Variance-Constrained $\mathcal{H}_\infty$ Filtering for a Class of Nonlinear Time-Varying Systems With Multiple Missing Measurements: The Finite-Horizon Case

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***Abstract*—This paper is concerned with the robust $\mathcal{H}_\infty$ finite-horizon filtering problem for a class of uncertain nonlinear discrete time-varying stochastic systems with multiple missing measurements and error variance constraints. All the system parameters are time-varying and the uncertainty enters into the state matrix. The measurement missing phenomenon occurs in a random way, and the missing probability for each sensor is governed by an individual random variable satisfying a certain probabilistic distribution in the interval [0 1]. The stochastic nonlinearities under consideration here are described by statistical means which can cover several classes of well-studied nonlinearities. Sufficient conditions are derived for a finite-horizon filter to satisfy both the estimation error variance constraints and the prescribed $\mathcal{H}_\infty$ performance requirement. These conditions are expressed in terms of the feasibility of a series of recursive linear matrix inequalities (RLMIs). Simulation results demonstrate the effectiveness of the developed filter design scheme.**

***Index Terms*—Discrete time-varying systems, error variance constraint, recursive matrix inequalities, robust $\mathcal{H}_\infty$ filtering, stochastic nonlinearities, stochastic systems.**

I. Introduction

In the past three decades, the optimal filtering or state estimation problems have been extensively studied by academic researchers and successfully applied in many branches of engineering such as signal processing and control design. Among a variety of existing approaches, Kalman filtering has proven to be one of the most popular one that has found wide applications in signal processing [25]. One vital assumption with the traditional Kalman filtering is that an exact model is required for the system whose states are to be estimated and the noise sources are stationary white-noise signals with known statistics. Obviously, such an assumption is sometimes restrictive in applications and, therefore, the robust and/or $\mathcal{H}_\infty$ filtering approaches have been recently developed to improve the robustness of Kalman filters, see [1]–[3], [5], [6], [8], [14], [17], [18], [24], [26], [28], [30] and the references therein. Generally speaking, the robust filtering approach guarantees an upper bound to the quadratic cost (i.e., estimation error variance) in spite of various parameter uncertainties, and subsequently minimizing this upper bound locally, while the $\mathcal{H}_\infty$ filtering theory aims at designing an estimator that ensures a bound on the induced L_2-norm of the operator from the noise signals to the estimation error [21]. The Riccati matrix equation or linear matrix inequality approaches have been frequently exploited in designing robust filters.

It is quite common in practise that the filtering performance requirements are described in terms of the upper bounds on the estimation error variances. A typical example is the tracking problem for highly maneuvering targets where the estimation error variance is no longer required to be the minimum, but should satisfy the specified upper bound constraint [21]. As a result, the variance-constrained filter design problem has received much research attention, see, e.g. [11] and [21]. It should be pointed out that, in almost all literature mentioned so far, the systems under consideration are assume to be time-invariant and the *infinite-horizon* (or steady-state) filtering problems have been dealt with using linear matrix inequality (LMI) approach owing to the numerical efficiency of the Matlab LMI toolbox. It is well known that time-varying systems are very often encountered in engineering applications and therefore the finite-horizon filtering problem makes more sense for online implementation. Unfortunately, for time-varying systems, there have been very few results published on filtering problems with variance constraints despite their practical importance, not to mention the simultaneous consideration of the $\mathcal{H}_\infty$ and robustness constraints. Note that, in [11], by solving two discrete Riccati difference equations, the robust $\mathcal{H}_\infty$ filtering problem with error variance constraints has been investigated for *linear* discrete time-varying systems.

In practical systems within a networked environment, the measurement signals is usually subject to probabilistic information missing (data dropouts or packet losses), which may

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be caused for a variety of reasons, such as the high manoeuvrability of the tracked target, a fault in the measurement, intermittent sensor failures, network congestion, accidental loss of some collected data, or some of the data may be jammed or coming from a very noisy environment, etc. Such a missing measurement phenomenon that typically occurs in networked control systems has attracted considerable attention during the past few years, see [4], [9], [10], [12], [13], [19], [20], [23], and the references therein. Up to now, in most reported paper concerning missing measurements, a common assumption is that the measurement signal is either completely missing or completely available, and all the sensors have the same data missing probability [7], [22]. Such an assumption, however, does have its limitations since it cannot cover some practical cases where multiple missing measurements occur, for example, the case when only *partial* information is missing and the case when the *individual* sensor has different missing probability [12], [13], [16], [23]. It is also noticed that, although the *nonlinear filtering* problem has been a research focus for several decades [1], there has been little literature handling the filtering problem for *nonlinear systems* with partial missing measurements from individual sensors. Therefore, there is a practical need to deal with the robust finite-horizon $\mathcal{H}_\infty$ filtering problem for discrete nonlinear time-varying stochastic systems with both error variance constraints and multiple missing measurements.

Motivated by the above discussions, in this paper, we aim to investigate the robust $\mathcal{H}_\infty$ filtering problem for a class of uncertain nonlinear discrete time-varying stochastic systems with error variance constraints and multiple missing measurements, where all the system parameters are time-varying and the uncertainties enter into the state matrix. The measurement missing probability for each sensor is governed by an individual random variable satisfying a certain probabilistic distribution in the interval [0 1], and the stochastic nonlinearities under consideration here are described by statistical means. Note that a similar infinite-horizon problem has been added in [23] for time-invariant systems only without considering the variance constraints. The main contribution of this paper is mainly threefold: 1) The system model addressed is new, which is quite comprehensive to cover time-varying parameters, stochastic nonlinearities, multiple missing measurements as well as parameter uncertainties, hence reflecting the reality more closely; 2) the problem addressed is new in the sense that this paper represents the first of few attempts to deal with the *variance-constrained* finite-horizon filtering problem for stochastic systems with multiple missing measurements; and 3) the algorithm developed is new which is computationally appealing in terms of the recursive linear matrix inequalities (RLMIs), hence suitable for online application.

Notation: The notation used in the paper is fairly standard. The superscript "T" stands for matrix transposition, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}^{m\times n}$ is the set of all real matrices of dimension $m\times n$, and I and 0 represent the identity matrix and zero matrix, respectively. The notation $P>0$ means that P is real symmetric and positive definite; the notation $\|A\|$ refers to the norm of a matrix A defined by $\|A\|=\sqrt{\mathrm{tr}(A^TA)}$ and $\|\cdot\|_2$ stands for the usual l_2 norm. In symmetric block matrices or complex matrix expressions, we use an asterisk $*$ to represent a term that is induced by symmetry, and $\mathrm{diag}\{\ldots\}$ stands for a block-diagonal matrix. In addition, $\mathbb{E}\{x\}$ and $\mathbb{E}\{x|y\}$ will, respectively, mean expectation of x and expectation of x conditional on y. The set of all nonnegative integers is denoted by $\mathbb{I}^+$ and the set of all nonnegative real numbers is represented by $\mathbb{R}^+$. $\mathrm{Var}\{x_i\}$ means the variance of x_i. $\mathrm{tr}(A)$ represents the trace of a matrix A. If A is a matrix, $\lambda_{\max}(A)$ (respectively, $\lambda_{\min}(A)$) means the largest (respectively, smallest) eigenvalue of A. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

II. Problem Formulation

In this paper, we consider the following discrete uncertain nonlinear time-varying stochastic system defined on $k\in[0,N]$:

$$\begin{cases} x_{k+1}=(A_k+\Delta A_k)x_k+f_k+D_{1k}w_k\\ y_{ck}=B_kx_k+g_k+D_{2k}w_k\\ z_k=C_kx_k\end{cases}\tag{1}$$

where $x_k\in\mathbb{R}^n$ represents the state vector, $y_{ck}\in\mathbb{R}^r$ is the process output, $z_k\in\mathbb{R}^m$ is the signal to be estimated, $w_k\in\mathbb{R}^p$ is a zero mean Gaussian white noise sequence with covariance $W>0$, and $A_k,B_k,C_k,D_{1k},D_{2k}$ are known, real, time-varying matrices with appropriate dimensions. The parameter uncertainty ΔA_k is a real-valued time-varying matrix of the form

$$\Delta A_k=H_kF_kE_k\tag{2}$$

where H_k and E_k are known time-varying matrices with appropriate dimensions, and F_k is an unknown time-varying matrix satisfying $F_kF_k^T\le I$.

The functions $f_k=f(x_k,k)$ and $g_k=g(x_k,k)$ are stochastic nonlinear functions which are described by their statistical characteristics as follows:

$$\mathbb{E}\left\{\begin{bmatrix}f_k\\ g_k\end{bmatrix}|x_k\right\}=0,\tag{3}$$

$$\mathbb{E}\left\{\begin{bmatrix}f_k\\ g_k\end{bmatrix}\begin{bmatrix}f_j^{\mathrm T} & g_j^{\mathrm T}\end{bmatrix}|x_k\right\}=0,\quad k\neq j\tag{4}$$

and

$$\begin{aligned}\mathbb{E}&\left\{\begin{bmatrix}f_k\\ g_k\end{bmatrix}\begin{bmatrix}f_k^{\mathrm T} & g_k^{\mathrm T}\end{bmatrix}|x_k\right\}\\ &=\sum_{i=1}^{q}\pi_i\pi_i^{\mathrm T}\mathbb{E}\left\{x_k^{\mathrm T}\Gamma_ix_k\right\}\\ &:=\sum_{i=1}^{q}\begin{bmatrix}\pi_{1i}\\ \pi_{2i}\end{bmatrix}\begin{bmatrix}\pi_{1i}\\ \pi_{2i}\end{bmatrix}^T\mathbb{E}\left\{x_k^{\mathrm T}\Gamma_ix_k\right\}\\ &:=\sum_{i=1}^{q}\begin{bmatrix}\theta_{11}^i & \theta_{12}^i\\ \left(\theta_{12}^i\right)^{\mathrm T} & \theta_{22}^i\end{bmatrix}\mathbb{E}\left\{x_k^{\mathrm T}\Gamma_ix_k\right\}\end{aligned}\tag{5}$$

where π_{1i}, π_{2i}, θ_{jl}^i, and Γ_i $(j,l=1,2;\ i=1,2,\cdots,q)$ are known matrices.

Remark 1: As pointed out in [15], [27], and [29], the nonlinearity description in (3)–(5) encompasses many well-studied nonlinearities in stochastic systems such as: 1) linear system with state- and control-dependent multiplicative noise; 2) nonlinear systems with random vectors dependent on the norms of states and control input; and 3) nonlinear systems with a random sequence dependent on the sign of a nonlinear function of states and control inputs.

In this paper, the measurement with sensor data missing is paid special attention, where the multiple missing measurements are described by

$$\begin{aligned} y_k &= \Xi B_k x_k + g_k + D_{2k} w_k \\ &= \sum_{i=1}^{r} \alpha_i B_{ki} x_k + g_k + D_{2k} w_k \end{aligned} \tag{6}$$

where $y_k \in \mathbb{R}^r$ is the actual measurement signal of the plant (1), $\Xi := \mathrm{diag}\{\alpha_1, \ldots, \alpha_r\}$ with α_i $(i = 1, \ldots, r)$ being r unrelated random variables which are also unrelated with w_k. It is assumed that α_i has the probabilistic density function $q_i(s)$ $(i = 1, \ldots, r)$ on the interval [0,1] with mathematical expectation μ_i and variance σ_i^2. B_{ki} is defined by

$$B_{ki} := \mathrm{diag}\{\underbrace{0, \cdots, 0}_{i-1}, 1, \underbrace{0, \cdots, 0}_{r-i}\} B_k.$$

Note that α_i could satisfy any discrete probabilistic distributions on the interval [0,1], which includes the widely used Bernoulli distribution as a special case. In the sequel, we denote $\bar{\Xi} = \mathbb{E}\{\Xi\}$.

Remark 2: In real-time systems, the measurement data may be transferred through multiple sensors. For different sensor, if there exists the data loss (also called packet dropout or measurement missing) phenomenon, the data missing probability may be different [9], [10], [12], [13], [23]. In this sense, it would be more reasonable to assume that the data missing law for each individual sensor satisfies individual probabilistic distribution. In (6), α_i can take value on the interval [0 1] and the associated probability may differ from each other. It is easy to see that the widely adopted Bernoulli distribution is now included as a special case.

In this paper, we consider the following time-varying filter for system (1)

$$\begin{cases} \hat{x}_{k+1} = A_{fk}\hat{x}_k + B_{fk} y_k \\ \hat{z}_k = C_{fk}\hat{x}_k \end{cases} \tag{7}$$

where $\hat{x}_k \in \mathbb{R}^n$ represents the state estimate, $\hat{z}_k \in \mathbb{R}^m$ is the estimated output, and A_{fk}, B_{fk}, C_{fk} are appropriately dimensioned filter parameters to be determined.

Setting $\bar{x}_k = [x_k^T \ \hat{x}_k^T]^T$, we subsequently obtain an augmented system as follows:

$$\begin{cases} \bar{x}_{k+1} = (\bar{A}_k + \check{A}_k)\bar{x}_k + \bar{G}_k h_k + \bar{D}_k w_k \\ \bar{z}_k = \bar{C}_k \bar{x}_k \end{cases} \tag{8}$$

where

$$\begin{aligned} &h_k = \begin{bmatrix} f_k^T & g_k^T \end{bmatrix}^T, \quad \bar{z}_k = z_k - \hat{z}_k, \quad \bar{C}_k = [C_k \quad -C_{fk}] \\ &\bar{A}_k = \begin{bmatrix} A_k + \Delta A_k & 0 \\ B_{fk}\bar{\Xi}B_k & A_{fk} \end{bmatrix}, \quad \bar{G}_k = \begin{bmatrix} I & 0 \\ 0 & B_{fk} \end{bmatrix} \\ &\check{A}_k = \begin{bmatrix} 0 & 0 \\ B_{fk}(\Xi - \bar{\Xi})B_k & 0 \end{bmatrix}, \quad \bar{D}_k = \begin{bmatrix} D_{1k} \\ B_{fk}D_{2k} \end{bmatrix}. \end{aligned} \tag{9}$$

The state covariance matrix of the augmented system (8) can be defined as

$$\bar{X}_k := \mathbb{E}\left\{\bar{x}_k \bar{x}_k^T\right\} = \mathbb{E}\left\{ \begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix} \begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix}^T \right\} \tag{10}$$

Our aim in this paper is to design a finite-horizon filter in the form of (7) such that the following two requirements are satisfied simultaneously:

$(R1)$ For given scalar $\gamma > 0$, matrix $S > 0$ and the initial state $\bar{x}_0$, the $\mathcal{H}_\infty$ performance index

$$\begin{aligned} J := \mathbb{E}\left\{ \|\bar{z}_k\|^2_{[0,N-1]} - \gamma^2 \|\omega_k\|^2_{[0,N-1]} \right\} \\ -\gamma^2 (x_0 - \hat{x}_0)^T S (x_0 - \hat{x}_0) < 0 \end{aligned} \tag{11}$$

is achieved for all admissible parameter uncertainties and all stochastic nonlinearities.

$(R2)$ For a sequence of specified definite matrices $\{\Psi_k\}_{0<k\leqslant N}$, at each sampling instant k, the estimation error covariance satisfies

$$\mathbb{E}\left\{(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T\right\} \leqslant \Psi_k, \qquad \forall k. \tag{12}$$

Remark 3: In the desired performance requirement (R2), the estimation error variance at each sampling time point is required to be not more than an individual upper bound. Note that the specified error variance constraint may not be minimal but should meet engineering requirements, which gives rise to a practically acceptable "window" with the hope to keep the estimated states within such a "window". On the other hand, since the variance constraint is relaxed from the minimum to the acceptable one, there would exist much freedom that can be used to attempt to directly achieve other desired performance requirements, such as the robustness and $\mathcal{H}_\infty$ disturbance rejection attenuation level as discussed in this paper.

The finite-horizon filter problem in the presence of missing measurements addressed above is referred to as the robust finite-horizon $\mathcal{H}_\infty$ filter problem for uncertain nonlinear discrete time-varying stochastic systems with variance constraint and multiple missing measurements.

III. Analysis of $\mathcal{H}_\infty$ and Covariance Performances

A. $\mathcal{H}_\infty$ Performance

We start with analyzing the $\mathcal{H}_\infty$ performance, i.e., presenting sufficient conditions under which the $\mathcal{H}_\infty$ performance index is achieved for a given filter.

Theorem 1: Consider the system (1) and suppose that the filter parameters A_{fk}, B_{fk} and C_{fk} in (7) are given. For a positive scalar $\gamma > 0$ and a positive definite matrix $S > 0$, the $\mathcal{H}_\infty$ performance requirement defined in (11) is achieved for all nonzero ω_k if, with the initial condition

$$Q_0 \leqslant \gamma^2 [I \ -I]^T S [I \ -I],$$

there exists a sequence of positive definite matrices $\{Q_k\}_{1\leqslant k\leqslant N+1}$ satisfying the following recursive matrix inequalities:

$$\Lambda := \begin{bmatrix} \Lambda_1 & * \\ 0 & \bar{D}_k^T Q_{k+1}\bar{D}_k - \gamma^2 I \end{bmatrix} < 0 \tag{13}$$

where

$$\begin{aligned} \Lambda_1 &= \bar{A}_k^T Q_{k+1}\bar{A}_k + \sum_{i=1}^{q}\hat{\Gamma}_i \cdot \mathrm{tr}\left[\bar{G}_k^T Q_{k+1}\bar{G}_k\hat{\theta}_i\right] \\ &\quad + \bar{C}_k^T\bar{C}_k - Q_k + \sum_{i=1}^{r}\sigma_i^2\bar{B}_{ki}^T Q_{k+1}\bar{B}_{ki} \\ \hat{\theta}_i &= \begin{bmatrix} \theta_{11}^i & \theta_{12}^i \\ (\theta_{12}^i)^T & \theta_{22}^i \end{bmatrix}, \quad \hat{\Gamma}_i = \begin{bmatrix} \Gamma_i & 0 \\ 0 & 0 \end{bmatrix} \\ \bar{B}_{ki} &= \begin{bmatrix} 0 & 0 \\ B_{fk}B_{ki} & 0 \end{bmatrix}. \end{aligned}$$

Proof: Define

$$J_k := \bar{x}_{k+1}^{\mathrm{T}} Q_{k+1}\bar{x}_{k+1} - \bar{x}_k^{\mathrm{T}} Q_k\bar{x}_k. \tag{14}$$

Noticing (3) and the filter structure in (7), we have

$$\mathbb{E}\left\{ \begin{bmatrix} f_k \\ g_k \end{bmatrix} \middle| \bar{x}_k \right\} = \mathbb{E}\{h_k|\bar{x}_k\} = 0 \tag{15}$$

Substituting (8) into consideration, we have

$$\begin{aligned} \mathbb{E}\{J_k\} = \mathbb{E}\{&\bar{x}_k^T\bar{A}_k^T Q_{k+1}\bar{A}_k\bar{x}_k + \bar{x}_k^T\check{A}_k^T Q_{k+1}\check{A}_k\bar{x}_k \\ &+ w_k^T\bar{D}_k^T Q_{k+1}\bar{D}_k w_k + h_k^T\bar{G}_k^T Q_{k+1}\bar{G}_k h_k^T \\ &-\bar{x}_k^{\mathrm{T}} Q_k\bar{x}_k\}. \end{aligned} \tag{16}$$

Taking (5) into consideration, we have

$$\begin{aligned} &\mathbb{E}\left\{h_k^T\bar{G}_k^T Q_{k+1}\bar{G}_k h_k\right\} \\ &\quad = \mathbb{E}\left\{\mathrm{tr}\left[\bar{G}_k^T Q_{k+1}\bar{G}_k h_k h_k^T\right]\right\} \\ &\quad = \mathbb{E}\left\{\bar{x}_k^T\sum_{i=1}^{q}\hat{\Gamma}_i\cdot\mathrm{tr}\left[\bar{G}_k^T Q_{k+1}\bar{G}_k\hat{\theta}_i\right]\bar{x}_k\right\}. \end{aligned} \tag{17}$$

Adding the zero term $\bar{z}_k^T\bar{z}_k - \gamma^2\omega_k^{\mathrm{T}}\omega_k - \bar{z}_k^T\bar{z}_k + \gamma^2\omega_k^{\mathrm{T}}\omega_k$ to $\mathbb{E}\{J_k\}$ results in

$$\mathbb{E}\{J_k\} = \mathbb{E}\left\{\begin{bmatrix}\bar{x}_k^{\mathrm{T}} & \omega_k^{\mathrm{T}}\end{bmatrix}\Lambda\begin{bmatrix}\bar{x}_k \\ \omega_k\end{bmatrix} - \bar{z}_k^T\bar{z}_k + \gamma^2\omega_k^{\mathrm{T}}\omega_k\right\}. \tag{18}$$

Summing up (18) on both sides from 0 to $N-1$ with respect to k, we obtain

$$\begin{aligned} \sum_{k=0}^{N-1}\mathbb{E}\{J_k\} &= \mathbb{E}\left\{\bar{x}_N^T Q_N\bar{x}_N\right\} - \bar{x}_0^T Q_0\bar{x}_0 \\ &= \mathbb{E}\left\{\sum_{k=0}^{N-1}\begin{bmatrix}\bar{x}_k^{\mathrm{T}} & \omega_k^{\mathrm{T}}\end{bmatrix}\Lambda\begin{bmatrix}\bar{x}_k \\ \omega_k\end{bmatrix}\right\} \\ &\quad - \mathbb{E}\left\{\sum_{k=0}^{N-1}\left(\bar{z}_k^T\bar{z}_k - \gamma^2\omega_k^{\mathrm{T}}\omega_k\right)\right\}. \end{aligned} \tag{19}$$

Hence, the $\mathcal{H}_\infty$ performance index defined in (11) is given by

$$\begin{aligned} J = \mathbb{E}&\left\{\sum_{k=0}^{N-1}\begin{bmatrix}\bar{x}_k^{\mathrm{T}} & \omega_k^{\mathrm{T}}\end{bmatrix}\Lambda\begin{bmatrix}\bar{x}_k \\ \omega_k\end{bmatrix}\right\} - \mathbb{E}\left\{\bar{x}_N^T Q_N\bar{x}_N\right\} \\ &+\bar{x}_0^T\left(Q_0 - \gamma^2[I \quad -I]^T S[I \quad -I]\right)\bar{x}_0. \end{aligned} \tag{20}$$

Noting that $\Lambda < 0$, $Q_N > 0$ and the initial condition $Q_0 \leqslant \gamma^2[I \ -I]^T S[I \ -I]$, we know $J < 0$ and the proof is now complete. ∎

B. Variance Analysis

Let us now deal with the error variance analysis issue for the addressed nonlinear stochastic time-varying systems.

Theorem 2: Consider the system (1) and let the filter parameters A_{fk}, B_{fk} and C_{fk} in (7) be given. We have $P_k \geqslant \bar{X}_k$ $(\forall k \in \{1, 2, \cdots, N+1\})$ if, with initial condition $P_0 = \bar{X}_0$, there exists a sequence of positive definite matrices $\{P_k\}_{1\leq k\leq N+1}$ satisfying the following matrix inequality:

$$P_{k+1} \geqslant \Phi(P_k) \tag{21}$$

where

$$\begin{aligned} \Phi(P_k) = \bar{A}_k P_k\bar{A}_k^T &+ \sum_{i=1}^{r}\sigma_i^2\bar{B}_{ki}P_k\bar{B}_{ki}^T \\ &+ \sum_{i=1}^{q}\bar{G}_k\hat{\theta}_i\bar{G}_k^T\cdot\mathrm{tr}[\hat{\Gamma}_i P_k] + \bar{D}_k W_k\bar{D}_k^T. \end{aligned} \tag{22}$$

Proof: As we know from (12), the Lyapunov-type equation that governs the evolution of state covariance $\bar{X}_k$ is given by

$$\begin{aligned} \bar{X}_{k+1} &= \mathbb{E}\left\{\bar{x}_{k+1}\bar{x}_{k+1}^T\right\} \\ &= \mathbb{E}\left\{(\bar{A}_k\bar{x}_k + \check{A}_k\bar{x}_k + \bar{G}_k h_k + \bar{D}_k w_k)\right. \\ &\qquad \left.\times(\bar{A}_k\bar{x}_k + \check{A}_k\bar{x}_k + \bar{G}_k h_k + \bar{D}_k w_k)^T\right\}. \end{aligned} \tag{23}$$

Since

$$\begin{aligned} \mathbb{E}\left\{\bar{G}_k h_k h_k^T\bar{G}_k^T\right\} &= \bar{G}_k\sum_{i=1}^{q}\hat{\theta}_i\mathbb{E}\left(x_k^T\Gamma_i x_k\right)\bar{G}_k^T \\ &= \sum_{i=1}^{q}\bar{G}_k\hat{\theta}_i\bar{G}_k^T\cdot\mathrm{tr}[\hat{\Gamma}_i\bar{X}_k] \end{aligned}$$

we obtain

$$\begin{aligned} \bar{X}_{k+1} &= \bar{A}_k\bar{X}_k\bar{A}_k^T + \sum_{i=1}^{r}\sigma_i^2\bar{B}_{ki}\bar{X}_k\bar{B}_{ki}^T + \bar{D}_k W_k\bar{D}_k^T \\ &\quad + \sum_{i=1}^{q}\bar{G}_k\hat{\theta}_i\bar{G}_k^T\cdot\mathrm{tr}[\hat{\Gamma}_i\bar{X}_k] \\ &= \Phi(\bar{X}_k). \end{aligned}$$

We now complete the proof by induction. Obviously, $P_0 \geqslant \bar{X}_0$. Letting $P_k \geqslant \bar{X}_k$, we arrive at

$$P_{k+1} \geqslant \Phi(P_k) \geqslant \Phi(\bar{X}_k) = \bar{X}_{k+1} \tag{24}$$

and therefore the proof is finished. ∎

Furthermore, in light of Theorem 2, we have the following corollary.

Corollary 1: The inequality holds

$$\begin{aligned}\mathbb{E}\left\{(x_k-\hat{x}_k)(x_k-\hat{x}_k)^T\right\} &= [I \quad -I]\bar{X}_k[I \quad -I]^T\\ &\leq [I \quad -I]P_k[I \quad -I]^T, \forall k.\end{aligned}$$

To conclude the above analysis, we present a theorem which intends to take both the $\mathcal{H}_\infty$ performance index and the covariance constraint into consideration in a unified framework via the RLMI method.

Theorem 3: Consider the system (1) and let the filter parameters A_{fk}, B_{fk} and C_{fk} in (7) be given. For a positive scalar $\gamma > 0$ and a positive definite matrix $S > 0$, if there exist families of positive definite matrices $\{Q_k\}_{1\leqslant k\leqslant N+1}$, $\{P_k\}_{1\leqslant k\leqslant N+1}$ and $\{\eta_{ik}\}_{0\leqslant k\leqslant N}$ $(i = 1, 2, \cdots, q)$ satisfying the following recursive matrix inequalities:

$$\begin{bmatrix} -\eta_{ik} & * \\ \bar{G}_k\pi_i & -Q_{k+1}^{-1}\end{bmatrix} < 0 \quad (25)$$

$$\begin{bmatrix} \hat{\Lambda}_1 & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * \\ \bar{A}_k & 0 & -Q_{k+1}^{-1} & * & * \\ \bar{B}_k & 0 & 0 & -\bar{Q}_{k+1}^{-1} & * \\ 0 & \bar{D}_k & 0 & 0 & -Q_{k+1}^{-1}\end{bmatrix} < 0 \quad (26)$$

$$\begin{bmatrix} -P_{k+1} & * & * & * & * \\ P_k\bar{A}_k^T & -P_k & * & * & * \\ \Theta_{31} & 0 & \Theta_{33} & * & * \\ \bar{D}_k^T & 0 & 0 & -W_k^{-1} & * \\ \hat{B}_{kp}^T & 0 & 0 & 0 & -\bar{P}_k\end{bmatrix} \leq 0 \quad (27)$$

with the initial condition

$$\begin{cases} Q_0 \leq \gamma^2[I \quad -I]^T S[I \quad -I] \\ P_0 = \bar{X}_0 \end{cases} \quad (28)$$

where

$$\begin{aligned}\hat{\Lambda}_1 &= -Q_k + \sum_{i=1}^{q}\hat{\Gamma}_i\eta_{ik} + \bar{C}_k^T\bar{C}_k\\ \bar{Q}_{k+1} &= \mathrm{diag}\{\underbrace{Q_{k+1,}, \cdots, Q_{k+1}}_{r}\}\\ \bar{B}_k &= \left[\sigma_1\bar{B}_{k1}^T, \cdots, \sigma_r\bar{B}_{kr}^T\right]^T\\ \hat{B}_{kp} &= [\sigma_1\bar{B}_{k1}P_k, \cdots, \sigma_r\bar{B}_{kr}P_k]\\ \Theta_{31} &= [\bar{G}_k\pi_1, \bar{G}_k\pi_2, \cdots, \bar{G}_k\pi_q]^T\\ \Theta_{33} &= \mathrm{diag}\{-\rho_1 I, -\rho_2 I, \cdots, -\rho_q I\}\\ \rho_i &= \left(\mathrm{tr}[\hat{\Gamma}_i P_k]\right)^{-1}, \quad i = 1, 2\cdots, q\\ \bar{P}_k &= \mathrm{diag}\{\underbrace{P_{k,}, \cdots, P_k}_{r}\}\end{aligned}$$

then, for the filtering error system (8), we have $J < 0$ and $\mathbb{E}\{(x_k-\hat{x}_k)(x_k-\hat{x}_k)^T\} \leqslant [I \ -I]P_k[I \ -I]^T$, $\forall k \in \{0, 1, \cdots, N+1\}$.

Proof: Based on the previous analysis on the $\mathcal{H}_\infty$ performance and state estimation covariance, we just need to show that, under initial conditions (28), (25) and (26) imply (13), and (27) is equivalent to (21).

From Schur Complement, (25) is equivalent to

$$\pi_i^{\mathrm{T}}\bar{G}_k^T Q_{k+1}\bar{G}_k\pi_i < \eta_{ik}, \quad (i = 1, 2, \cdots, q) \quad (29)$$

which, by the property of matrix trace, can be rewritten as

$$\mathrm{tr}\left[\bar{G}_k^T Q_{k+1}\bar{G}_k\hat{\theta}_i\right] < \eta_{ik}$$

and (26) is equivalent to

$$\hat{\Lambda} = \begin{bmatrix}\hat{\Lambda}_{11} & * \\ 0 & \bar{D}_k^T Q_{k+1}\bar{D}_k - \gamma^2 I\end{bmatrix} < 0 \quad (30)$$

where

$$\hat{\Lambda}_{11} = \bar{A}_k^T Q_{k+1}\bar{A}_k - Q_k + \bar{C}_k^{\mathrm{T}}\bar{C}_k + \sum_{i=1}^{q}\hat{\Gamma}_i\eta_{ik} + \sum_{i=1}^{r}\sigma_i^2\bar{B}_{ki}^T Q_{k+1}\bar{B}_{ki}.$$

Hence, it is easy to see that (13) can be obtained by (25) and (26) under the same initial condition.

In the same way, we can easily obtain that (27) is equivalent to (21). Thus, according to Theorem 1, Theorem 2 and Corollary 1, the $\mathcal{H}_\infty$ index defined in (20) satisfies $J < 0$ and, at the same time, the system error covariance achieves $\mathbb{E}\{(x_k-\hat{x}_k)(x_k-\hat{x}_k)^T\} \leqslant [I \ -I]P_k[I \ -I]^T$, $\forall k \in \{0, 1, \cdots, N+1\}$. The proof is complete. ■

Up to now, the analysis problem has been dealt with for the $\mathcal{H}_\infty$ filtering problem for a class of uncertain nonlinear discrete time-varying stochastic systems with error variance constraints and multiple missing measurements. In the next section, we proceed to solve the filter design problem using the developed RLMI approach.

IV. Robust Finite Horizon Filter Design

In this section, an algorithm is proposed to cope with the addressed filter design problem for the uncertain discrete time-varying nonlinear stochastic system (1). It is shown that the filter matrices can be obtained by solving a certain set of RLMIs. In other words, at each sampling instant k $(k > 0)$, a set of LMIs will be solved to obtain the desired filter matrices and, at the same time, certain key parameters are obtained which are needed in solving the LMIs for the $(k+1)$th instant.

Theorem 4: For a given disturbance attenuation level $\gamma > 0$, a positive definite matrix $S > 0$ and a sequence of prespecified variance upper bounds $\{\Psi_k\}_{0\leqslant k\leqslant N+1}$, if there exist families of positive definite matrices $\{\hat{M}_k\}_{1\leqslant k\leqslant N+1}$, $\{\hat{N}_k\}_{1\leqslant k\leqslant N+1}$, $\{P_{1k}\}_{1\leqslant k\leqslant N+1}$, $\{P_{2k}\}_{1\leqslant k\leqslant N+1}$, positive scalars $\{\varepsilon_{1k}\}_{0\leqslant k\leqslant N}$, $\{\varepsilon_{2k}\}_{0\leqslant k\leqslant N}$, $\{\eta_{ik}\}_{0\leqslant k\leqslant N}$ $(i = 1, 2, \cdots, q)$ and families of real-valued matrices $\{P_{3k}\}_{1\leqslant k\leqslant N+1}$,

$\{A_{fk}\}_{0\leqslant k\leqslant N}$, $\{B_{fk}\}_{0\leqslant k\leqslant N}$ and $\{C_{fk}\}_{0\leqslant k\leqslant N}$, under initial conditions

$$\begin{cases}\begin{bmatrix} M_0-\gamma^2 S & \gamma^2 S\\ \gamma^2 S & N_0-\gamma^2 S\end{bmatrix}\leqslant 0\\ \mathbb{E}\left\{(x_0-\hat{x}_0)(x_0-\hat{x}_0)^T\right\}=P_{10}+P_{20}-P_{30}-P_{30}^T\leq \Psi_0\end{cases} \tag{31}$$

such that the following recursive LMIs:

$$\begin{bmatrix} -\eta_{ik} & * & * \\ \pi_{1i} & -\hat{M}_{k+1} & * \\ B_{fk}\pi_{2i} & 0 & -\hat{N}_{k+1}\end{bmatrix}<0 \tag{32}$$

$$\begin{bmatrix}\Upsilon_{11} & * \\ \Upsilon_{21} & \Upsilon_{22}\end{bmatrix}<0 \tag{33}$$

$$\begin{bmatrix}\hat{\Upsilon}_{11} & * \\ \hat{\Upsilon}_{21} & \hat{\Upsilon}_{22}\end{bmatrix}<0 \tag{34}$$

$$P_{1k+1}+P_{2k+1}-P_{3k+1}-P_{3k+1}^T-\Psi_{k+1}\leqslant 0 \tag{35}$$

are satisfied with the parameters updated by

$$M_{k+1}=\hat{M}_{k+1}^{-1},\quad N_{k+1}=\hat{N}_{k+1}^{-1} \tag{36}$$

where

$$\begin{aligned}
\Upsilon_{11}&=\begin{bmatrix}\bar{\Pi}_{11k} & * \\ \bar{\Pi}_{21k} & \bar{\Pi}_{22k}\end{bmatrix}\\
\bar{\Pi}_{11k}&=\operatorname{diag}\{\Phi_1,-N_k,-\gamma^2 I\}\\
\bar{\Pi}_{21k}&=\begin{bmatrix} A_k & 0 & 0\\ B_{fk}\bar{\Xi}B_k & A_{fk} & 0\end{bmatrix}\\
\bar{\Pi}_{22k}&=\operatorname{diag}\{-\hat{M}_{k+1},-\hat{N}_{k+1}\}\\
\Upsilon_{21}&=\begin{bmatrix}\bar{L} & 0 & 0 & 0 & 0\\ C_k & -C_{fk} & 0 & 0 & 0\\ 0 & 0 & D_{1k} & 0 & 0\\ 0 & 0 & B_{fk}D_{2k} & 0 & 0\\ 0 & 0 & 0 & H_k^T & 0\end{bmatrix}\\
\Upsilon_{22}&=\operatorname{diag}\{-\bar{M}_{k+1},-I,-\hat{M}_{k+1},-\hat{N}_{k+1},-\varepsilon_{1k}I\}\\
\hat{\Upsilon}_{11}&=\begin{bmatrix}\Phi_2 & * & * & * \\ -P_{3k+1} & -P_{2k+1} & * & * \\ P_{1k}A_k^T & \tilde{\Omega}_{1k} & -P_{1k} & * \\ P_{3k}A_k^T & \tilde{\Omega}_{2k} & -P_{3k} & -P_{2k}\end{bmatrix}\\
\hat{\Upsilon}_{21}&=\begin{bmatrix}\hat{L}_{1k} & \hat{L}_{2k} & 0 & 0\\ D_{1k}^T & D_{2k}^T B_{fk}^T & 0 & 0\\ 0 & \check{L} & 0 & 0\\ 0 & 0 & E_k P_{1k} & E_k P_{3k}^T\end{bmatrix}\\
\tilde{\Omega}_{1k}&=P_{1k}B_k^T\bar{\Xi}B_{fk}^T+P_{3k}^T A_{fk}^T\\
\tilde{\Omega}_{2k}&=P_{3k}B_k^T\bar{\Xi}B_{fk}^T+P_{2k}A_{fk}^T\\
\hat{\Upsilon}_{22}&=\operatorname{diag}\left\{\hat{\Theta}_{33},-W_k^{-1},-\hat{P}_k,-\varepsilon_{2k}I\right\}\\
\Phi_1&=-M_k+\sum_{i=1}^{q}\Gamma_i\eta_{ik}+\varepsilon_{1k}E_k^T E_k\\
\Phi_2&=-P_{1k+1}+\varepsilon_{2k}H_k H_k^T\\
\bar{L}&=\left[\sigma_1\hat{\Pi}_{k1}^T,\cdots,\sigma_r\hat{\Pi}_{kr}^T\right]^T\\
\bar{M}_{k+1}&=\operatorname{diag}\underbrace{\{-\bar{\Pi}_{22k},\cdots,-\bar{\Pi}_{22k}\}}_{r}
\end{aligned}$$

$$\begin{aligned}
\hat{\Pi}_{ki}&=\begin{bmatrix}0 & 0\\ B_{fk}B_{ki} & 0\end{bmatrix},\quad i=1,2\cdots,r\\
\hat{L}_{1k}&=[\pi_{11},\pi_{12},\cdots,\pi_{1q}]^T\\
\hat{L}_{2k}&=[B_{fk}\pi_{21},B_{fk}\pi_{22},\cdots,B_{fk}\pi_{2q}]^T\\
\hat{\Theta}_{33}&=\operatorname{diag}\{-\hat{\rho}_1 I,-\hat{\rho}_2 I,\cdots,-\hat{\rho}_q I\}\\
\hat{\rho}_i&=\left(\operatorname{tr}\begin{bmatrix}\Gamma_i P_{1k} & \Gamma_i P_{3k}^T\\ 0 & 0\end{bmatrix}\right)^{-1},\quad i=1,2\cdots,q.\\
\hat{P}_k&=\operatorname{diag}\underbrace{\left\{\begin{bmatrix}P_{1k} & *\\ P_{3k} & P_{2k}\end{bmatrix},\cdots,\begin{bmatrix}P_{1k} & *\\ P_{3k} & P_{2k}\end{bmatrix}\right\}}_{r}\\
\check{L}&=[\check{L}_1,\cdots,\check{L}_r]^T\\
\check{L}_j&=\left[\sigma_j B_{fk}B_{kj}P_{1k}\quad \sigma_j B_{fk}B_{kj}P_{3k}^T\right]\\
&\quad j=1,2\cdots,r
\end{aligned}$$

then the addressed robust $\mathcal{H}_\infty$ finite horizon filter design problem is solved for the stochastic nonlinear system (1).

Proof: The proof is based on Theorem 3. We suppose that the variables P_k and Q_k can be decomposed as follows:

$$P_k=\begin{bmatrix}P_{1k} & *\\ P_{3k} & P_{2k}\end{bmatrix},\quad Q_k=\begin{bmatrix}M_k & 0\\ 0 & N_k\end{bmatrix}$$
$$Q_k^{-1}=\begin{bmatrix}\hat{M}_k & 0\\ 0 & \hat{N}_k\end{bmatrix}.$$

It is easy to see that (32) and (25) are equivalent.

In order to eliminate the parameter uncertainty ΔA_k in (26), we rewrite it in the following form:

$$\begin{bmatrix}\hat{\Lambda}_1 & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * \\ \tilde{A}_k & 0 & -Q_{k+1}^{-1} & * & * \\ \bar{B}_k & 0 & 0 & -\bar{Q}_{k+1}^{-1} & * \\ 0 & \bar{D}_k & 0 & 0 & -Q_{k+1}^{-1}\end{bmatrix} +\bar{H}_k F_k\bar{E}_k+\bar{E}_k^T F_k^{\mathrm{T}}\bar{H}_k^{\mathrm{T}}<0 \tag{37}$$

where

$$\begin{aligned}
\tilde{A}_k&=\begin{bmatrix}A_k & 0\\ B_{fk}\bar{\Xi}B_k & A_{fk}\end{bmatrix}\\
\bar{H}_k&=\begin{bmatrix}0 & 0 & \hat{H}_k^T & 0\end{bmatrix}^T,\quad \hat{H}_k=\begin{bmatrix}H_k^T & 0\end{bmatrix}^T\\
\bar{E}_k&=\begin{bmatrix}\hat{E}_k & 0 & 0 & 0\end{bmatrix},\quad \hat{E}_k=[E_k\quad 0].
\end{aligned}$$

Then, from Schur Complement and S-procedure, it follows that (26) is equivalent to (33). Similarly, we can see that (27) is also equivalent to (34). Therefore, according to Theorem 3, we have $J<0$ and $\mathbb{E}\{(x_k-\hat{x}_k)(x_k-\hat{x}_k)^T\}\leqslant[I\ \ -I]P_k[I\ \ -I]^T$, $\forall k\in\{0,1,\cdots,N+1\}$. From (35), it is obvious that $\mathbb{E}\{(x_k-\hat{x}_k)(x_k-\hat{x}_k)^T\}\leqslant[I\ \ -I]P_k[I\ \ -I]^T\leq\Psi_k$, $\forall k\in\{0,1,\cdots,N\}$. It can now be concluded that the requirements $(R1)$ and $(R2)$ are simultaneously satisfied. The proof is complete. ■

By means of Theorem 4, we can summarize the Robust Filter Design (RFD) algorithm as follows.

Algorithm RFD

Step 1. Given the $\mathcal{H}_\infty$ performance index γ, the positive definite matrix S and the error initial condition $x_0 - \hat{x}_0$, select the initial values for matrices $\{P_{10}, P_{20}, P_{30}, M_0, N_0\}$ which satisfy the condition (31) and set $k=0$.

Step 2. Obtain the values of matrices $\{\hat{M}_{k+1}, \hat{N}_{k+1}, P_{1k+1}, P_{2k+1}, P_{3k+1}\}$ and the desired filter parameters $\{A_{fk}, B_{fk}, C_{fk}\}$ for the sampling instant k by solving the LMIs (32)–(35).

Step 3. Set $k = k+1$ and obtain $\{M_{k+1}, N_{k+1}\}$ by the parameter update formula (36).

Step 4. If $k < N$, then go to Step 2, else go to Step 5.

Step 5. Stop.

Remark 4: From an engineering viewpoint, the recursive Kalman filter is efficient because only the estimated state from the previous time step and the current measurement are needed to compute the estimate for the current state. In fact, the main aim of this paper is to modify the traditional Kalman filtering approach to handle a class of nonlinearities and missing measurements with variance constraints. For the techniques used, we propose to replace the traditional recursive Riccati equations by the recursive linear matrix inequalities (RLMI) for the computational convenience. On the other hand, it would be interesting to deal with the corresponding robust steady-state filtering problem when the system parameters become time-invariant. This is one of our future research topics.

Remark 5: In Theorem 4, the robust $\mathcal{H}_\infty$ finite-horizon filter is designed by solving a series of recursive linear matrix inequalities (RLMIs) where both the current measurement and the previous state estimation are employed to estimate the current state. Such a recursive filtering process is particularly useful for real-time implementation such as online tracking of highly maneuvering targets. On the other hand, we point out that our main results can be extended to the case of dynamic output feedback control for the same class of nonlinear stochastic time-varying systems and the results will appear in the near future.

V. Illustrative Examples

In this section, we present a numerical simulation example to illustrate the effectiveness of the developed filter design method.

First of all, let us discuss the practical application of the developed theory to the target tracking problem through networked transmission, which is an important branch of signal processing. Let the maneuvering target be accelerating with random bursts of gas from its reaction control system (RCS) thrusters and the state vector consist of the position and velocity. Obviously, due to the high maneuver of the tracked target, it is neither possible nor necessary to track the target in a precise way. Instead, as discussed in Remark 3, an acceptable compromise is to keep the target within a given "window" as frequent as possible, and such a requirement can be expressed as upper bounds on the estimation error covariance. For online tracking, the system parameters would have to be time-varying that contain some uncertainties. Also, because of the bandwidth limit of the signal transmission channel, there are inevitably probabilistic measurement missing and probabilistic nonlinearities. In such a case, there is an urgent need to investigate the robust $\mathcal{H}_\infty$ filtering problem for the uncertain nonlinear discrete time-varying stochastic systems with error variance constraints and multiple missing measurements.

Motivated by the background discussed above, we consider the following discretized maneuvering target tracking system that is uncertain, time-varying with stochastic nonlinearities:

$$\begin{cases} x_{k+1} = \left(\begin{bmatrix} 0 & -0.3 \\ 0.2+0.2\sin(3k) & -0.3 \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix} \delta_k \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}^T \right) x_k + f_k + \begin{bmatrix} \sin(3k) \\ -0.03 \end{bmatrix} w_k \\ y_k = \Xi \begin{bmatrix} -2+0.3\sin(5k) & 0.5 \\ 0 & 1 \end{bmatrix} x_k + g_k + \begin{bmatrix} 0.1\sin(3k) \\ 0.2 \end{bmatrix} w_k \\ z_k = [0.5 \quad 0.5\sin(3k)]\, x_k \end{cases} \tag{38}$$

with the state initial value $x_0 = [0.26 \ -0.2]^T$, $\hat{x}_0 = [0.2 \ -0.16]^T$ and $S = \mathrm{diag}\{2,2\}$. Where the uncertain parameter δ_k satisfies $|\delta_k| \le 2$.

Let ω_k have the unity covariance and the nonlinear functions f_k and g_k be given as follows:

$$f_k = \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix} \times (0.2x_{1k}\xi_{1k} + 0.3x_{2k}\xi_{2k})$$
$$g_k = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} \times (0.2x_{1k}\xi_{1k} + 0.3x_{2k}\xi_{2k}),$$

where x_{ik} $(i=1,2)$ is the ith element of x_k, and ξ_{ik} $(i=1,2)$ are zero mean, uncorrelated Gaussian white noise processes with unity variances that is also uncorrelated with ω_k. It can be easily checked that the above class of stochastic nonlinearities satisfies

$$\mathbb{E}\left\{ \begin{bmatrix} f_k \\ g_k \end{bmatrix} \middle| x_k \right\} = 0$$

$$\mathbb{E}\left\{ \begin{bmatrix} f_k \\ g_k \end{bmatrix} [f_k^{\mathrm{T}} \quad g_{k^{\mathrm{T}}}] \middle| x_k \right\} = \begin{bmatrix} 0.1 \\ 0.3 \\ 0.1 \\ 0.1 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.3 \\ 0.1 \\ 0.1 \end{bmatrix}^{\mathrm{T}} \times \mathbb{E}\left\{ x_k^{\mathrm{T}} \begin{bmatrix} 0.04 & 0 \\ 0 & 0.09 \end{bmatrix} x_k \right\}.$$

Assuming that the probabilistic density functions of α_1 and α_2 in [0 1] are described by

$$q_1(s_1) = \begin{cases} 0 & s_1 = 0 \\ 0.1 & s_1 = 0.5 \\ 0.9 & s_1 = 1 \end{cases}$$
$$q_2(s_2) = \begin{cases} 0 & s_2 = 0 \\ 0.2 & s_2 = 0.5 \\ 0.8 & s_2 = 1 \end{cases} \tag{39}$$

from which the expectations and variances can be easily calculated as $\mu_1 = 0.95$, $\mu_2 = 0.9$, $\sigma_1 = 0.15$, and $\sigma_2 = 0.2$.

TABLE I
RECURSIVE PROCESS

k	0	1	2	3	$\cdots$
P_{1k}	$\begin{bmatrix} 0.0095 & 0.0099 \\ 0.0099 & 0.0194 \end{bmatrix}$	$\begin{bmatrix} 0.0182 & 0.0056 \\ 0.0056 & 0.0306 \end{bmatrix}$	$\begin{bmatrix} 0.0061 & 0.0032 \\ 0.0032 & 0.0107 \end{bmatrix}$	$\begin{bmatrix} 0.0066 & 0.0026 \\ 0.0026 & 0.0118 \end{bmatrix}$	$\cdots$
P_{2k}	$\begin{bmatrix} 0.0005 & 0.0000 \\ 0.0000 & 0.0005 \end{bmatrix}$	$\begin{bmatrix} 0.0155 & -0.0001 \\ -0.0001 & 0.0153 \end{bmatrix}$	$\begin{bmatrix} 0.0037 & -0.0000 \\ -0.0000 & 0.0037 \end{bmatrix}$	$\begin{bmatrix} 0.0052 & -0.0001 \\ -0.0001 & 0.0050 \end{bmatrix}$	$\cdots$
P_{3k}	$\begin{bmatrix} -0.0000 & 0.0000 \\ 0.0000 & -0.0000 \end{bmatrix}$	$\begin{bmatrix} 0.0005 & -0.0001 \\ -0.0001 & 0.0003 \end{bmatrix}$	$\begin{bmatrix} 0.0000 & 0.0000 \\ 0.0000 & 0.0001 \end{bmatrix}$	$\begin{bmatrix} 0.0001 & -0.0000 \\ -0.0000 & 0.0000 \end{bmatrix}$	$\cdots$
A_{fk}	$\begin{bmatrix} 0.2636 & 0.2636 \\ 0.2617 & 0.2617 \end{bmatrix}$	$\begin{bmatrix} -0.1906 & 0.6516 \\ -0.1686 & 0.6284 \end{bmatrix}$	$\begin{bmatrix} -0.27116 & 0.3895 \\ -0.2849 & 0.4025 \end{bmatrix}$	$\begin{bmatrix} -0.2036 & 0.6706 \\ -0.1806 & 0.6477 \end{bmatrix}$	$\cdots$
B_{fk}	$\begin{bmatrix} -0.0980 & 0.2668 \\ 0.0972 & 0.2961 \end{bmatrix}$	$\begin{bmatrix} 0.3462 & 0.4978 \\ 0.3317 & 0.4740 \end{bmatrix}$	$\begin{bmatrix} 0.6405 & 0.8414 \\ 0.5276 & 0.7679 \end{bmatrix}$	$\begin{bmatrix} 0.3968 & 0.7865 \\ 0.3786 & 0.7658 \end{bmatrix}$	$\cdots$
C_{fk}	$\begin{bmatrix} 0.4545 & 0.4545 \end{bmatrix}$	$\begin{bmatrix} 0.2165 & -0.9380 \end{bmatrix}$	$\begin{bmatrix} 0.3982 & -0.32261 \end{bmatrix}$	$\begin{bmatrix} 0.3261 & -0.2509 \end{bmatrix}$	$\cdots$

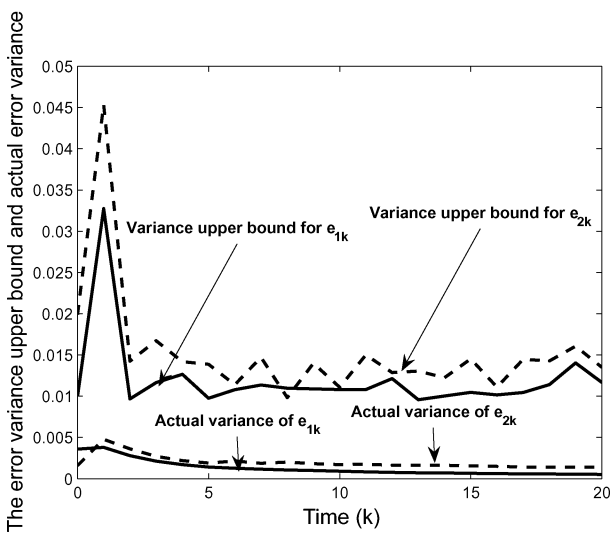

Fig. 1. The error variance upper bound and actual error variance.

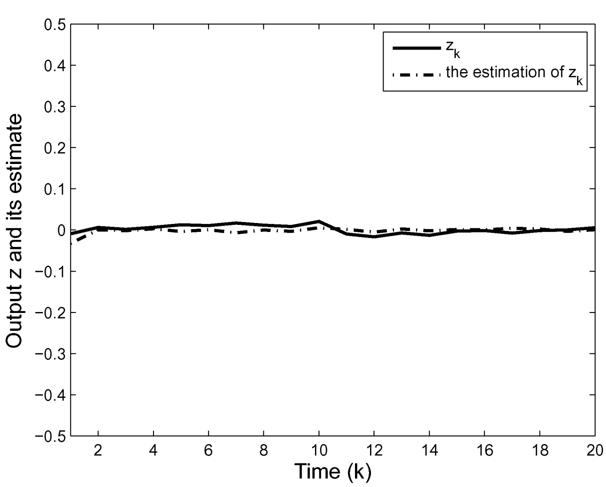

Fig. 2. Output z_k and its estimate.

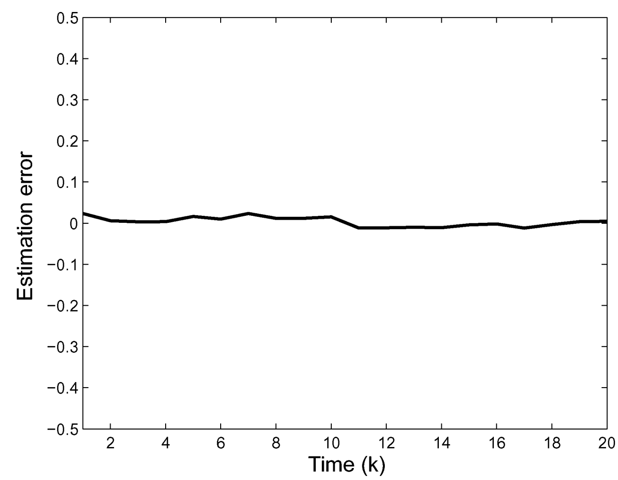

Fig. 3. Estimation error.

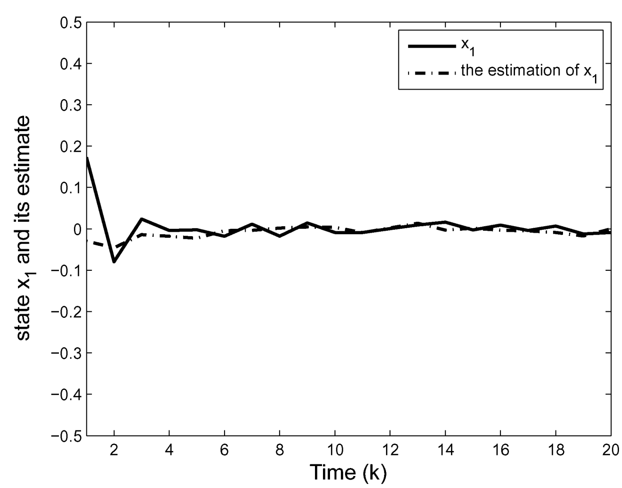

Fig. 4. The state x_1 and its estimate.

Set the disturbance attenuation level as $\gamma = 0.8$ and $\{\Psi_k\}_{1\leqslant k\leqslant N} = diag\{0.3, 0.3\}$, and choose the parameters' initial values satisfying (31). According to *Algorithm RFD*, the time-varying LMIs in Theorem 4 can be solved recursively subject to given initial conditions and prespecified performance indices. Table I lists the matrix variables P_{1k}, P_{2k}, P_{3k}, and the desired parameters of filter A_{fk}, B_{fk}, C_{fk} from the time $k = 0$ to $k = 3$.

The simulation results are shown in Figs. 1–5. Fig. 1 gives the error variance upper bound and actual error variance with $e_{1k} = x_{1k} - \hat{x}_{1k}$ and $e_{2k} = x_{2k} - \hat{x}_{2k}$. Fig. 2 plots the output z_k and its estimation $\hat{z}_k$, whereas the estimation error $\bar{z}_k$ is shown in Fig. 3. The actual states x_{1k}, x_{2k} and their estimates $\hat{x}_{1k}, \hat{x}_{2k}$ are depicted in Fig. 4 and Fig. 5, respectively. All the simulation results confirm that the desired finite-horizon performance is well achieved and the proposed RFD algorithm is indeed effective.

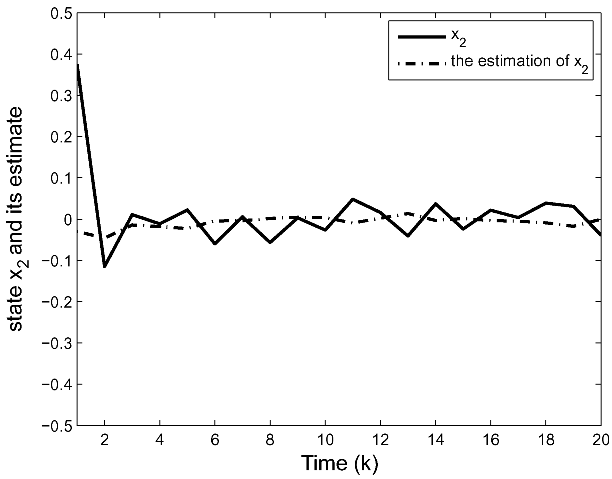

Fig. 5. The state x_2 and its estimate.

Remark 6: As with the use of traditional Kalman filtering, we can seek to use the *stationary* value of the initial parameters M_0, N_0, P_{i0} $(i=1,2,3)$ in our design. That is, letting the time-varying parameters be fixed as $k=0$, the matrix inequalities (31)–(35) become LMIs of M_0, N_0 and P_{i0} $(i=1,2,3)$ which can then be solved and chosen as the initial values.

VI. Conclusion

In this paper, the problem of robust $\mathcal{H}_\infty$ filtering problem has been discussed for a class of uncertain nonlinear discrete time-varying stochastic systems with error variance constraints and multiple missing measurements. The measurement missing phenomenon is assumed to occur in a random way, and the missing probability for each sensor is governed by an individual random variable satisfying a certain probabilistic distribution in the interval [0 1]. The stochastic nonlinearities under consideration here has been widely used in engineering applications. Sufficient conditions for the finite-horizon filter to satisfy state estimation error variance constraints and prescribed $\mathcal{H}_\infty$ performance have been given in terms of the feasibility of a series of RLMIs. Simulation results have demonstrated the effectiveness of the developed filtering technique in a target tracking example.

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