Spectra of massive and massless QCD Dirac operators: A novel link

G. Akemann[1] and E. Kanzieper[2]

[1] *Max–Planck–Institut für Kernphysik, Saupfercheckweg 1, D-69117 Heidelberg, Germany*

[2] *The Abdus Salam International Centre for Theoretical Physics, P.O. Box 586, I-34100 Trieste, Italy*

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We show that integrable structure of chiral random matrix models incorporating global symmetries of QCD Dirac operators (labeled by the Dyson index $\beta = 1, 2$, and 4) leads to emergence of a connection relation between the spectral statistics of massive and massless Dirac operators. This novel link established for β–fold degenerate massive fermions is used to explicitly derive (and prove the random matrix universality of) statistics of low–lying eigenvalues of QCD Dirac operators in the presence of SU(2) massive fermions in the fundamental representation ($\beta = 1$) and SU($N_c \geq 2$) massive adjoint fermions ($\beta = 4$). Comparison with available lattice data for SU(2) dynamical staggered fermions reveals a good agreement.

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Explicit knowledge of spectral statistics of low–lying eigenvalues of the Dirac operator is required to understand the phenomenon of chiral symmetry breaking (χSB) in quantum chromodynamics (QCD). It has first been conjectured by Verbaarschot and collaborators [1] that extreme infrared limit of the QCD Dirac operator spectrum can be described by the large–N chiral Random Matrix Theory (RMT) that models the true Dirac operator $\mathcal{D}$ by $N \times N$ block offdiagonal matrix $\mathcal{D}^{\rm RMT} = {\rm offdiag}(iW, iW^\dagger)$, W being $n \times m$ rectangular random matrix [see Eq. (1) below]. In such a formulation, $N = n+m$ is an analog of dimensionless space–time volume V, while $\nu = |n-m|$ is equivalent to the topological charge (equal to the number of zero modes of $\mathcal{D}$). If, in addition, the entries of W are chosen to be real, complex, or quaternion real, the random matrix $\mathcal{D}^{\rm RMT}$ possesses proper antiunitary symmetry (labeled by the Dyson index $\beta = 1, 2$, or 4) and, hence, correctly reproduces both the underlying symmetries of the Dirac operator and the χSB pattern associated with it. On the language of the chiral QCD Lagrangian, the above approach corresponds to the limit [2] $1/\Lambda \ll V^{1/4} \ll 1/m_\pi$ (Λ is a typical hadronic scale and m_π is the pion mass) in which the kinetic term in Lagrangian can be neglected, and only the global symmetries of the Dirac operator become important. Recently, RMT phenomenology has been put onto a firm field theoretic ground represented by the framework of finite–volume partition functions [3] and the partially quenched perturbation theory [4].

On the microscopic scale $\sim 1/V\Sigma$, chiral RMT (defined for a given topological charge ν [5]) leads to parameter–free predictions for the unfolded microscopic spectral density $\rho_S(\lambda) = \lim_{V\to\infty}(V\Sigma)^{-1}\rho(\lambda/V\Sigma)$ of the Dirac operator. Here, the absolute value Σ of the chiral condensate (the order parameter of χSB) is related to the Dirac spectral density $\rho(0)$ at zero virtuality through the Banks–Casher relation $\Sigma = \pi\rho(0)/V$ [6]. In a series of papers [7], RMT predictions have been confronted to the lattice data, and good agreement has been found for the spectral density, two–level correlation function, and distribution of the smallest Dirac eigenvalue for *massless* lattice data of all χSB patterns. The lattice data for *massive* fermions have recently appeared [8] as well.

Unfortunately, available theoretical results for microscopic spectral correlators of massive QCD Dirac operators are rather poor, being restricted to the $\beta = 2$ symmetry class [9,10] associated with the gauge group SU($N_c \geq 3$) in the fundamental representation. It is the aim of the present Letter to show that integrable structure of chiral RMT results in a simple but powerful link between the spectral statistics of massive and massless QCD Dirac operators for all three symmetry classes $\beta = 1, 2$, and 4. The connection relation [Eq. (6)], established below for β–fold degenerate massive fermions within the framework of chiral RMT, relates partially unknown massive spectral correlation functions to the massless ones (taken at both positive and fictitious negative energies) [11]. As the latters have already received a detailed study in the literature, this link not only solves the problem posed but also provides a particularly simple proof of RMT–universality of massive correlation functions, that becomes a consequence of celebrated universality [12–14] proven for the massless case.

Let us start with the definitions [10,15]. The joint probability distribution function of chiral random matrix ensemble associated with N_f massive fermions in the sector of topological charge ν is given by

$$P_n^{(N_f,\nu,\beta)}(\lambda_1,\ldots,\lambda_n) = \frac{1}{Z_n^{(N_f,\nu,\beta)}(\{m\})} \times |\Delta_n(\{\lambda\})|^\beta \prod_{i=1}^n [w_{\beta,\nu}(\lambda_i) \prod_{f=1}^{N_f} m_f^\nu(\lambda_i + m_f^2)]. \qquad (1)$$

Here, $\{\lambda\} \geq 0$ are the eigenvalues of the matrix $WW^\dagger$, $\Delta_n(\{\lambda\}) = \prod_{i<j}^n(\lambda_i - \lambda_j)$ is the Vandermonde determinant, the weight function $w_{\beta,\nu}$ is $w_{\beta,\nu}(\lambda) = \lambda^{\frac{\beta}{2}\nu + \frac{\beta}{2} - 1} e^{-\beta V(\lambda)}$, $V(\lambda)$ is the finite–polynomial confinement potential, and the topological charge ν is taken to be positive integer or zero.

The p–point correlation function in the above ensemble is expressed as [16]

$$R_{n,p}^{(N_f,\nu,\beta)}(\lambda_1,\dots,\lambda_p) = \frac{n!}{(n-p)!} \times \int_0^{+\infty} d\lambda_{p+1}\dots d\lambda_n P_n^{(N_f,\nu,\beta)}(\lambda_1,\dots,\lambda_p,\lambda_{p+1},\dots,\lambda_n). \quad (2)$$

For $p=0$ this yields the mass–dependent partition function $Z_n^{(N_f,\nu,\beta)}(\{m\})$ appearing in Eq. (1). The unfolded spectra of the Dirac operator are then obtained from the appropriately unfolded spectra $\hat{R}_p^{(N_f,\nu,\beta)}(\{\lambda\})$ of associated random matrix model Eq. (1) by a simple transformation of variables:

$$\rho_S(\lambda_1,\dots,\lambda_p) = 2^p \prod_{k=1}^{p} |\lambda_k| \hat{R}_p^{(N_f,\nu,\beta)}(\lambda_1^2,\dots,\lambda_p^2). \quad (3)$$

We notice that for massive correlation functions this also demands to rescale the quark masses, $\mu_f = m_f V\Sigma$. This completes our definition of the model.

In what follows, we assume that the massive fermions are β–fold degenerate, $\mathcal{M}_\beta = (m_1 \mathbb{1}_\beta,\dots,m_{N_f}\mathbb{1}_\beta)$, so that appropriate matrix ensemble is given by the joint probability distribution function $P_n^{(\beta N_f,\nu,\beta)}(\{\lambda\})$. Since in this case, $P_n^{(\beta N_f,\nu,\beta)}(\{\lambda\})$ contains a positive definite factor $\prod_{f=1}^{N_f}(\lambda_i+m_f^2)^\beta$, it can conveniently be absorbed into a Vandermonde determinant of a larger dimension

$$\Delta_{n+N_f}(\{\lambda\},\{-m^2\}) \equiv \Delta_n(\{\lambda\})\Delta_{N_f}(\{-m^2\}) \times \prod_{i=1}^{n}\prod_{f=1}^{N_f}(\lambda_i+m_f^2). \quad (4)$$

This immediately results in a pretty fact that the partition function of the model Eq. (1) with βN_f massive fermions can be expressed in terms of associated *massless* N_f–point correlation function $R_{n+N_f,N_f}^{(0,\nu,\beta)}(\{-m^2\})$ of the matrix ensemble of larger dimension, $(n+N_f)\times(n+N_f)$, taken at fictitious *negative* energies:

$$\frac{Z_n^{(\beta N_f,\nu,\beta)}(\{m\})}{Z_{n+N_f}^{(0,\nu,\beta)}} = \frac{n!}{(n+N_f)!}\left(\prod_{f=1}^{N_f}\frac{m_f^{\beta\nu}}{w_{\beta,\nu}(-m_f^2)}\right) \times \frac{R_{n+N_f,N_f}^{(0,\nu,\beta)}(-m_1^2,\dots,-m_{N_f}^2)}{\left|\Delta_{N_f}(\{-m^2\})\right|^\beta}. \quad (5)$$

The same strategy is applied to the p–point correlator, Eq. (2). After a few transformations, we arrive at the following remarkable relationship:

$$R_{n,p}^{(\beta N_f,\nu,\beta)}(\lambda_1,\dots,\lambda_p) = \frac{R_{n+N_f,p+N_f}^{(0,\nu,\beta)}(\lambda_1,\dots,\lambda_p,-m_1^2,\dots,-m_{N_f}^2)}{R_{n+N_f,N_f}^{(0,\nu,\beta)}(-m_1^2,\dots,-m_{N_f}^2)}. \quad (6)$$

Finite–n Eq. (6) establishes a link between massive and massless spectral correlators via associated chiral random matrix ensemble, and represents a basic relation to be examined in the rest of the Letter, where we consider the most interesting symmetry classes $\beta=1$ and 4. It should be stressed that, in spite of seeming simplicity, the link Eq. (6) is not obvious as it involves correlation functions of massless chiral ensemble taken at both positive and fictitious *negative* energies. We also wish to emphasize that, in the microscopic limit, Eq. (6) immediately leads to the RMT–universality of the microscopic massive correlators which becomes a simple consequence of the universality phenomenon established for massless correlation functions [12,13].

(i) Let us turn to the $\beta=4$ symmetry class, associated with SU($N_c\geq 2$) massive adjoint fermions. In the vicinity of the hard edge, the unfolded p–point spectral correlators in the *massless* chiral model Eq. (1), $N_f=0$, admit quaternion determinant representation [16]

$$\hat{R}_p^{(0,\nu,4)}(\{\lambda\}) = \mathrm{Qdet}\left[f_4(\lambda_i,\lambda_j)\right]_{1\leq i,j\leq p}. \quad (7)$$

Here, the 2×2 matrix kernel $f_4\equiv f_{\beta=4}$ [17]

$$f_\beta(X,Y) = \begin{pmatrix} S_\beta(X,Y) & D_\beta(X,Y) \\ I_\beta(X,Y) & S_\beta(Y,X)\end{pmatrix},\ D_\beta(X,Y) = -\partial_Y S_\beta(X,Y),\ I_\beta(X,Y) = \int_Y^X dZ S_\beta(Z,Y) - \epsilon(X-Y)\delta_{\beta,1}, \quad (8)$$

$\epsilon(X) = (1/2)\mathrm{sgn}(X)$, is uniquely specified by the function [18]

$$S_4(X,Y) = 2K_{2\nu+1}(2X^{1/2},2Y^{1/2}) - \frac{J_{2\nu}(2X^{1/2})}{4X^{1/2}}\int_0^{2Y^{1/2}} dt J_{2\nu+2}(t) \quad (9)$$

with $K_\alpha(X,Y)$ being the Bessel kernel [19]:

$$K_\alpha(X,Y) = \frac{XJ_{\alpha+1}(X)J_\alpha(Y) - YJ_{\alpha+1}(Y)J_\alpha(X)}{2(X^2-Y^2)}. \quad (10)$$

It is important to stress that Eqs. (9) and (10) hold generically [13] for arbitrary finite–polynomial confinement potential $V(\lambda)$ in Eq. (1).

As long as at $\beta=4$ the definitions Eqs. (1) and (2) do not discriminate between the positive and (fictitious) negative eigenvalues λ_k, Eqs. (9) and (10) remain valid at negative arguments also, provided one makes use of the fact that $J_\alpha(iX) = i^\alpha I_\alpha(X)$. This circumstance allows us, at once, to derive closed expressions for p–point *massive* correlation functions in the microscopic limit. Straightforward calculations based on Eqs. (3), (6) and (7) yield:

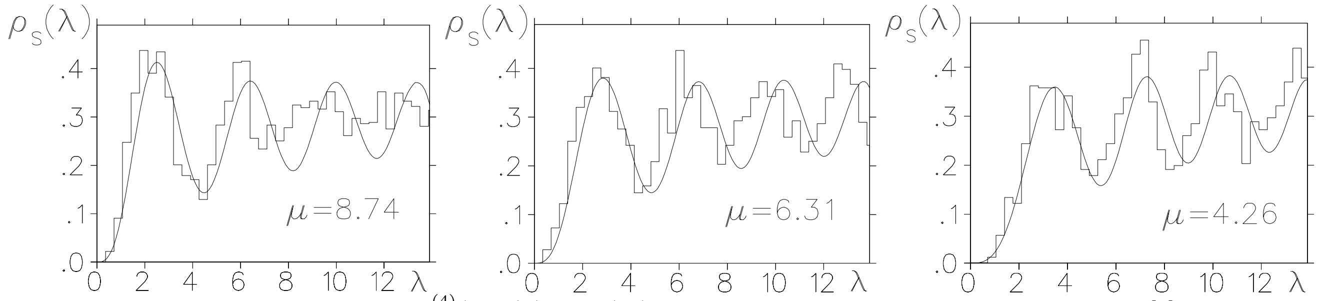

FIG. 1. Microscopic massive density $\rho_S^{(4)}(\lambda;\mu 1\!\!1_4)$, Eq. (12), plotted versus lattice data simulated in Ref. [8] for 8 flavors of masses $\mu = 8.74, 6.31$, and 4.26. For discussion of species doubling and systematic deviations at large λ we refer to Ref. [8].

$$\rho_S^{(4N_f)}(\lambda_1,\dots,\lambda_p;\mu_1 1\!\!1_4,\dots,\mu_{N_f} 1\!\!1_4) = 2^p \prod_{k=1}^{p} |\lambda_k| \times \frac{\mathrm{Qdet}\left[\begin{array}{cc} f_4(\lambda_i^2,\lambda_j^2) & f_4(\lambda_i^2,-\mu_{f'}^2) \\ f_4(-\mu_f^2,\lambda_j^2) & f_4(-\mu_f^2,-\mu_{f'}^2)\end{array}\right]}{\mathrm{Qdet}[f_4(-\mu_f^2,-\mu_{f'}^2)]}, \quad (11)$$

$1 \le i,j \le p$, $1 \le f,f' \le N_f$. We remind that this result applies to 4–fold degenerate quark masses, $\mu_f = m_f V\Sigma$.

Microscopic density for the particular case of 4 degenerate fermions of mass μ is of special interest as it can be compared to available lattice data for dynamical SU(2) staggered fermions in the fundamental representation simulated in Ref. [8] at $\nu = 0$. [Because of the lattice symmetry of staggered fermions they belong to the symmetry class $\beta = 4$]. Computing quaternion determinants in Eq. (11), we come down to

$$\rho_S^{(4)}(\lambda;\mu 1\!\!1_4) = 2|\lambda| \left(S_4(\lambda^2,\lambda^2) - \frac{S_4(-\mu^2,\lambda^2)S_4(\lambda^2,-\mu^2) - I_4(\lambda^2,-\mu^2)D_4(\lambda^2,-\mu^2)}{S_4(-\mu^2,-\mu^2)} \right). \quad (12)$$

We have explicitly checked that the limit $\mu \to 0$ reproduces the known massless result [1] at a shifted topological charge $\nu \to \nu + 2$ [20]. Theoretical results plotted in Fig. 1 for three different values of μ show reasonable agreement with numerical data.

We close our consideration of $\beta = 4$ symmetry class by giving a compact expression for the previously unknown massive RMT (or finite–volume [21]) partition function with 4–fold degenerate massive fermions [see Eq. (5)]:

$$\tilde{Z}_\nu^{(4N_f)}(\mu_1 1\!\!1_4,\dots,\mu_{N_f} 1\!\!1_4) = \frac{(-1)^{N_f}\,\mathrm{Qdet}[f_4(-\mu_f^2,-\mu_{f'}^2)]}{|\Delta_{N_f}(-\mu_f^2)|^4 \prod_{f=1}^{N_f} \mu_f^2}. \quad (13)$$

Here, only nontrivial mass dependence has been displayed.

(ii) Now we turn to the $\beta = 1$ symmetry class associated with SU(2) massive fermions in the fundamental representation. In this case, the modulus of the Vandermonde determinant in Eqs. (1) and (2) makes all p–point correlation functions to be nonanalytic functions of their arguments. This is exactly the reason of why one cannot use known expressions for massless correlation functions $\hat{R}_p^{(0,\nu,1)}(\{\lambda\})$ to naively compute them at negative energies. Below we show how to circumvent this obstacle for the simplest situation of the spectral density with a single quark mass. Extension to higher order correlation functions and/or larger number of masses is straightforward.

In accordance with the connection relation Eq. (6), the finite–n massive spectral density equals

$$R_{2n-1,1}^{(1,\nu,1)}(\lambda) = \frac{R_{2n,2}^{(0,\nu,1)}(\lambda,-m^2)}{R_{2n,1}^{(0,\nu,1)}(-m^2)}, \quad (14)$$

where, for definiteness, we have fixed the dimension of the massless random matrix ensemble to be even, $2n$; from now on, the superscripts are omitted for brevity. We observe that the function $R_{2n,2}(\lambda,-m^2)$ can be evaluated through the functional derivative of $R_{2n,1}(-m^2;[W]) \equiv R_{2n,1}(-m^2)$ with respect to the confinement potential W, $\exp\{-W(\lambda)\} = \lambda^{(\nu-1)/2}\exp\{-V(\lambda)\}$:

$$R_{2n,2}(\lambda,-m^2) \equiv R_{2n,1}(-m^2)\left(R_{2n,1}(\lambda) - \delta(\lambda+m^2)\right) - \frac{\delta}{\delta W(\lambda)} R_{2n,1}(-m^2;[W]). \quad (15)$$

To facilitate taking the functional derivative in Eq. (15), we utilize the approach of Ref. [22] [see Eq. (A6) of second reference], but express $R_{2n,1}(-m^2;[W])$ in terms of arbitrary polynomials $p_j(x)$ rather than in terms of the skew orthogonal ones:

$$R_{2n,1}(-m^2;[W]) = \frac{1}{2} e^{-W(-m^2)} \sum_{j,k=0}^{2n-1} p_j(-m^2)\mu_{jk}[W] \times \int_0^{+\infty} dZ\; e^{-W(Z)} p_k(Z). \quad (16)$$

The $2n \times 2n$ real antisymmetric matrix $\mu_{jk}[W]$ is the inverse to the matrix [17]

$$M_{jk} = \int_0^{+\infty} dx dy\; e^{-W(x)-W(y)} \epsilon(x-y) p_j(x) p_k(y), \quad (17)$$

$\epsilon(x) = (1/2)\mathrm{sgn}(x)$. Substituting Eq. (16) into Eq. (15), and then into Eq. (14), we are able to express the finite–n massive spectral density $R^{(1,\nu,1)}_{2n-1,1}(\lambda)$ in the form

$$R^{(1,\nu,1)}_{2n-1,1}(\lambda) = S_1^{(2n)}(\lambda,\lambda) - \frac{S_1^{(2n)}(-m^2,\lambda)S_1^{(2n)}(\lambda,0) - I_1^{(2n)}(0,\lambda)D_1^{(2n)}(-m^2,\lambda)}{S_1^{(2n)}(-m^2,0)} \quad (18)$$

that contains the entries of finite–n, 2×2 matrix kernel $f_1^{(2n)}(X,Y) \equiv f_{\beta=1}^{(2n)}(X,Y)$ of the massless ensemble [see Eq. (8)]. In deriving Eq. (18) we have used both the representation [17] $S_1^{(2n)}(X,Y) = -e^{-W(X)}\sum_{j,k=0}^{2n-1} p_j(X)\mu_{jk}\int_0^{+\infty} dZ\epsilon(Y-Z)e^{-W(Z)}p_k(Z)$, and Eq. (8). Finally, taking into account Eqs. (3), (18), and the universal [13] formula [18]

$$S_1(X,Y) = K_{\nu-1}(X^{1/2},Y^{1/2}) - \frac{J_\nu(X^{1/2})}{4X^{1/2}}\left(\int_0^{Y^{1/2}} dt J_{\nu-2}(t) - 1\right) \quad (19)$$

(valid in the vicinity of the hard edge), we deduce a closed expression for the microscopic single–mass spectral density $\rho_S^{(1)}(\lambda;\mu)$. It exhibits the quaternion determinant structure of Eq. (12) with obvious changes in arguments of S, D, and I functions as is given by Eq. (18). As a consistency check, we have verified that for $\mu = 0$ it reduces to the known result [1] for one massless flavor. Let us stress, that universal form [13] of the function S_1 thus confirms the universality of the massive spectral density following in a more general context directly from the microscopic limit of the connection relation Eq. (6).

In conclusion, we have derived universal expressions for spectral correlators of massive chiral matrix ensembles corresponding to β–fold degenerate massive fermions, by establishing a new link between the statistics of massive and massless random matrices. The results obtained have been compared to the available lattice data associated with $\beta = 4$ χSB pattern in low–energy QCD.

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Note added.–After completing this work, the preprint [23] by T. Nagao and S.M. Nishigaki on finite–volume partition functions has appeared. In particular, these authors give alternative representations of our Eqs. (11) and (13).

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