Robust mixed $H_2/H_\infty$ control for a class of nonlinear stochastic systems

F. Yang, Z. Wang and D.W.C. Ho

Abstract: The problem of mixed $H_2/H_\infty$ control is considered for a class of uncertain discrete-time nonlinear stochastic systems. The nonlinearities are described by statistical means of the stochastic variables and the uncertainties are represented by deterministic norm-bounded parameter perturbations. The mixed $H_2/H_\infty$ control problem is formulated in terms of the notion of exponentially mean-square quadratic stability and the characterisations of both the $H_2$ control performance and the $H_\infty$ robustness performance. A new technique is developed to deal with the matrix trace terms arising from the stochastic nonlinearities and the well-known S-procedure is adopted to handle the deterministic uncertainties. A unified framework is established to solve the addressed mixed $H_2/H_\infty$ control problem using a linear matrix inequality approach. Within such a framework, two additional optimisation problems are discussed, one is to optimise the $H_\infty$ robustness performance, and the other is to optimise the $H_2$ control performance. An illustrative example is provided to demonstrate the effectiveness of the proposed method.

1 Introduction

In engineering practice, it is always welcome to design a controller that achieves multiple objectives. A typical example is the mixed $H_2/H_\infty$ control scheme, which attempts to capture the benefits of both the $H_2$ control performance and the $H_\infty$ robustness performance simultaneously. In general, a pure $H_2$ controller is designed for a good measure of transient performance [1], whereas a pure $H_\infty$ control framework is developed for robustness with respect to disturbances and system uncertainties. Therefore the mixed $H_2/H_\infty$ multiobjective design framework has a better and clearer physical interpretation and has received much attention from the control research community in the past few decades.

For linear deterministic systems, the mixed $H_2/H_\infty$ control problems have been extensively studied. For example, algebraic approaches to mixed $H_2/H_\infty$ control problems have been proposed in [2] and a time domain Nash game approach has been provided to solve the mixed $H_2/H_\infty$ in [1, 3]. Moreover, some efficient numerical methods for mixed $H_2/H_\infty$ control problems have been developed based on a convex optimisation approach in [4–6]. In particular, since the linear matrix inequality (LMI) approach has proven to be a very effective numerical optimisation algorithm [7], it has been employed to design both linear state feedback and output feedback controllers subject to $H_2/H_\infty$ criterion, see, for example, [8]. It is noted that the mixed $H_2/H_\infty$ control theories have already been applied to various engineering fields [9–11]. Parallel to the mixed $H_2/H_\infty$ control problem, the mixed $H_2/H_\infty$ filtering problem has also been well studied, see [12–14] and the references therein.

For nonlinear deterministic systems, the mixed $H_2/H_\infty$ control problem has gained some research interests, see, for example, [15], where the solutions have been characterised in terms of the cross-coupled Hamilton–Jacobi–Isacs (HJI) partial differential equations. Since it is difficult to solve the cross-coupled HJI partial differential equations either analytically or numerically, Chen et al. [16] have used the Takagi and Sugenio (T–S) fuzzy linear model to approximate the nonlinear system, and solutions to the mixed $H_2/H_\infty$ fuzzy output feedback control problem have been obtained via an LMI approach.

On the other hand, since stochastic modelling has been playing a more and more important role in engineering designs [17, 18], the stochastic $H_\infty$ control problem has attracted growing research attention recently. Many research results have been available which, unfortunately, are mainly for linear stochastic systems. In [17], a stochastic-bounded real lemma has been developed to solve the $H_\infty$ control problem for stochastic linear systems with state- and control-dependent noises. The results have been extended to the $H_\infty$ control problem for discrete-time stochastic linear systems with the state- and control-dependent noises [19]. A robust stochastic $H_\infty$ control problem has been addressed in [20] to deal with the systems in the presence of stochastic uncertainty. Very recently, a stochastic mixed $H_2/H_\infty$ control problem has been considered for the system with the state-dependent noises [21], where sufficient conditions have been provided in terms of the existence of the solutions of cross-coupled Riccati equations. However, there are very few results on the mixed $H_2/H_\infty$ control problem for nonlinear stochastic systems. In [22], an elegant LMI approach has been developed to deal with the analysis problem for a class of systems with stochastic nonlinearities, where the nonlinearities characterised by statistical means were first introduced in [23]. Unfortunately, the robustness issue in the presence of parameter uncertainties has not been addressed.

In this paper, we aim to substantially extend part of the analysis results in [22] to the uncertain systems, derive...
the explicit expressions of the upper bounds for the robust $H_2$ and $H_{\infty}$ performances and deal with the corresponding robust mixed $H_2/H_{\infty}$ control problem using the LMI approach. Specifically, we are interested in designing a state feedback controller such that, for all admissible stochastic nonlinearities and deterministic uncertainties, the closed-loop system is exponentially mean-square quadratically stable, the $H_2$ control performance is achieved and the prescribed disturbance attenuation level is guaranteed in an $H_{\infty}$ sense. The nonlinearities considered in this paper, which are characterised by statistical means of the stochastic variables, are shown to be more general than many well-studied nonlinearities in the literature concerning nonlinear stochastic systems. The parameter uncertainties are assumed to be norm-bounded and enter the system matrices. A new technique is developed to handle the deterministic uncertainties. The solution to the mixed $H_2/H_{\infty}$ control problem is enforced within a unified LMI framework. In order to demonstrate the feasibility of the proposed framework, we will examine two types of the optimisation problems that optimise either the $H_2$ control performance or the $H_{\infty}$ robustness performance, and a numerical example is provided to illustrate the effectiveness of the proposed design method.

The remainder of this paper is organised as follows. In Section 2, a class of uncertain discrete-time nonlinear stochastic systems is described and the mixed $H_2/H_{\infty}$ control problem for the systems is formulated. In Section 3, the system analysis problem is considered, where the existence conditions for the solution to the mixed $H_2/H_{\infty}$ control problem are derived, by introducing the notion of exponentially mean-square quadratic stability and by characterising the $H_2$ control performance and the $H_{\infty}$ robustness performance. An LMI algorithm is developed in Section 4 to design the mixed $H_2/H_{\infty}$ controller for the systems with stochastic nonlinearities and deterministic norm-bounded parameter uncertainties. An illustrative example is presented in Section 5 to demonstrate the applicability of the method and some concluding remarks are provided in Section 6.

Notation: The notation used here is fairly standard. $\mathbb{R}^n$ and $\mathbb{R}^{n \times m}$ denote, respectively, the $n$-dimensional Euclidean space and the set of all $n \times m$ real matrices, and $\mathbb{Z}^+$ stands for the set of non-negative integers. The notation $X \succ Y$ (respectively, $X \succ Y$), where $X$ and $Y$ are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). $\text{tr}(A)$ represents the trace of matrix $A$. $\mathbb{E}[x]$ stands for the expectation of stochastic variable $x$ and $\mathbb{E}[x|y]$ for the expectation of $x$ conditional on $y$. The superscript ‘T’ denotes the transpose. $\lambda_{\text{max}}(M)$ stands for the maximum eigenvalue of matrix $M$. $\text{diag}(M_1, M_2, \ldots, M_n)$ denotes a block diagonal matrix whose diagonal blocks are given by $M_1, M_2, \ldots, M_n$, and in symmetric block matrices, $*$ is used as an ellipsis for terms induced by symmetry.

2 Problem formulation

Consider the following class of discrete-time systems with stochastic nonlinearities and deterministic norm-bounded parameter uncertainties

\[ x_{k+1} = (A + H_1 F)x_k + f(x_k, u_k) + B_1 w_k + B_2 u_k \]  
\[ z_{nk} = L_{\infty} x_k \]  
\[ z_{2k} = L_2 x_k \]  

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^r$ is the control input, $z_{nk} \in \mathbb{R}^{p_1}$ is a combination of the states to be controlled (with respect to $H_\infty$-norm constraints), $z_{2k} \in \mathbb{R}^{p_2}$ is another combination of the states to be controlled (with respect to $H_2$-norm constraints), $w_k \in \mathbb{R}^m$ is the process noise, which is a zero mean Gaussian white noise sequences with covariance $R$ and $A, B_1, B_2, L_\infty, L_2, H_1$ and $E$ are known real matrices with appropriate dimensions.

The matrix $F \in \mathbb{R}^{r \times n}$, which may be time-varying, represents the deterministic parameter uncertainties, that is

\[ FF^T \leq I \]  

The deterministic uncertain matrix $F$ is said to be admissible if it satisfies the condition (4).

The function $f(x_k, u_k) : \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}^m$ is a stochastic nonlinear function of the states and control inputs, which is assumed to have the following first moment for all $x_k$ and $u_k$

\[ \mathbb{E}[f_k | x_k, u_k] = 0 \]  

with its covariance given by

\[ \mathbb{E}[f_k f_k^T | x_k, u_k] = \sum_{i=1}^q \theta_i \theta_i^T (x_k^T \Gamma x_k + u_k^T \Pi u_k) \]  

where $\theta_i$ ($i = 1, \ldots, q$) is known column vector, $\Gamma$ and $\Pi$, ($i = 1, \ldots, q$) are known positive-definite matrices with appropriate dimensions.

Remark 1: Note that the output matrices $L_\infty$ and $L_2$ can be chosen to be identical in practical design. Furthermore, the structure of the deterministic uncertainties in (4) has been used in many works concerning robust control and filtering problems, see, for example, [24, 25].

We are now in a position to discuss the generality of the nonlinear description in (5) and (6). As pointed out in [23], such a description encompasses many well-studied nonlinearities in stochastic systems, which enables the designer to deal with:

- Linear systems with state- and control-dependent multiplicative noises $D_1 x_k \xi_{k1} + D_2 u_k \xi_{k2}$, where $\xi_{k1}$ and $\xi_{k2}$ are zero mean, uncorrelated noise sequences.
- Nonlinear systems with random vectors dependent on the norms of states and control inputs, that is, $\|x_k\|D_1 x_k + \|u_k\|D_2 u_k$, where $\xi_{k1}$ and $\xi_{k2}$ are zero mean, uncorrelated noise sequences.
- Nonlinear systems with a random sequence dependent on the sign of a nonlinear function of states and control inputs, that is, $\text{sign}(\phi(x_k, u_k))D_1 x_k \xi_{k1} + D_2 u_k \xi_{k2}$, where $\xi_{k1}$ and $\xi_{k2}$ are zero mean, uncorrelated noise sequences.
- Other models that have been discussed in [23].

One can see that some of the most important uncertain nonlinear stochastic models can be special cases of the system given in (1)–(6).

We now consider the following state feedback controller for the system (1)

\[ u_k = K x_k \]  

where $K$ is the state feedback gain to be determined.

The closed-loop system is governed as follows by substituting (7) into (1)

\[ x_{k+1} = A_K x_k + f(x_k, K x_k) + B_1 w_k \]  

where

\[ A_K = A + B_2 K + H_1 FE \]
Before giving our design goal, we introduce the following notion of exponentially quadratic stability in the mean-square sense for the closed-loop system (8).

**Definition 1:** The system (8) is said to be exponentially mean-square quadratically stable if, with \( w_k = 0 \), there exist constants \( \alpha \geq 1 \) and \( \tau \in (0, 1) \) such that

\[
\mathbb{E}[[x_k^2]] \leq \alpha \tau^t \mathbb{E}[[x_0^2]], \quad \forall \ x_0 \in \mathbb{R}^n, \quad k \in 1^+ \tag{10}
\]

for all admissible uncertainties satisfying (4).

The purpose of this paper is to seek a state feedback controller of the form (7), for the system (1), such that for all stochastic nonlinearities and all admissible deterministic uncertainties, the closed-loop system is exponentially mean-square quadratically stable, and additional \( H_2 \) control performance constraint and \( H_\infty \) robust performance constraint are also satisfied. In other words, we aim to design a controller such that the closed-loop system satisfies the following requirements (Q1) and (Q2), simultaneously:

(Q1) For a given constant \( \beta > 0 \), the system (8) is exponentially mean-square quadratically stable and the following constraint is satisfied

\[
J_2 = \lim_{k \to \infty} \mathbb{E}[[z_{2k}^2]] < \beta \tag{11}
\]

(Q2) For a given \( \gamma > \gamma_0 > 0 \), the system (8) is exponentially mean-square quadratically stable and the following constraint is achieved

\[
\sum_{k=0}^{\infty} \mathbb{E}[[z_{mk}^2]] < \gamma^2 \sum_{k=0}^{\infty} \mathbb{E}[[w_k^2]] \tag{12}
\]

for all non-zero \( w_k \) under zero initial condition, where \( \gamma_0 \) is the minimum attenuation level.

The design problem stated above will be referred to as the robust-mixed \( H_2/H_\infty \) control problem for the nonlinear stochastic system (1)–(6).

### 3 Robust mixed \( H_2/H_\infty \) analysis problem

To facilitate our discussion on the \( H_2 \) control problem (Q1) and the \( H_\infty \) control problem (Q2), we need the following technical results.

**Lemma 1 [22, 26]:** Given the feedback gain matrix \( K \). The system (8) is exponentially mean-square quadratically stable if, for all admissible uncertainties, there exists a positive definite matrix \( P \) satisfying

\[
A_k^T P A_k - P + \sum_{i=1}^{q} (I_i + K^T \Pi_i K) \text{tr}(\theta_i \theta_i^T P) < 0 \tag{13}
\]

**Lemma 2 [22]:** If the system (8) is exponentially mean-square quadratically stable, then

\[
\rho \left( A_K \otimes A_K + \sum_{i=1}^{q} \text{st}(\theta_i \theta_i^T) \text{st}^T (I_i + K^T \Pi_i K) \right) < 1 \tag{14}
\]

or equivalently

\[
\rho \left( A_k^T A_k + \sum_{i=1}^{q} \text{st}(I_i + K^T \Pi_i K) \text{st}^T (\theta_i \theta_i^T) \right) < 1 \tag{15}
\]

where \( \otimes \) is the Kronecker product of matrices, \( \rho \) is the spectral radius of a matrix and \( \text{st} \) stands for the stack of a matrix that forms a vector out of the columns of the matrix.

**Lemma 3:** Consider the system

\[
\dot{\xi}_{k+1} = M \xi_k + f(\xi_k) \tag{16}
\]

where \( \mathbb{E}(f(\xi_k) \xi_k) = 0 \), and \( \mathbb{E}(f(\xi_k) \xi_k) = \sum_{i=1}^{q} \theta_i \theta_i^T (\xi_i^T \xi_k), \theta_i (i = 1, \ldots, g) \) are known column vectors, \( \xi_i (i = 1, \ldots, q) \) are known positive-definite matrices with appropriate dimensions. If the system (16) is exponentially mean-square stable, and there exists a symmetric matrix \( Y \) satisfying

\[
M^T Y M - Y + \sum_{i=1}^{q} \xi_i \text{tr}(\theta_i \theta_i^T Y) < 0 \tag{17}
\]

then \( Y \geq 0 \).

Lemma 3 can be easily proved by using the Lyapunov method, hence the proof is omitted.

### 3.1 \( H_2 \) control problem

Define the state covariance by

\[
Q_k := \mathbb{E}(x_k x_k^T) = \mathbb{E}(x_{1,k} x_{1,k}^T \cdots x_{n,k} x_{n,k}^T) \tag{18}
\]

and then the Lyapunov-type equation that governs the evolution of the state covariance matrix \( Q_k \) can be derived from the system (8) and the relation (7) as follows

\[
Q_{k+1} = A_k Q_k A_k^T + \sum_{i=1}^{q} \theta_i \theta_i^T \text{tr}(\theta_i \theta_i^T Q_k) + B_i R B_i^T \tag{19}
\]

We rewrite (19) in the form of the stack matrix by

\[
\text{st}(Q_{k+1}) = \Psi \cdot \text{st}(Q_k) + \text{st}(B_i R B_i^T) \tag{20}
\]

where

\[
\Psi := A_K \otimes A_K + \sum_{i=1}^{q} \text{st}(\theta_i \theta_i^T) \text{st}^T (I_i + K^T \Pi_i K)
\]

If the system (8) is exponentially mean-square quadratically stable, it follows from Lemma 2 that \( \rho(\Psi) < 1 \) and \( Q_k \) in (20) converges to a constant matrix \( \bar{Q} \) when \( k \to \infty \), that is

\[
Q = \lim_{k \to \infty} Q_k \tag{21}
\]

Therefore \( H_2 \) performance can be written by

\[
J_2 = \lim_{k \to \infty} \mathbb{E}[[z_{2k}^2]] = \lim_{k \to \infty} \text{tr}(L_z Q_k L_z^T) = \text{tr}(L_z \bar{Q} L_z^T) \tag{22}
\]

In order to make sure that the \( H_2 \) performance and \( H_\infty \) performance can be tackled within the same framework by using a unified LMI approach, we will need to derive an alternative expression of the \( H_2 \) performance (22). Suppose now that there exists a matrix \( P_k > 0 \) such that the following backward recursion is satisfied

\[
\hat{P}_k = A_k^T \hat{P}_{k+1} A_k + \sum_{i=1}^{q} (I_i + K^T \Pi_i K) \text{tr}(\theta_i \theta_i^T \hat{P}_{k+1}) + L_2^T L_2 \tag{23}
\]
which can be rearranged in terms of the stack operator as follows
\[
\text{st}(\hat{P}_k) = \Phi \cdot \text{st}(\hat{P}_{k+1}) + \text{st}(L_2^T L_2)
\]
where
\[
\Phi := A_K^T \otimes A_K^T + \sum_{i=1}^{q} \text{st}(I_i + K^T \Pi_i K) \text{st}(\theta_i \theta_i^T)
\]

If the system (8) is exponentially mean-square quadratically stable, then it follows from Lemma 2 that \(\rho(\Phi) < 1\) and \(P_k\) in (24) converges to \(P\) when \(k \to \infty\), that is
\[
\hat{P} = \lim_{k \to \infty} P_k
\]
Hence, in the steady state, (23) becomes
\[
\hat{P} = A_K^T P A_K + \sum_{i=1}^{q} (I_i + K^T \Pi_i K) \text{tr}(\theta_i \theta_i^T) + L_2^T L_2
\]
(26)

Summing up (23)–(26), we obtain the following result that gives an alternative to the \(H_\infty\) performance formulation, we

**Theorem 1:** If the system (8) is exponentially mean-square quadratically stable, \(H_2\) performance can be expressed in terms of \(\hat{P}\) as follows
\[
J_2 = \text{tr}[RB_i^T PB_i]
\]
where \(\hat{P} > 0\) is the solution to (26).

**Proof:** Noting that
\[
\lim_{k \to \infty} \text{tr}(Q_k + \hat{P}_{k+1} - Q_k \hat{P}_k) = \lim_{k \to \infty} \left\{ A_K Q_k A_K^T + \sum_{i=1}^{q} \theta_i \theta_i^T \text{tr}(Q_k (I_i + K^T \Pi_i K)) \\
+ B_i R B_i^T \right\} \hat{P}_{k+1} - Q_k \left[ A_K^T \hat{P}_{k+1} A_K + \sum_{i=1}^{q} (I_i + K^T \Pi_i K) \\
\times \text{tr}(\theta_i \theta_i^T) + L_2^T L_2 \right] \right\} = 0
\]
(28)

Therefore we have
\[
\text{tr}(L_2 Q L_2^T) = \text{tr}[RB_i^T PB_i]
\]
(29)

and the proof follows from (22) immediately. \(\square\)

**Remark 2:** We use (27) to compute the \(H_2\) performance instead of (22). The reason is that the \(H_2\) control performance and \(H_\infty\) robustness performance need to be characterised as a similar structure so that the solution to the mixed \(H_2/H_\infty\) control problem can be obtained by using a unified LMI approach. We will see in the next subsection that the structure of (27) is similar to that for the \(H_\infty\) robustness performance.

Notice that the system model in (1)–(3) involves parameter uncertainties, and hence the exact \(H_2\) performance (27) cannot be obtained by simply solving (26). One way to deal with this problem is to provide an upper bound for the \(H_2\) performance. Suppose that there exists a positive definite matrix \(P\) such that the following matrix inequality is satisfied
\[
A_K^T P A_K - P + \sum_{i=1}^{q} (I_i + K^T \Pi_i K) \text{tr}(\theta_i \theta_i^T) + L_2^T L_2 < 0
\]
(30)

Now we are ready to give the upper bound for \(\hat{P}\). Comparing (26) to (30), we obtain the following main result in this subsection.

**Theorem 2:** If there exists a positive definite matrix \(P\) satisfying (30), then the system (8) is exponentially mean-square quadratically stable
\[
\hat{P} \leq P
\]
(31)
and
\[
\text{tr}[RB_i^T PB_i] \leq \text{tr}[RB_i^T PB_i]
\]
(32)
where \(\hat{P} > 0\) satisfies (26).

**Proof:** It is obvious that (30) implies (13), and then it follows directly from Lemma 1 that the system (8) is exponentially mean-square quadratically stable. Hence, the solution \(P > 0\) to (26) exists. Subtracting (30) from (26) yields
\[
A_K^T (P - \hat{P}) A_K - (P - \hat{P}) + \sum_{i=1}^{q} (I_i + K^T \Pi_i K) \text{tr}(\theta_i \theta_i^T) (P - \hat{P}) < 0
\]
(33)
which indicates from Lemma 3 that \(P - \hat{P} \geq 0\). Furthermore, (31) implies (32), and this completes the proof. \(\square\)

The corollary given below follows immediately from Theorem 2 and (11).

**Corollary 1:** If there exists a positive definite matrix \(P\) satisfying (30) and \(\text{tr}[RB_i^T PB_i] < \beta\), where \(\beta > 0\) is a given scalar, then the system (8) is exponentially mean-square quadratically stable, and (11) is satisfied for \(\beta > 0\).

### 3.2 \(H_\infty\) control problem

Contrary to the standard \(H_\infty\) performance formulation, we shall use the expression (12) to describe the \(H_\infty\) performance of the stochastic system, where the expectation operator is utilised on both the controlled output and the disturbance input, see [18] for more details.

The following lemma can be proved along a similar line in [21].

**Lemma 4:** Given a scalar \(\gamma > 0\) and a feedback gain matrix \(K\), the system (8) is exponentially mean-square quadratically stable and the \(H_\infty\)-norm constraint (12) is achieved for all non-zero \(w_k\), if there exists a positive-definite matrix \(P\) satisfying
\[
\left[ A_K^T P A_K - P + \sum_{i=1}^{q} (I_i + K^T \Pi_i K) \text{tr}(\theta_i \theta_i^T) P + L_2^T L_2 \right] B_i^T PA_K
\]
\[
B_i^T PB_i - \gamma^2 I \right] < 0
\]
(34)
for all admissible uncertainties.

Up to now, the \(H_2\) control problem and the \(H_\infty\) control problem have been considered separately.
proceeding to the next Section, we will need to discuss the mixed \( H_2/H_{\infty} \) analysis problem.

### 3.3 Robust mixed \( H_2/H_{\infty} \) analysis problem

In order to realise our design goals (Q1) and (Q2) simultaneously, it can be easily seen that the robust mixed \( H_2/H_{\infty} \) control problem addressed in Section 2 can be restated as follows.

**Problem A:** Design a controller \( (7) \) such that there exists a positive definite matrix \( P \) satisfying the following inequalities

\[
\text{tr}(RB^TPB_1) < \beta 
\]

\[
A^T_KPA_K - P + \sum_{i=1}^q (G_i + K^TII_iK) \text{tr}(\theta_i\theta_i^TP) + L_1^T L_2 < 0 
\]

\[
\left[
A^T_KPB_K - \sum_{i=1}^q (G_i + K^TII_iK) \text{tr}(\theta_i\theta_i^TP) + L_1^T L_\infty \\
B^T_iPB_K \\
B^T_iPB_1 - \gamma^2 I
\right] < 0
\]

The purpose of Problem A is to find a controller \( (7) \) so as to ensure that (35)–(37) are satisfied for all admissible uncertainties, and subsequently the stability, the \( H_2 \) and \( H_{\infty} \) constraints are all achieved. Note that at this stage, a problem is still complicated since the matrix trace terms and the uncertainty \( F \) are involved in (35)–(37).

Our goal in the next Section is therefore to develop an LMI approach to designing the desired controller based on (35)–(37).

### 4 Robust mixed \( H_2/H_{\infty} \) controller design

In this Section, we will present the solution to the robust \( H_2/H_{\infty} \) state feedback controller design problem for the discrete-time systems with stochastic nonlinearities and deterministic norm-bounded parameter uncertainty. In other words, we aim to design the controller that satisfies the performance requirements (Q1) and (Q2) simultaneously. In order to develop a unified LMI framework, the main task at this stage is to deal with the matrix trace terms (nonlinear term) and handle the uncertainties in the matrix inequalities (35)–(37), such that Problem A can be converted into a convex optimisation problem that is easy to be solved.

Before giving our main result, we recall the following useful lemmas.

**Lemma 5 (Schur complement) [7]:** Given constant matrices \( L_1, L_2, L_3 \) where \( L_1 = L_1^T \) and \( 0 < L_2 = L_2^T \), then

\[
L_1 + L_2 L_2^{-1} L_3 < 0
\]

if and only if

\[
\begin{bmatrix}
L_1 & L_2 \\
L_3 & -L_2
\end{bmatrix} < 0
\]

or equivalently

\[
\begin{bmatrix}
-L_2 & L_3 \\
L_3^T & -L_1
\end{bmatrix} < 0
\]

**Lemma 6 (S-procedure) [7, 14]:** Let \( M = M^T \), \( H \) and \( E \) be real matrices of appropriate dimensions, with \( F \) satisfying (4), then

\[
M + HE + E^TF^TH < 0
\]

if and only if, there exists a positive scalar \( \varepsilon > 0 \) such that

\[
M + \varepsilon HH^T + \frac{1}{\varepsilon}F^TE < 0
\]

or equivalently

\[
\begin{bmatrix}
M & \varepsilon H \\
E & 0
\end{bmatrix} < 0
\]

In order to recast Problem A into a convex optimisation problem, we first tackle the matrix trace terms in (35)–(37) by introducing new variables, which is actually one of the technical contributions in this paper. The following theorem presents sufficient conditions for solving Problem A.

**Theorem 3:** Given constants \( \gamma > 0 \), \( \beta > 0 \) and the feedback gain matrix \( K \). If there exists positive-definite matrix \( P > 0 \) and \( \Theta > 0 \), and positive scalars \( \alpha_i > 0 \) \((i = 1, \ldots, q)\) such that the following matrix inequalities

\[
\text{tr}(\Theta) < \beta
\]

\[
\begin{bmatrix}
-\Theta & R^{1/2}B_1^T \\
P & -P^{-1}
\end{bmatrix} < 0
\]

\[
\begin{bmatrix}
-\alpha_i & \alpha_i \theta_i^T \\
\alpha_i \theta_i & -P^{-1}
\end{bmatrix} < 0 \quad (i = 1, \ldots, q)
\]

\[
\begin{bmatrix}
-P & A_K^T & I_{1/2} & \cdots & I_{q/2} \\
A_K & -P^{-1} & 0 & \cdots & 0 \\
I_{1/2} & 0 & -\alpha_i I & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
I_{q/2} & 0 & 0 & \cdots & -\alpha_q I \\
K & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
L_2 & 0 & 0 & \cdots & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
K^T & \cdots & K^T & L_2^T
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & 0 \\
-\alpha_i I_{1/2} & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & \cdots & -\alpha_q I_{1/2} & 0 \\
0 & \cdots & 0 & -I
\end{bmatrix}
\]

\[
< 0
\]
The proof is complete.

**Remark 3:** In Theorem 3, we provide sufficient conditions for satisfying (35)–(37), where the nonlinear matrix trace terms are handled so as to form a convex optimisation problem. The possible conservatism caused by such a transformation can be reduced by making the values of \( \alpha_i \) as close as possible to the value of \( \alpha_i \) when solving the LMIs. This will be demonstrated later in Section 5.

In the following, we will continue to eliminate the uncertainty \( F \) contained in (44) and (45) by using the well-known S-procedure technique, and then the desired robust mixed \( H_2/H_{\infty} \) controller could be obtained via an LMI approach by solving Problem A.

**Theorem 4:** Given constants \( \gamma > 0 \) and \( \beta > 0 \). If there exists positive-definite matrix \( X > 0 \) and \( \Theta > 0 \), a real matrix \( G \), positive scalars \( \alpha_i > 0 \) \( (i = 1, \ldots, q) \) and \( \varepsilon_i > 0 \)
\((i = 1, 2)\) such that the following linear matrix inequalities

\[
\begin{bmatrix}
-\Theta & R^{1/2}B_1^T \\
B_1R^{1/2} & -X
\end{bmatrix} < 0
\]

\[\left[ -\alpha_i \alpha_i \theta_i^T \right] < 0 \quad (i = 1, \ldots, q) \] (56)

\[\frac{1}{2} \left[ \begin{array}{cccc}
-X & * & * & * \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
G & 0 & \cdots & 0 \\
L_2X & 0 & \cdots & 0 \\
0 & \varepsilon_1 H_1^T & 0 & \cdots & 0 \\
EX & 0 & \cdots & 0 & 0 \\
\end{array} \right] < 0 \] (57)

are feasible, then there exists a state feedback controller of the form (7) such that the requirements (Q1) and (Q2) are satisfied for all stochastic nonlinearities and all admissible deterministic uncertainties. Moreover, the desired controller (7) can be determined by

\[K = GX^{-1} \] (59)

**Proof:** In view of Theorem 3, we just need to show that (44) holds if and only if there exists a positive scalar \(\varepsilon_1\) such that (57) holds, and (45) holds if and only if there exists a positive scalar \(\varepsilon_2\) such that (58) holds.

Rewrite the condition (44) in the form of (38) as follows

\[
\begin{bmatrix}
-\varepsilon_1 I & * & * & * \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
0 & -\varepsilon_2 I & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
\end{bmatrix} \begin{bmatrix}
-\varepsilon_1 I & * & * & * \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
0 & -\varepsilon_2 I & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
\end{bmatrix} \begin{bmatrix}
-\varepsilon_1 I & * & * & * \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
0 & -\varepsilon_2 I & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
\end{bmatrix} < 0 \]

where

\[\hat{M} = \begin{bmatrix}
-\varepsilon_1 I & * & * & * \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
0 & -\varepsilon_2 I & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
\end{bmatrix} \begin{bmatrix}
-\varepsilon_1 I & * & * & * \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
0 & -\varepsilon_2 I & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
\end{bmatrix} \begin{bmatrix}
-\varepsilon_1 I & * & * & * \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
0 & -\varepsilon_2 I & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
\end{bmatrix} < 0 \]
Applying Lemma 6 to (60) to ‘eliminate’ the uncertainty $F$, we know that (44) holds if and only if there exists a positive scalar parameter $e_1$ such that the following LMI holds

$$
\begin{bmatrix}
-P & * & * & * & * \\
A + B_2 K - P^{-1} & -P^{-1} & * & * & * \\
I^{1/2} & 0 & -\alpha_1 I & * & * \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
I^{1/2} & 0 & 0 & -\alpha_2 I & * \\
K & 0 & 0 & 0 & -\alpha_1 I^{1} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
K & 0 & 0 & 0 & 0 \\
L_2 & 0 & 0 & 0 & 0 \\
0 & e_1 I & 0 & \cdots & 0 \\
E & 0 & 0 & \cdots & 0 \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
\cdots & * & * & * & * \\
\cdots & * & * & * & * \\
0 & 0 & 0 & e_1 I & * \\
0 & 0 & 0 & 0 & -e_1 I
\end{bmatrix} < 0
$$

(61)

Letting

$$X = P^{-1}
$$

and

$$G = KX
$$

(62)

and performing the congruence transformation $\text{diag}(P^{1/2}, I, I, \ldots, I, I, I, I, I, I, I)$ to (61), we obtain (57).

Similarly, applying Lemma 6 to (45) and performing the congruence transformation $\text{diag}(P^{1/2}, I, I, I, \ldots, I, I, I, I, I, I, I, I)$, we get (58) from (62) and (63). Furthermore, (55) and (56) are obtained by solving (42) and (43) by using (62). Using the property of matrix trace, (54) is derived from (41). This completes the proof.

**Remark 4:** The robust mixed $H_2/H_\infty$ controller can be obtained by solving LMIs (54)–(58) in Theorem 4. The LMIs can be solved efficiently via interior point method [7]. Note that LMIs (54)–(58) are affine in the scalar positive parameters $e_1$ and $e_2$. Hence, they can be defined as LMI variables in order to increase the possibility of the solutions and decrease conservatism with respect to the uncertainty $F$.

So far the controller has been designed which satisfies the requirements (Q1) and (Q2). Because of the advantages of LMI formulations, the results in Theorem 4 also suggest the following two optimisation problems that would be interesting to control engineers:

(P1) The optimal $H_\infty$ control problem with $H_2$ performance constraints for uncertain nonlinear stochastic systems

$$
\min_{Q>0, G, \alpha_1, \alpha_2, e_1>0, e_2>0} \gamma \text{ subject to (54)–(58)}
$$

for some given $\beta
$$

(64)

(P2) The optimal $H_2$ control problem with $H_\infty$ performance constraints for uncertain nonlinear stochastic systems

$$
\min_{Q>0, G, \alpha_1, \alpha_2, e_1>0, e_2>0} \beta \text{ subject to (54)–(58)}
$$

for some given $\gamma
$$

(65)

**Remark 5:** In many engineering applications, the performances constraints are often specified a priori. For example, in Theorem 4, the controller is designed after $H_\infty$ performance and $H_2$ performance are prescribed. In fact, however, we can obtain an improved performance by optimisation method. The problem (P1) will help exploit the design freedom to meet the optimal $H_\infty$ performance under a prescribed $\beta$. The problem (P2) will find an optimal solution among them to achieve the $H_2$ performance under a prescribed $\gamma_2$. These are certainly attractive because the addressed multiobjective problems can be solved while a local optimal performance can also be achieved, and the computation is efficient by using the Matlab LMI toolbox.

## 5 Illustrative example

Consider a discrete-time system described by (1)–(3) with stochastic nonlinearities and deterministic norm-bounded parameter uncertainties as follows

$$A = \begin{bmatrix}
-0.5 & 0 & -0.8 \\
0 & -1.2 & 0 \\
0.6 & 0 & 0.6
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
0.3 \\
0 \\
0.2
\end{bmatrix}
$$

$$B_2 = \begin{bmatrix}
-1 \\
2 \\
1
\end{bmatrix}, \quad L_\infty = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}
$$

$$H_1 = \begin{bmatrix}
0.6 \\
0 \\
0.5
\end{bmatrix}, \quad E = \begin{bmatrix}
0.8 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
$$

where $w_k$ is a zero mean Gaussian white noise sequence with covariance $R = 0.1$. The deterministic uncertainty $F$ satisfies the condition (4), and the stochastic nonlinear function $f(x_k, u_k)$ satisfies

$$E[f_k(x_k, u_k) = 0]
$$

$$E[f_k^T(x_k, u_k) = 0]
$$

$$H_1 = \begin{bmatrix}
0.1 & 0.1 \\
0.2 & 0.2
\end{bmatrix} \begin{bmatrix}
x_k^T \\
0
\end{bmatrix} = \begin{bmatrix}
0.5 & 0 & 0 \\
0.6 & 0 & 0 \\
0 & 0 & 0.7
\end{bmatrix} x_k
$$

$$+ 0.6 u_k^T u_k + x_k^T \begin{bmatrix}
0.6 & 0 & 0 \\
0 & 0 & 0.7 \\
0 & 0 & 0.8
\end{bmatrix} x_k + 0.8 u_k^T u_k
$$

Now, let us examine the following three cases.

**Case 1:** $\gamma = 0.5, \beta = 0.1$.

This case is exactly concerned with the addressed robust $H_2/H_\infty$ control problem, hence can be tackled by using Theorem 4 with $q = 2$. In theory, the solution set is large, and we just provide one solution by employing the Matlab
LMI toolbox

\[ X = \begin{bmatrix} 0.3628 & 0.1928 & -0.1127 \\ 0.1928 & 7.1889 & 5.4349 \\ -0.1127 & 5.4349 & 5.2622 \end{bmatrix} \]

\[ G = [-0.0864 \quad -4.6579 \quad -4.0586] \]

\[ \varepsilon_1 = 0.3826, \quad \varepsilon_2 = 0.3813 \]

\[ \alpha_1 = 26.3259, \quad \alpha_2 = 26.3549, \quad \Theta = 0.0716 \]

\[ \text{tr}(\theta_1 \theta_1^T X^{-1}) = 0.0377 < \alpha_1^{-1} = 0.0380 \]

\[ \text{tr}(\theta_2 \theta_2^T X^{-1}) = 0.0377 < \alpha_2^{-1} = 0.0379 \]

\[ K = [-0.2732 \quad -0.2422 \quad -0.5270] \]

**Case 2:** \( \beta = 0.1 \)

In this case, we wish to design the controller which minimises the \( H_\infty \) performance under the \( H_2 \) performance constraints. That is, we want to solve the problem (P1). Solving the optimisation problem (64) using LMI toolbox yields the minimum value \( \gamma_{\text{min}}^2 = 0.3958 \)

\[ X = \begin{bmatrix} 0.5657 & 0.3112 & -0.1701 \\ 0.3112 & 9.8502 & 7.5724 \\ -0.1701 & 7.5724 & 7.2331 \end{bmatrix} \]

\[ G = [-0.1399 \quad -6.4779 \quad -5.9296] \]

\[ \varepsilon_1 = 0.9033, \quad \varepsilon_2 = 0.9008 \]

\[ \alpha_1 = 37.4043, \quad \alpha_2 = 37.4862, \quad \Theta = 0.0661 \]

\[ \text{tr}(\theta_1 \theta_1^T X^{-1}) = 0.0263 < \alpha_1^{-1} = 0.0267 \]

\[ \text{tr}(\theta_2 \theta_2^T X^{-1}) = 0.0263 < \alpha_2^{-1} = 0.0267 \]

\[ K = [-0.2583 \quad -0.2583 \quad -0.5089] \]

**Case 3:** \( \gamma = 0.5 \)

We now deal with the problem (P2). Solving the optimisation problem (65), we obtain the minimum \( H_2 \) performance \( \beta_{\text{min}} = 0.00319 \), and

\[ X = \begin{bmatrix} 0.5829 & 0.3100 & -0.1853 \\ 0.3100 & 10.4074 & 8.0742 \\ -0.1853 & 8.0742 & 7.6853 \end{bmatrix} \]

\[ G = [-0.1368 \quad -6.8501 \quad -5.9134] \]

\[ \varepsilon_1 = 0.9687, \quad \varepsilon_2 = 0.9673 \]

\[ \alpha_1 = 38.4558, \quad \alpha_2 = 38.4895, \quad \Theta = 0.0319 \]

\[ \text{tr}(\theta_1 \theta_1^T X^{-1}) = 0.0259 < \alpha_1^{-1} = 0.0260 \]

\[ \text{tr}(\theta_2 \theta_2^T X^{-1}) = 0.0259 < \alpha_2^{-1} = 0.0260 \]

\[ K = [-0.2516 \quad -0.2652 \quad -0.4969] \]

The results show that the designed system can satisfy \( H_2 \) control performance and \( H_\infty \) disturbance rejection performance simultaneously. In Case 2, in order to achieve a better disturbance rejection performance, the optimisation algorithm (P1) is employed to obtain the optimal solution. Similarly, to get a better \( H_2 \) control performance, the optimisation algorithm (P2) is applied to obtain the optimal solution in Case 3. Furthermore, we can see from the results that the values of \( \text{tr}(\theta_i \theta_i^T X^{-1}) \) \( (i = 1, 2) \) is very close to \( \alpha_i^{-1} \) \( (i = 1, 2) \) in all three cases, hence the possible conservatism could be significantly reduced.

**Remark 6:** Within the LMI framework developed in this paper, we can show that there are some trade-off that can be used for satisfying specific performance requirements. For example, the \( H_\infty \) performance will be improved if the \( H_2 \) performance constraints become more relaxed (larger). Also, if the value of the \( H_\infty \) performance constraint is allowed to be increased, then the \( H_2 \) performance can be further reduced. Hence, the proposed approach allows much flexibility in making compromise between the \( H_2 \) performance and the \( H_\infty \) performance, while the essential multiple objectives can all be met simultaneously.

**6 Conclusions**

A robust mixed \( H_2/H_\infty \) controller has been designed in this paper for a class of uncertain discrete-time nonlinear stochastic systems. A key technique has been used to deal with the matrix trace terms arising from the stochastic nonlinearities, and the well-known S-procedure has been adopted to handle the deterministic uncertainties. A unified framework has been established to solve the addressed mixed \( H_2/H_\infty \) control problem, and sufficient conditions for the solvability of the mixed \( H_2/H_\infty \) control problem have been given in terms of a set of feasible LMI. Two types of the optimisation problems have been proposed by optimising either the \( H_2 \) performance or the \( H_\infty \) performance. Our method can also be extended to output feedback case, and the results will appear in the near future.

**7 Acknowledgments**

This work was supported in part by the Engineering and Physical Sciences Research Council (EPSRC) of the UK under Grant GR/S27658/01, the Nuffield Foundation of the UK under Grant NAL/00630/G and the Alexander von Humboldt Foundation of Germany, the National Natural Science Foundation of China under Grant 60474049 and the Fujian provincial Natural Science Foundation of China under Grant A0410012.

**8 References**


*IEE Proc.-Control Theory Appl., Vol. 153, No. 2, March 2006*