Robust $H_\infty$ Filtering for a Class of Nonlinear Networked Systems With Multiple Stochastic Communication Delays and Packet Dropouts

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Abstract—In this paper, the robust $H_\infty$ filtering problem is studied for a class of uncertain nonlinear networked systems with both multiple stochastic time-varying communication delays and multiple packet dropouts. A sequence of random variables, all of which are mutually independent but obey Bernoulli distribution, are introduced to account for the randomly occurred communication delays. The packet dropout phenomenon occurs in a random way and the occurrence probability for each sensor is governed by an individual random variable satisfying a certain probabilistic distribution in the interval [0 1]. The discrete-time system under consideration is also subject to parameter uncertainties, state-dependent stochastic disturbances and sector-bounded nonlinearities. We aim to design a linear full-order filter such that the estimation error converges to zero exponentially in the mean square while the disturbance rejection attenuation is constrained to a given level by means of the $H_\infty$ performance index. Intensive stochastic analysis is carried out to obtain sufficient conditions for ensuring the exponential stability as well as prescribed $H_\infty$ performance for the overall filtering error dynamics, in the presence of random delays, random dropouts, nonlinearities, and the parameter uncertainties. The filtering conditions are characterized in terms of the feasibility of a set of linear matrix inequalities (LMIs), and then the explicit expression is given for the desired filter parameters. Simulation results are employed to demonstrate the effectiveness of the proposed filter design technique in this paper.

Index Terms—Networked systems, nonlinear systems, packet dropout, robust $H_\infty$ filtering, stochastic systems, stochastic time-varying communication delays.

I. INTRODUCTION

The objective of $H_\infty$ filtering is to design an estimator for a given system such that the $L_2$ gain from the exogenous disturbance to the estimation error is less than a given level $\gamma$. $H_\infty$ filtering is closely related to many robustness problems such as stabilization and sensitivity minimization of uncertain systems, and has therefore gained persistent attention from the early 1980s; see [1], [2], [6], [8], [22], [27], [29], [33], and the references therein.

To address the robustness issue, in recent years, the robust $H_\infty$ filtering problems have been extensively investigated for a variety of complex dynamical systems, such as Markovian jumping systems [28], fuzzy systems [5], [26], time-varying systems [12], [18], [30], stochastic systems [32], and nonlinear systems [7], [21]. Furthermore, recognizing that both nonlinearity and stochasticity are commonly encountered in engineering practice, the robust $H_\infty$ filtering problems for nonlinear stochastic systems have stirred a great deal of research interests. For example, the stochastic $H_\infty$ filtering problem for time-delay systems subject to sensor nonlinearities have been dealt with in [23] and [24]. In [32], the robust $H_\infty$ filtering problem for affine nonlinear stochastic systems with state and external disturbance-dependent noise has been studied, where the filter can be designed by solving second-order nonlinear Hamilton–Jacobi inequalities. So far, in comparison with the fruitful literature available for continuous-time systems, the corresponding $H_\infty$ filtering results for discrete-time systems have been relatively few.

On another research frontier, in the past decade, networked control systems (NCSs) have attracted much attention owing to their successful applications in a wide range of areas for the advantage of decreasing the hardwiring, the installation cost and implementation difficulties. Nevertheless, the NCS-related challenging problems arise inevitably that would degrade the system performances. Such network-induced problems include, but are not limited to, the communication delays (also called network-induced time-delays) [3], [7], [10], [22], [23], [30], [35], and packet dropouts (probabilistic information missing, missing measurement) [4], [13], [17], [19], [20], [25]. In most relevant literature, the network-induced time-delays have been commonly assumed to be deterministic, which is fairly unrealistic as, by nature, delays resulting from network transmissions are typically time-varying and random. Very recently, researchers have started to model the communication delays in various probabilistic ways and, accordingly, some initial results have appeared [31], [34], among which the binary random communication delay has received much research attention due to its practicality and simplicity in describing network-induced delays.

The packet dropout (often named as missing measurement) serves as one of the most frequently occurred phenomenon with

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the networked control systems that has attracted considerable research attention during the past few years. In most results reported until now, however, it has been implicitly assumed that the measurement signal is either completely available (denoted by 1) or completely missing (denoted by 0), and all the sensors have the same missing probability [8], [22]. Unfortunately, such an assumption cannot cover some practical cases where partial/multiple missing measurements take place for an array of sensors, such as the case when the individual sensor has different missing probability and the case when only partial information is missing [19], [20], [25]. Note that the nonlinear filtering problem has been intensively investigated by researchers for several decades. However, to the best of the authors’ knowledge, the filtering problem has not yet been addressed for uncertain stochastic nonlinear systems with multiple randomly occurred communication delays and partial missing measurements from individual sensors. It is, therefore, the main purpose of this paper to shorten such a gap by investigating the robust $H_{\infty}$ filtering for discrete nonlinear networked systems with multiple stochastic communication delays and multiple missing measurements.

Motivated by the above analysis, in this paper, we aim to investigate the robust $H_{\infty}$ filtering problem for discrete uncertain nonlinear networked systems with multiple stochastic time-varying communication delays and multiple missing measurements.

The main contributions of this paper are summarized as follows: 1) a new model is proposed to describe the multiple network communication delays, each of which satisfies an individual Bernoulli distribution; 2) a combination of important factors contributing to the complexity of NCSs are investigated in a unified framework which comprises partial measurement missing, sector nonlinearities and parameter uncertainties; and 3) stochastic analysis is conducted to enforce the $H_{\infty}$ performance for the addressed “complex” systems in addition to the usual stability requirement. By means of LMIs, a sufficient condition for the robustly exponential stability of the filtering error dynamics is obtained and a prescribed $H_{\infty}$ disturbance rejection attenuation level is guaranteed, and the explicit expression of the desired filter parameters is also derived. A numerical simulation example is used to demonstrate the effectiveness of the presented filtering scheme in this paper.

The rest of this paper is organized as follows. Section II formulates the problem under consideration. The exponentially stability condition and robust $H_{\infty}$ performance of the filter error system are given in Section III. The filter design problem is solved in Section IV. An illustrative example is given in Section V and we conclude the paper in Section VI.

Notation: The notation used in the paper is fairly standard. The superscript “$T$” stands for matrix transposition, $\mathbb{R}^{n}$ denotes the $n$-dimensional Euclidean space, $\mathbb{R}^{m \times n}$ is the set of all real matrices of dimension $m \times n$; $I$ and 0 represent the identity matrix and zero matrix, respectively. The notation $P > 0$ means that $P$ is real symmetric and positive definite; the notation $\|A\|$ refers to the norm of a matrix $A$ defined by $\|A\| = \sqrt{\text{tr}(A^{T}A)}$ and $\|\cdot\|_{2}$ stands for the usual $L_{2}$ norm. In symmetric block matrices or complex matrix expressions, we use an asterisk (*) to represent a term that is induced by symmetry, and $\text{diag}\{\cdots\}$ stands for a block-diagonal matrix. In addition, $E\{x\}$ and $E\{x|y\}$ will, respectively, mean expectation of $x$ and expectation of $x$ conditional on $y$. The set of all nonnegative integers is denoted by $\mathbb{N}^{+}$ and the set of all nonnegative real numbers is represented by $\mathbb{R}^{+}$. If $A$ is a matrix, $\lambda_{\text{max}}(A)$ (respectively, $\lambda_{\text{min}}(A)$) means the largest (respectively, smallest) eigenvalue of $A$. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

II. PROBLEM FORMULATION

To start with, let us denote the following for presentation clarity:

$$\bar{x}(k) := \sum_{i=1}^{q} \alpha_{i}(k)x(k - \tau_{i}(k))$$  \hspace{1cm} (1)

where $\tau_{i}(k)$ (for $i = 1, 2, \cdots, q$) are the random communication delays to be discussed in detail.

Consider the following discrete-time uncertain nonlinear networked system with multiple stochastic communication delays:

\[
\begin{align*}
\dot{x}(k+1) &= (A + \Delta A)x(k) + A_{d}\vartheta(k) + Ff(x(k)) + g(x(k), \hat{x}(k), k)u(k) + D_{1}v(k) \\
\dot{y}(k) &= Cx(k) + D_{2}v(k) \\
z(k) &= Lx(k) \\
x(j) &= \varphi(j), j = -d_{M}, -d_{M} + 1, \cdots, 0
\end{align*}
\]

where $x(k) \in \mathbb{R}^{n}$ represents the state vector; $\vartheta(k) \in \mathbb{R}^{n}$ is defined in (1); $\vartheta(k) \in \mathbb{R}^{n}$ is the process output; $z(k) \in \mathbb{R}^{m}$ is the signal to be estimated; $v(k) \in \mathbb{R}^{m}$ is the exogenous disturbance signal belonging to $L_{2}[0, \infty)$; $\varphi(k)$ (for $j = -d_{M}, -d_{M} + 1, \cdots, 0$) are the initial conditions. $u(k)$ is a scalar Wiener process (Brownian motion) satisfying $E\{u(k)\} = 0$, $E\{u^2(k)\} = 1$, $E\{u(k)u(j)\} = 0$ ($k \neq j$), and $g: \mathbb{R}^{n} \times \mathbb{R}^{n} \times \mathbb{N} \rightarrow \mathbb{R}^{n}$ is the continuous function quantifying the noise intensity which satisfies

$$g^{T}(x(k), \hat{x}(k), k)g(x(k), \hat{x}(k), k) \leq \rho_{1}x^{T}(k)x(k) + \rho_{2}\hat{x}^{T}(k)\hat{x}(k)$$

where $\rho_{1} > 0$ and $\rho_{2} > 0$ are known constant scalars. The parameter uncertainties $\Delta A$ is a real-valued matrix of the form

$$\Delta A = HA(k)E$$  \hspace{1cm} (3)

where $H$ and $E$ are known constant matrices with appropriate dimensions, and $F(k)$ is the unknown time-varying matrix function satisfying $F^{T}(k)F(k) \leq I$.

The vector-valued nonlinear functions $f$ is assumed to satisfy the following sector-bounded conditions with $f(0) = 0$:

\[
[f(x) - f(y) - R_{1}(x-y)^{T}f(x) - f(y) - R_{2}(x-y)]^{T} \leq 0
\]

(4)

where $R_{1}, R_{2} \in \mathbb{R}^{m \times n}$ and $R_{1} - R_{2}$ is a positive definite matrix.

Remark 1: It is customary that the nonlinear function $f$ is said to belong to $[R_{1}, R_{2}]$ (see [14]). The nonlinear description in (4) is quite general that include the usual Lipschitz conditions.
as a special case. Note that both the control analysis and model reduction problems for systems with sector nonlinearities have been intensively studied; see, e.g., [9] and [15].

The random variables $\alpha_i(k) \in \mathbb{R}$ ($i = 1, 2, \cdots, q$) in (2) are mutually uncorrelated Bernoulli distributed white sequences obeying the following probability distribution law:

$$\text{Prob}\{\alpha_i(k) = 1\} = \mathbb{E}\{\alpha_i(k)\} = \bar{\alpha}_i,$$

$$\text{Prob}\{\alpha_i(k) = 0\} = 1 - \bar{\alpha}_i.$$

The following assumption is needed on the random communication time-delays considered.

**Assumption 1:** The communication delays $\tau_i(k) (i = 1, 2, \cdots, q)$ are time-varying and satisfy $d_m \leq \tau_i(k) \leq d_M$, where $d_m$ and $d_M$ are constant positive scalars representing the lower and upper bounds on the communication delays, respectively.

**Remark 2:** The way the communication delays are described in (1) is believed to be new because, different from most existing literature, such a description exhibits the following two features: 1) the communication delays are allowed to occur in any fashion, either discretely, successively, or even distributionally; and 2) each possible delay could occur independently and randomly according to an individual probability distribution which can be specified a priori through statistical test.

In this paper, the packet dropout (missing measurement) phenomenon constitutes another focus of our present research. The multiple packet dropouts are described by

$$y(k) = \Xi C x(k) + D_2 v(k)$$

$$= \sum_{j=1}^{r} \beta_j C_j x(k) + D_2 v(k)$$

where $y(k) \in \mathbb{R}^n$ is the actual measurement signal of (2), $\Xi := \text{diag}\{\beta_1, \beta_2, \ldots, \beta_r\}$ with $\beta_j$ ($j = 1, \ldots, r$) being $r$ unrelated random variables which are also unrelated with $\alpha_i(k)$ and $u(k)$. It is assumed that $\beta_j$ has the probabilistic density function $q_j(\cdot) (j = 1, \ldots, r)$ on the interval $[0, 1]$ with mathematical expectation $\mu_j$ and variance $\sigma_j^2$. $C_j$ is defined by

$$C_j := \text{diag}\{0, 0, 0, 1, 0, \ldots, 0\} C.$$

Note that $\beta_j$ could satisfy any discrete probabilistic distributions on the interval $[0, 1]$, which includes the widely used Bernoulli distribution as a special case. In the sequel, we denote $\bar{\Xi} = \mathbb{E}\{\Xi\}$.

**Remark 3:** In real systems, the measurement data may be transferred through multiple sensors. On one hand, for different sensor, the data missing probability may be different. On the other hand, due to various reasons such as sensor aging and sensor temporal failure, the data missing at one moment might be partial, and therefore the missing probability cannot be simply described by 0 or 1. In (5), the diagonal matrix $\Xi$ accounts for the probabilistic missing status of the array of sensors where the random variable $\beta_j$ corresponds to the $j$th sensor. Also, $\beta_j$ can take value on the interval $[0, 1]$ and the probability for $\beta_j$ to take different values may differ from each other. It is easy to see that the widely used Bernoulli distribution is included here as a special case.

According to the above analysis, we have the following system to be investigated:

$$\begin{align*}
\dot{x}(k+1) &= (A + \Delta A) x(k) + A_d \hat{x}(k) + F f(x(k)) + g(x(k), \hat{x}(k), k) u(k) + D_1 v(k) \\
y(k) &= \Xi C x(k) + D_2 v(k) \\
\bar{z}(k) &= \sum_{j=1}^{r} \beta_j C_j x(k) + D_2 v(k) \\
\dot{\hat{x}}(k+1) &= A_f \hat{x}(k) + B_f y(k) \\
\hat{z}(k) &= C_f \hat{x}(k)
\end{align*}$$

(6)

In this paper, we are interested in obtaining $\hat{z}(k)$, the estimate of the signal $z(k)$, from the actual measured output $y(k)$. The full-order filter to be considered is given as follows:

$$\begin{align*}
\dot{\hat{x}}(k+1) &= A_f \hat{x}(k) + B_f y(k) \\
\hat{z}(k) &= C_f \hat{x}(k)
\end{align*}$$

(7)

where $\hat{x}(k) \in \mathbb{R}^n$ represents the state estimate, $\hat{z}(k) \in \mathbb{R}^m$ is the estimated output, and $A_f, B_f, C_f$ are appropriately dimensioned filter matrices to be determined.

Augmenting the model of (6) to include the states of the filter (7), the filtering error system is given by

$$\begin{align*}
\bar{x}(k+1) &= (\bar{A} + \bar{A}_d) \bar{x}(k) + \sum_{i=1}^{q} (\bar{A}_{di} + \bar{A}_d) \bar{x}(k - \tau_i(k)) \\
&\quad + \bar{D}_u u(k) + \bar{F} f(x(k)) + \bar{D}_v v(k) \\
\bar{z}(k) &= \bar{L} \bar{x}(k)
\end{align*}$$

(8)

where

$$\bar{x}(k) = \begin{bmatrix} x^T(k) & \hat{x}^T(k) \end{bmatrix}^T, \quad \bar{z}(k) = z(k) - \hat{z}(k),$$

$$\bar{L} = [L \quad -C_f], \quad \bar{F} = [F^T \quad 0]^T,$$

$$\bar{A} = \begin{bmatrix} A + \Delta A^T & 0 \\ B_f \Xi C & A_f \end{bmatrix}, \quad \bar{A}_d = \begin{bmatrix} 0 & 0 \\ B_f (\Xi - \Xi) C & 0 \end{bmatrix},$$

$$\bar{A}_{di} = \begin{bmatrix} \bar{\alpha}_i k \quad 0 \\ 0 \quad 0 \end{bmatrix}, \quad \bar{L}_d = \begin{bmatrix} 0 & \bar{\alpha}_d k \quad 0 \\ 0 \quad 0 \quad 0 \end{bmatrix},$$

$$\bar{D}_u = \begin{bmatrix} g(x(k), \hat{x}(k), k) \quad 0 \quad 0 \end{bmatrix}, \quad \bar{D}_v = \begin{bmatrix} 0 & \bar{D}_1 \quad \bar{D}_2 \end{bmatrix}.$$
the system (6) such that, for all admissible parameter uncertainties, nonlinearity, multiple communication delays and packet dropouts, the following two requirements are satisfied simultaneously:

R1) the filter error system (8) is exponentially stable in the mean square;

R2) under zero initial condition, the filtering error \( \overline{z}(k) \) satisfies

\[
\sum_{k=0}^\infty \mathbb{E}\left\{ \| \overline{z}(k) \|^2 \right\} \leq \gamma^2 \sum_{k=0}^\infty \mathbb{E}\left\{ \| v(k) \|^2 \right\}
\]  

(9)

for all nonzero \( v(k) \), where \( \gamma > 0 \) is a prescribed scalar.

III. ROBUST \( H_\infty \) FILTERING PERFORMANCE ANALYSIS

Before proceeding further, we introduce the following lemmas which will be needed for the derivation of our main results.

**Lemma 1:** (Schur Complement) Given constant matrices \( S_1, S_2, S_3 \) where \( S_1 = S_1^T \) and \( 0 < S_2 = S_2^T \), then \( S_1 + S_2^T S_3 \) is positive definite if and only if

\[
\begin{bmatrix} S_1 & S_2 \\ S_2^T & S_3 \end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix} -S_2 & S_1 \\ S_1^T & -S_3 \end{bmatrix} < 0.
\]  

(10)

**Lemma 2:** (S-procedure) Let \( L = L^T \) and \( H \) be real matrices of appropriate dimensions with \( F \) satisfying \( F^T F \leq I \), then \( L + HFE + E^T F^T H \leq 0 \), if and only if there exists a positive scalar \( \varepsilon > 0 \) such that \( L + \varepsilon^{-1} HE + \varepsilon F^T E \leq 0 \) or equivalently,

\[
\begin{bmatrix} L & H & \varepsilon E^T \\ H^T & -\varepsilon I & 0 \\ \varepsilon E & 0 & -\varepsilon I \end{bmatrix} < 0.
\]  

(11)

Let us first consider the robust exponential stability analysis problem for the filter error system (8) with \( v(k) = 0 \).

**Theorem 1:** Let the filter parameters \( A_f, B_f, \) and \( C_f \) be given and the admissible conditions hold. Then, the filtering error system (8) with \( v(k) = 0 \) is robustly exponentially stable in the mean square if there exist matrices \( P > 0, Q_j > 0 \) \( (j = 1, 2, \ldots, q) \) and positive constants \( \lambda_1, \lambda_2 \) satisfying

\[
\Omega = \begin{bmatrix} \Omega_{11} & \ast & \ast \\ \ast & \Omega_{22} & \ast \\ \ast & \ast & \Omega_{33} \end{bmatrix} < 0,
\]  

(12)

\[
P \leq \lambda_1 I,
\]  

(13)

where

\[
\begin{aligned}
\Omega_{11} &= \lambda_1 A_f - P + \sum_{j=1}^q (d_m - d_m + 1) Q_j + A_f^T P A_f \\
&+ \sum_{j=1}^q \rho_j C_f P C_f - \lambda_2 G^T \hat{R}_1 G, \\
\Omega_{22} &= \text{diag}\left\{ -Q_1 + \hat{A}_1, -Q_2 + \hat{A}_2, \ldots, -Q_q + \hat{A}_q \right\} \\
&+ \sum_{j=1}^q \rho_j C_f P C_f - \lambda_2 G^T \hat{R}_1 G, \\
\Omega_{33} &= F^T P A_f - \lambda_2 \hat{R}_2 G, \quad \Omega_{33} = F^T P A_f - \lambda_2 \hat{R}_2 G, \quad \Omega_{33} = F^T P A_f - \lambda_2 \hat{R}_2 G, \\
\hat{A}_i &= \hat{\alpha}_i (1 - \hat{\alpha}_k) A_f P A_f + \lambda_1 \rho_2 \hat{W}_i, \quad i = 1, 2, \ldots, q, \\
\hat{R}_1 &= (\hat{R}_1^T \hat{R}_2 + \hat{R}_2^T \hat{R}_1) / 2, \\
\hat{G} &= [I \ 0], \quad \hat{Z} = [\hat{A}_d \ \hat{A}_d \ 0 \ \hat{A}_d],
\end{aligned}
\]

\[
\hat{Z}_n = [T_1 \ T_2 \ \cdots \ T_q], \quad \hat{C}_j = \begin{bmatrix} 0 & B_f C_j \\ 0 & 0 \end{bmatrix},
\]

\[
\hat{A}_d = \begin{bmatrix} A_d & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{P}_d = \begin{bmatrix} \rho \hat{I} & 0 \\ 0 & 0 \end{bmatrix},
\]

\[
W_i = \begin{bmatrix} \hat{\alpha}_i (1 - \hat{\alpha}_k) I & 0 \\ 0 & 0 \end{bmatrix}, \quad T_i = \begin{bmatrix} \hat{\alpha}_i I & 0 \\ 0 & 0 \end{bmatrix}.
\]

**Proof:** Let

\[
\Theta_j(k) = \{ x(k - \tau_j(k)) \times x(k - \tau_j(k) + 1), \ldots, x(k) \} \quad (j = 1, \ldots, q)
\]

\[
\chi(k) = \{ \Theta_1(k) \cup \Theta_2(k) \cup \cdots \cup \Theta_q(k) \} = \bigcup_{j=1}^q \Theta_j(k).
\]

Choose the following Lyapunov functional for system (8):

\[
V(\chi(k)) = \sum_{i=1}^3 V_i(k)
\]

where

\[
V_1(k) = \pi^T(k) \pi(k), \quad V_2(k) = \sum_{j=1}^q \sum_{i=k-\tau_j(k)}^k \pi^T(i) Q_j \pi(i),
\]

\[
V_3(k) = \sum_{j=1}^q \sum_{m=-d_m+1}^{d_m-1} \sum_{i=k+m}^k \pi^T(i) Q_j \pi(i)
\]

with \( P > 0, Q_j > 0 \) \( (j = 1, 2, \ldots, q) \) being matrices to be determined. Then, along the trajectory of system (8) with \( v(k) = 0 \), we have

\[
\mathbb{E}\{\Delta V|\chi(k)\} = \mathbb{E}\{\mathbb{V}(\chi(k+1)) - \mathbb{V}(\chi(k))\} = \mathbb{E}\{(\mathbb{V}(\chi(k+1)) - \mathbb{V}(\chi(k))|\chi(k)\}
\]

\[
= \sum_{i=1}^3 \mathbb{E}\{\Delta V_i|\chi(k)\}.
\]  

(14)

From (8), we can obtain that

\[
\mathbb{E}\{\Delta V_1|\chi(k)\}
\]

\[
= \mathbb{E}\{\pi^T(k+1) P \pi(k + 1) - \pi^T(k) P \pi(k) \| \chi(k)\}
\]

\[
= \mathbb{E}\{\pi^T(k) (\pi^T P A_f + A_f^T P \pi) \| \chi(k)\}
\]

\[
+ 2 \pi^T(k) \pi P \left( \sum_{i=1}^q \hat{A}_d \pi (k - \tau_i(k)) \right)
\]

\[
+ 2 \pi^T(k) \pi P f(x(k))
\]

\[
+ \sum_{i=1}^q \pi^T(k - \tau_i(k)) \hat{A}_d \pi A_f \hat{R}_1 \pi (k - \tau_i(k))
\]

\[
+ \left( \sum_{i=1}^q \hat{A}_d \pi (k - \tau_i(k)) \right)^T
\]

\[
	imes P \left( \sum_{i=1}^q \hat{A}_d \pi (k - \tau_i(k)) \right)
\]

\[
+ 2 \left( \sum_{i=1}^q \hat{A}_d \pi (k - \tau_i(k)) \right)^T P f(x(k))
\]

\[
+ \pi^T(k) \pi f(x(k))\}
\]
Letting
\[ \xi(k) = [\pi^T(k) \pi^T(k - \tau_1(k)) \cdots \pi^T(k - \tau_q(k)) f^T(x(k))]^T \]
the combination of (16) and (17) results in
\[ E\{\Delta V[\pi(k)]\} \leq \xi^T(k) \Omega_1 \xi(k) \] (18)
where
\[ \Omega_1 = \begin{bmatrix} \Omega_{11} + \lambda_2 G T \hat{R}_1 G & * & * \\ Z T PA & \Omega_{22} & * \\ F^T P D \hat{Z} & F^T P F & \end{bmatrix}. \]
Notice that (4) implies
\[ \begin{bmatrix} \pi(k) \\ f(x(k)) \end{bmatrix}^T \begin{bmatrix} G T \hat{R}_1 G & \hat{R}_2 \\ \hat{R}_2 G & I \end{bmatrix} \begin{bmatrix} \pi(k) \\ f(x(k)) \end{bmatrix} \leq 0, \] (19)
From (18) and (19), it follows that
\[ E\{\Delta V[\pi(k)]\} \leq \xi^T(k) \Omega \xi(k). \]
According to Theorem 1, we have \( \Omega < 0 \). Hence, for all \( \xi(k) \neq 0 \), \( E\{\Delta V[\pi(k)]\} \leq \xi^T(k) \Omega \xi(k) < 0 \). Furthermore, from Theorem 1 in [22], the robustly exponential stability of system (8) can be confirmed in the mean square sense. The proof is complete.

Next, we will analyze the \( H_\infty \) performance of the filtering error system (8).

**Theorem 2:** Let the filter parameters \( A_f, B_f \) and \( C_f \) be given and \( \gamma \) be a prespecified positive constant. Then the filtering error system (8) is robustly exponentially stable in the mean square for \( \gamma(k) = 0 \) and satisfies \( \|\hat{v}(k)\|_2 \leq \gamma \|v(k)\|_2 \) under the zero initial condition for any nonzero \( v(k) \in L_2(0, +\infty) \), if there exist matrices \( P > 0, Q_j > 0 \) (\( j = 1, 2, \cdots q \)) and positive constant scalars \( \lambda_1, \lambda_2 \) satisfying
\[ \Phi < 0 \]
(20)
\[ P \leq \lambda_1 I \]
(21)
where
\[ \Phi = \begin{bmatrix} \Phi_{11} & * & * \\ \Omega_{11} & \Phi_{22} & * \\ D_f^T P A & D_f^T P D \hat{Z} & \end{bmatrix} \]
\[ \Phi_{11} = \Omega_{11} + L^T L, \]
with \( \Omega_{11}, \Omega_{22}, \Omega_{31}, \Omega_{32}, C_f, A_f, A_{\nu}, W_i, T_i, \hat{Z}, \hat{Z}_i, \hat{R}_1, \hat{R}_2, \) and \( \hat{A}_j \) being defined as in Theorem 1.

**Proof:** It is clear that \( \Phi < 0 \) implies \( \Omega < 0 \). According to Theorem 1, the filtering error system (8) is robustly exponentially stable in the mean square.

Let us now deal with the \( H_\infty \) performance of the system (8). Construct the same Lyapunov-Krasovskii functional as in Theorem 1. A similar calculation as in the proof of Theorem 1 leads to
\[ E\{\Delta V[\pi(k)]\} \leq \xi^T(k) \Omega \xi(k) \] (22)
where
\[
\xi_0(k)=[\bar{\xi}^T(k)\bar{\xi}^T(k-\tau_1(k))\cdots\bar{\xi}^T(k-\tau_p(k)) f^T(x(k)) \nu^T(k)]^T
\]
\[
\Omega_2 = \begin{bmatrix}
\Omega_{11} + \lambda_2 G^T R_1 G & * & * & * \\
F^T P \bar{A} & \Omega_{22} & F^T P \bar{Z} & * \\
F^T P \bar{A} & D^T \bar{P} \bar{A} & D^T \bar{P} \bar{Z} & D^T \bar{P} F \\
D^T \bar{P} \bar{A} & D^T \bar{P} \bar{Z} & D^T \bar{P} F & D^T \bar{P} D_1
\end{bmatrix}.
\]

In order to deal with the $H_{\infty}$ performance of the filtering system (8), we introduce the following index:
\[
J(n) = \mathbb{E} \sum_{k=0}^{n} [\bar{\xi}(k) - \gamma^2 \nu^T(k) \nu(k)]^T 
\]
where $n$ is non-negative integer. Obviously, our aim is to show $J(n) < 0$ under the zero initial condition. From (19), (22), and (23), one has
\[
J(n) = \mathbb{E} \sum_{k=0}^{n} [\bar{\xi}(k) - \gamma^2 \nu(k)] + \Delta V(\chi(k)) 
\]
\[
\leq \mathbb{E} \sum_{k=0}^{n} \left[ \bar{\xi}^T(k) \Xi^T \Xi(k) - \gamma^2 \nu^T(k) \nu(k) \right] + \Delta V(\chi(k)) 
\]
\[
= \mathbb{E} \left[ \bar{\xi}_0(k) \nu_0(k) - \lambda_2 [\bar{\xi}(k) f(x(k))] + \bar{\xi}^T(k) \Omega_2 \xi_0(k) \right] 
\]
\[
= \xi_0(k) \nu_0(k).
\]

According to Theorem 2, we have $J(n) \leq 0$. Letting $n \to \infty$, we obtain
\[
\|ar{\xi}(k)\|_2 \leq \gamma \|
u(k)\|_2
\]
which completes the proof of Theorem 2.

IV. ROBUST $H_{\infty}$ FILTER DESIGN

In this section, we aim at solving the $H_{\infty}$ filter design problem for the system (6), that is, we are interested in determining the filter matrices in (7) such that the filtering error system in (8) is exponentially stable with a guaranteed $H_{\infty}$ performance. The following theorem provides sufficient conditions for the existence of such $H_{\infty}$ filters for system (8).

**Theorem 3:** Let $\gamma > 0$ be a given positive constant and the admissible conditions hold. Then, for the nonlinear system (6) with multiple communication delays and packet dropouts, there exists an admissible $H_{\infty}$ filter of the form (7) such that the filtering error system (8) is robustly exponentially stable in the mean square for $\gamma(k) = 0$ and also satisfies $\|ar{\xi}(k)\|_2 \leq \gamma \|ar{\nu}(k)\|_2$ under the zero initial condition for any nonzero $\nu(k) \in L_2[0, +\infty]$, if there exist positive definite matrices $\hat{P}, Q_j \geq 0$ ($j = 1, 2, \cdots, q$), positive constant scalars $\varepsilon, \lambda_1, \lambda_2$ and matrices $X, C_f$, where
\[
A \leq 0, \quad P \leq \lambda_1 I
\]
where

$$
\hat{\Omega}_{11} = -P + \sum_{j=1}^{q}(d_{M} - d_{m} + 1)Q_{j} = \lambda_{2}C_{2}^{T} \hat{R}_{1}G,
$$

$$
\hat{C} = [\sigma_{1}C_{1}^{T} \sigma_{2}C_{2}^{T} \cdots \sigma_{r}C_{r}^{T}]^{T}.
$$

In order to avoid partitioning the positive definite matrices $P$ and $Q_{j}$, we rewrite the parameters in Theorem 2 in the following form:

$$
\check{A} = \hat{A}_0 + H_{0}F(k)E_{0} + \hat{E}K\hat{R}_{3}, \quad \check{C} = \hat{E}K\hat{R}_{2},
$$

$$
\check{L} = \hat{L}_0 + C_{f}\hat{R}_{3}, \quad \check{D}_1 = \hat{D}_0 + \hat{E}K\hat{D}_1,
$$

$$
K = [A_f \quad B_f], \quad X = P\hat{E}K,
$$

and therefore we can get (26). Then, from Lemma 2, we can obtain (24). This completes the proof of this theorem. 

Remark 4: In Theorem 3, the robust $H_{\infty}$ filtering problem is solved for a class of discrete-time nonlinear networked systems with multiple stochastic communication delays and multiple packet dropouts by using an LMI approach. Obviously, our main results can be easily specialized to many special cases, for example, the cases when there are no nonlinearities, or no stochastic disturbances, or no parameter uncertainties, etc. These specialized results are not listed here to keep the exposition concise. It is also worth pointing out that the main results in this paper can be easily extended to the delayed jumping systems with sensor nonlinearities [24] and other more complicated systems. Note that we mainly focus on the effects brought by multiple stochastic communication delays and packet dropouts, which are two of the most important network-induced characteristics.

Remark 5: Lemma 2 is used to tackle the norm-bounded parameter uncertainties in the proof of Theorem 3. Comparing to existing literature, the system we consider is more comprehensive since the random delays, partial measurement missing, sector nonlinearities, parameter uncertainties and stochastic disturbances are simultaneously taken into account. For deterministic time-delay system, a lot of research attention has been paid on the selection of Lyapunov functionals to reduce the conservatism; see, e.g., [11]. Similarly, for the discrete-time stochastic system considered in this paper, we could further reduce the conservativeness of the main results by paying an effort towards the construction of more general Lyapunov functionals (e.g., the one used in [16]), which leaves a relatively minor research issue for further investigation.

Remark 6: Our main results are based on the LMI conditions. The LMI Control Toolbox implements state-of-the-art interior-point LMI solvers. While these solvers are significantly faster than classical convex optimization algorithms, it should be kept in mind that the complexity of LMI computations remains higher than that of solving, say, a Riccati equation. For instance, problems with a thousand design variables typically take over an hour on today’s workstations. However, research on LMI optimization is a very active area in the applied math, optimization and the operations research community, and substantial speed-ups can be expected in the future.

V. AN ILLUSTRATIVE EXAMPLE

In this section, we present an illustrative example to demonstrate the effectiveness of the developed method. The system data of (2) are given as follows:

$$
A = \begin{bmatrix}
0.2 & 0 & 0.1 \\
0.1 & -0.3 & 0.1 \\
0.1 & 0 & -0.2
\end{bmatrix}, \quad H = \begin{bmatrix}
0.1 \\
0.2 \\
0.1
\end{bmatrix},
$$

$$
F(k) = \sin(0.6k), \quad E = \begin{bmatrix}
0.1 & 0.1 \\
0.1 & 0.1 \\
0.1 & 0.1
\end{bmatrix},
$$

$$
A_d = \begin{bmatrix}
0.2 & 0 & 0.1 \\
0.1 & -0.3 & 0.1 \\
0.1 & 0 & -0.2
\end{bmatrix}, \quad F = \begin{bmatrix}
0.2 & 0 & 0.1 \\
0.1 & 0.3 & 0.1 \\
0.1 & 0 & 0.2
\end{bmatrix},
$$

$$
D_1 = \begin{bmatrix}
-0.2 & 0 & 0.1 \\
-0.1 & 0.1 & 0.1 \\
0 & 0.2 & 0.1
\end{bmatrix}, \quad C = \begin{bmatrix}
-0.6 & 0.9 & 0.6 \\
0.2 & 0.1 & 0.1
\end{bmatrix},
$$

$$
D_2 = \begin{bmatrix}
0.9 & -0.6 & 0.1 \\
0.5 & 0.8 & 0.1 \\
0.2 & 0.3 & 0.1
\end{bmatrix}, \quad L = \begin{bmatrix}
-0.1 & 0 & 0.1
\end{bmatrix}.
$$

Let $\gamma = 0.9, f(x(k)) = 0.4\sin(x(k))$ and $g(x(k),\hat{x}(k),k) = 0.5x(k) + 0.5\hat{x}(k)$. Assume that the time-varying communication delays satisfy $2 \leq \tau_i(k) \leq 6 (i = 1, 2)$ and $\alpha_1 = E\{\alpha_1(k)\} = 0.8, \quad \alpha_2 = E\{\alpha_2(k)\} = 0.6$.

Suppose that the probabilistic density functions of $\beta_1, \beta_2$ and $\beta_3$ in $[0 1]$ are described by

$$
q_1(s_1) = \begin{cases}
0s_1 = 0 \\
0.1s_1 = 0.5, \quad q_2(s_2) = \begin{cases}
0.1s_2 = 0 \\
0.1s_2 = 0.5, \quad 0.8s_2 = 1
\end{cases} \\
0s_3 = 0, \\
0.2s_3 = 0.5, \\
0.8s_3 = 1
\end{cases}
$$

from which the expectations and variances can be easily calculated as $\mu_1 = 0.95, \mu_2 = 0.85, \mu_3 = 0.9, \sigma_1 = 0.15, \sigma_2 = 0.32$ and $\sigma_3 = 0.2$. The initial condition is set to be $x_0 = [1.0 - 1.0]^T, \hat{x}_0 = [0 0 0]^T$ and the external disturbance $v_k$ is described by

$$
v_k = \begin{cases}
0.1, \quad 20 \leq k \leq 50 \\
-0.1, \quad 70 \leq k \leq 100 \\
0, \quad \text{else}
\end{cases}
$$

We would like to design a filter in the form of (7) so that the filtering error system in (8) is exponentially stable with a guaranteed $H_{\infty}$ norm bound $\gamma$. By applying Theorem 3 with help from Matlab LMI toolbox, we can obtain the desired $H_{\infty}$ filter parameters as follows (other matrices are omitted for space saving):

$$
A_f = \begin{bmatrix}
0.3170 & 0.2021 & 0.1123 \\
0.3169 & 0.3169 & 0.3169 \\
0.3170 & 0.1106 & 0.3170
\end{bmatrix},
$$

$$
B_f = \begin{bmatrix}
-0.0079 & -0.0407 & 0.0941 \\
-0.0080 & -0.0408 & 0.0948 \\
-0.0079 & -0.0406 & 0.0942
\end{bmatrix},
$$

$$
C_f = \begin{bmatrix}
0.3280 & 0.1220 & 0.4231
\end{bmatrix}.
$$
The simulation results are shown in Figs. 1–6. Fig. 1 plots the estimation error $z(k) - \hat{z}(k)$. The actual state response $x_i(k)$ and the estimate $\hat{x}_i(k)$ ($i = 1, 2, 3$) are depicted in Figs. 2–4, respectively. Fig. 5 shows the time-varying delays $\tau_i(k)$ ($i = 1, 2$). The Bernoulli sequences $\alpha_i(k)$ ($i = 1, 2$) are drawn in Fig. 6. All the simulation have confirmed our theoretical analysis for the problems of robust $H_{\infty}$ filtering for discrete nonlinear networked systems with multiple time-varying random communication delays and multiple packet dropouts.
VI. CONCLUSION

In this paper, we have studied the robust $H_{\infty}$ filtering problem for nonlinear networked systems with multiple time-varying random communication delays and multiple packet dropouts. The discrete-time system under study involves parameter uncertainties, state-dependent stochastic disturbances (multiplicative noises or Itô-type noises), multiple stochastic time-varying delays, sector-bounded nonlinearities and multiple packet dropouts. By means of LMIs, sufficient conditions for the robustly exponential stability of the filtering error dynamics have been obtained and, at the same time, the prescribed $H_{\infty}$ disturbance rejection attenuation level has been guaranteed. Then, the explicit expression of the desired filter parameters has been derived. A numerical example has been provided to show the usefulness and effectiveness of the proposed design method.

REFERENCES


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