y-averaging maps are clear. To see equiproperness of f_t we note first that from equiproperness of g_t it follows:

$$d_H\left(\phi^{-1}\left(\operatorname*{conv}_{i\in\underline{m}} y^i(\phi(g_t(x)))\right), \phi^{-1}\left(\operatorname*{conv}_{i\in\underline{m}} y^i(\phi(x))\right)\right) \geq \delta(x)$$

$$\Rightarrow d_H\left(\phi^{-1}\left(\operatorname*{conv}_{i\in\underline{m}} y^i(f_t(\xi))\right), \phi^{-1}\left(\operatorname*{conv}_{i\in\underline{m}} y^i(\xi)\right)\right) \geq \delta(\phi^{-1}(\xi))$$

while the second line holds for all $\xi = \phi(x) \in S^n$ and $t \geq 0$. Due to (3) we can express the Hausdorff distance as

$$\begin{aligned} &d_H\left(\phi^{-1}\left(\operatorname*{conv}_{i\in\underline{m}} y^i(f_t(\xi))\right), \phi^{-1}\left(\operatorname*{conv}_{i\in\underline{m}} y^i(\xi)\right)\right) \\ &= \max_{z\in\phi^{-1}\left(\operatorname{conv}_{i\in\underline{m}} y^i(f_t(\xi))\right)} \ \min_{w\in\phi^{-1}\left(\operatorname{conv}_{i\in\underline{m}} y^i(\xi)\right)} \|z-w\| \\ &= \max_{\phi(z)\in\operatorname{conv}_{i\in\underline{m}} y^i(f_t(\xi))} \ \min_{\phi(w)\in\operatorname{conv}_{i\in\underline{m}} y^i(\xi)} \|z-w\|. \end{aligned}$$

With this preparation we show equiproperness of the f_t's

$$\begin{aligned} &d_H\left(\operatorname*{conv}_{i\in\underline{m}} y^i(f_t(\xi)), \operatorname*{conv}_{i\in\underline{m}} y^i(\xi)\right) \\ &= \max_{\zeta\in\operatorname{conv}_{i\in\underline{m}} y^i(f_t(\xi))} \ \min_{\omega\in\operatorname{conv}_{i\in\underline{m}} y^i(\xi)} \|\zeta-\omega\| \\ &= \max_{\phi(z)\in\operatorname{conv}_{i\in\underline{m}} y^i(f_t(\xi))} \ \min_{\phi(w)\in\operatorname{conv}_{i\in\underline{m}} y^i(\xi)} \|\phi(z)-\phi(w)\| \\ &\geq L \max_{\phi(z)\in\operatorname{conv}_{i\in\underline{m}} y^i(f_t(\xi))} \ \min_{\phi(w)\in\operatorname{conv}_{i\in\underline{m}} y^i(\xi)} \|z-w\| \\ &\geq L\delta(\phi^{-1}(\xi)) \end{aligned}$$

where L is the Lipschitz constant of ϕ^{-1}. Now for $\xi(t) = \phi(x(t))$ it follows $\xi(t+1) = \phi(g_t(x(t)) = \phi(g_t(\phi^{-1}(\xi(t)))) = f_t(\xi(t))$. By virtue of Theorem 2.4, $\xi(t) \to c$ where $c \in S^n$ is a consensus and hence, $x(t) \to \phi^{-1}(c) \in T^n$ which is also a consensus. ■

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Exponential Stabilization of a Class of Stochastic System With Markovian Jump Parameters and Mode-Dependent Mixed Time-Delays

Zidong Wang, Yurong Liu, and Xiaohui Liu

***Abstract*—In this technical note, the globally exponential stabilization problem is investigated for a general class of stochastic systems with both Markovian jumping parameters and mixed time-delays. The mixed mode-dependent time-delays consist of both discrete and distributed delays. We aim to design a memoryless state feedback controller such that the closed-loop system is stochastically exponentially stable in the mean square sense. First, by introducing a new Lyapunov-Krasovskii functional that accounts for the mode-dependent mixed delays, stochastic analysis is conducted in order to derive a criterion for the exponential stabilizability problem. Then, a variation of such a criterion is developed to facilitate the controller design by using the linear matrix inequality (LMI) approach. Finally, it is shown that the desired state feedback controller can be characterized explicitly in terms of the solution to a set of LMIs. Numerical simulation is carried out to demonstrate the effectiveness of the proposed methods.**

***Index Terms*—Discrete time-delays, distributed time-delays, Markovian jumping parameters, mixed mode-dependent (MDD) time-delays, stochastic systems.**

I. INTRODUCTION

It is now well known that time-delays are frequently encountered in practical systems such as engineering and biological systems, and their existence may induce instability, oscillation, and poor performances [1], [4], [5]. Time delays may also arise in several signal processing areas such as multipath propagation, telemanipulation systems, data communication in high-speed internet and network control systems [2]. According to the way time-delays occur, they can be classified as discrete (point) delays [16] and distributed delays [10]. In the past few years, considerable attention has been devoted to the robust stabilization and H_∞ control problem for linear and nonlinear time-delay systems, and a great number of papers have appeared on this general topic, see [2] for a survey.

Markovian jump systems (MJSs) involve both time-evolving and event-driven mechanisms, which can be employed to model the abrupt phenomena such as random failures and repairs of the components, changes in the interconnections of subsystems, sudden environment changes, etc. The issues of stability, stabilization, control and filtering have been well investigated, see e.g. [1], [3], [7], [9], [13], [15], [16], [19]. On another research forefront, since stochastic phenomenon typically exhibits in many branches of science and engineering applica-

Manuscript received July 17, 2009; revised November 08, 2009, February 09, 2010, March 06, 2010, and March 10, 2010. First published March 22, 2010; current version published July 08, 2010. This work was supported in part by the Engineering and Physical Sciences Research Council (EPSRC) of the U.K. under Grant GR/S27658/01, the Royal Society of the U.K., the National 973 Program of China under Grant 2009CB320600, and the Alexander von Humboldt Foundation of Germany. Recommended by Associate Editor G. Chesi.

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Digital Object Identifier 10.1109/TAC.2010.2046114

tions, a great deal of attention has been devoted to the stochastic systems governed by Itô stochastic differential equations, and a variety of works have been published with respect to the stability and stabilization of Itô-type stochastic systems. Naturally, stochastic systems with Markovian jumping parameters have also received considerable research attention, see e.g. [11], [16], [18], [19].

Although the stabilization problem for stochastic Markovian jumping systems with discrete time-delays has been well investigated, there has been very little literature on the mixed mode-dependent (MDD) time-delays comprising distributed ones. MDD time-delays are of practical significance since the signal may switch between different modes and also propagate in a distributed way during a certain time period with the presence of an amount of parallel pathways. It is, therefore, the purpose of this technical note to close such a gap by making one of the first few attempts to deal with the control problem for a class of stochastic systems with MDD delays. The main contributions of this technical note lie in the following aspects: 1) mode-dependent distributed delays are introduced in the system model; 2) a new Lyapunov-Krasovskii functional is proposed to account for the mode-dependent distributed delay; 3) a unified delay-dependent LMI framework is developed that tackles the "complexity" consisting of Markovian jumping parameters, MDD time-delays, external disturbances and Itô-type Brownian motions.

Notations: Throughout this technical note, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript "T" denotes the transpose and the notation $X \geq Y$ (respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). I is the identity matrix with compatible dimension. For $h > 0$, $C([-h,0];\mathbb{R}^n)$ denotes the family of continuous functions φ from $[-h,0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{-h\leq\theta\leq 0}|\varphi(\theta)|$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. If A is a matrix, denote by $\|A\|$ its operator norm, i.e., $\|A\| = \sup\{\|Ax\| : |x| = 1\} = \sqrt{\lambda_{\max}(A^TA)}$ where $\lambda_{\max}(\cdot)$ (respectively, $\lambda_{\min}(\cdot)$) means the largest (respectively, smallest) eigenvalue of A. $l_2[0,\infty]$ is the space of square integrable vector. Moreover, let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., the filtration contains all $\mathcal{P}$-null sets and is right continuous). Denote by $L^p_{\mathcal{F}_0}([-h,0];\mathbb{R}^n)$ the family of all $\mathcal{F}_0$-measurable $C([-h,0];\mathbb{R}^n)$-valued random variables $\xi = \{\xi(\theta) : -h \leq \theta \leq 0\}$ such that $\sup_{-h\leq\theta\leq 0}\mathbb{E}|\xi(\theta)|^p < \infty$ where $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure $\mathcal{P}$.

II. Problem Formulation

Let $r(t)$ $(t \geq 0)$ be a right-continuous Markov chain on the probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathcal{P})$ taking values in a finite state space $\mathcal{S} = \{1,2,\ldots,N\}$ with generator $\Pi = (\pi_{ij})_{N\times N}$ given by

$$\mathcal{P}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases}\pi_{ij}\Delta + o(\Delta), & \text{if } i \neq j;\\ 1 + \pi_{ii}\Delta + o(\Delta), & \text{if } i = j.\end{cases}$$

Here $\Delta > 0$ and $\pi_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\pi_{ii} = -\sum_{j\neq i}\pi_{ij}$.

Consider, on a probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathcal{P})$, the following stochastic system with mode-dependent mixed time delays and Markovian switching of the form:

$$\begin{aligned} dx(t) = \Big[&A(r(t))x(t) + B(r(t))x\left(t-\tau_{1,r(t)}\right)\\ &+ C(r(t))\int_{t-\tau_{2,r(t)}}^{t} f(x(s))\,ds\\ &+g\left(x(t), x\left(t-\tau_{1,r(t)}\right), t\right) + D(r(t))u(t)\Big]dt\\ &+\sigma\left(x(t), x\left(t-\tau_{1,r(t)}\right), t, r(t)\right)dw(t), \end{aligned} \tag{1a}$$

$$x(t) = \phi(t), t \in [-\tau, 0] \tag{1b}$$

where $x(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^T \in \mathbb{R}^n$ is the state vectors; the $n\times n$ matrices $A(i)$, $B(i)$, $C(i)$ and $n \times q$ matrices $D(i)$ $(i \in \mathcal{S})$ are known constant matrices; $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear vector function and $g(\cdot,\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ serves as an external, mode-dependent, nonlinear disturbance. In (1a), (1b), $u(t) \in \mathbb{R}^q$ is the control input; $\tau_{1,r(t)}$ stands for the *discrete* mode-dependent time-delay, while $\tau_{2,r(t)}$ describes the *distributed* mode-dependent time-delay; $\tau := \max\{\tau_{i,j} | i = 1,2, j = 1,2,\ldots,N\}$ and $\phi \in L^2_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$ is the initial condition; $w(t)$ represents a scalar Wiener process (Brownian motion) on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathcal{P})$ that is independent of the Markov chain $r(\cdot)$ and satisfies $\mathbb{E}[w(t)] = 0$, $\mathbb{E}[w^2(t)] = t$. $\sigma(\cdot,\cdot,\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{S} \to \mathbb{R}^n$ is a Borel measurable n-dimensional vector function.

In this technical note, we make the following assumptions:

$$|\sigma(x,y,t,r(t))| \leq \rho|x|, \quad \sigma(0,0,0,r(t)) \equiv 0 \tag{2}$$

$$(f(x) - F_1x)^T(f(x) - F_2x) \leq 0, \quad f(0) \equiv 0 \tag{3}$$

$$|g(x,y,t)|^2 \leq |G_1x|^2 + |G_2y|^2, \quad g(0,0,0) \equiv 0 \tag{4}$$

where ρ is a positive scalar constant and F_1, F_2, G_1 and G_2 are known constant matrices.

Note that the sector-like nonlinear description in (3) is quite general that includes the usual Lipschitz conditions as a special case [8]. According to (3), (4), it can be deduced that $f(0) = g(0,0,t) = 0$, and then the system (1) admits a trivial solution, i.e., $x(t) \equiv 0$ corresponding to the initial data $\phi = 0$.

Definition 1: The system (1) with $u(t) \equiv 0$ is said to be exponentially stable in mean square sense if there exist positive constants $\gamma > 0$ and $\mu > 0$ such that every solution $x(t;\phi)$ of (1) satisfies

$$\mathbb{E}|x(t;\phi)|^2 \leq \gamma e^{-\mu t}\sup_{-\tau\leq s\leq 0}\mathbb{E}|\phi(s)|^2, \quad \forall t > 0.$$

Let $u(t) = K(r(t))x(t)$ be the state feedback controller where $K(i) \in \mathbb{R}^{q\times n}$ $(i \in \mathcal{S})$ are controller gains to be designed. Then, the closed-loop system can be given as follows:

$$\begin{aligned} dx(t) = \Big[&(A(r(t)) + D(r(t))K(r(t)))x(t)\\ &+ B(r(t))x\left(t-\tau_{1,r(t)}\right)\\ &+ C(r(t))\int_{t-\tau_{2,r(t)}}^{t} f(x(s))\,ds\\ &+g\left(x(t), x\left(t-\tau_{1,r(t)}\right), t\right)\Big]dt\\ &+\sigma\left(x(t), x\left(t-\tau_{1,r(t)}\right), t, r(t)\right)dw(t), \end{aligned} \tag{5a}$$

$$x(t) = \phi(t), \quad t \in [-\tau, 0]. \tag{5b}$$

Definition 2: The system (1) is said to be exponentially stabilizable in the mean square sense if there exists a state controller $u(t) = K(r(t))x(t)$ such that the closed-loop (5) is exponentially stable in the mean square sense.

The main purpose of this technical note is to deal with the stability analysis and the exponential stabilization problems for a class of nonlinear system with Markovian jump parameters and mode-dependent mixed time-delays.

III. Exponential Stability

Lemma 1: [6] For any matrix $M > 0$, scalar $\gamma > 0$, vector function $\omega : [0,\gamma] \to \mathbb{R}^n$ such that the integrations concerned are well defined, the following inequality holds:

$$\left(\int_0^\gamma \omega(s)ds\right)^T M\left(\int_0^\gamma \omega(s)ds\right) \leq \gamma\left(\int_0^\gamma \omega^T(s)M\omega(s)ds\right). \tag{6}$$

Before proceeding, it is worth pointing out that the joint process $(x(t), r(t))$ is not Markovian. By definition, a Markov process is a stochastic process which assumes that, in a series of random events, the probability of an occurrence of each event depends only on the immediately preceding outcome. This captures the idea that its future state is independent of its past states. However, from (1a), (1b), it is easy to see that the derivative of the system state x at time t is expressed in terms of x at t and earlier instants. Consequently, the evolution of the system state x is dependent on not only its present state (at time t) but also its past states (over the interval $[t-\tau\ t]$). Therefore, the joint process $(x(t), r(t))$ is not a Markov process.

Let us now consider the exponential stability of the system (1) with $u(t) \equiv 0$. Denote

$$\overline{\tau}_i = \max\{\tau_{i,j}, j \in \mathcal{S}\},\ \underline{\tau}_i = \min\{\tau_{i,j}, j \in \mathcal{S}\}\ (i=1,2),$$
$$\breve{F}_1 = \left(F_1^T F_2 + F_2^T F_1\right)/2,\ \breve{F}_2 = \left(F_1^T + F_2^T\right)/2,$$
$$\bar{\pi} = \max\{-\pi_{ii}, i \in \mathcal{S}\}.$$

Theorem 1: System (1) with $u(t) \equiv 0$ is exponentially stable in the mean square sense if there exist a set of positive scalar constant $\{\lambda_i^*, i \in \mathcal{S}\}$, two positive definite matrices Q and R, and a set of positive definite matrices $\{P_i, i \in \mathcal{S}\}$ such that the following LMIs hold:

$$P_i < \lambda_i^* I, \tag{7}$$

$$\Omega_i = \begin{bmatrix} \Xi_i & P_iB(i) & \breve{F}_2 & P_iC(i) & P_i \\ * & -Q+G_2^TG_2 & 0 & 0 & 0 \\ * & * & \Delta_i & 0 & 0 \\ * & * & * & -\frac{1}{\tau_{2,i}}R & 0 \\ * & * & * & * & -I \end{bmatrix} < 0 \tag{8}$$

where

$$\bar{P}_i = \sum_{j=1}^{m} \pi_{ij} P_j,$$
$$\Xi_i = P_iA(i) + A^T(i)P_i + \bar{P}_i + (1+\bar{\pi}(\overline{\tau}_1 - \underline{\tau}_1))\,Q + \lambda_i^*\rho^2 I - \breve{F}_1 + G_1^TG_1, \tag{9}$$
$$\Delta_i = \left[\tau_{2,i} + \frac{1}{2}\bar{\pi}\left(\overline{\tau}_2^2 - \underline{\tau}_2^2\right)\right]R - I. \tag{10}$$

Proof: By (8), we can find a scalar $\varepsilon_0 > 0$ such that

$$\tilde{\Omega}_i := \Omega_i + \left[\begin{array}{c|c} \varepsilon_0 I_n & 0 \\ \hline 0 & 0 \end{array}\right] < 0. \tag{11}$$

As discussed previously, $\{(x(t), r(t)), t \geq 0\}$ is not a Markov process. In order to cast our model into the framework for a Markov system, let us define a new Markov process $\{(x_t, r(t)), t \geq 0\}$ with $x_t(s) = x(t+s), -\tau \leq s \leq 0$ [14], where the evolution of x_t is dependent on its present state only. Note that x_t is fundamentally different from $x(t)$ in that x_t is a functional but $x(t)$ is a function of t.

Consider the following stochastic Lyapunov functional candidate for the system (1) with $u(t) \equiv 0$:

$$\begin{aligned} V&(x_t, t, r(t)) \\ &= x^T(t)P_{r(t)}x(t) + \int_{t-\tau_{1,r(t)}}^{t} x^T(s)Qx(s)ds \\ &\quad + \bar{\pi}\int_{\underline{\tau}_1}^{\overline{\tau}_1}\int_{t-s}^{t} x^T(\theta)Qx(\theta)d\theta ds \\ &\quad + c_0\int_{0}^{\tau}\int_{t-s}^{t} x^T(\theta)x(\theta)d\theta ds \\ &\quad + \int_{0}^{\tau_{2,r(t)}}\int_{t-s}^{t} f^T(x(\theta))Rf(x(\theta))d\theta ds \\ &\quad + \bar{\pi}\int_{\underline{\tau}_2}^{\overline{\tau}_2}\int_{0}^{u}\int_{t-s}^{t} f^T(x(\theta))Rf(x(\theta))d\theta ds du \end{aligned} \tag{12}$$

where $c_0 = \varepsilon_0/\tau$.

Let $\mathcal{L}$ be the weak infinitesimal generator of the random process $\{(x_t, r(t)), t \geq 0\}$ along the system (1) with $u(t) = 0$ (see [11], [18]). Then, we have

$$\begin{aligned} \mathcal{L}&V(x_t, t, i) \\ &:= \lim_{\Delta\to 0^+}\sup\left[\mathbb{E}\left\{V(x_{t+\Delta}, t+\Delta, r(t+\Delta))|x_t, r(t)=i\right\} - V(x_t, t, r(t)=i)\right] \\ &= 2x^T(t)P_i\Big(A(i)x(t) + B(i)x(t-\tau_{1,i}) + C(i)\int_{t-\tau_{2,i}}^{t} f(x(s))ds + g(x(t), x(t-\tau_{1,i}), t)\Big) \\ &\quad + \sigma^T(x(t), x(t-\tau_{1,i}), t, i)P_i\sigma(x(t), x(t-\tau_{1,i}), t, i) \\ &\quad + \sum_{j=1}^{m}\pi_{ij}x^T(t)P_jx(t) - x^T(t-\tau_{1,i})Qx(t-\tau_{1,i}) \\ &\quad + \sum_{j=1}^{m}\pi_{ij}\int_{t-\tau_{1,j}}^{t} x^T(s)Qx(s)ds \\ &\quad + \bar{\pi}(\overline{\tau}_1 - \underline{\tau}_1)x^T(t)Qx(t) \\ &\quad - \bar{\pi}\int_{\underline{\tau}_1}^{\overline{\tau}_1} x^T(t-s)Qx(t-s)ds \\ &\quad + \tau_{2,i}f^T(x(t))Rf(x(t)) \\ &\quad - \int_{0}^{\tau_{2,i}} f^T(x(t-s))Rf(x(t-s))ds + x^T(t)Qx(t) \\ &\quad + \sum_{j=1}^{m}\pi_{i,j}\int_{0}^{\tau_{2,j}}\int_{t-s}^{t} f^T(x(\theta))Rf(x(\theta))d\theta ds \\ &\quad + \bar{\pi}\int_{\underline{\tau}_2}^{\overline{\tau}_2}\int_{0}^{u} f^T(x(t))Rf(x(t))ds du \\ &\quad - \bar{\pi}\int_{\underline{\tau}_2}^{\overline{\tau}_2}\int_{0}^{u} f^T(x(t-s))Rf(x(t-s))ds du \\ &\quad + c_0\tau x^T(t)x(t) - c_0\int_{0}^{\tau} x^T(t-s)x(t-s)ds. \end{aligned} \tag{13}$$

Notice that

$$-\bar{\pi}\int_{\underline{\tau}_1}^{\overline{\tau}_1} x^T(t-s)Qx(t-s)ds = -\bar{\pi}\int_{t-\overline{\tau}_1}^{t-\underline{\tau}_1} x^T(s)Qx(s)ds, \tag{14}$$

$$-\int_{0}^{\tau_{2,i}} f^T(x(t-s))Rf(x(t-s)) = -\int_{t-\tau_{2,i}}^{t} f^T(x(s))Rf(x(s))ds, \quad (15)$$

$$\bar{\pi}\int_{\underline{\tau}_2}^{\overline{\tau}_2}\int_{0}^{u} f^T(x(t))Rf(x(t))dsdu = \frac{1}{2}\bar{\pi}\left(\overline{\tau}_2{}^2-\underline{\tau}_2{}^2\right)f^T(x(t))Rf(x(t)), \quad (16)$$

$$-\bar{\pi}\int_{\underline{\tau}_2}^{\overline{\tau}_2}\int_{0}^{u} f^T(x(t-s))Rf(x(t-s))dsdu = -\bar{\pi}\int_{\underline{\tau}_2}^{\overline{\tau}_2}\int_{t-s}^{t} f^T(x(\theta))Rf(x(\theta))d\theta ds, \quad (17)$$

$$-c_0\int_{0}^{\tau} x^T(t-s)x(t-s)ds = -c_0\int_{t-\tau}^{t} x^T(s)x(s)ds. \quad (18)$$

Also, it is not difficult to see that

$$\begin{aligned}\sum_{j=1}^{m}\pi_{ij}\int_{t-\tau_{1,j}}^{t} x^T(s)Qx(s)ds &\le \sum_{j\neq i}\pi_{ij}\int_{t-\overline{\tau}_1}^{t} x^T(s)Qx(s)ds+\pi_{ii}\int_{t-\underline{\tau}_1}^{t} x^T(s)Qx(s)ds\\ &\le \bar{\pi}\int_{t-\overline{\tau}_1}^{t-\underline{\tau}_1} x^T(s)Qx(s)ds \end{aligned}\quad (19)$$

and

$$\begin{aligned}\sum_{j=1}^{m}\pi_{ij}\int_{0}^{\tau_{2,j}}\int_{t-s}^{t} f^T(x(\theta))Rf(x(\theta))d\theta ds &\le \sum_{j\neq i}\pi_{ij}\int_{0}^{\overline{\tau}_2}\int_{t-s}^{t} f^T(x(\theta))Rf(x(\theta))d\theta ds\\ &\quad+\pi_{ii}\int_{0}^{\underline{\tau}_2}\int_{t-s}^{t} f^T(x(\theta))Rf(x(\theta))d\theta ds\\ &\le \bar{\pi}\int_{\underline{\tau}_2}^{\overline{\tau}_2}\int_{t-s}^{t} f^T(x(\theta))Rf(x(\theta))d\theta ds. \end{aligned}\quad (20)$$

Furthermore, it follows readily from (2) and (7) that

$$\begin{aligned}&\sigma^T(x(t),x(t-\tau_{1,i}),t,i)P_i\sigma(x(t),x(t-\tau_{1,i}),t,i)\\ &\le \lambda_{\max}(P_i)\sigma^T(x(t),x(t-\tau_{1,i}),t,i)\\ &\quad\times\sigma(x(t),x(t-\tau_{1,i}),t,i)\\ &\le \lambda_i^*\rho^2x^T(t)x(t). \end{aligned}\quad (21)$$

Substituting (14)–(21) into (13) results in

$$\begin{aligned}\mathcal{L}V(x_t,t,i) &\le 2x^T(t)P_i\Big(A(i)x(t)+B(i)x(t-\tau_{1,i})+C(i)\\ &\quad\times\int_{t-\tau_{2,i}}^{t} f(x(s))ds+g(x(t),x(t-\tau_{1,i}),t)\Big)\\ &\quad+\sum_{j=1}^{m}\pi_{ij}x^T(t)P_jx(t)+(1+\bar{\pi}(\overline{\tau}_1-\underline{\tau}_1))x^T(t)Qx(t)\\ &\quad-x^T(t-\tau_{1,i})Qx(t-\tau_{1,i})\\ &\quad-\int_{t-\tau_{2,i}}^{t} f^T(x(s))Rf(x(s))ds\\ &\quad+\left(\tau_{2,i}+\frac{1}{2}\bar{\pi}\left(\overline{\tau}_2^2-\underline{\tau}_2^2\right)\right)f^T(x(t))Rf(x(t))\\ &\quad+\lambda_i^*\rho^2x^T(t)x(t)+\varepsilon_0x^T(t)x(t)\\ &\quad-c_0\int_{t-\tau}^{t} x^T(s)x(s)ds. \end{aligned}\quad (22)$$

From Lemma 1, it follows that:

$$-\int_{t-\tau_{2,i}}^{t} f^T(x(s))Rf(x(s))ds \le -\frac{1}{\tau_{2,i}}\left(\int_{t-\tau_{2,i}}^{t} f(x(s))ds\right)^T R\int_{t-\tau_{2,i}}^{t} f(x(s))ds. \quad (23)$$

Moreover, we can obtain from (3) and (4) that

$$\begin{bmatrix} x(t)\\ f(x(t))\end{bmatrix}^T\begin{bmatrix}\breve{F}_1 & -\breve{F}_2\\ -\breve{F}_2^T & I\end{bmatrix}\begin{bmatrix} x(t)\\ f(x(t))\end{bmatrix}\le 0 \quad (24)$$

and

$$g^T(x(t),x(t-\tau_{1,i}),t)g(x(t),x(t-\tau_{1,i}),t)- x^T(t)G_1^TG_1x(t)-x^T(t-\tau_{1,i})G_2^TG_2x(t-\tau_{1,i})\le 0. \quad (25)$$

To this end, we can conclude from (22)–(25) that

$$\mathcal{L}V(x_t,t,i)\le \xi_i^T(t)\tilde{\Omega}_i\xi_i(t)-c_0\int_{t-\tau}^{t} x^T(s)x(s)ds \quad (26)$$

where

$$\xi_i(t):=\left[x^T(t)\quad x^T(t-\tau_{1,i})\quad f^T(x(t))\quad \int_{t-\tau_{2,i}}^{t} f^T(x(s))ds\quad g^T(x(t),x(t-\tau_{1,i}),t)\right]^T.$$

Since $\tilde{\Omega}_i<0$, one has $\xi_i^T(t)\tilde{\Omega}_i\xi_i(t)\le\lambda_{\max}(\tilde{\Omega}_i)\xi^T(t)\xi(t)\le\lambda_{\max}(\tilde{\Omega}_i)x^T(t)x(t)\le\tilde{\lambda}x^T(t)x(t)$, where $\tilde{\lambda}=\max_{i\in\mathcal{S}}\{\lambda_{\max}(\tilde{\Omega}_i)\}$, and then it follows from (26) that

$$\mathcal{L}V(x_t,t,i)\le\tilde{\lambda}x^T(t)x(t)-c_0\int_{t-\tau}^{t} x^T(s)x(s)ds. \quad (27)$$

In order to deal with the exponential stability of (1), we consider $\mathcal{L}[e^{\mu t}V(x_t,t,i)]$ with $\mu>0$ being a constant to be determined later. It is obvious that

$$\mathcal{L}\left[e^{\mu t}V(x_t,t,i)\right]=e^{\mu t}\left[\mu V(x_t,t,i)+\mathcal{L}V(x_t,t,i)\right]. \quad (28)$$

Letting $\overline{\lambda}_0 = \max_{i\in\mathcal{S}}\{\lambda_{\max}(P_i)\}$ and $\underline{\lambda}_0 = \min_{i\in\mathcal{S}}\{\lambda_{\min}(P_i)\}$, we can verify that

$$\begin{aligned}V(x_t,t,i) \le\ & \overline{\lambda}_0 x^T(t)x(t) \\ & + (c_0\tau + (1+\bar{\pi}(\overline{\tau}_1 - \underline{\tau}_1))\lambda_{\max}(Q)) \\ & \times \int_{t-\tau}^{t} x^T(s)x(s)ds + \left(\overline{\tau}_2 + \frac{\bar{\pi}}{2}\left(\overline{\tau}_2^2 - \underline{\tau}_2^2\right)\right) \\ & \times \lambda_{\max}(R)\int_{t-\tau}^{t} f^T(x(s))f(x(s))ds. \end{aligned} \quad (29)$$

From (3), it can be inferred that there exists a constant ϱ such that $f^T(x(t))f(x(t)) \le \varrho x^T(t)x(t)$ which, together with (29), implies that

$$V(x_t,t,i) \le \overline{\lambda}_0 x^T(t)x(t) + \varrho_0\int_{t-\tau}^{t} x^T(s)x(s)ds \quad (30)$$

where

$$\varrho_0 = c_0\tau + (1+\bar{\pi}(\overline{\tau}_1 - \underline{\tau}_1))\lambda_{\max}(Q) + \left(\overline{\tau}_2 + \frac{\bar{\pi}}{2}\left(\overline{\tau}_2^2 - \underline{\tau}_2^2\right)\right)\varrho\lambda_{\max}(R).$$

Since $\tilde{\lambda} < 0$ and $c_0 > 0$, we can always choose a constant μ small enough such that the following inequalities hold:

$$\begin{cases}\tilde{\lambda} + \mu\overline{\lambda}_0 < 0,\\ -c_0 + \mu\varrho_0 < 0.\end{cases} \quad (31)$$

As a result, it follows from (27), (28) and (29) that $\mathcal{L}[\mathrm{e}^{\mu t}V(x_t,t,i)] \le 0$. By the generalized Itô formula (cf. [12]), one has

$$\begin{aligned}&\mathrm{e}^{\mu t}\mathbb{E}V(x_t,t,r(t)) \\ &\le \mathbb{E}V(x_0,0,r(0)) + \mathbb{E}\int_0^t \mathrm{e}^{\mu s}\mathcal{L}V(x_s,s,r(s))ds \\ &\le \mathbb{E}V(x_0,0,r(0)).\end{aligned} \quad (32)$$

On the other hand, it follows from (30) that

$$\mathbb{E}V(x_0,0,r(0)) \le (\overline{\lambda}_0 + \varrho_0\tau)\sup_{-\tau\le s\le 0}\mathbb{E}|x(s)|^2. \quad (33)$$

It is obvious that $\mathbb{E}V(x_t,t,r(t)) \ge \underline{\lambda}_0\mathbb{E}|x(t)|^2$. Then, from (32) and (33), it follows that:

$$\mathbb{E}|x(t;\phi)|^2 \le \frac{\overline{\lambda}_0 + \varrho_0\tau}{\underline{\lambda}_0}e^{-\mu t}\sup_{-\tau\le s\le 0}\mathbb{E}|\phi(s)|^2 \quad (34)$$

which completes the proof of Theorem 1. ■

In order to facilitate the controller design problem, we provide a variation of Theorem 1 as follows.

Theorem 2: System (1) with $u(t) \equiv 0$ is exponentially stable in the mean square sense if there exist a set of positive scalar constant $\{\nu_i, i\in\mathcal{S}\}$, two positive definite matrices Z and R, and a set of positive definite matrices $\{X_i, i\in\mathcal{S}\}$ such that the following LMIs hold:

$$X_i > \nu_i I \quad (35)$$

and (36), shown at the bottom of page, where

$$\begin{aligned}\Pi_i =& A(i)X_i + X_iA^T(i) + \pi_{ii}X_i + B(i)ZB^T(i) + I, \quad (37)\\ \bar{\mathcal{X}}_i =& [\sqrt{\pi_{i1}}X_i,\ldots,\sqrt{\pi_{ii-1}}X_i,\sqrt{\pi_{ii+1}}X_i,\ldots,\sqrt{\pi_{in}}X_i], \quad (38)\\ \hat{\mathcal{X}}_i =& \mathrm{diag}\{X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_n\},\\ \kappa =& 1+\bar{\pi}(\overline{\tau}_1 - \underline{\tau}_1) \quad (39)\end{aligned}$$

and Δ_i is defined in Theorem 1.

Proof: By Schur Complement, the inequality (36) is equivalent to $\overline{\Omega}_i < 0$, where

$$\begin{aligned}\overline{\Omega}_i =& \Pi_i + B(i)ZG_2^T\left(I - G_2ZG_2^T\right)^{-1}G_2ZB^T(i) \\ & - X_i\breve{F}_2\Delta_i^{-1}\breve{F}_2^TX_i + C(i)\left(\frac{1}{\tau_{2,i}}R\right)^{-1}C^T(i) \\ & - X_i\breve{F}_1X_i + X_iG_1^TG_1X_i + \frac{\rho^2}{\nu_i}X_iX_i \\ & + \kappa X_iZ^{-1}X_i + \sum_{j\ne i}\pi_{ij}X_iX_j^{-1}X_i. \end{aligned} \quad (40)$$

Also, it follows from the matrix inversion lemma that:

$$ZG_2^T\left(I - G_2ZG_2^T\right)^{-1}G_2Z = \left(Z^{-1} - G_2^TG_2\right)^{-1} - Z. \quad (41)$$

Substituting the above inequality and (37) into (40) yields that

$$\begin{aligned}\overline{\Omega}_i =& A(i)X_i + X_iA^T(i) + \pi_{ii}X_i + I \\ & + B(i)\left(Z^{-1} - G_2^TG_2\right)^{-1}B^T(i) - X_i\breve{F}_2\Delta_i^{-1}\breve{F}_2^TX_i \\ & + C(i)\left(\frac{1}{\tau_{2,i}}R\right)^{-1}C^T(i) - X_i\breve{F}_1X_i + X_iG_1^TG_1X_i \\ & + \frac{\rho^2}{\nu_i}X_iX_i + \kappa X_iZ^{-1}X_i + \sum_{j\ne i}\pi_{ij}X_iX_j^{-1}X_i. \end{aligned} \quad (42)$$

Letting $P_i = X_i^{-1}, Q = Z^{-1}$ and $\lambda_i^* = 1/\nu_i$, we know that $\overline{\Omega}_i < 0$ is equivalent to $P_i\overline{\Omega}_iP_i < 0$. It is easy to verify that

$$\begin{aligned}P_i\overline{\Omega}_iP_i =& P_iA(i) + A^T(i)P_i + \pi_{ii}P_i + P_iP_i \\ & + P_iB(i)\left(Q - G_2^TG_2\right)^{-1}B^T(i)P_i - \breve{F}_2\Delta_i^{-1}\breve{F}_2^T\end{aligned}$$

$$\hat{\Omega}_i = \begin{bmatrix} \Pi_i & B(i)ZG_2^T & X_i\breve{F}_2 & C(i) & X_i & X_iG_1^T & \rho X_i & X_i & \bar{\mathcal{X}}_i \\ * & -I + G_2ZG_2^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \Delta_i & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\frac{1}{\tau_{2,i}}R & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \breve{F}_1^{-1} & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & -I & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & -\nu_i & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 & -\frac{1}{\kappa}Z & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 & -\hat{\mathcal{X}}_i \end{bmatrix} < 0 \quad (36)$$

$$
\begin{aligned}
&+ P_iC(i)\left(\frac{1}{\tau_{2,i}}R\right)^{-1} C^T(i)P_i - \breve{F}_1 \\
&+ G_1^TG_1 + \lambda_i^*\rho^2 I + \kappa Q + \sum_{j\neq i}\pi_{ij}P_j, \\
&= \Xi_i + P_iP_i + P_iB(i)\left(Q - G_2^TG_2\right)^{-1} B^T(i)P_i \\
&- \breve{F}_2\Delta_i^{-1}\breve{F}_2^T + P_iC(i)\left(\frac{1}{\tau_{2,i}}R\right)^{-1} C^T(i)P_i \qquad (43)
\end{aligned}
$$

which, by Schur Complement, implies that the inequality $P_i\overline{\Omega}_iP_i < 0$ is equivalent to (8), that is to say, $\hat{\Omega}_i < 0$ is equivalent to (8). On the other hand, the inequality (35) is equivalent to the inequality (7). Therefore, by Theorem 1, the system (1) is exponentially stable in the mean square sense, and the proof of this theorem is complete. ■

In Theorem 2, LMI-based conditions are provided for synthesis purpose for the open-loop system. Next, we aim to characterize the controller gain in Theorem 3, which is an easy consequence of Theorem 2, and therefore the proof of Theorem 3 is omitted.

Theorem 3: System (1) is exponentially stabilizable in the mean square sense if there exist a set of positive scalar constant $\{\nu_i, i \in \mathcal{S}\}$, two positive definite matrices Z and R, a set of matrices $\{Y_i, i \in \mathcal{S}\}$ and a set of positive definite matrices $\{X_i, i \in \mathcal{S}\}$ such that the LMIs

$$X_i > \nu_i I \qquad (44)$$

and (45), shown at the bottom of page, hold, where $\Upsilon_i = A(i)X_i + X_iA^T(i) + D(i)Y_i + Y_i^TD^T(i) + \pi_{ii}X_i + B(i)ZB^T(i) + I$, and κ, $\bar{\mathcal{X}}_i$, $\hat{\mathcal{X}}_i$ and Δ_i are defined in Theorem 2. Furthermore, if the LMIs (44), (45) are feasible, the state feedback gain matrix can be taken as $K(i) = Y_iX_i^{-1}, i \in \mathcal{S}$.

Remark 1: The features of the main results can be summarized as follows: 1) a new Lyapunov-Krasovskii functional is introduced to account for the mode-dependent distributed delay; 2) a sector-like nonlinearity is imposed on the function concerning the distributed delays; and 3) a new delay-dependent approach is developed to obtain the LMI-based stabilizability conditions.

IV. A Numerical Example

In this section, an example is presented here to demonstrate the effectiveness of our main results. Consider the system (1), where the nominal system matrix A is taken from the linearized model of an F-404 aircraft engine system in [17]

$$A(t) = \begin{bmatrix} -1.46 & 0 & 2.428 \\ 0.1643 + 0.5\delta(t) & -0.4 + \delta(t) & -0.3788 \\ 0.3107 & 0 & -2.23 \end{bmatrix}$$

with $\delta(t)$ being an uncertain model parameter. Let $A(t)$ be subject to a Markov Process $r(t)$ with $N = 2$ and the transition rate be given as $\pi_{11} = -3$, $\pi_{12} = 3$, $\pi_{21} = 4$, $\pi_{22} = -4$. The uncertainty $\delta(t)$ is assumed to be 0.4 when $r(t) = 1$ and 0.2 otherwise. Therefore, we have

$$A(1) = \begin{bmatrix} -1.46 & 0 & 2.428 \\ 0.3643 & -0.2 & -0.3788 \\ 0.3107 & 0 & -2.23 \end{bmatrix},$$
$$A(2) = \begin{bmatrix} -1.46 & 0 & 2.428 \\ 0.2643 & -0.3 & -0.3788 \\ 0.3107 & 0 & -2.23 \end{bmatrix}.$$

In stochastic system (1), we let

$$B(1) = \begin{bmatrix} -1 & 0.1 & 0.2 \\ 0.2 & -2 & 0 \\ 0 & -0.1 & -0.5 \end{bmatrix}, B(2) = \begin{bmatrix} -1 & 0.1 & 0.2 \\ 0 & -2 & 0.2 \\ 0 & -0.1 & -0.5 \end{bmatrix},$$
$$C(1) = \begin{bmatrix} 0.5 & 0.2 & -0.1 \\ 0 & 0.8 & 0.3 \\ -0.1 & 0 & 0.5 \end{bmatrix}, D(1) = \begin{bmatrix} 0.1 & 0.2 \\ 0.5 & 0.2 \\ 0 & 0.5 \end{bmatrix},$$
$$C(2) = \begin{bmatrix} 0.5 & 0.2 & 0 \\ 0 & 0.8 & 0 \\ 0.3 & 0 & 0.5 \end{bmatrix}, \quad D(2) = \begin{bmatrix} 0.3 & 0.4 \\ -0.5 & 0.9 \end{bmatrix}$$

and $\tau_{11} = 0.4$, $\tau_{21} = 0.6$, $\tau_{12} = 0.3$, $\tau_{22} = 0.5$. Moreover, for the nonlinear functions, we let

$$F_1 = \begin{bmatrix} -1.6 & 0.2 & 0.1 \\ 0.2 & -1.7 & 0 \\ 0.1 & 0 & -1.7 \end{bmatrix}, F_2 = \begin{bmatrix} 0.4 & 0.2 & 0.1 \\ 0.2 & 0.3 & 0 \\ 0.1 & 0 & 0.3 \end{bmatrix},$$
$$G_1 = G_2 = \frac{\sqrt{2}}{2}I, \quad \rho = 0.3. \qquad (46)$$

With the above parameters and by using Matlab LMI Toolbox, we solve the LMIs (44) and (45) to obtain a feasible solution for X_1, X_2, Y_1, Y_2, Z, R, ν_1, ν_2 (the values are omitted for space saving). Therefore, it follows from Theorem 3 that the stochastic system (1) with the given parameters is exponentially stabilizable in the mean square sense, and the resulting feedback gain matrices are

$$K_1 = Y_1X_1^{-1} = \begin{bmatrix} 8.4485 & -29.4250 & 3.5847 \\ -22.4767 & 10.0976 & -14.9234 \end{bmatrix},$$
$$K_2 = Y_2X_2^{-1} = \begin{bmatrix} 4.2836 & -25.1256 & 1.7919 \\ -23.3595 & 3.6641 & -12.7366 \end{bmatrix}.$$

Numerical simulation further confirms the obtained results. Figs. 1–3 show that the states x_1, x_2 and x_3 of the closed-loop system asymptotically approach zero indeed.

Remark 2: The nonlinear function $f(x)$ in (3) belongs to the so-called sector $[F_1, F_2]$. Note that

$$[f(x) - F_1x]^T[f(x) - F_2x] = \left|f(x) - \frac{F_1+F_2}{2}x\right|^2 - \frac{1}{4}x^T(F_2-F_1)^T(F_2-F_1)x \qquad (47)$$

$$\check{\Omega}_i = \begin{bmatrix}
\Upsilon_i & B(i)ZG_2^T & X_i\breve{F}_2 & C(i) & X_i & X_iG_1^T & \rho X_i & X_i & \bar{\mathcal{X}}_i \\
* & -I + G_2ZG_2^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & \Delta_i & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -\frac{1}{\tau_{2,i}}R & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & \breve{F}_1^{-1} & 0 & 0 & 0 & 0 \\
* & * & * & * & 0 & -I & 0 & 0 & 0 \\
* & * & * & * & 0 & 0 & -\nu_i & 0 & 0 \\
* & * & * & * & 0 & 0 & 0 & -\frac{1}{\kappa}Z & 0 \\
* & * & * & * & 0 & 0 & 0 & 0 & -\hat{\mathcal{X}}_i
\end{bmatrix} < 0 \qquad (45)$$

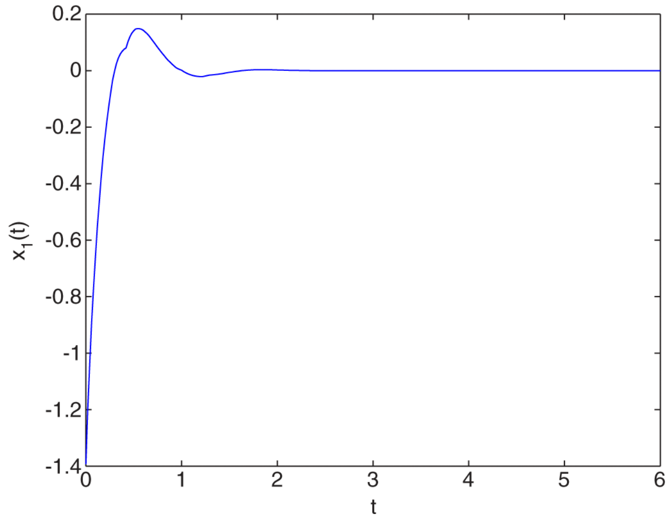

Fig. 1. Trajectory of x_1 of the closed-loop system.

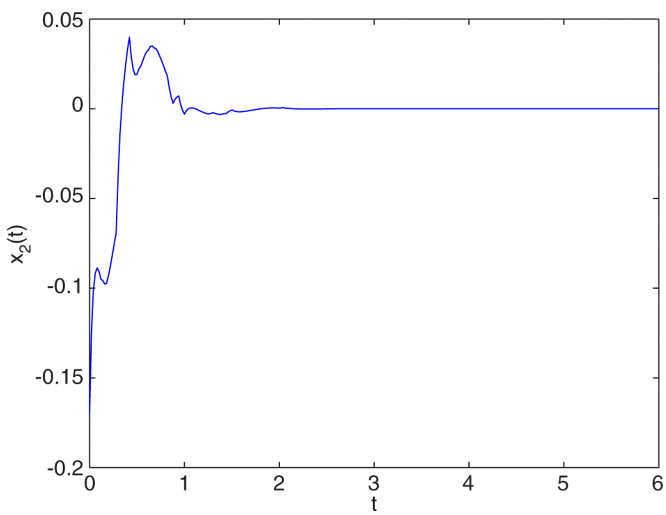

Fig. 2. Trajectory of x_2 of the closed-loop system.

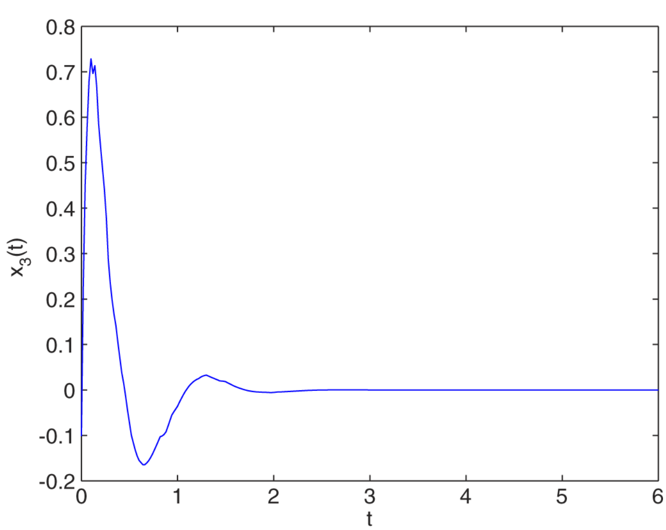

Fig. 3. Trajectory of x_3 of the closed-loop system.

Usually, $F_2 - F_1$ is a symmetric positive semi-definite matrix, i.e., $F_2 - F_1 \geq 0$. In this case, the relationship (47) reduces to

$$[f(x) - F_1 x]^T [f(x) - F_2 x] = \left| f(x) - \frac{F_1 + F_2}{2} x \right|^2 - \frac{1}{4} x^T (F_2 - F_1)^2 x. \quad (48)$$

Also

$$[f(x) - F_1 x]^T [f(x) - F_2 x] = \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix}^T \begin{bmatrix} \breve{F}_1 & -\breve{F}_2 \\ -\breve{F}_2^T & I \end{bmatrix} \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix} \quad (49)$$

where $\breve{F}_1 = (F_1^T F_2 + F_2^T F_1)/2$ and $\breve{F}_2 = (F_1^T + F_2^T)/2$.

Eq. (26), together with (48) and (49), implies that the less is $F_2 - F_1$, the less is $\mathcal{L}V(x_t, t, i)$, which is better for the system (1) to maintain its stability.

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