrix}$	$\begin{bmatrix} 6.5249 & -1.5846 \\ -1.5846 & 6.5494 \end{bmatrix}$	0.4990
2	$\begin{bmatrix} 2.7170 & -0.0017 \\ -0.0017 & 2.7147 \end{bmatrix}$	$\begin{bmatrix} 2.7140 & -0.0033 \\ -0.0033 & 2.7117 \end{bmatrix}$	$\begin{bmatrix} 2.7162 & -0.0021 \\ -0.0021 & 2.7155 \end{bmatrix}$	$\begin{bmatrix} 2.7168 & -0.0009 \\ -0.0009 & 2.7160 \end{bmatrix}$	0.4987
3	$\begin{bmatrix} 13.3071 & -0.5008 \\ -0.5008 & 13.3598 \end{bmatrix}$	$\begin{bmatrix} 11.0833 & -1.8382 \\ -1.8382 & 12.5355 \end{bmatrix}$	$\begin{bmatrix} 10.9266 & -2.2962 \\ -2.2962 & 11.9898 \end{bmatrix}$	$\begin{bmatrix} 13.1279 & -0.6367 \\ -0.6367 & 13.2111 \end{bmatrix}$	0.4986
4	$\begin{bmatrix} 41.7495 & -1.2458 \\ -1.2458 & 41.6259 \end{bmatrix}$	$\begin{bmatrix} 38.2524 & -3.7762 \\ -3.7762 & 39.5097 \end{bmatrix}$	$\begin{bmatrix} 40.6841 & -1.6534 \\ -1.6534 & 41.3739 \end{bmatrix}$	$\begin{bmatrix} 42.3505 & -0.5179 \\ -0.5179 & 42.2990 \end{bmatrix}$	0.4984
5	$\begin{bmatrix} 5.0967 & -0.0018 \\ -0.0018 & 5.0968 \end{bmatrix}$	$\begin{bmatrix} 5.0951 & -0.0022 \\ -0.0022 & 5.0956 \end{bmatrix}$	$\begin{bmatrix} 5.0962 & -0.0017 \\ -0.0017 & 5.0968 \end{bmatrix}$	$\begin{bmatrix} 5.0968 & -0.0013 \\ -0.0013 & 5.0972 \end{bmatrix}$	0.4980
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
21	$\begin{bmatrix} 14.6042 & 0.0066 \\ 0.0066 & 14.6504 \end{bmatrix}$	$\begin{bmatrix} 14.5695 & 0.0003 \\ 0.0003 & 14.6490 \end{bmatrix}$	$\begin{bmatrix} 14.5482 & -0.0053 \\ -0.0053 & 14.6513 \end{bmatrix}$	$\begin{bmatrix} 14.6075 & 0.0079 \\ 0.0079 & 14.6509 \end{bmatrix}$	0.4942
22	$\begin{bmatrix} 75.0034 & 0.4867 \\ 0.4867 & 77.0554 \end{bmatrix}$	$\begin{bmatrix} 72.4101 & 0.8626 \\ 0.8626 & 76.8426 \end{bmatrix}$	$\begin{bmatrix} 70.4913 & 0.2790 \\ 0.2790 & 77.1546 \end{bmatrix}$	$\begin{bmatrix} 75.2171 & 0.5131 \\ 0.5131 & 77.0856 \end{bmatrix}$	0.4940
23	$\begin{bmatrix} 12.1434 & 0.0001 \\ 0.0001 & 12.1438 \end{bmatrix}$	$\begin{bmatrix} 12.1444 & -0.0004 \\ -0.0004 & 12.1438 \end{bmatrix}$	$\begin{bmatrix} 12.1445 & -0.0001 \\ -0.0001 & 12.1439 \end{bmatrix}$	$\begin{bmatrix} 12.1434 & 0.0002 \\ 0.0002 & 12.1438 \end{bmatrix}$	0.4938
24	$\begin{bmatrix} 98.8228 & 5.9542 \\ 5.9542 & 116.0138 \end{bmatrix}$	$\begin{bmatrix} 98.8232 & 4.0458 \\ 4.0458 & 116.3961 \end{bmatrix}$	$\begin{bmatrix} 91.1770 & 1.8811 \\ 1.8811 & 117.8131 \end{bmatrix}$	$\begin{bmatrix} 101.6412 & 5.5271 \\ 5.5271 & 116.3522 \end{bmatrix}$	0.4937
25	$\begin{bmatrix} 39.6120 & 0.0099 \\ 0.0099 & 39.6777 \end{bmatrix}$	$\begin{bmatrix} 39.6102 & -0.0011 \\ -0.0011 & 39.6776 \end{bmatrix}$	$\begin{bmatrix} 39.5736 & -0.0083 \\ -0.0083 & 39.6794 \end{bmatrix}$	$\begin{bmatrix} 39.6328 & 0.0087 \\ 0.0087 & 39.6789 \end{bmatrix}$	0.4934

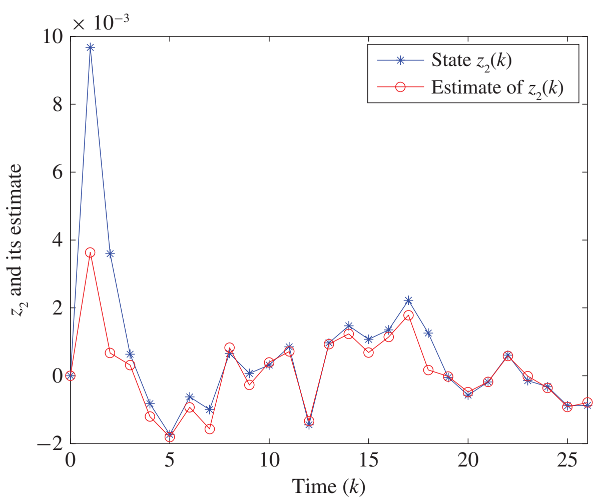

Fig. 3. Output $z_2(k)$ and its estimate $\hat{z}_2(k)$.

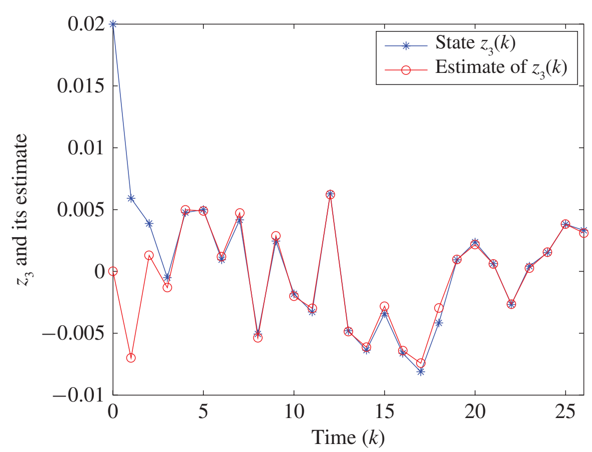

Fig. 4. Output $z_3(k)$ and its estimate $\hat{z}_3(k)$.

level γ, but is not affected by the initial values. Although a small attenuation level leads to smaller synchronization error, there does exist a lowest bound for the attenuation level γ especially when certain complexities such as parameter uncertainties are present. For the complex network in Example 1, the minimum γ can be computed as $\gamma = 0.4425$. On the other hand, for the H_∞ estimation algorithm, it can be seen from (19) that the estimation algorithm depends not only on the attenuation level γ but also on the initial values of the complex network. In order to show the effects on the filtering performance caused by different initial values and attenuation levels, some comparative simulation results are presented in Figs. 6–13. Figs. 6–9 plot the filtering errors $\tilde{z}_i$ $(i = 1, 2, 3, 4)$ with different attenuation levels ($\gamma = 1$ and $\gamma = 3$), which shows that a smaller attenuation level indeed results in better filtering performance. Moreover, the filtering errors $\tilde{z}_i$ $(i = 1, 2, 3, 4)$ with different initial values are depicted in Figs. 10–13.

Remark 7: Note that the RLMI approach developed in this paper is based on LMIs. The standard LMI system has a polynomial-time complexity, which is bounded by $O(\mathcal{M}\mathcal{N}3\log(\mathcal{V}/\varepsilon))$, where $\mathcal{M}$ is the total row size of the LMI system, $\mathcal{N}$ is the total number of scalar decision variables, $\mathcal{V}$ is a data-dependent scaling factor, and ε is relative accuracy set for algorithm. The computational complexity of the developed RLMI-based algorithm can be easily obtained via the time

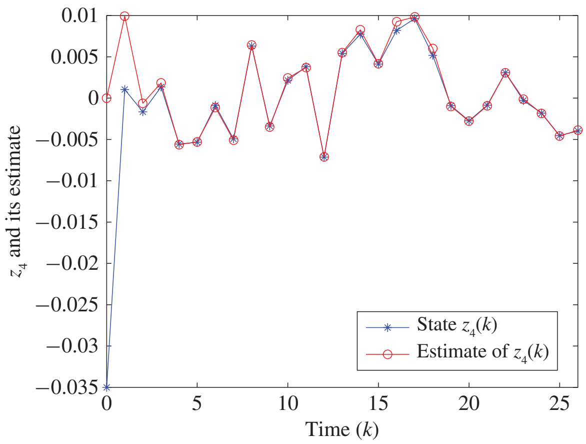

Fig. 5. Output $z_4(k)$ and its estimate $\hat{z}_4(k)$.

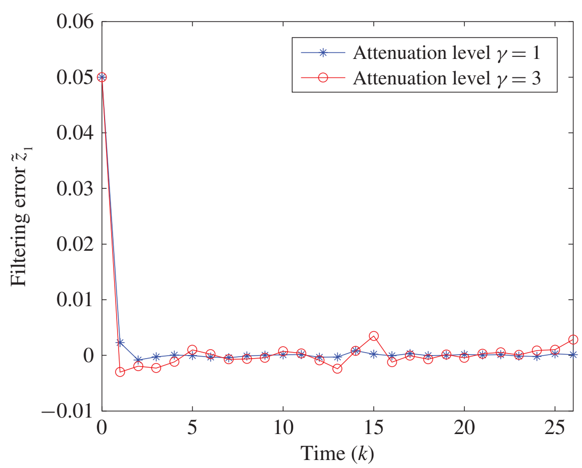

Fig. 6. Filtering error $\tilde{z}_1(k)$ with different attenuation levels.

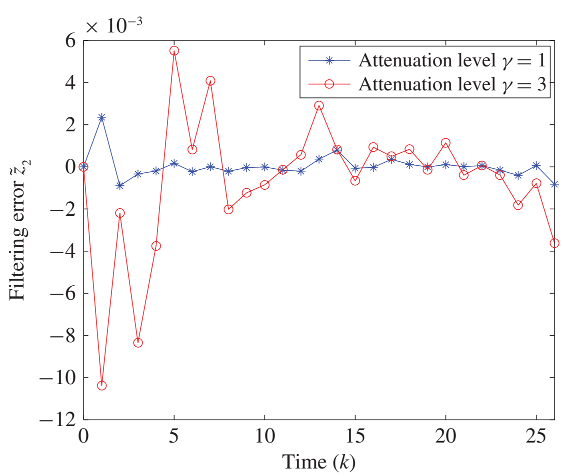

Fig. 7. Filtering error $\tilde{z}_2(k)$ with different attenuation levels.

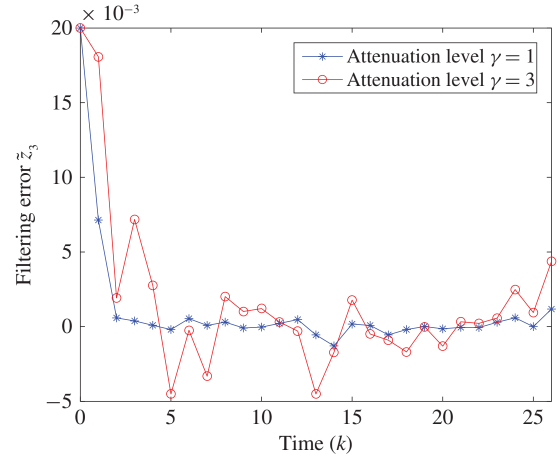

Fig. 8. Filtering error $\tilde{z}_3(k)$ with different attenuation levels.

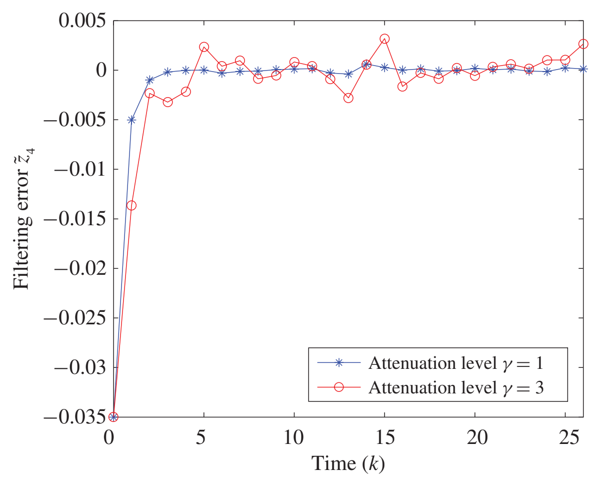

Fig. 9. Filtering error $\tilde{z}_4(k)$ with different attenuation levels.

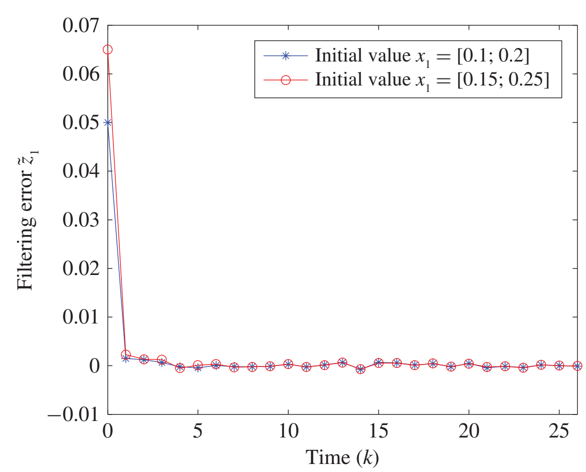

Fig. 10. Filtering error $\tilde{z}_1(k)$ with different initial values.

complexity of the standard LMI system. For example, let us look at the bounded H_∞-synchronization criterion for the complex network (1) (as described in Theorem 1), where the number of network nodes is M, the length of finite time horizon is $N+1$, and the dimensions of network variables can be seen from $x_i(k) \in \mathbb{R}^n$, $z_i(k) \in \mathbb{R}^m$ $(i = 1, 2, \ldots, M)$, $v(k) \in \mathbb{R}^q$, and $\omega(k) \in \mathbb{R}$. The RLMI-based synchronization criterion is implemented recursively for $N+1$ steps and, at every step, $M(M-1)/2$ standard LMIs given by (11) need to be solved. For each of these LMIs, we have $\mathcal{M} = 3n + q$ and $\mathcal{N} = (n^2 + n + 4)/2$. Therefore, the

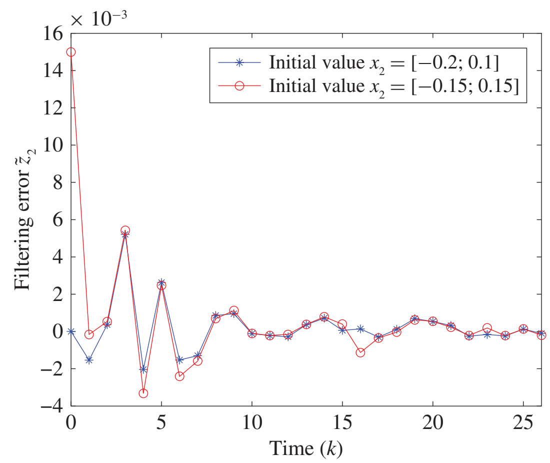

Fig. 11. Filtering error $\tilde{z}_2(k)$ with different initial values.

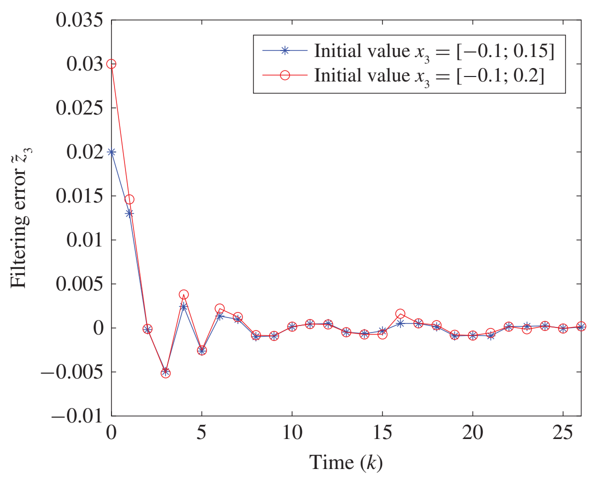

Fig. 12. Filtering error $\tilde{z}_3(k)$ with different initial values.

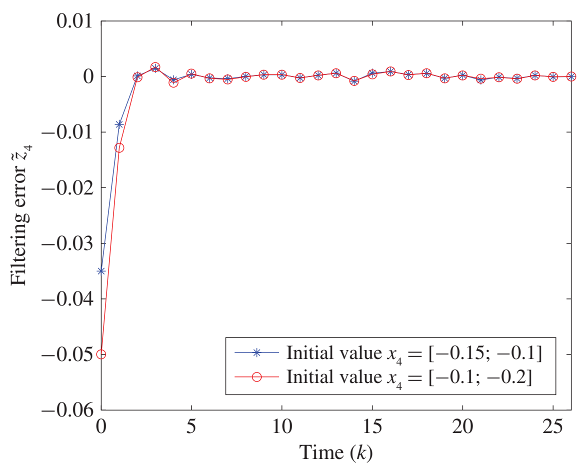

Fig. 13. Filtering error $\tilde{z}_4(k)$ with different initial values.

computational complexity of the RLMI-based synchronization criterion algorithm can be represented as $O(n^3M^2N + n^2M^2Nq)$. Similarly, it is not difficult to calculate that the time complexity of the finite-horizon H_∞ state estimation algorithm is $O(n^3M^2N + n^2qMN + n^2rM^2N + nqrMN)$. Obviously, the computational complexity of the RLMI-based algorithms depends linearly on the length of finite time horizon and polynomially on the dimensions of network variables, which means that the overall computational burden is mainly caused by the complexity of LMI computation. Fortunately, research on LMI optimization is a very active area in the applied mathematics, optimization, and the operations research community, and substantial speedups can be expected in the future.

VI. Conclusion

In this paper, we have addressed a novel synchronization problem for a class of discrete time-varying stochastic complex networks over a finite horizon. A notion of bounded H_∞ synchronization has been first defined to characterize the transient performance of synchronization. Then a testing criterion on the bounded H_∞-synchronization has been established for the considered complex networks in terms of a set of RLMIs. Subsequently, the finite-horizon H_∞ state estimation problem has been considered for the complex networks under consideration. By using the RLMI approach, a sufficient condition under which the filtering error satisfies the H_∞ performance constraint has been obtained, and then all the desired finite-horizon H_∞ estimators have been designed. Finally, two simulation examples have been employed to demonstrate the effectiveness of the results derived in this paper. Further research topics include the extension of our results to more general complex networks with various time delays and also to the H_∞ estimation problem for complex networks with multiple coupled sensors.

Appendix I
Proof of Theorem 1

Proof: Define the real-valued function

$$V(k, x(k)) = x^T(k)\Big(\mathcal{U} \otimes P(k)\Big)x(k) \tag{25}$$

where $\{P(k)\}_{0\le k\le N+1}$ is the solution to the RLMIs (11) with the initial condition (10) and $\mathcal{U} = (\alpha_{ij})_{M\times M}$ with

$$\alpha_{ij} = \begin{cases} M-1, & i = j \\ -1, & i \neq j\,. \end{cases}$$

We can calculate

$$\begin{aligned}
&\mathbb{E}\{V(k+1, x(k+1))\} - \mathbb{E}\{V(k, x(k))\} \\
&\quad + \sum_{1\le i<j\le M} \mathbb{E}\{\|z_i(k) - z_j(k)\|^2\} - \gamma^2\mathbb{E}\{\|v(k)\|^2\} \\
&= \mathbb{E}\Bigg\{ \mathcal{F}^T(k, x(k))\Big(\mathcal{U} \otimes P(k+1)\Big)\mathcal{F}(k, x(k)) \\
&\quad + x^T(k)(W\otimes\Gamma)^T\Big(\mathcal{U} \otimes P(k+1)\Big)(W\otimes\Gamma)x(k)
\end{aligned}$$

$$\begin{aligned}&+\mathcal{G}^T(k,x(k))\Big(\mathcal{U}\otimes P(k+1)\Big)\mathcal{G}(k,x(k))\\&+v^T(k)B^T(k)\Big(\mathcal{U}\otimes P(k+1)\Big)B(k)v(k)\\&+2\mathcal{F}^T(k,x(k))\Big(\mathcal{U}\otimes P(k+1)\Big)(W\otimes\Gamma)x(k)\\&+2\mathcal{F}^T(k,x(k))\Big(\mathcal{U}\otimes P(k+1)\Big)B(k)v(k)\\&+2x^T(k)(W\otimes\Gamma)^T\Big(\mathcal{U}\otimes P(k+1)\Big)B(k)v(k)\\&-x^T(k)\Big(\mathcal{U}\otimes P(k)\Big)x(k)\\&\left.+\sum_{1\le i<j\le M}\|z_i(k)-z_j(k)\|^2-\gamma^2\|v(k)\|^2\right\}.\end{aligned}\tag{26}$$

For the purpose of notation simplicity, set

$$\begin{aligned}\mathbf{x}_{ij}(k)&=x_i(k)-x_j(k)\\\mathbf{f}_{ij}(k)&=f(k,x_i(k))-f(k,x_j(k))\\\mathbf{g}_{ij}(k)&=g_i(k,x_i(k))-g_j(k,x_j(k)).\end{aligned}\tag{27}$$

By using Lemma 1 and noting (6), we can obtain that

$$\begin{aligned}&\mathbb{E}\{V(k+1,x(k+1))\}-\mathbb{E}\{V(k,x(k))\}\\&\quad+\sum_{1\le i<j\le M}\mathbb{E}\left\{\|z_i(k)-z_j(k)\|^2\right\}-\gamma^2\mathbb{E}\left\{\|v(k)\|^2\right\}\\&=\sum_{1\le i<j\le M}\mathbb{E}\Big\{\mathbf{f}_{ij}^T(k)P(k+1)\mathbf{f}_{ij}(k)\\&\quad-Mw_{ij}^{(2)}\mathbf{x}_{ij}^T(k)\Gamma^TP(k+1)\Gamma\mathbf{x}_{ij}(k)\\&\quad+\mathbf{g}_{ij}^T(k)P(k+1)\mathbf{g}_{ij}(k)\\&\quad+v^T(k)\mathbf{B}_{ij}^T(k)P(k+1)\mathbf{B}_{ij}(k)v(k)\\&\quad-2Mw_{ij}\mathbf{f}_{ij}^T(k)P(k+1)\Gamma\mathbf{x}_{ij}(k)\\&\quad+2\mathbf{f}_{ij}^T(k)P(k+1)\mathbf{B}_{ij}(k)v(k)\\&\quad-2Mw_{ij}\mathbf{x}_{ij}^T(k)\Gamma^TP(k+1)\mathbf{B}_{ij}(k)v(k)\\&\quad-\mathbf{x}_{ij}^T(k)P(k)\mathbf{x}_{ij}(k)\\&\quad+\mathbf{x}_{ij}^T(k)E^T(k)E(k)\mathbf{x}_{ij}(k)-\frac{2\gamma^2}{M(M-1)}\|v(k)\|^2\Big\}\\&=\sum_{1\le i<j\le M}\mathbb{E}\Big\{\xi_{ij}^T(k)\bar{\Phi}_{ij}(k)\xi_{ij}(k)\Big\}\end{aligned}\tag{28}$$

where

$$\xi_{ij}(k)=\begin{bmatrix}\mathbf{x}_{ij}^T(k) & \mathbf{f}_{ij}^T(k) & \mathbf{g}_{ij}^T(k) & v^T(k)\end{bmatrix}^T$$

$$\bar{\Phi}_{ij}(k)=\begin{bmatrix}\bar{\Theta}_{ij}^{(1)}(k) & -Mw_{ij}\Gamma^TP(k+1) & 0 & \Theta_{ij}^{(3)}(k)\\ * & P(k+1) & 0 & \Theta_{ij}^{(4)}(k)\\ * & * & P(k+1) & 0\\ * & * & * & \Theta_{ij}^{(5)}(k)\end{bmatrix}$$

and $\bar{\Theta}_{ij}^{(1)}(k)=-Mw_{ij}^{(2)}\Gamma^TP(k+1)\Gamma-P(k)+E^T(k)E(k)$.

Subsequently, using the notations in (27), we rewrite (4) as

$$\begin{bmatrix}\mathbf{x}_{ij}(k)\\\mathbf{f}_{ij}(k)\end{bmatrix}^T\begin{bmatrix}\tilde{U}_1(k) & \tilde{U}_2(k)\\ * & I\end{bmatrix}\begin{bmatrix}\mathbf{x}_{ij}(k)\\\mathbf{f}_{ij}(k)\end{bmatrix}\le 0.\tag{29}$$

Similarly, (5) can also be rewritten as

$$\begin{bmatrix}\mathbf{x}_{ij}(k)\\\mathbf{g}_{ij}(k)\end{bmatrix}^T\begin{bmatrix}-V^T(k)V(k) & 0\\ * & I\end{bmatrix}\begin{bmatrix}\mathbf{x}_{ij}(k)\\\mathbf{g}_{ij}(k)\end{bmatrix}\le 0.\tag{30}$$

Therefore, by noting (11), it follows from (28)–(30) that

$$\begin{aligned}&\mathbb{E}\{V(k+1,x(k+1))\}-\mathbb{E}\{V(k,x(k))\}\\&\quad+\sum_{1\le i<j\le M}\mathbb{E}\{\|z_i(k)-z_j(k)\|^2\}-\gamma^2\mathbb{E}\{\|v(k)\|^2\}\\&\le\sum_{1\le i<j\le M}\mathbb{E}\Big\{\xi_{ij}^T(k)\bar{\Phi}_{ij}(k)\xi_{ij}(k)\\&\quad-\lambda_1(k)\begin{bmatrix}\mathbf{x}_{ij}(k)\\\mathbf{f}_{ij}(k)\end{bmatrix}^T\begin{bmatrix}\tilde{U}_1(k) & \tilde{U}_2(k)\\ * & I\end{bmatrix}\begin{bmatrix}\mathbf{x}_{ij}(k)\\\mathbf{f}_{ij}(k)\end{bmatrix}\\&\quad-\lambda_2(k)\begin{bmatrix}\mathbf{x}_{ij}(k)\\\mathbf{g}_{ij}(k)\end{bmatrix}^T\begin{bmatrix}-V^T(k)V(k) & 0\\ * & I\end{bmatrix}\begin{bmatrix}\mathbf{x}_{ij}(k)\\\mathbf{g}_{ij}(k)\end{bmatrix}\Big\}\\&=\sum_{1\le i<j\le M}\mathbb{E}\Big\{\xi_{ij}^T(k)\Phi_{ij}(k)\xi_{ij}(k)\Big\}\\&\le 0.\end{aligned}\tag{31}$$

Summing up (31) from 0 to N with respect to k yields

$$\begin{aligned}\sum_{1\le i<j\le M}\|z_i-z_j\|_{[0\ \ N]}^2\le\gamma^2\|v\|_{[0\ \ N]}^2\\+\mathbb{E}\{x^T(0)\left(\mathcal{U}\otimes P(0)\right)x(0)\}.\end{aligned}\tag{32}$$

By considering the initial condition (10), the inequality (8) follows from (32) immediately and, consequently, the proof of this theorem is complete. ■

Appendix II

Proof of Theorem 2

Proof: Let the real-valued function be

$$V(k,e(k))=e^T(k)P(k)e(k)+\mu(k)\tag{33}$$

where $\{P(k)\}_{0\le k\le N+1}$ and $\{\mu(k)\}_{0\le k\le N+1}$ are the solutions to the RLMIs (20) with the initial condition (19).

For notation simplicity, we denote

$$\begin{aligned}\zeta(k)&=\begin{bmatrix}e^T(k) & \tilde{\mathcal{F}}^T(k,e(k)) & v^T(k) & \mathcal{G}^T(k,e(k)+\hat{x}(k)) & 1\end{bmatrix}^T\\\mathcal{A}(k)&=\begin{bmatrix}-K(k)C(k)+W\otimes\Gamma & I & B(k)-K(k)D(k) & 0 & 0\end{bmatrix}\\\mathcal{H}&=\begin{bmatrix}0 & 0 & 0 & I & 0\end{bmatrix}.\end{aligned}\tag{34}$$

Tedious but straightforward calculation shows that

$$\begin{aligned}&\mathbb{E}\{V(k+1,e(k+1))\}-\mathbb{E}\{V(k,e(k))\}\\&\quad+\mathbb{E}\{\|\tilde{z}(k)\|^2\}-\gamma^2\mathbb{E}\{\|v(k)\|^2\}\\&=\mathbb{E}\Big\{\Big(\mathcal{A}(k)\zeta(k)+\mathcal{H}\zeta(k)\omega(k)\Big)^T\\&\quad\times P(k+1)\Big(\mathcal{A}(k)\zeta(k)+\mathcal{H}\zeta(k)\omega(k)\Big)\\&\quad-e^T(k)P(k)e(k)+e^T(k)E_\Lambda^T(k)E_\Lambda(k)e(k)\\&\quad-\gamma^2v^T(k)v(k)+\mu(k+1)-\mu(k)\Big\}\\&=\mathbb{E}\Big\{\zeta^T(k)\Big(\mathbf{\Omega}_1(k)+\mathcal{A}^T(k)P(k+1)\mathcal{A}(k)\\&\quad+\mathcal{H}^TP(k+1)\mathcal{H}\Big)\zeta(k)\Big\}\end{aligned}\tag{35}$$

$$\Omega_2(k) = \begin{bmatrix} \Xi_1(k) & -\varepsilon_1(k)\tilde{U}_{2\Lambda}(k) & 0 & 0 & \varepsilon_2(k)V_\Lambda^T(k)V_\Lambda(k)\hat{x}(k) \\ * & -\varepsilon_1(k)I & 0 & 0 & 0 \\ * & * & -\gamma^2 I & 0 & 0 \\ * & * & * & -\varepsilon_2(k)I & 0 \\ * & * & * & * & \Xi_4(k) \end{bmatrix}.$$

where

$$\Omega_1(k) = \begin{bmatrix} -P(k)+E_\Lambda^T(k)E_\Lambda(k) & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & * & -\gamma^2 I & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & \mu(k+1)-\mu(k) \end{bmatrix}.$$

From (4) and (5), we have

$$\begin{bmatrix} e(k) \\ \tilde{\mathcal{F}}(k,e(k)) \end{bmatrix}^T \begin{bmatrix} \tilde{U}_{1\Lambda}(k) & \tilde{U}_{2\Lambda}(k) \\ * & I \end{bmatrix} \begin{bmatrix} e(k) \\ \tilde{\mathcal{F}}(k,e(k)) \end{bmatrix} \le 0 \quad (36)$$

and

$$\begin{bmatrix} e(k) \\ \mathcal{G}(k,e(k)+\hat{x}(k)) \\ 1 \end{bmatrix}^T \times \begin{bmatrix} -V_\Lambda^T(k)V_\Lambda(k) & 0 & -V_\Lambda^T(k)V_\Lambda(k)\hat{x}(k) \\ * & I & 0 \\ * & * & -\hat{x}^T(k)V_\Lambda^T(k)V_\Lambda(k)\hat{x}(k) \end{bmatrix} \times \begin{bmatrix} e(k) \\ \mathcal{G}(k,e(k)+\hat{x}(k)) \\ 1 \end{bmatrix} \le 0 \quad (37)$$

respectively.

By considering (35) and (37), we can obtain

$$\begin{aligned}
&\mathbb{E}\{V(k+1,e(k+1))\} - \mathbb{E}\{V(k,e(k))\} + \mathbb{E}\{\|\tilde{z}(k)\|^2\} \\
&-\gamma^2\mathbb{E}\{\|v(k)\|^2\} \\
&\le \mathbb{E}\Bigg\{\zeta^T(k)\Big(\Omega_1(k) + \mathcal{A}^T(k)P(k+1)\mathcal{A}(k) \\
&+\mathcal{H}^T P(k+1)\mathcal{H}\Big)\zeta(k) \\
&-\varepsilon_1(k)\begin{bmatrix} e(k) \\ \tilde{\mathcal{F}}(k,e(k)) \end{bmatrix}^T \begin{bmatrix} \tilde{U}_{1\Lambda}(k) & \tilde{U}_{2\Lambda}(k) \\ * & I \end{bmatrix} \begin{bmatrix} e(k) \\ \tilde{\mathcal{F}}(k,e(k)) \end{bmatrix} \\
&-\varepsilon_2(k)\begin{bmatrix} e(k) \\ \mathcal{G}(k,e(k)+\hat{x}(k)) \\ 1 \end{bmatrix}^T \\
&\times \begin{bmatrix} -V_\Lambda^T(k)V_\Lambda(k) & 0 & -V_\Lambda^T(k)V_\Lambda(k)\hat{x}(k) \\ * & I & 0 \\ * & * & -\hat{x}^T(k)V_\Lambda^T(k)V_\Lambda(k)\hat{x}(k) \end{bmatrix} \\
&\times \begin{bmatrix} e(k) \\ \mathcal{G}(k,e(k)+\hat{x}(k)) \\ 1 \end{bmatrix}\Bigg\} \\
&= \mathbb{E}\Big\{\zeta^T(k)\Big(\Omega_2(k) + \mathcal{A}^T(k)P(k+1)\mathcal{A}(k) \\
&+\mathcal{H}^T P(k+1)\mathcal{H}\Big)\zeta(k)\Big\}, \qquad (38)
\end{aligned}$$

where $\Omega_2(k)$ is shown at the top of the page.

By using the Schur complement formula and noting (20), we can easily obtain from (38)

$$\begin{aligned}
&\mathbb{E}\{V(k+1,e(k+1))\} - \mathbb{E}\{V(k,e(k))\} + \mathbb{E}\{\|\tilde{z}(k)\|^2\} \\
&\quad -\gamma^2\mathbb{E}\{\|v(k)\|^2\} \le 0.
\end{aligned} \qquad (39)$$

Then, the rest of this paper can be easily accomplished by following the methods used in the proof of Theorem 1 and is therefore omitted. ■

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Bo Shen received the B.Sc. degree in mathematics from Northwestern Polytechnical University, Xi'an, China, in 2003. He is currently pursuing the Ph.D. degree in the School of Information Science and Technology, Donghua University, Shanghai, China. He is also now a visiting Ph.D. student in the Department of Information Systems and Computing, Brunel University, West London, U.K.

He was a Research Assistant in the Department of Electrical and Electronic Engineering, University of Hong Kong, Hong Kong, China, from August 2009 to February 2010. His current research interests include nonlinear control and filtering, stochastic control and filtering, complex networks, and genetic regulatory networks.

Dr. Shen is a very active reviewer for many international journals.

Zidong Wang (SM'03) was born in Jiangsu, China, in 1966. He received the B.Sc. degree in mathematics from Suzhou University, Suzhou, China, in 1986, the M.Sc. degree in applied mathematics in 1990, and the Ph.D. degree in electrical and computer engineering in 1994, both from Nanjing University of Science and Technology, Nanjing, China.

He is currently a Professor of Dynamical Systems and Computing at Brunel University, West London, U.K. He has published more than 120 papers in refereed international journals. His current research interests include dynamical systems, signal processing, bioinformatics, and control theory and applications.

Prof. Wang is currently serving as an Associate Editor for 12 international journals including the IEEE TRANSACTIONS ON AUTOMATIC CONTROL, the IEEE TRANSACTIONS ON NEURAL NETWORKS, the IEEE TRANSACTIONS ON SIGNAL PROCESSING, the IEEE TRANSACTIONS ON SYSTEMS, MAN, AND CYBERNETICS—PART C: APPLICATIONS AND REVISIONS, and the IEEE TRANSACTIONS ON CONTROL SYSTEMS TECHNOLOGY.

Xiaohui Liu received the B.E. degree in computing from Hohai University, Nanjing, China, in 1982 and the Ph.D. degree in computer science from Heriot-Watt University, Edinburg, U.K., in 1988.

He is currently a Professor of Computing at Brunel University, West London, U.K. He leads the Intelligent Data Analysis (IDA) Group, performing interdisciplinary research involving artificial intelligence, dynamic systems, image and signal processing, and statistics, particularly for applications in biology, engineering and medicine.

Prof. Liu serves on editorial boards of four computing journals, founded the biennial international conference series on IDA in 1995, and has given numerous invited talks in bioinformatics, data mining, and statistics conferences.