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On Finite-Volume Gauge Theory Partition Functions

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Abstract

We prove a Mahoux-Mehta–type theorem for finite-volume partition functions of $SU(N_c \geq 3)$ gauge theories coupled to fermions in the fundamental representation. The large-volume limit is taken with the constraint $V << 1/m_\pi^4$. The theorem allows one to express any k-point correlation function of the microscopic Dirac operator spectrum entirely in terms of the 2-point function. The sum over topological charges of the gauge fields can be explicitly performed for these k-point correlation functions. A connection to an integrable KP hierarchy, for which the finite-volume partition function is a τ-function, is pointed out. Relations between the effective partition functions for these theories in 3 and 4 dimensions are derived. We also compute analytically, and entirely from finite-volume partition functions, the microscopic spectral density of the Dirac operator in $SU(N_c)$ gauge theories coupled to quenched fermions in the adjoint representation. The result coincides exactly with earlier results based on Random Matrix Theory.

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1 Introduction

While it has been known for some years that Random Matrix Theory provides universality classes that describe the microscopic spectrum of the Dirac operator in theories with spontaneous breaking of chiral symmetry [1, 2, 3], it is only more recently that the precise relationship is become unraveled [4, 5, 6]. The essential point is that the Random Matrix Theory partition function, in a particular scaling limit, becomes *identical* to the effective field theory partition function in an analogous scaling limit [1, 7]. This scaling limit is best thought of in finite-size scaling terms, as it requires sending the space-time volume V to infinity with the constraint that $V \ll 1/m_\pi^4$, where m_π generically indicates the pseudo-Goldstone masses. At the level of fermion masses m_i, one keeps the product $\mu_i = m_i\Sigma V$ fixed, Σ being the infinite-volume chiral condensate. Using the universality proof of ref. [3] one easily establishes that the relation between the partition functions holds universally, independently of the chosen Random Matrix Theory potential [8]. The established identity between the two partition functions is not sufficient to establish that the microscopic Dirac operator spectrum can be computed in Random Matrix Theory, but a series of surprisingly simple relations that express all microscopic spectral correlators of on Random Matrix Theory in terms of the universal partition functions [4] indicate that one must be able to compute the microscopic Dirac operator spectrum directly from an effective field theory that is suitably extended by additional fermionic species. The supersymmetric formulation of partially quenched effective lagrangians provides an analytical framework where this can be established [5, 6]. This is an important point, because it *proves*, starting directly from the effective field theory partition function that the microscopic Dirac operator spectrum coincides with that obtained universally from Random Matrix Theory.

In the derivation of microscopic spectral correlators of the Dirac operator one makes efficient use of the fact that one can insert factors of unity inside the field theory path integral by means of cancelling fermionic and bosonic degrees of freedom. Thus, for the one-point spectral function, the spectral density itself, it suffices to insert one such pair, and hence consider a partially quenched field theory partition function of the form

$$Z_\nu \;=\; \left(\prod_{f=1}^{N_f} m_f^\nu\right)\left(\frac{m_v^F}{m_v^B}\right)^\nu \int [dA]_\nu \, \frac{\det(i\rlap{\,/}D - m_v^F)}{\det(i\rlap{\,/}D - m_v^B)} \prod_{f=1}^{N_f} \det(i\rlap{\,/}D - m_f)\; e^{-S_{YM}[A]} \; . \qquad (1.1)$$

Here ν is the topological charge, and the additional ("quenched") fermion-boson species have masses m_v^F, m_v^B that eventually are taken to be equal (which makes the two determinants cancel). It is also clear [5] that in order to derive higher k-point spectral correlation functions one will have to insert k such additional factors, which pairwise cancel in the end:

$$Z_\nu \;=\; \left(\prod_{f=1}^{N_f} m_f^\nu\right)\left(\prod_{i=1}^{k}\frac{m_{vi}^F}{m_{vi}^B}\right)^\nu \int [dA]_\nu \, \prod_{j=1}^{k}\frac{\det(i\rlap{\,/}D - m_{vj}^F)}{\det(i\rlap{\,/}D - m_{vj}^B)} \prod_{f=1}^{N_f} \det(i\rlap{\,/}D - m_f)\; e^{-S_{YM}[A]} \; . \qquad (1.2)$$

Let us for clarity here briefly restrict ourselves to the universality class of the chiral Unitary Ensemble (chUE) in Random Matrix Theory language, which is the appropriate universality class for SU($N_c \geq 3$) gauge groups with N_f fermions in the fundamental representation (as was implicitly assumed when we wrote down the partially quenched partition functions above). To obtain the spectral correlation function from the effective finite-volume partition function one needs to take a discontinuity at a cut in a kth order chiral susceptibility. Using the technique of the first of ref. [5] it should be possible to rewrite this, at the level of the effective lagrangian, in terms of a partition function extended with $2k$

additional species, all of purely imaginary masses. While this has not yet been explicitly established beyond the one-point function, it seems beyond any doubt that it will be possible to carry such a program through. The reason one can say this with such confidence is that it is known, if one accepts the use of Random Matrix Theory for *all* spectral correlators, that the k-point spectral function in the chUE universality class can be be written [4]

$$\rho_S^{(\nu)}(\xi_1,\ldots,\xi_k;\{\mu\}) = C_2^{(k)}\prod_{i=1}^{k}\left(|\xi_i|\prod_{f=1}^{N_f}(\xi_i^2+\mu_f^2)\right)\prod_{j<l}^{k}(\xi_j^2-\xi_l^2)^2\,\frac{\mathcal{Z}_\nu^{(N_f+2k)}(\{\mu\},\{i\xi_1\},\ldots,\{i\xi_k\})}{\mathcal{Z}_\nu^{(N_f)}(\{\mu\})}, \tag{1.3}$$

where on the right hand side each additional mass $i\xi_j$ is two-fold degenerate. Indeed, the supersymmetric coset needed to derive the k-point spectral function is $Gl(N_f+k|k)$ [5], which precisely involves a finite-volume partition function of $k+k$ additional species.

However, it is known from Random Matrix Theory that there exists a different, and far more compact, expression for the spectral k-point function:

$$\rho_S^{(\nu)}(\xi_1,\ldots,\xi_k;\{\mu\}) \;=\; \det_{a,b} K_S^{(\nu)}(\xi_a,\xi_b;\{\mu\})\ , \tag{1.4}$$

where the microscopic kernel $K_S^{(\nu)}(\xi_a,\xi_b;\{\mu_i\})$ also can be expressed in terms of finite-volume partition functions alone [4]:

$$K_S^{(\nu)}(\xi,\xi';\mu_1,\ldots,\mu_{N_f}) \;=\; (-1)^{\nu+[N_f/2]}\sqrt{\xi\xi'}\prod_{f=1}^{N_f}\sqrt{(\xi^2+\mu_f^2)(\xi'^2+\mu_f^2)}\;\frac{\mathcal{Z}_\nu^{(N_f+2)}(\{\mu\},i\xi,i\xi')}{\mathcal{Z}_\nu^{(N_f)}(\{\mu\})}\ . \tag{1.5}$$

The formula (1.4) is just one out of three compact expressions for the k-point spectral correlators of all three classical matrix ensembles, which we generically (although their history date much further back) shall denote Mahoux-Mehta relations [9]. Taken together with the expressions (1.3) and (1.5) it implies a surprising identity for the partition function, which was noted in the last of reference [4], and which we shall denote Consistency Condition I (to be stated in precise form in section 2 below).

Although Consistency Condition I is valid without any doubt, it has been derived through the rather tortuous route of going through a Random Matrix Theory representation for the partition function. It is of interest to see if this identity can be proven *directly* from the effective partition function itself, without recourse to Random Matrix Theory. In this way one can logically replace the expression for the k-point spectral correlation function derived through the supersymmetric technique (1.3) by the much more compact Master Formula (1.5) and the relation (1.4), without having to make use the Random Matrix Theory formulation. One of the purposes of this paper is to provide such a direct algebraic proof.

Random Matrix Theory implies a number of other identities among the effective finite-volume partition functions, that cannot easily be guessed from these partition functions themselves. One of these, which we shall denote Consistency Condition II below, can be derived in the Random Matrix Theory language from the relationship between orthogonal polynomials and the kernel. We shall prove this relation, too. Curiously, if read conversely this one single relation allows one to derive unambiguously the effective partition function for any number of flavors N_f, starting with just two "boundary conditions", such as the effective partition function for $N_f=0$ and $N_f=1$ (which are both trivial).

Yet other partition function identities arise from the rather simple relationship between orthogonal polynomials in the chUE and UE Random Matrix Theories, relevant for QCD-like theories in 4 and

3 space-time dimensions, respectively. As we shall show in this paper (see section 4), these relations imply surprising identities among the well-known Harish-Chandra integral and the external field problem, both for gauge groups $U(N_f)$.

The existence of a long list of "miraculous" identities involving the effective partition function that is relevant for the microscopic Dirac operator spectrum is undoubtedly related to the fact that this partition function can be written as a τ-function of the integrable KP hierarchy, as we shall discuss in section 5. The connection between the microscopic Dirac operator spectrum and this integrable system is certainly interesting in its own right, and may in addition be used to shed new light on the universal analytical expressions that have been obtained. It is also intriguing that this connection suggests that the effective theories relevant for describing the microscopic part of the Dirac operator spectrum may be "topological" in the sense of Witten [10], *i.e.* having an entirely different formulation in terms of a BRST-exact field theory action. Intuitively this is perhaps understandable from the fact that the effective partition function in this regime stems from the zero modes of the pseudo-Goldstone bosons alone, with no kinetic energy contribution. No degrees of freedom can therefore propagate in this effective theory.

Having established what we call Consistency Condition I, we get as a by-product the ν-dependent normalization factor in front of the spectral k-point function. With this factor explicitly known, one can then perform the sum over topological charges ν of the gauge field configurations to provide a compact expression for this k-point function in terms of the full effective partition functions that are already summed over topological charges. We do this simple derivation of the full k-point function in section 6. In section 7 we turn to one of the other universality classes, labeled chSE, which is the one relevant for SU($N_c \geq 2$) gauge theories with N_f fermions in the adjoint representation. The analogous formula for the microscopic spectral density, as derived through Random Matrix Theory [4], involves the partition function of four additional species. We check that the resulting formula is correct at least in the fully quenched case by explicitly evaluating the relevant effective partition function for four flavors. Finally, section 8 contains our brief conclusions. Some technical details are relegated to the appendices.

Before starting with the main part of the paper let us fix the notation for clarity. The finite-volume partition function for QCD-like theories in even space-time dimensions can be written [11]

$$\mathcal{Z}_\nu^{(N_f)}(\{\mu\}) \;=\; \det A(\{\mu\})/\Delta(\{\mu^2\}) \tag{1.6}$$

where the matrix A is defined either by

$$A(\{\mu\})_{ij} = \mu_i^{j-1} I_{\nu+j-1}(\mu_i) \;\;,\quad i,j = 1,\ldots,N_f \;, \tag{1.7}$$

or alternatively (by making use of standard Bessel function identities, and invariance properties of the determinant),

$$A(\{\mu\})_{ij} = \mu_i^{j-1} I_\nu^{(j-1)}(\mu_i) \;\;,\quad i,j = 1,\ldots,N_f \;. \tag{1.8}$$

The denominator is given by the Vandermonde determinant of the squared masses

$$\Delta(\{\mu^2\}) \;\equiv\; \prod_{i>j}^{N_f}(\mu_i^2-\mu_j^2) \;=\; \det_{i,j}\left[(\mu_i^2)^{j-1}\right] \;. \tag{1.9}$$

In the literature a different sign is very often chosen inside the product, which leads to an overall N_f-dependent sign compared with the determinant $\det_{i,j}[(\mu_i^2)^{j-1}] = (-1)^{N_f(N_f-1)/2}\prod_{i<j}^{N_f}(\mu_i^2-\mu_j^2)$.

This just gives an overall sign in the partition function eq. (1.6), which normally is irrelevant. Let us stress, however, that only with the definition above the partition function $\mathcal{Z}_\nu^{(N_f)}(\{\mu\})$ is a positive quantity. We choose the form (1.9) in what follows.

2 Consistency Condition I

First of all let us recall the consistency condition relating the determinant of partition functions with N_f+2 flavors $\mathcal{Z}_\nu^{(N_f+2)}$ and a partition functions with N_f+2k flavors as it has been stated in [4]. Using properties of determinants eq. (11) of the third ref. [4] is equivalent to

$$\det_{1\leq a,b\leq k}\left[\frac{\mathcal{Z}_\nu^{(N_f+2)}(\{\mu\},\xi_a,\xi_b)}{\mathcal{Z}_\nu^{(N_f)}(\{\mu\})}\right] \;=\; \prod_{i<j}^{k}\left(\xi_i^2-\xi_j^2\right)^2\;\frac{\mathcal{Z}_\nu^{(N_f+2k)}(\{\mu\},\xi_1,\xi_1,\ldots,\xi_k,\xi_k)}{\mathcal{Z}_\nu^{(N_f)}(\{\mu\})}\;. \tag{2.1}$$

The proportionality constant that remained undetermined in [4] is therefore fixed[1] as we will see below to be $C_2^{(k)}=(-1)^{k(\nu+[N_f/2])}$.

In order to prove eq. (2.1) we make a more general statement which will be more easy to prove due to the lack of degeneracy of the fermion masses, which is present on the right hand side of eq. (2.1):

$$\det_{1\leq a,b\leq k}\left[\frac{\mathcal{Z}_\nu^{(N_f+2)}(\{\mu\},\xi_a,\eta_b)}{\mathcal{Z}_\nu^{(N_f)}(\{\mu\})}\right] \;=\; \prod_{i<j}^{k}(\xi_i^2-\xi_j^2)(\eta_i^2-\eta_j^2)\;\frac{\mathcal{Z}_\nu^{(N_f+2k)}(\{\mu\},\xi_1,\ldots,\xi_k,\eta_1,\ldots,\eta_k)}{\mathcal{Z}_\nu^{(N_f)}(\{\mu\})}\;. \tag{2.2}$$

Taking the degeneracy limit $\eta_i=\xi_i$, $i=1,\ldots,k$, we recover the original claim eq. (2.1).

We will not be able to give a proof of the above statement in full generality. In a first step we will prove that it holds in the asymptotic regime where $\mu_f,\xi_i,\eta_j\to\infty$. In particular this fixes the proportionality constant in eq. (2.2) which may depend on N_f and ν. In a second step we can prove eq. (2.2) for finite arguments and any $k\in\mathrm{N}$ in the quenched case $N_f=0$ in an arbitrary topological sector ν. Using the established flavor-topology duality [12] in this finite-volume scaling regime, we have then automatically also proven the identity for *any* number of massless flavors N_f in a sector of *any* topological charge ν. The statement eq. (2.2) is this case reads

Theorem - Consistency Condition I (massless):

Let $\mathcal{Z}_\nu^{(N_f)}(\{\mu\})=\det A(\{\mu\})/\Delta(\{\mu^2\})$ be defined as in eqs.(1.6)–(1.9). For N_f massless flavors the following identity holds, where we have chosen the constant $\mathcal{Z}_\nu^{(N_f)}(0)$ to be unity:

$$\det_{1\leq a,b\leq k}\left[\mathcal{Z}_\nu^{(N_f+2)}(\xi_a,\eta_b)\right] \;=\; \prod_{i<j}^{k}(\xi_i^2-\xi_j^2)(\eta_i^2-\eta_j^2)\;\mathcal{Z}_\nu^{(N_f+2k)}(\xi_1,\ldots,\xi_k,\eta_1,\ldots,\eta_k)\;. \tag{2.3}$$

Proof: (i) asymptotic region (massive, eq. (2.2)):

[1] There is a misprint in eq. (11) of the third paper in ref. [4], where the overall factor should read $C_2^{(k)}(-1)^{k(\nu+[N_f/2])}$.

Inserting the explicit form of the partition function into eq. (2.2) and factorizing out common parts of the Vandermonde determinants eq. (2.2) can be brought into the following form:

$$\det_{1\leq a,b\leq k}\left[\frac{1}{\eta_b^2-\xi_a^2}\frac{\det A(\{\mu\},\xi_a,\eta_b)}{\det A(\{\mu\})}\right] \;=\; \prod_{a,b=1}^{k}\frac{1}{(\eta_a^2-\xi_b^2)}\;\frac{\det A(\{\mu\},\xi_1,\ldots,\xi_k,\eta_1,\ldots,\eta_k)}{\det A(\{\mu\})}\;. \tag{2.4}$$

We can now apply the asymptotics of Bessel functions

$$\lim_{x\to\infty} I_n(x) \;=\; \frac{e^x}{\sqrt{2\pi x}}(1+O(x^{-1}))\;, \tag{2.5}$$

which leads to

$$\det A(\{\mu\},\{\xi\},\{\eta\}) \;\longrightarrow\; \prod_{i=1}^{k}\frac{e^{\xi_i+\eta_i}}{\sqrt{2\pi\xi_i\eta_i}}\prod_{f=1}^{N_f}\frac{e^{\mu_f}}{\sqrt{2\pi\mu_f}}\;\Delta(\{\mu\},\{\xi\},\{\eta\}) \tag{2.6}$$

where we have taken all arguments to infinity and where $\Delta(\{\mu\},\{\xi\},\{\eta\})$ is now the Vandermonde of the *unsquared* sets of variables. Inserting the result eq. (2.6) into eq. (2.4) we obtain for the left hand side (l.h.s.):

$$\begin{aligned}
\text{l.h.s.}\quad\longrightarrow\quad & \det_{1\leq a,b\leq k}\left[\frac{1}{\eta_b^2-\xi_a^2}\frac{e^{\xi_a+\eta_b}}{\sqrt{2\pi\xi_a\eta_b}}\frac{\Delta(\{\mu\},\xi_a,\eta_b)}{\Delta(\{\mu\})}\right]\\
=\quad & \prod_{i=1}^{k}\frac{e^{\xi_i+\eta_i}}{\sqrt{2\pi\xi_i\eta_i}}\prod_{f=1}^{N_f}\prod_{a=1}^{k}(\mu_f-\xi_a)(\mu_f-\eta_a)\det_{1\leq a,b\leq k}\left[\frac{1}{\eta_b+\xi_a}\right]
\end{aligned} \tag{2.7}$$

and for the right hand side (r.h.s.)

$$\begin{aligned}
\text{r.h.s.}\quad\longrightarrow\quad & \prod_{a,b=1}^{k}\frac{1}{(\eta_a^2-\xi_b^2)}\prod_{i=1}^{k}\frac{e^{\xi_i+\eta_i}}{\sqrt{2\pi\xi_i\eta_i}}\frac{\Delta(\{\mu\},\{\xi\},\{\eta\})}{\Delta(\{\mu\})}\\
=\quad & \prod_{i=1}^{k}\frac{e^{\xi_i+\eta_i}}{\sqrt{2\pi\xi_i\eta_i}}\prod_{f=1}^{N_f}\prod_{a=1}^{k}(\mu_f-\xi_a)(\mu_f-\eta_a)\frac{\prod_{a<b}^{k}(\xi_a-\xi_b)(\eta_a-\eta_b)}{\prod_{a,b=1}^{k}(\eta_a+\xi_b)}\;.
\end{aligned} \tag{2.8}$$

Putting both sides together and dropping common factors we obtain

$$\det_{1\leq a,b\leq k}\left[\frac{1}{\eta_b+\xi_a}\right] \;=\; \frac{\prod_{a<b}^{k}(\xi_a-\xi_b)(\eta_a-\eta_b)}{\prod_{a,b=1}^{k}(\eta_a+\xi_b)}\;, \tag{2.9}$$

which is nothing else than Cauchy's Lemma. The asymptotic analysis performed so far determines the mass independent overall proportionality constant in eq. (2.2) to be unity.

(ii) proof for N_f massless fermions with arbitrary ν (eq. (2.3)):

Due to the flavor-topology duality it is sufficient to prove the statement for $N_f = 0$ with arbitrary $\nu \neq 0$, and then shifting $\nu \to N_f + \nu$. Let us first give an outline of the proof. We will proceed with the simplified version eq. (2.4) of the theorem for $N_f = 0$, where the Vandermonde determinants have been already cancelled. In a first step we shall further simplify $\det A(\{\xi\},\{\eta\})$ using Lemma 1 in the Appendix A. There it is shown that $\det A(\{\xi\},\{\eta\})$ is given by a determinant similar to a Vandermonde containing powers of ξ_i and η_i as well as first derivatives with respect to these variables acting on a product of Bessel functions of the same index ν. In this form we can prove the theorem by performing a Laplace expansion of the right hand side of eq. (2.4) and using Lemma 2 for the left hand side.

Starting with the right hand side the determinant can be rewritten using Lemma 1:

$$\det A(\xi_1,\ldots,\xi_k,\eta_1,\ldots,\eta_k) \;=\; (-1)^{\frac{k(k-1)}{2}} \begin{vmatrix} B(\xi) & C(\xi) \\ B(\eta) & C(\eta) \end{vmatrix} \prod_{i=1}^{k} I_\nu(\xi_i) I_\nu(\eta_i) \tag{2.10}$$

where $B(\xi)$ is the matrix of the Vandermonde determinant of squared arguments $B(\xi)_{ij} = (\xi_i^2)^{j-1}$ and $C(\xi)$ is the same as B with an additional $\xi_i \vec{\partial}_{\xi_i}$ in each row, which is defined to act only on the Bessel functions. If we perform a Laplace expansion with respect to the first k columns into $k \times k$ blocks we can use the fact that the resulting determinants $\det B \det C$ can all be rewritten as the product of two Vandermonde determinants times k linear differential operators that can be taken out of $\det C$. We obtain

$$\begin{aligned} \text{r.h.s.} \;&=\; (-1)^{\frac{k(k-1)}{2}} \prod_{a,b=1}^{k} \frac{1}{(\eta_a^2 - \xi_b^2)} \sum_\sigma (-1)^\sigma \Delta(x_{\sigma(1)}^2,\ldots,x_{\sigma(k)}^2) \;\times \\ &\quad\times\; \Delta(x_{\sigma(k+1)}^2,\ldots,x_{\sigma(2k)}^2) \prod_{i=k+1}^{2k} x_{\sigma(i)} \partial_{x_{\sigma(i)}} \prod_{j=1}^{2k} I_\nu(x_j) \end{aligned} \tag{2.11}$$

where we have renamed

$$x_i \;=\; \xi_i \;,\quad x_{i+k} = \eta_i \quad \text{for} \quad i = 1,..,k \;. \tag{2.12}$$

The permutations σ run over all possible $\binom{2k}{k}$ permutations to put $2k$ variables into 2 sets of k variables with ordered indices respecting $\sigma(1) < \ldots < \sigma(k)$ and $\sigma(k+1) < \ldots < \sigma(2k)$. The sign of the permutation is defined by $(-1)^\sigma = (-1)^{1+\ldots+k+\sigma(1)+\ldots+\sigma(k)}$.

The building blocks of the left hand side of eq. (2.4) are the 2×2 determinants

$$\det A(\xi_a,\eta_b) \;=\; (\eta_b \partial_{\eta_b} - \xi_a \partial_{\xi_a}) I_\nu(\xi_a) I_\nu(\eta_b) \tag{2.13}$$

where we have used again Lemma 1 eq. (A.2). Therefore we can rewrite the left hand side as

$$\text{l.h.s.} \;=\; \det_{1\le a,b\le k}\left[\frac{1}{\eta_b^2 - \xi_a^2} \det A(\xi_a,\eta_b)\right] \;=\; \det_{1\le a,b\le k}\left[\frac{1}{\eta_b^2 - \xi_a^2}\left(\eta_b \vec{\partial}_{\eta_b} - \xi_a \vec{\partial}_{\xi_a}\right)\right] \prod_{i=1}^{k} I_\nu(\xi_i) I_\nu(\eta_i) \;, \tag{2.14}$$

with the derivatives only acting on the Bessel functions.

Using again the fact that determinants differing only by a single column can be added, eq. (2.14) can be rewritten as a sum of determinants containing only one derivative $\eta_b \vec{\partial}_{\eta_b}$ or $\xi_a \vec{\partial}_{\xi_a}$, $a = 1,\ldots,k$. Due to the structure of the determinant in the columns with $\eta_b \vec{\partial}_{\eta_b}$ the derivative can be taken out as a common factor. After reordering columns we end up with

$$\det_{1\le a,b\le k}\left[\frac{1}{\eta_b^2 - \xi_a^2}\left(\eta_b \vec{\partial}_{\eta_b} - \xi_a \vec{\partial}_{\xi_a}\right)\right] \;=\; \sum_{i=0}^{k} \sum_{\gamma_i} (-1)^{\gamma_i + i} \det G(i;\gamma_i) \prod_{j=i+1}^{k} \eta_{\gamma_i(j)} \partial_{\eta_{\gamma_i(j)}} \;, \tag{2.15}$$

where the matrices $G(i;\gamma_i)$ are defined in Lemma 2 eq. (A.11) to contain derivatives $\xi_a \vec{\partial}_{\xi_a}$ in the first i columns. The γ_i are all $\binom{k}{i}$ permutations for different $G(i;\gamma_i)$ with i fixed, where the ordering is such that $\gamma_i(1) < \ldots < \gamma_i(i)$ and $\gamma_i(i+1) < \ldots < \gamma_i(k)$. The factor $(-1)^i$ stems from taking out the common factor minus one of all columns containing derivatives with respect to the ξ's.

We can now apply Lemma 2 where $\det G(i;\gamma_i)$ is evaluated by Laplace expansion and Cauchy's Lemma, to obtain

$$\begin{aligned}
&\det_{1\le a,b\le k}\left[\frac{1}{\eta_b^2-\xi_a^2}\left(\eta_b\vec{\partial}_{\eta_b}-\xi_a\vec{\partial}_{\xi_a}\right)\right] = \\
&= (-1)^{\frac{k(k-1)}{2}}\prod_{a,b=1}^{k}\frac{1}{(\eta_a^2-\xi_b^2)}\sum_{i=0}^{k}\sum_{\gamma_i,\bar\gamma_i}(-1)^{\gamma_i+\bar\gamma_i+i}\Delta(\xi^2_{\bar\gamma_i(1)},\ldots,\xi^2_{\bar\gamma_i(k-i)},\eta^2_{\gamma_i(1)},\ldots,\eta^2_{\gamma_i(i)})\times \\
&\quad\times\Delta(\xi^2_{\bar\gamma_i(k-i+1)},\ldots,\xi^2_{\bar\gamma_i(k)},\eta^2_{\gamma_i(i+1)},\ldots,\eta^2_{\gamma_i(k)})\prod_{j=k-i+1}^{k}\xi_{\bar\gamma_i(j)}\partial_{\xi_{\bar\gamma_i(j)}}\prod_{l=i+1}^{k}\eta_{\gamma_i(l)}\partial_{\eta_{\gamma_i(l)}} \qquad (2.16)
\end{aligned}$$

This equation is nothing else than the operator in eq. (2.11) acting on $\prod_{i=1}^k I_\nu(\xi_i)I_\nu(\eta_i)$, where here in the first sum the number of ξ's in the first Vandermonde is made explicit to be i.

To see this we observe that in eq. (2.16) we have $\sum_{i=0}^{k}\binom{k}{i}^2=\binom{2k}{k}$ different terms, which matches to the number of permutations in eq. (2.11). In order to map individual permutations including signs we use again the notation eq. (2.12). The index $\gamma(j)$ of the η's thus changes according to

$$\gamma(j)\ \to\ k+\gamma(j)\ \equiv\ \sigma(j+k)\ , \qquad (2.17)$$

which implies$(-1)^{\gamma(1)+\ldots+\gamma(i)}\ =\ (-1)^{\sigma(1+k)+\ldots+\sigma(i+k)-ik}$. Consequently we obtain

$$\begin{aligned}
(-1)^{\gamma_i\,+\,\bar\gamma_i\,+\,i} &= (-1)^{\frac{(i)(i+1)}{2}\,+\,\sum_{l=1}^{k-i}\sigma_i(k-l)\,-\,ik\,+\,\frac{(k-i)(k-i+1)}{2}\,+\sum_{l=1}^{k-i}\bar\gamma_i(l)\,+\,i} \\
&= (-1)^{\frac{k(k+1)}{2}\,+\,\sum_{j=1}^{k}\sigma_i(j)}\ =\ (-1)^{\sigma_i}\ , \qquad (2.18)
\end{aligned}$$

where σ_i is a permutation with i ξ's with ordered indices $\{\bar\gamma_i(j)=\sigma_i(j);\ j=1,\ldots,k-i\}\in\{1,\ldots,k\}$ and $(k-i)$ η's with ordered indices $\{\sigma_i(k+j);\ j=1,\ldots,i\}\in\{k+1,\ldots,2k\}$ as in the first Vandermonde in eq. (2.11). We have thus completed the matching of both sides of theorem eq. (2.3).

3 Consistency Condition II

In this section we will prove the following theorem relating partition functions with N_f, N_f+1 and N_f+2 massive flavors, as it has been stated in the last of reference [4].

Theorem - Consistency Condition II:

For $\mathcal{Z}_\nu^{(N_f)}(\{\mu\})=\det A(\{\mu\})/\Delta(\{\mu^2\})$ as defined in eqs.(1.6)–(1.9) it holds

$$\begin{aligned}
\mathcal{Z}_\nu^{(N_f+2)}(\{\mu\},\xi,\eta) &= \frac{1}{(\xi^2-\eta^2)\mathcal{Z}_\nu^{(N_f)}(\{\mu\})}\times \\
&\times\left[\left((\sum_{i=1}^{N_f}\mu_i\partial_{\mu_i}+\xi\partial_\xi)\mathcal{Z}_\nu^{(N_f+1)}(\{\mu\},\xi)\right)\mathcal{Z}_\nu^{(N_f+1)}(\{\mu\},\eta)\ -\ (\xi\leftrightarrow\eta)\right] \qquad (3.1)
\end{aligned}$$

A few remarks can be made here. Taking the inverse statement, the differential equation eq. (3.1) together with the boundary conditions $\mathcal{Z}_\nu^{(0)}=1$ and $\mathcal{Z}_\nu^{(1)}(\mu)=I_\nu(\mu)$ can be seen as a generating equation for all $\mathcal{Z}_\nu^{(N_f)}(\{\mu\})$. In this way we can actually *derive* the precise form of the partition function eqs. (1.6)–(1.9) instead of taking it as a starting point. In ref. [13], an even more compact recursive relation for the partition functions was derived in the Random Matrix Theory formulation, using the supersymmetric formalism.

The consistency condition eq. (3.1) had been conjectured in the third reference of [4] (eq. (16)) with an arbitrary constant, which has been determined here to be $C = 1$. This result follows independently from inserting the asymptotics of the Bessel functions.

PROOF: Before going into the details let us give the general outline of the proof. We will first investigate the action of the power-counting operator $\sum_{i=1}^{N_f} \mu_i \partial_{\mu_i} + \xi\partial_\xi$ on the partition function $\mathcal{Z}_\nu^{(N_f+1)}(\{\mu\},\xi)$. Most of the outcome will be again proportional to the same partition function which then drops out due to the antisymmetry with respect to ξ and η in the bracket in eq. (3.1). The remainder times the partition functions $\mathcal{Z}_\nu^{(N_f+1)}(\{\mu\},\eta)$ expanded once will then precisely arrange to the Laplace expansion of the left hand side with respect to the last two columns. Throughout the proof we will make use of 3 different Lemmas collected in Appendix B.

Let us start with the action of the power-counting differential operator on the Vandermonde in the denominator. It is easy to show that

$$(\sum_{i=1}^{N_f} \mu_i \partial_{\mu_i} + \xi\partial_\xi)\ \Delta(\{\mu^2\},\xi^2) \ = \ (N_f+1)N_f\ \Delta(\{\mu^2\},\xi^2) \tag{3.2}$$

since the operator counts the sum of all powers. From the product rule every factor in Δ gets differentiated twice and is thus reproduced with a factor of 2 in front, furthermore there are $(N_f+1)N_f/2$ such factors. Hence the differentiation of the Δ's drops out of eq. (3.1) due to the $(\xi \leftrightarrow \eta)$ antisymmetry. Inserting the explicit form of the partition function we obtain

$$\det A(\xi,\eta) \ = \ -(\det A)^{-1}\left[\left((\sum_{i=1}^{N_f} \mu_i \partial_{\mu_i} + \xi\partial_\xi) \det A(\xi)\right) \det A(\eta) \ - \ (\xi \leftrightarrow \eta)\right] \tag{3.3}$$

after cancelling the Vandermonde determinants. Here and in the rest of this section we have omitted the dependence of the matrix A on the set of variables μ_i. Next we apply the differential operator to the determinant of $A(\xi)$. Since it is linear and the matrix A depends on each variable only in one row we can differentiate row-wise, using

$$\mu\partial_\mu(\mu^n I_{\nu+n}(\mu)) \ = \ (2n+\nu)\mu^n I_{\nu+n}(\mu) \ + \ \mu^{n+1} I_{\nu+n+1}(\mu). \tag{3.4}$$

In fact this is the only place where the properties of Bessel functions enter, the rest of the argument being valid for general matrices A. Determinants that only differ by one row can be added and we thus have

$$\sum_{i=1}^{N_f} \mu_i \partial_{\mu_i} \det A \ = \ \sum_{k=1}^{N_f}\left[2\sum_{L=2}^{N_f} \det A_k^L + \nu \det A \ + \ \det A_k\right] . \tag{3.5}$$

Here in the last term A_k is the matrix A with the k-th row shifted to the left by one unit, $(A_k)_{kl} = \mu_k^l I_{\nu+l}(\mu_k)$ and $A_k = A$ else, which results from the last term in eq. (3.4). This last term which can be further simplified using Lemma 3 in Appendix B will be the only term that survives in eq. (3.3). The second term $\nu \det A$ vanishes immediately due to the antisymmetry in eq. (3.3).

We will now show that also the first term in eq. (3.5) is proportional to $\det A$ and thus drops out. The matrix A_k^L is defined to be the matrix A with the first L entries in the k-th row vanishing, $(A_k^L)_{km} = 0$ for $m = 1, \ldots, L$ and $A_k^L = A$ else. The sum over L in eq. (3.5) thus reproduces the $2n\mu^n I_{\nu+n}(\mu)$ from eq. (3.4). Applying Lemma 4 from Appendix B we obtain

$$2\sum_{L=2}^{N_f}\sum_{k=1}^{N_f} \det A_k^L \;=\; 2\sum_{L=2}^{N_f}(N_f - L)\det A \;=\; (N_f-1)(N_f-2)\det A \ . \tag{3.6}$$

Together with Lemma 3 Appendix B we finally obtain

$$\sum_{i=1}^{N_f} \mu_i \partial_{\mu_i} \det A \;=\; ((N_f-1)(N_f-2) + \nu N_f)\det A + \sum_{j=1}^{N_f}(-1)^{N_f+j}\mu_j^{N_f} I_{\nu+N_f}(\mu_j)\det A^*_{jN_f} \tag{3.7}$$

where $A^*_{jN_f}$ is the matrix A with row j and column N_f missing (algebraic complement of matrix element A_{jN_f}). Inserting eq. (3.7) for the derivative of the $(N_f+1)\times(N_f+1)$ determinant of $A(\xi)$ in eq. (3.3) we obtain

$$\begin{aligned}\det A(\xi,\eta)\det A \;=\;& \Big(\sum_{j=1}^{N_f}(-1)^{N_f+1+j}\mu_j^{N_f+1} I_{\nu+N_f+1}(\mu_j)\det A^*_{jN_f+1}(\xi) \\ &- \ \xi^{N_f+1} I_{\nu+N_f+1}(\xi)\det A\Big)\det A(\eta) \ - \ (\xi\leftrightarrow\eta) \ .\end{aligned} \tag{3.8}$$

Expanding $\det A(\eta)$ and $\det A(\xi)$ on the right hand side with respect to the last column and multiplying out we obtain for the right hand side of eq. (3.8)

$$\begin{aligned}\text{r.h.s.} =& \sum_{\substack{i,j=1\\ i\neq j}}^{N_f} (-1)^{i+j}\mu_j^{N_f+1} I_{\nu+N_f+1}(\mu_j)\mu_i^{N_f} I_{\nu+N_f}(\mu_i)\Big[\det A^*_{jN_f+1}(\xi)\det A^*_{iN_f+1}(\eta) - (\xi\leftrightarrow\eta)\Big] \\ +& \Bigg[\sum_{j=1}^{N_f}(-1)^{N_f+1+j}\mu_j^{N_f+1} I_{\nu+N_f+1}(\mu_j)\Big(\eta^{N_f} I_{\nu+N_f}(\eta)\det A^*_{jN_f+1}(\xi) - \xi^{N_f} I_{\nu+N_f}(\xi)\det A^*_{jN_f+1}(\eta)\Big) \\ &+ \sum_{j=1}^{N_f}(-1)^{N_f+1+j}\mu_j^{N_f} I_{\nu+N_f}(\mu_j)\Big(\xi^{N_f+1} I_{\nu+N_f+1}(\xi)\det A^*_{jN_f}(\eta) - \eta^{N_f+1} I_{\nu+N_f+1}(\eta)\det A^*_{jN_f}(\xi)\Big) \\ &+ \Big(\xi^{N_f+1} I_{\nu+N_f+1}(\xi)\eta^{N_f} I_{\nu+N_f}(\eta) \ - \ \eta^{N_f+1} I_{\nu+N_f+1}(\eta)\xi^{N_f} I_{\nu+N_f}(\xi)\Big)\det A\ \Bigg]\det A\end{aligned} \tag{3.9}$$

We are now ready to apply Lemma 5 from Appendix B which relates products of determinants of matrices which just differ by the last two rows. The bracket in the first line then reads

$$\Big[\det A^*_{jN_f+1}(\xi)\det A^*_{iN_f+1}(\eta) - (\xi\leftrightarrow\eta)\Big] \;=\; \det A^*_{iN_f+1;jN_f+2}(\xi,\eta)\det A \begin{cases} \cdot(+1) & i<j \\ \cdot(-1) & i>j \end{cases} \tag{3.10}$$

where we have introduced the matrix $A^*_{iN_f+1;jN_f+2}$ where i-th and j-th row and the last two columns are missing. The obvious symmetry $A^*_{iN_f+1;jN_f+2} = A^*_{jN_f+1;iN_f+2}$ leads to a sum over $i<j$ only in the first line of eq. (3.9). Since we can trivially rewrite $A^*_{jN_f+1}(\xi) = A^*_{iN_f+1;N_f+2N_f+2}(\xi,\eta)$ one can immediately see that eq. (3.9) is nothing else than the Laplace expansion of $\det A(\xi,\eta)$ in 2×2 times $N_f\times N_f$ blocks choosing the last two columns for the expansion. Thus we have obtained the right hand side of eq. (3.8) and completed the proof.

4 Relations between QCD_3 and QCD_4 partition functions

Let us first recall how the Consistency Condition I and II of the the previous sections have been originally derived in [4]. The important point is that every finite volume partition function equals a partition function of a massive chiral random matrix model in the microscopic large-N scaling limit. The fact that in matrix models the orthogonal polynomials, the associated kernel, as well as all correlation functions can be expressed in terms of matrix model partition functions translates the well known relations between these objects to relations among finite volume partition functions. Consistency Condition II reflects the Christoffel-Darboux identity between the kernel and the orthogonal polynomials, and Consistency Condition I is the Mahoux-Mehta relation among k-point correlation functions and the determinant of the kernel.

In this section we will follow the same reasoning to translate relations between matrix model quantities for QCD_3 and QCD_4 to the corresponding finite volume partition functions. In contrast to the previous section we will not provide a proof solely based on finite volume partition functions. While the identities are very easily derived in terms of the Random matrix Theory formulation, they translate, in the microscopic large-N limit into highly non-trivial relations between group theory integrals of Harish-Chandra type (for unitary groups) and what can be called the external field problem for unitary groups. These surprising identities deserve to be understood in their own right.

Let us start with a relation between the orthogonal polynomials the chUE (QCD_4) and those of the UE (QCD_3):

$$P_{N,\ chUE}^{(N_f,\ \nu=-1/2)}(z^2;\{\mu\}) \ = \ P_{2N,\ UE}^{(2N_f)}(z;\{\mu\}) \ . \qquad (4.1)$$

This simple identity is very easily derived from the defintion of the two Random Matrix Theories as given in [16]. Since in ref. [16] no explicit use was made of the measure, eq. (4.1) also holds in the massive case[2]. On the right hand side the same N_f masses appear in pairs with opposite sign. Expressing the orthogonal polynomials by partition functions as given in the second of ref. [4] we obtain

$$C(i\xi)^{1/2}\frac{\mathcal{Z}_{\nu=-1/2}^{(N_f+1)}(\{\mu\},i\xi)}{\mathcal{Z}_{\nu=-1/2}^{(N_f)}(\{\mu\})} \ = \ \frac{\mathcal{Z}_{QCD3}^{(2N_f+1)}(\{\mu\},i\xi)}{\mathcal{Z}_{QCD3}^{(2N_f)}(\{\mu\})} \ . \qquad (4.2)$$

Here the unknown proportionality constant only reflects the choice of normalization for the polynomials, and it can easily be fixed. Since we have started with an even polynomial the odd-flavor partition function on the right hand side is given by [2]

$$\mathcal{Z}_{QCD3}^{(2N_f+1)} \ = \ \int dU \cosh[\mathrm{Tr}(\mathcal{M}U\Gamma U^\dagger)] \ , \qquad (4.3)$$

where $\mathcal{M}$=diag $(\mu_1,\ldots,\mu_{N_f},-\mu_1,\ldots,-\mu_{N_f},i\xi)$ and Γ=diag($\mathbf{1}_{N_f}$,-$\mathbf{1}_{N_f+1}$). The denominator is given by the 3-dimensional even-flavor partition function [2],

$$\mathcal{Z}_{QCD3}^{(2N_f)} \ = \ \int dU \exp[\mathrm{Tr}(\mathcal{M}U\Gamma U^\dagger)] \ , \qquad (4.4)$$

with $\mathcal{M}$=diag $(\mu_1,\ldots,\mu_{N_f},-\mu_1,\ldots,-\mu_{N_f})$ and Γ=diag($\mathbf{1}_{N_f}$,-$\mathbf{1}_{N_f}$). These group integrals are both of the Harish-Chandra type, while the left hand side of eq. (4.2) is given by an entirely different ratio of unitary group theory integrals of the external-field kind. This is the first of such relations.

[2]In contrast to [16] we have shifted the non-integer part to the left, $\nu = -1/2$, in order to deal with a physical (even) number of massive flavors for QCD_3 on the right hand side.

The relation between the orthogonal polynomials eq. (4.1) can be exploited furthermore in order to relate also the corresponding kernels of the chUE and the UE. For convenience, let us introduce the "wave functions"

$$\Psi_n(\lambda) \;\equiv\; \sqrt{\omega(\lambda)} P_n(\lambda) \;, \tag{4.5}$$

where $\omega(\lambda)$ is the measure function (so that the wave functions $\Psi_n(\lambda)$ are orthogonal with respect to a weight of unity). Now, from eq. (3.11) of ref. [16] we can use the wave functions of the UE inside the Christoffel-Darboux identity for the chUE kernel[3]. Doing this, we readily derive the following identities:

$$\begin{aligned}
K_{N,\;chUE}^{(N_f,\;\nu=-1/2)}(z^2,w^2) &= \frac{c_{2N}}{z^2-w^2}\left(\Psi_{2N}^{UE}(z)\;w\;\Psi_{2N-1}^{UE}(w) \;-\; z\;\Psi_{2N-1}^{UE}(z)\Psi_{2N}^{UE}(w)\right)\\
&= \frac{1}{2}\frac{c_{2N}}{(z-w)}\left(\Psi_{2N}^{UE}(z)\Psi_{2N-1}^{UE}(w) \;-\; \Psi_{2N-1}^{UE}(z)\Psi_{2N}^{UE}(w)\right)\\
&\quad - \frac{1}{2}\frac{c_{2N}}{(z+w)}\left(\Psi_{2N}^{UE}(z)\Psi_{2N-1}^{UE}(w) \;+\; \Psi_{2N-1}^{UE}(z)\Psi_{2N}^{UE}(w)\right)\\
&= \frac{1}{2}\left(K_{2N,\;UE}^{(2N_f)}(z,w) \;+\; K_{2N,\;UE}^{(2N_f)}(-z,w)\right)\;,
\end{aligned} \tag{4.6}$$

where for clarity we have not explcitly indicated the mass dependence of the wave functions. The coefficients c_{2N} are defined in ref. [16]. We have here made use of the fact that in the UE the polynomials $P_n(\lambda)$ are of parity $(-1)^n$. As one can easily convince oneself the last equation is, despite its appearance, symmetric in the arguments z and w. Expressing the kernels in terms of the corresponding finite volume partition functions [4] eq. (4.6) leads to the following relation:

$$(-1)^{[\frac{N_f}{2}]-\frac{1}{2}}\sqrt{\xi\omega}\;\frac{\mathcal{Z}_{\nu=-1/2}^{(N_f+2)}(\{\mu\},i\xi,i\omega)}{\mathcal{Z}_{\nu=-1/2}^{(N_f)}(\{\mu\})} \;=\; \frac{1}{4\pi}\left(\frac{\mathcal{Z}_{QCD3}^{(2N_f+2)}(\{\mu\},i\xi,i\omega)}{\mathcal{Z}_{QCD3}^{(2N_f)}(\{\mu\})} \;-\; \frac{\mathcal{Z}_{QCD3}^{(2N_f+2)}(\{\mu\},-i\xi,i\omega)}{\mathcal{Z}_{QCD3}^{(2N_f)}(\{\mu\})}\right)\;. \tag{4.7}$$

The proportionality constant is known in this case as it can be determined for both kernels from the matching condition with the microscopic spectral density.

Let us finally mention that also in the case of QCD_3-like theories with an *odd* number of massive flavors a relation similar to eq. (4.6) exists. In ref.[14] the random matrix model kernel for odd-flavored QDC_3 has been derived from a chUE, which in the microscopic scaling limit reads:

$$\frac{\xi+\omega}{\sqrt{\xi\omega}}K_{S,\;chUE}^{(N_f,\;\nu=+1/2)}(\xi,\omega;\{\mu\}) \;=\; K_{S,\;UE}^{(2N_f+1)}(\xi,\omega;\{\mu\},0)\;. \tag{4.8}$$

Inserting again the representation in terms of partition functions [4], we obtain the following relation:

$$\frac{\xi+\omega}{\sqrt{\xi\omega}}\;(-1)^{\frac{1}{2}+[\frac{N_f}{2}]}\frac{\mathcal{Z}_{\nu=1/2}^{(N_f+2)}(\{\mu\},i\xi,i\omega)}{\mathcal{Z}_{\nu=1/2}^{(N_f)}(\{\mu\})} \;=\; \frac{1}{2\pi}\frac{\mathcal{Z}_{QCD3}^{(2N_f+1+2)}(\{\mu\},0,i\xi,i\omega)}{\mathcal{Z}_{QCD3}^{(2N_f+1)}(\{\mu\},0)}\;. \tag{4.9}$$

For more details on the odd-flavor partition function we refer to ref. [14].

We have explicitly checked all of the above relations for a few (small) number of flavors, starting from the finite-volume partition functions alone. Apart from those given above, we have empirically found other non-linear relations between these Harish-Chandra type integrals and those of the unitary external field problem. These relations, however, do not seem to follow easily from the Random Matrix Theory formulation.

[3]Eq. (4.1) trivially holds also for the wave functions instead of the polynomials.

5 The effective QCD_4 partition function as a τ-function

The identities derived above may have their origin in a surprising relation to integrable systems, which we shall now briefly discuss. The starting point is the following observation. Suppose we define an $N_f \times N_f$ hermitian matrix integral by

$$\tau(X) \;=\; \int dY \exp\left[\mathrm{Tr}[XY + V(Y)]\right] \ . \tag{5.1}$$

With X itself being an $N_f \times N_f$ hermitian matrix, and the potential $V(Y)$ as yet unspecified, this is a generalized "external field problem" of Random Matrix Theory. The universality of its correlation functions have been proved in the microscopic large-N_f limit in [15] for polynomial potentials $V(Y)$. Surprisingly, a closed solution of the integral eq. (5.1) can be written down for any value of N_f, and for any potential $V(Y)$ that satisfies suitable convergence criteria [17, 18]. After diagonalizing the Y-matrix, $Y = U\mathrm{diag}(y_1, \ldots, y_{N_f})U^\dagger$, one obtains the standard Jacobian of $\Delta(y)^2$, and one can then make use of the Harish-Chandra integral to obtain

$$\tau(X) \;=\; \int \prod dy_i \frac{\Delta(y)}{\Delta(x)} \exp\left[\sum_j (x_j y_j + V(y_j))\right] \ , \tag{5.2}$$

where the x_i are the N_f eigenvalues of X. If one now introduces the function

$$\phi(x) \;\equiv\; \int dy e^{xy+V(y)} \ , \tag{5.3}$$

as well as the derivatives

$$\phi_k(x) \;\equiv\; \frac{\partial^k}{\partial x^k}\phi(x) \;=\; \int dy y^k e^{xy+V(y)} \ , \tag{5.4}$$

one sees that the integral is simply

$$\tau(X) \;=\; \frac{\det[\phi_{j-1}(x_i)]}{\Delta(x)} \ . \tag{5.5}$$

This shows that $\tau(X)$ is a τ-function of the integrable KP hierarchy.

The expression (5.5) has an uncanny resemblance to the finite-volume partition function if one identifies $x_i = \mu_i^2$. This is particularly clear if one considers the form (1.7), and starts with the case $\nu = 0$. Using the Bessel function identity

$$\frac{d^k}{d(x^2)^k}\left(x^{N_f} I_{N_f}(x)\right) \;=\; \frac{1}{2^k} x^{N_f - k} I_{N_f - k}(x) \ , \tag{5.6}$$

we see that the partition function (1.6) can be written in the form (5.5) if we identify

$$\phi(\mu^2) \;=\; \left(\sqrt{\mu^2}\right)^{N_f} I_{N_f}(\sqrt{\mu^2}) \ , \tag{5.7}$$

and ignore irrelevant overall factors. The case of non-zero ν can then be treated by using again the flavor-topology duality [12], thus obtaining the $\nu \neq 0$ case by simply setting ν of the N_f masses to zero. In fact, eq. (5.7) alone suffices to prove that the effective partition function is a τ-function of the integrable KP hierarchy [17]. It is nevertheless interesting to note that in addition an integral representation actually exists such that the partition function explicitly can be written in the form (5.1) [19]. This turns out to correspond to a potential $V(Y) = 1/Y - N_f \ln(Y)$, and an integration contour for the eigenvalues encircling the origin. (Strictly speaking this is outside the scope of the hermitian matrix formulation (5.1), so the notion of "hermiticity" is here simply taken to mean "of flat measure" – see ref. [19] for a discussion of this point).

Knowing that the partition function is a τ-function immediately implies a number of identities. The most general of these is the following set of Hirota equations, which read [17]

$$\begin{aligned} 0 &= (x_a - x_b)\tau(X; p_a, p_b, p_c + 1)\tau(X; p_a + 1, p_b + 1, p_c) \\ &+ (x_b - x_c)\tau(X; p_a + 1, p_b, p_c)\tau(X; p_a, p_b + 1, p_c + 1) \\ &+ (x_c - x_a)\tau(X; p_a, p_b + 1, p_c)\tau(X; p_a + 1, p_b, p_c + 1) \ . \end{aligned} \tag{5.8}$$

Here the p_i denote the multiplicities of the parameters x_i for $i = a, b, c$, where in our notation $x_i = \mu_i^2$. Due to the Jacobi identity for determinants yet another identity holds for τ-functions as given in *e.g.* eq. (2.43) of ref.[17]:

$$\begin{aligned} \tau^{(N_f+2)}(\{x\}, x_{N_f+1}, x_{N_f+2}) &= \frac{1}{(x_{N_f+1} - x_{N_f+2})\tau^{(N_f)}(\{x\})} \times \\ &\times \left[\tau^{(N_f+1)}(\{x\}, x_{N_f+1})\hat{\tau}^{(N_f+1)}(\{x\}, x_{N_f+2}) \ - \ (x_{N_f+1} \leftrightarrow x_{N_f+2})\right] \ . \end{aligned} \tag{5.9}$$

The upper index indicates the number of parameters of the corresponding τ-function and $\hat{\tau}$ means that in the last row the index of the functions $\phi_k(x)$ in eq. (5.4) has been shifted by $+1$. This relation looks remarkably similar to our Consistency Condition II eq. (3.1). However, in eq. (3.1) the derivatives, which shift the indices of the ϕ_k are taken with respect to all variables. In the derivation of the Consistency Condition II we have used properties of the Bessel-functions in only one step, namely in eq. (3.4). The fact that all τ-functions eq. (5.5) obey the property

$$2\mu\partial_\mu\phi_k(x) \ = \ \partial_x\phi_k(x)|_{x=\mu^2} \ = \ \phi_{k+1}(x) \tag{5.10}$$

can probably be used to show that all τ-functions obey our Consistency Condition II.

There exists another set of relations for τ-functions of the KP hierarchy which apparently can be related to our finite-volume partition functions. According to ref. [20] these relations read

$$\det_{1\leq a,b\leq k}\left[\frac{\tau(\mathbf{t} + [\xi_a^{-1}] - [\eta_b^{-1}])}{(\xi_a - \eta_b)\tau(\mathbf{t})}\right] \ = \ \prod_{a<b}^{k}(\xi_a - \xi_b)(\eta_b - \eta_a)\frac{\tau\left(\mathbf{t} + \sum_{a=1}^{k}([\xi_a^{-1}] - [\eta_a^{-1}])\right)}{\prod_{a,b}^{k}(\xi_a - \eta_b)\tau(\mathbf{t})} \ . \tag{5.11}$$

At first sight they look remarkably similar to our Consistency Condition I in the form of eq. (2.2). In order to to explain the differences let us give the notation of eq. (5.11) from ref. [20]. The argument of the τ-function $\mathbf{t}$ stands for all the coupling constants or times in a matrix potential $V(\lambda) = \sum_{k=1}^{\infty} t_k\lambda^k$. The bracket [ ] then is a shorthand notation for

$$\tau(\mathbf{t} \pm [\xi]) \ = \ \tau(t_1 \pm \xi, t_2 \pm \frac{1}{2}\xi^2, t_3 \pm \frac{1}{3}\xi^3, \ldots) \ . \tag{5.12}$$

Performing the sum over the additional parameter $[\xi^{-1}]$ in the potential leads to an additinal logarithmic term $V(\lambda) \to V(\lambda) - \ln(1 - \lambda/\xi)$, which resembles an extra "mass term" if we could consider this as an ordinary hermitian Random Matrix Theory in which one has taken the microscopic limit. Such an identification is, however, far from obvious. Moreover, in eq. (5.11) these extra terms occur in pairs ξ and η with *opposite* signs and thus one of them appears as a bosonic "mass term". Due to this difference, apart from other additional factors, we have not been able to explicitly match our Consistency Condition I eq. (2.2) with eq. (5.11). Moreover, no direct proof has been given in ref. [20] for the relation eq. (5.11), and we have not been able to find it elsewhere.

6 Summing over topological charges

So far our discussion has been restricted to finite-volume partition functions $\mathcal{Z}_\nu^{(N_f)}(\{\mu\})$ in sectors of fixed topological index ν. These partition functions can be thought of as Fourier coefficients of the full partition function, which for given vacuum angle θ, is given by

$$\mathcal{Z}^{(N_f)}(\theta,\{\mu\}) \;=\; \sum_{\nu=-\infty}^{\infty} e^{i\nu\theta}\,\mathcal{Z}_\nu^{(N_f)}(\{\mu\}) \ , \tag{6.1}$$

or, in terms of the effective partition function on the coset of chiral symmetry breaking in this case,

$$\mathcal{Z}^{(N_f)}(\theta,\{\mu\}) \;=\; \int_{SU(N_f)} dU \exp\left[V\Sigma\mathrm{Re}\,[e^{i\theta/N_f}\,\mathrm{Tr}\,\mathcal{M}U^\dagger]\right] \ . \tag{6.2}$$

Contrary to the effective partition functions $\mathcal{Z}_\nu^{(N_f)}(\{\mu\})$ in sectors of fixed gauge field topology, the group integral of eq. (6.2) is not known in closed form for $N_f \geq 3$. We are not aware of any analogue of the theorems discussed above for the full partition functions, and in view of the non-linearity of the relations it seems unlikely that they could be established. Nevertheless, as an interesting by-product of the above analysis we are now able to provide a simple compact formula for any k-point spectral correlation function after having summed over all topological charges.

The first observation is that for any observable $\langle\mathcal{O}\rangle_\nu$ in the fixed-ν theory one finds the same observable in the full theory by summing over ν with weight factor $e^{i\nu\theta}\mathcal{Z}_\nu^{(N_f)}(\{\mu\})$:

$$\langle\langle\mathcal{O}\rangle\rangle \;=\; \sum_{\nu=-\infty}^{\infty} e^{i\nu\theta}\,\mathcal{Z}_\nu^{(N_f)}(\{\mu\})\ \langle\mathcal{O}\rangle_\nu \ . \tag{6.3}$$

Next, noting that the k-point spectral correlation function by itself is just an expectation value, also this function can be summed over topological charges:

$$\bar{\rho}_S(\xi_1,\ldots,\xi_k;\theta,\{\mu\}) = \mathcal{Z}^{(N_f)}(\theta;\{\mu\})^{-1}\sum_{\nu=-\infty}^{\infty} e^{i\nu\theta}\,\mathcal{Z}_\nu^{(N_f)}(\{\mu\})\ \rho_S^{(\nu)}(\xi_1,\ldots,\xi_k;\{\mu\}) \ , \tag{6.4}$$

where we have *not* included the zero-mode contributions in the spectral sums, and where we have already taken the microscopic limit.

We now make us of the fact that the k-point function in a sector of fixed topological charge ν can be expressed in terms of a partition functions with $2k$ additional species as in eq. (1.3). As a side result of proving the Consistency Condition I in section 2 we have already fixed the constant

$C_2^{(k)} = (-1)^{k(\nu+[N_f/2])}$. Only the ν-dependence is important for the summation over topological charges, where it leads to a shift in the θ-angle:

$$\begin{aligned}
\bar{\rho}_S(\xi_1,\ldots,\xi_k;\theta,\{\mu\}) &= \mathcal{Z}^{(N_f)}(\theta;\{\mu\})^{-1}\sum_{\nu=-\infty}^{\infty} e^{i\nu\theta}(-1)^{k(\nu+[N_f/2])}\prod_i^k\left(|\xi_i|\prod_{f=1}^{N_f}(\xi_i^2+\mu_f^2)\right)\times\\
&\quad\times\prod_{j<l}^k(\xi_j^2-\xi_l^2)^2\mathcal{Z}_\nu^{(N_f+2k)}(\{\mu\},\{i\xi_1\},\ldots,\{i\xi_k\})\\
&= (-1)^{k[N_f/2]}\prod_{i=1}^k\left(|\xi_i|\prod_{f=1}^{N_f}(\xi_i^2+\mu_f^2)\right)\prod_{j<l}^k(\xi_j^2-\xi_l^2)^2\ \times\\
&\quad\times\frac{\mathcal{Z}^{(N_f+2k)}(\theta+k\pi;\{\mu\},\{i\xi_1\},\ldots,\{i\xi_k\})}{\mathcal{Z}^{(N_f)}(\theta;\{\mu\})}\ .
\end{aligned}\tag{6.5}$$

Due to the periodicity of the angle θ we need to know the partition function at either a shifted or unshifted vacuum angle θ for the k-point correlation function with k either even or odd [4].

7 Fermions in the adjoint representation

With N_f fermions taken in the adjoint representation of the gauge group SU($N_c \geq 2$) the pattern of spontaneous chiral symmetry breaking, if it occurs at all, is believed to proceed according to SU(N_f) $\rightarrow$ SO(N_f). In the Random Matrix Theory classification this corresponds to the chiral Symplectic Ensemble chSE. Fermions in the adjoint representation of the gauge group occur in *e.g.* supersymmetric gauge theories even without matter fields, but its interest in the present context stems more from the fact that staggered fermions in the *fundamental* representation and gauge group SU(2) actually also fall into this universality class away from the continuum limit. In ref. [22] the effective partition function in the same finite-volume limit as above was written, in a sector of fixed topological charge $\bar{\nu} = N_c\nu$, in terms of a relatively simple-looking group integral over the unitary group U(N_f):

$$\mathcal{Z}_{\bar{\nu}}^{(N_f)}(\mathcal{M}) = \int dU(\det U)^{-2\bar{\nu}}\exp\left[\Sigma V\mathrm{Re}\ \mathrm{Tr}\mathcal{M}UU^T\right]\ ,\tag{7.1}$$

where as before $\mathcal{M}$ denotes the mass matrix. This group integral is surprisingly difficult to perform explicitly, and it has in fact until now only been evaluated for $N_f = 2$ fermions of equal mass, and any topological charge $\bar{\nu}$ [23]. In general, the equal-mass partition functions are much easier to evaluate due to an interesting rewriting of the group integral (7.1), which is valid for those cases [22]. We shall here use that form to explicitly evaluate the group integral for $N_f = 4$ equal-mass fermions in a sector of zero topological charge (the derivation extends straightforwardly to any topological charge, but we have not considered that extension in detail).

The partition function (7.1) of N_f equal masses has been be rewritten by Smilga and Verbaarschot as (for N_f even) [22]:

$$\mathcal{Z}_{\bar{\nu}}^{(N_f)}(\mathcal{M}) = \mathrm{Pf}(A)\ ,\tag{7.2}$$

where, in our normalization, the $N_f \times N_f$ matrix A has elements

$$A_{pq} = -\frac{i\pi}{2}\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}\int_{-\pi}^{\pi}\frac{d\phi}{2\pi}\epsilon(\theta-\phi)e^{i(p\phi+q\theta)}e^{\mu\cos\phi+\mu\cos\theta+i\bar{\nu}(\phi+\theta)}\ ,\tag{7.3}$$

[4]In [21] a factor of $(-1)^{[N_f/2]}$ is missing in the formula for the density (corresponding here to $k = 1$).

and the indices p and q run from $-\frac{N_f}{2}+\frac{1}{2}$ to $\frac{N_f}{2}-\frac{1}{2}$. Again $\mu \equiv m\Sigma V$ is the (common) rescaled mass. A for our purposes convenient infinite-sum representation was also given in ref. [22], based on a Fourier series expansion of the sign function in eq. (7.3):

$$A_{pq} \ = \ \sum_{k=-\infty}^{\infty} \frac{1}{2k+1} I_{\bar{\nu}+p+k+\frac{1}{2}}(\mu) I_{\bar{\nu}+q-k-\frac{1}{2}}(\mu) \ . \tag{7.4}$$

It is with this form of the matrix A that we have managed to evaluate the partition function (7.1) for $N_f = 4$ equal masses and $\bar{\nu} = 0$ (this last restriction can readily be lifted). After some tedious algebra we find (technical details can be found in Appendix C):

$$\mathcal{Z}_0^{(4)}(\mu) \ = \ \frac{1}{\mu^2}\left[I_1(2\mu)^2 - I_0(2\mu)^2\right] + \frac{1}{2\mu^3} I_0(2\mu) \int_0^{2\mu} dt I_0(2\mu) \ . \tag{7.5}$$

The last integral is explicitly known in terms of a combination of Struve and Bessel functions, but we leave the result like this in order to facilitate a comparison to be discussed below.

The reason for our interest in the partition function (7.1) is a general relation derived in the third of ref. [4], which expresses the microscopic spectral density of the Dirac operator for this case in terms of the partition function itself and the partition function with 4 additional fermion species of imaginary (degenerate) masses:

$$\rho_S^{(\nu)}(\xi;\{\mu\}) \ = \ C_4 \ \xi^3 \ (\xi^2+\mu^2)^4 \ \frac{\mathcal{Z}_\nu^{(N_f+4)}(\{\mu\},\{i\xi\})}{\mathcal{Z}_\nu^{(N_f)}(\{\mu\})} \ . \tag{7.6}$$

The normalization coefficient C_4 can be fixed as soon as one settles on the normalization of the partition functions. The general formula (7.6) as derived in ref. [4] in the Random Matrix Theory formulation (and the partition functions involved were therefore those of Random Matrix Theory, too). But in the microscopic limit these coincide, modulo uninteresting mass-independent normalization factors, with the field theory partition functions (7.1), thus giving explicitly the microscopic spectral density in terms of the field theory partition functions, as indicated. Although this general formula may provide a simple way of deriving the massive double-microscopic spectral density for this universality class, it has not yet been tested due to the lack of a simple analytical expression for the partition functions (7.1). Now, with the analytical result (7.5) we can for the first time check the formula, since for the *quenched* case (formally defined by taking N_f to zero) the partition function itself becomes an uninteresting constant (which we take to be unity), while the partition function in the numerator of eq. (7.6) is that of just four fermions (of imaginary and degenerate masses).

Using well-known relations between Bessel functions and modified Bessel functions, we thus derive the quenched microscopic spectral density of the Dirac operator for this case:

$$\begin{aligned} \rho_S^{(0)}(\xi) \ &= \ C_4 \xi^3 \mathcal{Z}_0^{(4)}(\{i\xi\}) \\ &= \ C_4 \left\{ \xi[J_0(2\xi)^2 + J_1(2\xi)^2] - \frac{1}{2} J_0(2\xi) \int_0^{2\xi} dt J_0(t) \right\} \ . \end{aligned} \tag{7.7}$$

If we next impose the matching condition $\rho_s(\xi \to \infty) = 1/\pi$, then the overall constant in front is fixed to $C_4 = 1$. This finally gives

$$\rho_S^{(0)}(\xi) \ = \ \xi[J_0(2\xi)^2 + J_1(2\xi)^2] - \frac{1}{2} J_0(2\xi) \int_0^{2\xi} dt J_0(t) \ , \tag{7.8}$$

which agrees exactly with the result obtained directly from Random Matrix Theory [24].

In a derivation of the formula (7.6) directly from a partially quenched chiral Lagrangian, the four-fold mass-degenerate additional fermion species should come out from coset of the supergroup chiral symmetry breaking of that case. The fact that one is led to *four* additional quarks is a somewhat surprising feature, as in a very naive counting one could have expected *two*: one from the additional quenched quark, and one from its supersymmetric partner. It is therefore very comforting to see that the recent explicit computation of the partially quenched effective lagrangian by Toublan and Verbaarschot [6] in this case precisely leads to four additional species in total. By compactifying variables after taking the discontinuity that gives the spectral density it should therefore now be possible to derive the formula (7.6) directly from the effective Lagrangian, following the steps of the first of ref. [5].

8 Conclusions

Our main purpose here has been to show that the surprising relations among the effective partition functions relevant for describing the microscopic Dirac operator spectrum can be derived directly, without recourse to the Random Matrix Theory formulation. We have noted that these identities most likely have as their origin the fact that the effective partition functions of the chUE universality class are τ-functions of an integrable KP hierarchy. We believe that one of these identities, here called Consistency Condition II, holds in general for all these τ-functions.

As a by-product of our analysis, we have computed the ν-dependent normalization factor of the k-point spectral correlation function of the same universality class. This has allowed us to perform the sum over topological charges ν explicitly, and express the k-point function of the full theory entirely in terms of the full effective partition functions, without the restriction to fixed topological charge.

We have noted a series of relations between the effective finite-volume partition functions for QCD_3-like theories and QCD_4-like theories. These relations translate into surprising relations between the external $U(N_f)$ field problem and the Harish-Chandra integral for unitary groups.

Finally, we have considered an analogous formula for the microscopic spectral density of the chSE universality class, which expresses this spectral density in terms of the effective field theory partition function with four additional (imaginary-mass) fermion species. We have explicitly shown that this formula yields the same analytical result as the Random Matrix Theory approach in the quenched case of $N_f = 0$. Considering the analytical difficulties in extending the corresponding Random Matrix Theory calculation to the case of massive fermions, this may provide the most economical way of deriving all the microscopic spectral correlators of that universality class.

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A Some lemmas for Consistency Condition I

Lemma 1: Let $A(\{\xi\},\{\eta\})$ be the following $2k \times 2k$ matrix

$$(A(\{\xi\},\{\eta\}))_{ij} \;=\; \left\{ \begin{array}{ll} \xi_i^{j-1} I_{\nu+j-1}(\xi_i) & i=1,\ldots,k \\ \eta_i^{j-1} I_{\nu+j-1}(\eta_i) & i=k+1,\ldots,2k \end{array} \right. \quad \forall\, j \;. \tag{A.1}$$

Then the following statement holds:

$$\det A(\{\xi\},\{\eta\}) \;=\; (-1)^{k(k-1)/2} \left| \begin{array}{cc} B(\xi) & C(\xi) \\ B(\eta) & C(\eta) \end{array} \right| \prod_{i=1}^{k} I_\nu(\xi_i) I_\nu(\eta_i) \tag{A.2}$$

where the two $k \times k$ matrices B and C are given by

$$B(\xi)_{ij} \;=\; (\xi_i^2)^{j-1} \;, \quad C(\xi)_{ij} \;=\; (\xi_i^2)^{j-1} \xi_i \vec{\partial}_{\xi_i} \;, \quad i,j=1,\ldots,k \;. \tag{A.3}$$

In other words B is the matrix inside the Vandermonde with squared arguments whereas C contains in addition the power counting operator in each row, which acts on Bessel functions outside the determinant only.

Before proving the lemma let us add a remark. The above statement is not specific for a matrix with variables split into 2 equal groups, as we will see in the proof below. Furthermore eq. (A.2) trivially extends to matrices A of odd size, where one additional column has to be added in the matrix B. We have just presented the statement in the precise form as we need it in the proof of theorem 2.3.

Proof: Let us first for simplicity set

$$x_i \;\equiv\; \xi_i \;, \quad x_{i+k} \;\equiv\; \eta_i \;\; \text{for} \;\; i=1,\ldots,k \;. \tag{A.4}$$

We will in a first step use the property of the Bessel functions

$$x I_{n+1}(x) \;+\; 2n I_n(x) \;=\; x I_{n-1} \tag{A.5}$$

to reduce all indices of the Bessel functions in $\det A(\{x\})$ down to ν or $\nu+1$. Starting with the last column we can add $2(\nu+2k-2)$ times the last but one column to it to reduce the index of the Bessel function by 2:

$$x_i^{2k-1} I_{\nu+2k-1}(x_i) \;+\; 2(\nu+2k-2) x_i^{2k-2} I_{\nu+2k-2}(x_i) \;=\; x_i^{2k-1} I_{\nu+2k-3}(x_i) \;. \tag{A.6}$$

We proceed similarly with the last but one column and go down till the third column. All indices of Bessel functions have been reduced by 2 except in the first 2 columns. We then start again with the last column reducing the index by 2 (eq. (A.6) for $\nu \to \nu-2$) and continue down to the fifth column. So after going down through all columns $k-1$ times always starting from the right we have achieved

$$\det A(\{x\}) \;=\; \det_{1\leq i,j\leq 2k} \left(x_i^{j-1} I_{\nu+\frac{1+(-1)^j}{2}}(x_i) \right) \;. \tag{A.7}$$

In a second step we write all functions $I_{\nu+1}(x)$ which appear in every second column as a derivative $\partial_x I_\nu(x)$ so that we can factor out all Bessel functions to the right of the determinant. Using eq. (3.4) for $n=0$ we have

$$x^{2l+1} I_{\nu+1}(x) \;=\; x^{2l+1} \partial_x I_\nu(x) \;-\; \nu x^{2l} I_\nu(x) \;. \tag{A.8}$$

Inserting this expression into every second column we can again compensate the last term in eq. (A.8) by adding ν times the column to the left. We thus have

$$\det A(\{x\}) \;=\; \det \tilde{A}(\{x\}) \prod_{i=1}^{k} I_\nu(x_i) \ , \tag{A.9}$$

where

$$(\tilde{A}(\{x\}))_{ij} \;=\; \left\{ \begin{array}{cc} x_i^{j-1} & j=1,3,\ldots,2k-1 \\ x_i^{j-1}\vec{\partial}_{x_i} & j=2,4,\ldots,2k \end{array} \right. \quad \forall\ i=1,\ldots,2k\ . \tag{A.10}$$

Here we have pulled out all common factors $I_\nu(x_i)$ out of each row i and again the derivatives only act on these. After reordering those columns with even powers to the left and those with odd powers times a derivative to the right, which results into a factor $(-1)^{k(k-1)/2}$, we obtain eq. (A.2) when renaming back the variables from eq. (A.4).

<u>Lemma 2:</u> Let

$$(G(i:\gamma))_{jl} \;\equiv\; \left\{ \begin{array}{cc} \frac{1}{\eta_{\gamma(j)}^2-\xi_l^2}\xi_l\vec{\partial}_{\xi_l} & l=1,\ldots,i \\ \frac{1}{\eta_{\gamma(j)}^2-\xi_l^2} & l=i+1,\ldots,k \end{array} \right. \ \forall\ j=1,\ldots,k \ \ , \tag{A.11}$$

where γ is a permutation of the indices $1,\ldots,k$ with the ordering $\gamma(1)<\ldots<\gamma(i)$ and $\gamma(i+1)<\ldots<\gamma(k)$ for $1\leq i\leq k$ fixed. It then holds

$$\begin{aligned} \det G(i;\gamma) \;&=\; (-1)^{\frac{k(k-1)}{2}} \prod_{a,b=1}^{k} \frac{1}{(\eta_a^2-\xi_b^2)} \sum_{\bar{\gamma}} (-1)^{\bar{\gamma}} \Delta(\xi_{\bar{\gamma}(1)}^2,\ldots,\xi_{\bar{\gamma}(k-i)}^2,\eta_{\gamma(1)}^2,\ldots,\eta_{\gamma(i)}^2) \times \\ &\times\ \Delta(\xi_{\bar{\gamma}(k-i+1)}^2,\ldots,\xi_{\bar{\gamma}(k)}^2,\eta_{\gamma(i+1)}^2,\ldots,\eta_{\gamma(k)}^2) \prod_{j=k-i+1}^{k} \xi_{\bar{\gamma}}(j)\partial_{\xi_{\bar{\gamma}(j)}}\ , \end{aligned} \tag{A.12}$$

where $\bar{\gamma}$ is a permutation as γ of the indices of the ξ's.

<u>Proof:</u> We will prove Lemma 2 by doing a Laplace expansion with respect to the first i columns into i times $(k-i)$ dimensional determinants. Each of the subdeterminants can then be evaluated by using Cauchy's Lemma eq. (2.9) for squared variables, which reads

$$\det_{1\leq a,b\leq k}\left[\frac{1}{\eta_b^2-\xi_a^2}\right] \;=\; (-1)^{\frac{k(k-1)}{2}} \frac{\prod_{a<b}^{k}(\xi_a^2-\xi_b^2)(\eta_a^2-\eta_b^2)}{\prod_{a,b=1}^{k}(\eta_a^2-\xi_b^2)}\ . \tag{A.13}$$

We thus obtain for the first upper block

$$\det_{1\leq j,l\leq i}\left[\frac{1}{\eta_{\gamma(j)}^2-\xi_l^2}\xi_l\vec{\partial}_{\xi_l}\right] \;=\; (-1)^{\frac{i(i-1)}{2}} \frac{\prod_{a<b}^{i}(\xi_a^2-\xi_b^2)(\eta_{\gamma(a)}^2-\eta_{\gamma(b)}^2)}{\prod_{a,b=1}^{i}(\eta_{\gamma(a)}^2-\xi_b^2)} \prod_{j=1}^{i}\xi_j\partial_{\xi_j}\ , \tag{A.14}$$

and similarly for the lower blocks, where we can directly use eq. (A.13). Taking all the permutations $\bar{\gamma}$ according to the Laplace expansion we arrive after a few steps at

$$\begin{aligned} \det G(i;\gamma) \;&=\; (-1)^{\frac{k(k-1)}{2}}(-1)^{i(i-k)} \prod_{a,b=1}^{k} \frac{1}{(\eta_a^2-\xi_b^2)} \sum_{\bar{\gamma}} (-1)^{\bar{\gamma}} \Delta(\xi_{\bar{\gamma}(1)}^2,\ldots,\xi_{\bar{\gamma}(i)}^2,\eta_{\gamma(i+1)}^2,\ldots,\eta_{\gamma(k)}^2) \times \\ &\times\ \Delta(\xi_{\bar{\gamma}(i+1)}^2,\ldots,\xi_{\bar{\gamma}(k)}^2,\eta_{\gamma(1)}^2,\ldots,\eta_{\gamma(i)}^2) \prod_{j=1}^{i} \xi_{\bar{\gamma}}(j)\partial_{\xi_{\bar{\gamma}(j)}}\ , \end{aligned} \tag{A.15}$$

where the sign of the permutation is defined by

$$(-1)^{\bar\gamma} \;=\; (-1)^{\sum_{j=1}^{i}(j+\bar\gamma(j))} \;=\; (-1)^{\sum_{j=i+1}^{k}(j+\bar\gamma(j))}\ . \tag{A.16}$$

To obtain the final form of eq. (A.12) we still have to perform a cyclic shift by $-i$ places in the indices permuted by $\bar\gamma$. Using the second form of eq. (A.16) we obtain

$$\begin{aligned}(-1)^{\bar\gamma\,+\,i(i-k)} \;&\to\; (-1)^{\frac{k(k+1)}{2}\,-\,\frac{i(i+1)}{2}\,+\,\sum_{j=1}^{k-i}\bar\gamma(j)\,+\,i(i-k)}\\ &=\; (-1)^{\sum_{j=1}^{k-i}(j+\bar\gamma(j))}\ ,\end{aligned} \tag{A.17}$$

which leads us from eq. (A.15) to eq. (A.12).

B Some lemmas for Consistency Condition II

<u>LEMMA 3:</u> Let A be an $n\times n$ matrix with elements a_{kl} and A_j be the same matrix with the j-th row shifted by one unit to the left: $(A_j)_{kl}=a_{kl}$ for $k\neq j$ and $(A_j)_{jl}=a_{jl+1}$. Expanding the determinant of A with respect to the last column we have

$$\det A \;=\; \sum_{l=1}^{n}(-1)^{n+l}a_{ln}\det A^*_{ln} \tag{B.1}$$

where A^*_{ln} is the algebraic complement of a_{ln}. It then holds

$$\sum_{j=1}^{n}\det A_j \;=\; \sum_{j=1}^{n}(-1)^{n+j}a_{jn+1}\det A^*_{jn} \tag{B.2}$$

<u>PROOF:</u> We have to show that when summing over the determinants of the shifted matrices A_j and expanding each of them with respect to the last column only determinants of the algebraic complement of A remain, all the other terms containing determinants of shifted matrices cancel.

Starting from the definition we obtain after rearranging terms

$$\sum_{j=1}^{n}\det A_j \;=\sum_{j=1}^{n}(-1)^{n+j}a_{jn+1}\det A^*_{jn} \;+\; \sum_{i=1}^{n}(-1)^{n+i}a_{in}\sum_{j=1\ ;\ j\neq i}^{n}\det(A_j)^*_{in}\ . \tag{B.3}$$

Here we have used the same notation for the shifted matrices and applied the fact that $(A_j)^*_{jn}=A^*_{jn}$. We will now show by induction that the double sum in eq. (B.3) vanishes. For $n=2$ this holds trivially. Inside the double sum we sum over shifted matrices of size $n-1$ so we can apply the statement eq. (B.2) for $n-1$. Introducing the $(n-2)\times(n-2)$ submatrix $A^*_{jn-1;in}$ of A, where now the last 2 columns and rows i and j are missing we are left with

$$0 \;=\; \sum_{i=1}^{n}(-1)^{n+i}a_{in}\left[\sum_{j=1}^{i-1}(-1)^{n-1+j}a_{jn}\det A^*_{jn-1;in} \;+\; \sum_{j=i+1}^{n}(-1)^{n+j}a_{jn}\det A^*_{jn-1;in}\right] \tag{B.4}$$

where the split is due to the missing row when expanding twice. It then follows from the obvious symmetry $A^*_{in-1;jn}=A^*_{jn-1;in}$ that the two sums inside the bracket cancel.

Lemma 4: Let A and A^*_{ln} be the same as in Lemma 3 and let A^L_k be the matrix A with the first L entries vanishing in the k-th row: $(A^L_k)_{ij} = 0$ for $i = 1, \ldots, L,\ j = k$ and a_{ij} else-wise. Then it holds for any fixed L

$$\sum_{k=1}^{n} \det A^L_k \; = \; (n-L) \det A \; . \tag{B.5}$$

Proof: Let us first state the trivial cases apart from $L = 0, n$. For $L = 1$ the determinants just differ by the first column and can thus be added up, giving $(n-1)\det A$ after pulling out a common factor. For $L = n-1$ each A^L_k can be expanded with respect to the row k which then just gives $\det A$.

We will now proceed by induction. Due to the above remarks the cases $n = 2, 3$ are trivial. If we expand the determinants on the left hand side of eq. (B.5) with respect to the last column we obtain

$$\begin{aligned} \sum_{k=1}^{n} \det A^L_k \; &= \; \sum_{k=1}^{n} \sum_{i=1; i\neq k}^{n} (-1)^{n+i} a_{in} \det (A^L_k)^*_{in} \; + \; \sum_{i=1}^{n} (-1)^{n+i} a_{in} \det A^*_{in} \\ &= \; \sum_{i=1}^{n} (-1)^{n+i} a_{in} (n-1-L) \det A^*_{in} \; + \; \det A \quad . \end{aligned} \tag{B.6}$$

In the first step we have used that $(A^L_i)^*_{in} = A^*_{in}$ and in the second step we have employed induction for $n-1$. The sum in the last line then gives $(n-1-L)\det A$ and adds up to the statement eq. (B.5).

Lemma 5: Let A be a nonsingular $(n-2) \times n$ matrix and b, c, ξ and η be n-vectors. Then the following property holds for determinants of $n \times n$ matrices which differ by the last two rows or columns:

$$\begin{vmatrix} A \\ b \\ \xi \end{vmatrix} \begin{vmatrix} A \\ c \\ \eta \end{vmatrix} - \begin{vmatrix} A \\ b \\ \eta \end{vmatrix} \begin{vmatrix} A \\ c \\ \xi \end{vmatrix} = \begin{vmatrix} A \\ b \\ c \end{vmatrix} \begin{vmatrix} A \\ \xi \\ \eta \end{vmatrix} \; . \tag{B.7}$$

Proof: Using properties of determinants $\det(BC) = \det B \det C$ and $\det A = \det A^T$ the statement eq. (B.7) is equivalent to

$$\begin{vmatrix} AA^T & Ac & A\eta \\ bA^T & bc & b\eta \\ \xi A^T & \xi c & \xi\eta \end{vmatrix} - \begin{vmatrix} AA^T & Ac & A\xi \\ bA^T & bc & b\xi \\ \eta A^T & \eta c & \eta\xi \end{vmatrix} = \begin{vmatrix} AA^T & A\xi & A\eta \\ bA^T & b\xi & b\eta \\ cA^T & c\xi & c\eta \end{vmatrix} \; . \tag{B.8}$$

We will now use extensively that determinants that differ by one row can be added. We obtain for the left hand side

$$\begin{aligned} \text{l.h.s.} \; &= \begin{vmatrix} AA^T & Ac & A\eta \\ 0 & bc & 0 \\ \xi A^T & \xi c & \xi\eta \end{vmatrix} + \begin{vmatrix} AA^T & Ac & A\eta \\ bA^T & 0 & b\eta \\ \xi A^T & \xi c & \xi\eta \end{vmatrix} - \begin{vmatrix} AA^T & Ac & A\xi \\ 0 & bc & 0 \\ \eta A^T & \eta c & \eta\xi \end{vmatrix} - \begin{vmatrix} AA^T & Ac & A\xi \\ bA^T & 0 & b\xi \\ \eta A^T & \eta c & \eta\xi \end{vmatrix} \\ &= \begin{vmatrix} AA^T & Ac & A\eta \\ bA^T & 0 & b\eta \\ 0 & 0 & \xi\eta \end{vmatrix} + \begin{vmatrix} AA^T & Ac & A\eta \\ bA^T & 0 & b\eta \\ \xi A^T & \xi c & 0 \end{vmatrix} - \begin{vmatrix} AA^T & Ac & A\xi \\ bA^T & 0 & b\xi \\ 0 & 0 & \eta\xi \end{vmatrix} - \begin{vmatrix} AA^T & Ac & A\xi \\ bA^T & 0 & b\xi \\ \eta A^T & \eta c & 0 \end{vmatrix} \\ &= \begin{vmatrix} AA^T & A\xi & Ac \\ bA^T & b\xi & 0 \\ \eta A^T & 0 & 0 \end{vmatrix} - \begin{vmatrix} AA^T & A\eta & Ac \\ bA^T & b\eta & 0 \\ \xi A^T & 0 & 0 \end{vmatrix} + \begin{vmatrix} AA^T & Ac & A\eta \\ bA^T & 0 & b\eta \\ 0 & c\xi & 0 \end{vmatrix} + \begin{vmatrix} AA^T & A\xi & Ac \\ bA^T & b\xi & 0 \\ 0 & 0 & c\eta \end{vmatrix} \end{aligned} \tag{B.9}$$

In the first line the 1. and 3. term cancel after expanding with respect to the last but one row, the same happens in the second line expanding the last row. The last line gives the right hand side of eq. (B.8) expanded with respect to the last row if we can show that

$$\begin{vmatrix} AA^T & A\xi & A\eta \\ bA^T & b\xi & b\eta \\ cA^T & 0 & 0 \end{vmatrix} = \begin{vmatrix} AA^T & A\xi & Ac \\ bA^T & b\xi & 0 \\ \eta A^T & 0 & 0 \end{vmatrix} - \begin{vmatrix} AA^T & A\eta & Ac \\ bA^T & b\eta & 0 \\ \xi A^T & 0 & 0 \end{vmatrix} . \tag{B.10}$$

Performing similar steps as before this can be shown to be equivalent to

$$\begin{vmatrix} AA^T & A\xi & A\eta \\ bA^T & 0 & 0 \\ cA^T & 0 & 0 \end{vmatrix} = \begin{vmatrix} AA^T & A\xi & Ac \\ bA^T & 0 & 0 \\ \eta A^T & 0 & 0 \end{vmatrix} - \begin{vmatrix} AA^T & A\eta & Ac \\ bA^T & 0 & 0 \\ \xi A^T & 0 & 0 \end{vmatrix} . \tag{B.11}$$

We will now apply the property $\det \begin{pmatrix} B & C \\ D & E \end{pmatrix} = \det B \det(E - DB^{-1}C)$ for matrices where B, E are quadratic and B is nonsingular. Having $E = 0_{2\times 2}$ and $B = AA^T$ nonsingular in the upper left corner eq. (B.11) is equivalent to

$$\begin{aligned} \det\left(\begin{pmatrix} bA^T \\ cA^T \end{pmatrix} \left(AA^T\right)^{-1} (A\xi \; A\eta)\right) &= \det\left(\begin{pmatrix} bA^T \\ \eta A^T \end{pmatrix} \left(AA^T\right)^{-1} (A\xi \; Ac)\right) \\ &- \det\left(\begin{pmatrix} bA^T \\ \xi A^T \end{pmatrix} \left(AA^T\right)^{-1} (A\eta \; Ac)\right) \end{aligned}$$

and thus to

$$\det\left((A\xi \; A\eta)\begin{pmatrix} bA^T \\ cA^T \end{pmatrix}\right) = \det\left((A\xi \; Ac)\begin{pmatrix} bA^T \\ \eta A^T \end{pmatrix}\right) - \det\left((A\eta \; Ac)\begin{pmatrix} bA^T \\ \xi A^T \end{pmatrix}\right) , \tag{B.12}$$

which can be seen to hold by writing out the 2×2 determinants.

C The partition function for $N_f = 4$ adjoint rep. fermions

We shall here give some technical details on the derivation of eq. (7.5) in the main text. Our starting point is the general relation (7.2) with, for equal masses, the matrix A given by eq. (7.4). The first observation is that the matrix A, apart from being antisymmetric, also has a mirror symmetry along the line perpendicular to (the conventionally defined) diagonal. In other words:

$$A_{-\frac{3}{2},-\frac{1}{2}} = A_{\frac{1}{2},\frac{3}{2}} \quad , \qquad A_{-\frac{3}{2},\frac{1}{2}} = A_{-\frac{1}{2},\frac{3}{2}} . \tag{C.1}$$

This means that we only have to evaluate 4 independent matrix elements. We choose these to be $a_{12} \equiv A_{-\frac{3}{2},-\frac{1}{2}}, a_{13} \equiv A_{-\frac{3}{2},\frac{1}{2}}, a_{14} \equiv A_{-\frac{3}{2},\frac{3}{2}}$ and $a_{23} \equiv A_{-\frac{1}{2},\frac{1}{2}}$. (To avoid the cumbersome indices we have changed the notation from matrix A to matrix a as indicated).

Our trick is to first Taylor expand all relevant expressions by means of the Taylor expansion of Bessel functions,

$$I_b(x) = \left(\frac{1}{2}x\right)^b \sum_{k=0}^{\infty} \frac{(\frac{1}{2}x)^{2k}}{k!\Gamma(b+k+1)} , \tag{C.2}$$

and then re-express the result for each matrix element as far as possible in terms of Bessel functions again, using the same Taylor expansion. The first steps of this procedure give

$$\begin{aligned} a_{12} &= \sum_{n=0}^{\infty} \frac{\mu^{2(n+1)}}{(n!)^2(2n+3)(n^2+3n+2)} \\ &= \frac{1}{\mu} \sum_{n=0}^{\infty} \frac{\mu^{2n+3}}{n!(n+2)!(2n+3)} \\ &= \frac{1}{\mu} \sum_{n=0}^{\infty} \int_0^{\mu} dt \frac{t^{2n+2}}{n!(n+2)!} \\ &= \frac{1}{2\mu} \int_0^{2\mu} dt I_2(t) \\ &= \frac{1}{\mu} I_1(2\mu) - \frac{1}{2\mu} \int_0^{2\mu} dt I_0(t) \ , \end{aligned} \tag{C.3}$$

after using a simple Bessel function identity. We next evaluate a_{23}:

$$\begin{aligned} a_{23} &= \sum_{n=0}^{\infty} \frac{\mu^{2n}}{(n!)^2(2n+1)} \\ &= \frac{1}{\mu} \sum_{n=0}^{\infty} \int_0^{\mu} dt \frac{t^{2n}}{(n!)^2} \\ &= \frac{1}{2\mu} \int_0^{2\mu} dt I_0(t) \ , \end{aligned} \tag{C.4}$$

which shows that we have the relation

$$a_{12} + a_{23} = \frac{1}{\mu} I_1(\mu) \ . \tag{C.5}$$

Next, we evaluate a_{13} using the same procedure:

$$\begin{aligned} a_{13} &= \sum_{n=0}^{\infty} \frac{(2n+2)\mu^{2n+1}}{((n+1)!)^2(2n+3)} \\ &= \frac{1}{\mu} \left[\sum_{n=0}^{\infty} \frac{\mu^{2(n+1)}}{((n+1)!)^2} - \sum_{n=0}^{\infty} \frac{\mu^{2(n+1)}}{((n+1)!)^2(2n+3)} \right] \\ &= \frac{1}{\mu} \left[\sum_{n=0}^{\infty} \frac{\mu^{2n}}{((n)!)^2} - \left(1 + \sum_{n=0}^{\infty} \frac{\mu^{2(n+1)}}{((n+1)!)^2(2n+3)} \right) \right] \\ &= \frac{1}{\mu} \left[I_0(2\mu) - \sum_{n=0}^{\infty} \frac{\mu^{2n}}{(n!)^2(2n+1)} \right] \\ &= \frac{1}{\mu} \left[I_0(2\mu) - a_{23} \right] \ , \end{aligned} \tag{C.6}$$

which shows that also a_{13} and a_{23} are simply related. Using the Taylor expansion of a_{14},

$$a_{14} = \sum_{n=0}^{\infty} \frac{(6n+1)\mu^{2n}}{(n!)^2(2n+3)(2n+1)} \ , \tag{C.7}$$

we find that the following identity holds:

$$a_{14} - a_{23} + \frac{1}{\mu} a_{13} = 2a_{12} \ . \tag{C.8}$$

PROOF: We simply note that the Taylor series for the matrix elements a_{14}, a_{23} and a_{13} combine in a simple way:

$$
\begin{aligned}
& \sum_{n=0}^{\infty}\left[\frac{(6n+1)\mu^{2n}}{(n!)^2(2n+3)(2n+1)} - \frac{\mu^{2n}}{(n!)^2(2n+1)} + \frac{1}{\mu}\frac{(2n+2)\mu^{2n+1}}{((n+1)!)^2(2n+3)}\right] \\
= & \sum_{n=0}^{\infty}\frac{\mu^{2n}}{(n!)^2}\left[\frac{6n+1}{(2n+3)(2n+1)} - \frac{1}{(2n+1)} + \frac{2}{(n+1)(2n+3)}\right] \\
= & \sum_{n=0}^{\infty}\frac{\mu^{2n}}{(n!)^2}\frac{4n^2+6n}{(2n+3)(2n+1)(n+1)} \\
= & \sum_{n=0}^{\infty}\frac{2\mu^{2(n+1)}}{(n!)^2(n+1)(2n+3)(n+2)} \\
= & \; 2a_{12} \; , \qquad (C.9)
\end{aligned}
$$

which proves the identity eq. (C.8).

We are now ready to evaluate the Pfaffian of eq. (7.2), using the fact the a Pfaffian of a 4×4 antisymmetric matrix is

$$
\begin{aligned}
\mathrm{Pf}(A) &= a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} \\
&= a_{12}^2 - a_{13}^2 + a_{14}a_{23} \; , \qquad (C.10)
\end{aligned}
$$

where in the second line we have used our additional symmetry (C.1).

It is handy to evaluate instead $\mu^3\mathrm{Pf}(A)$. We then get successively, using relations (C.5), (C.6) and (C.8):

$$
\begin{aligned}
\mu^3\mathrm{Pf}(A) &= \mu^3[a_{12}^2 - a_{13}^2 + a_{14}a_{23}] \\
&= \mu I_1(2\mu)^2 + \mu^3 a_{23}^2 - 2\mu^2 I_1(2\mu)a_{23} - \mu I_0(2\mu)^2 \\
&\quad + \mu I_0(2\mu)a_{23} + \mu^2 a_{23}a_{13} + a_{14}a_{23}\mu^3 \\
&= \mu[I_1(2\mu)^2 - I_0(2\mu)^2] + \mu I_0(2\mu)a_{23} \\
&\quad + a_{23}[\mu^3 a_{23} - 2\mu^2 I_1(2\mu) + \mu^2 a_{13} + \mu^3 a_{14}] \\
&= \mu[I_1(2\mu)^2 - I_0(2\mu)^2] + \mu I_0(2\mu)a_{23} \\
&\quad + a_{23}[\mu^3 a_{23} - 2\mu^2(\mu[a_{12} + a_{23}]) + \mu^2 a_{13} + \mu^3 a_{14}] \\
&= \mu[I_1(2\mu)^2 - I_0(2\mu)^2] + \mu I_0(2\mu)a_{23} \\
&\quad + \mu^3 a_{23}[a_{14} - 2a_{12} - a_{23} + \frac{1}{\mu}a_{13}] \\
&= \mu[I_1(2\mu)^2 - I_0(2\mu)^2] + \mu I_0(2\mu)a_{23} \\
&= \mu[I_1(2\mu)^2 - I_0(2\mu)^2] + \frac{1}{2}I_0(2\mu)\int_0^{2\mu} dt I_0(t) \; . \qquad (C.11)
\end{aligned}
$$

This is the result quoted in the main text.

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