H_∞ Filtering for Nonlinear Discrete-time Stochastic Systems with Randomly Varying Sensor Delays ⋆

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Abstract

This paper is concerned with the H_∞ filtering problem for a general class of nonlinear discrete-time stochastic systems with randomly varying sensor delays, where the delayed sensor measurement is governed by a stochastic variable satisfying the Bernoulli random binary distribution law. In terms of the Hamilton-Jacobi-Isaacs inequalities, preliminary results are first obtained that ensure the addressed system to possess an l_2-gain less than a given positive scalar γ. Next, a sufficient condition is established under which the filtering process is asymptotically stable in the mean square and the filtering error satisfies the H_∞ performance constraint for all nonzero exogenous disturbances under the zero-initial condition. Such a sufficient condition is then decoupled into four inequalities for the purpose of easy implementation. Furthermore, it is shown that our main results can be readily specialized to the case of linear stochastic systems. Finally, a numerical simulation example is used to demonstrate the effectiveness of the results derived.

Key words: Nonlinear systems; stochastic systems; discrete-time systems; H_∞ filtering; random sensor delay; Hamilton-Jacobi-Isaacs inequality.

1 Introduction

Filtering problem has long been one of the fundamental problems in signal processing, communications and control application. The filtering problem can be briefly described as the design of an estimator from the measured output to estimate the state of the given system. There have been many different kinds of filters designed under different conditions, see e.g. [2, 3] and the references therein. In particular, Kalman filtering has proven to be the most representative one among varieties of filters. In general, the assumption that measured data contains information about the current state of the system is needed in the Kalman filtering approach. However, in practical application such as engineering, biological and economic systems, the measured output may be delayed. Therefore, the problem of filtering with *delayed measurements* has been attracting considerable research interests, see [7, 12] for some recent publications, where the time-delays in the measurement is customarily assumed to be *deterministic*.

It is quite common in practice that the time-delays occur in a random way, rather than a deterministic way, for a number of engineering applications such as real-time distributed decision-making and multiplexed data communication networks. Hence, there is a great need to develop new filtering approaches for the system with randomly varying delayed measurements, and some efforts have been made in this regard so far, see e.g. [11, 14, 15]. On the other hand, nonlinear H_∞ filtering or H_∞ state estimation has been an active branch within the general research area of nonlinear filtering problems. As nonlinear H_∞ control theory develops, the nonlinear H_∞ filtering technologies have been extensively developed. Especially, the H_∞ filtering problems for nonlinear and/or stochastic systems have received increasing research attention, see e.g. [8, 12, 13], where the nonlinearities have been assumed to be bounded by a linearity-like form (e.g., Lipschitz and sector conditions), and the filters have been designed by solving a set of LMIs. With respect to general stochastic systems, the nonlinear H_∞ filtering problem has also been paid great efforts in [9, 16]. Unfortunately, the H_∞ filtering problem for *general nonlinear discrete-time stochastic systems with randomly varying sensor delays* has not been properly investigated yet, and the purpose of this paper is therefore to tackle such a problem by establishing a rather general framework.

In this paper, the H_∞ filtering problem is addressed for a general class of nonlinear discrete-time stochastic systems with randomly varying sensor delays. We first obtain a theorem which provides a Hamilton-Jacobi-Isaacs (HJI) inequality guaranteeing that the system under investigation has an l_2-gain less than a given scalar $\gamma > 0$. Second, we derive a sufficient condition under which the filtering process is asymptotically stable in the mean square and the filtering error satisfies H_∞ performance constraint for all nonzero exogenous disturbances under the zero-initial condition. We then decouple such a sufficient condition into four inequalities that can be checked more easily. Moreover, as we expect, our main results are specialized to linear system case without any difficulty. Finally, a numerical simulation example is exploited to

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show the effectiveness of the results derived.

2 Problem Formulation and Preliminaries

The notation used here is fairly standard except where otherwise stated.

Let $(\Omega, \mathscr{F}, \text{Prob})$ be a complete probability space, where Prob, the probability measure, has total mass 1. $\mathbb{E}\{x\}$ stands for the expectation of the stochastic variable x with respect to the given probability measure Prob.

Consider the following class of nonlinear discrete-time stochastic systems

$$\begin{cases} \vec{x}_{k+1} = \vec{f}(\vec{x}_k) + \vec{g}(\vec{x}_k)\vec{v}_k \\ \qquad\qquad + \vec{s}(\vec{x}_k)w_k^1 + \vec{h}(\vec{x}_k)\vec{v}_k w_k^2 \\ z_k = \vec{m}(\vec{x}_k), \end{cases} \tag{1}$$

where $\vec{x}_k \in \mathbb{R}^n$ is the state vector, $z_k \in \mathbb{R}^m$ is the state combination to be estimated, and $\{w_k\} \triangleq \{[(w_k^1)^T, (w_k^2)^T]^T\}$ is an $\mathbb{R}^{l+1}$-valued, zero-mean white-noise sequence on a probability space $(\Omega, \mathscr{F}, \text{Prob})$ with the covariance $\mathbb{E}\{w_k w_k^T\} = \widetilde{\Theta} = \text{diag}\{\theta_1, \theta_2, \cdots, \theta_l, \theta\} = \text{diag}\{\Theta, \theta\}$. Obviously, $\Theta = \mathbb{E}\{w_k^1 {w_k^1}^T\}$ and $\theta = \mathbb{E}\{(w_k^2)^2\}$. Let $(\Omega, \mathscr{F}, \{\mathscr{F}_k\}_{k\in\mathbb{I}^+}, \text{Prob})$ be a filtered probability space where $\{\mathscr{F}_k\}_{k\in\mathbb{I}^+}$ is the family of sub σ-algebras of $\mathscr{F}$ generated by $\{w_k\}_{k\in\mathbb{I}^+}$. In fact, each $\mathscr{F}_k$ is assumed to be the minimal σ-algebras generated by $\{w_i\}_{0\le i\le k-1}$ while $\mathscr{F}_0$ is assumed to be some given sub σ-algebras of $\mathscr{F}$, independent of $\mathscr{F}_k$ for all $k > 0$. The exogenous disturbance input $v_k \in \mathbb{R}^q$, which is assumed to satisfy $\{v_k\}_{k\in\mathbb{I}^+} \in l_2([0,\infty), \mathbb{R}^q)$, where $l_2([0,\infty), \mathbb{R}^q)$ is the space of nonanticipatory square-summable stochastic process $v = \{v_k\}_{k\in\mathbb{I}^+}$ with respect to $(\mathscr{F}_k)_{k\in\mathbb{I}^+}$ with the following norm:

$$\|v\|_2^2 = \mathbb{E}\left\{\sum_{k=0}^{\infty} \|v_k\|^2\right\} = \sum_{k=0}^{\infty} \mathbb{E}\left\{\|v_k\|^2\right\}.$$

The initial state $\vec{x}_0$ is assumed to be independent of the process $\{w_k\}_{k\in\mathbb{I}^+}$. The nonlinear functions $\vec{f}: \mathbb{R}^n \to \mathbb{R}^n$, $\vec{g}: \mathbb{R}^n \to \mathbb{R}^{n\times q}$, $\vec{h}: \mathbb{R}^n \to \mathbb{R}^{n\times q}$, $\vec{s}: \mathbb{R}^n \to \mathbb{R}^{n\times l}$, $\vec{m}: \mathbb{R}^n \to \mathbb{R}^m$ in (1) are all assumed to be smooth, time-invariant, matrix-valued functions with $\vec{f}(0) = 0$, $\vec{s}(0) = 0$ and $\vec{m}(0) = 0$.

Remark 1 *In model (1), v_k is an exogenous input that usually describes the external disturbance, and w_k represents both the exogenous random inputs and parameter uncertainty of the system (see [1]). In [16], the H_∞ filtering problem has been investigated for a class of continuous-time Itô-type stochastic nonlinear systems disturbed by the one-dimensional Wiener process. Actually, model (1) can be viewed as a discrete-time version of the system in [16] in the case of multidimensional Wiener process.*

The delayed sensor measurement is described by

$$\vec{y}_k = \vec{l}(\vec{x}_k) + \vec{k}(\vec{x}_k)\vec{v}_k \tag{2}$$

$$y_k = (1-\gamma_k)\vec{y}_k + \gamma_k \vec{y}_{k-1}, \tag{3}$$

where $\vec{y}_k \in \mathbb{R}^p$ is the ideal output vector, $y_k \in \mathbb{R}^p$ is the actual measured output vector, and $\gamma_k \in \mathbb{R}$ is a Bernoulli distributed white sequence taking the values of 1 and 0 with

$$\text{Prob}\{\gamma_k = 1\} = \mathbb{E}\{\gamma_k\} := \beta \tag{4}$$

$$\text{Prob}\{\gamma_k = 0\} = 1 - \mathbb{E}\{\gamma_k\} := 1 - \beta. \tag{5}$$

Here, $\gamma_k \in \mathbb{R}$ is assumed to be uncorrelated with w_k, v_k, and $\vec{x}_0$. Moreover, the nonlinear functions $\vec{l}: \mathbb{R}^n \to \mathbb{R}^p$, $\vec{k}: \mathbb{R}^n \to \mathbb{R}^{p\times q}$ in (2) are also assumed to be smooth, time-invariant, matrix-valued functions with $\vec{l}(0) = 0$. Note that the system measurement model (3) was used in [11, 14].

By setting

$$\begin{aligned} x_k &:= \begin{bmatrix} \vec{x}_k \\ \vec{x}_{k-1} \end{bmatrix}, \ v_k := \begin{bmatrix} \vec{v}_k \\ \vec{v}_{k-1} \end{bmatrix}, \\ f(x_k) &:= \begin{bmatrix} \vec{f}(\vec{x}_k) \\ \vec{x}_k \end{bmatrix}, \ g(x_k) := \begin{bmatrix} \vec{g}(\vec{x}_k) & 0 \\ 0 & 0 \end{bmatrix}, \\ h(x_k) &:= \begin{bmatrix} \vec{h}(\vec{x}_k) & 0 \\ 0 & 0 \end{bmatrix}, \ s(x_k) := \begin{bmatrix} \vec{s}(\vec{x}_k) \\ 0 \end{bmatrix}, \\ l(x_k) &:= \begin{bmatrix} \vec{l}(\vec{x}_k) \\ \vec{l}(\vec{x}_{k-1}) \end{bmatrix}, \ k(x_k) := \begin{bmatrix} \vec{k}(\vec{x}_k) & 0 \\ 0 & \vec{k}(\vec{x}_{k-1}) \end{bmatrix}, \\ m(x_k) &:= \vec{m}(\vec{x}_k), \qquad C_{\gamma_k} := \begin{bmatrix} (1-\gamma_k)I_p & \gamma_k I_p \end{bmatrix}, \end{aligned} \tag{6}$$

we can combine the nonlinear stochastic system (1) and the delayed sensor measurement (2)-(3) into a new form as follows

$$\begin{cases} x_{k+1} = f(x_k) + g(x_k)v_k \\ \qquad\qquad + s(x_k)w_k^1 + h(x_k)v_k w_k^2 \\ y_k = C_{\gamma_k}(l(x_k) + k(x_k)v_k) \\ z_k = m(x_k). \end{cases} \tag{7}$$

In this paper, we are interested in constructing a filter of the following form for system (7):

$$\begin{cases} \hat{x}_{k+1} = \hat{f}(\hat{x}_k) + \hat{G}(\hat{x}_k)y_k \\ \hat{z}_k = \hat{m}(\hat{x}_k), \ \hat{f}(0) = 0, \ \hat{m}(0) = 0, \ \hat{x}_0 = 0, \end{cases} \tag{8}$$

where $\hat{x}_k \in \mathbb{R}^{2n}$ is the state estimate of the stochastic system (7), $\hat{z}_k \in \mathbb{R}^m$ is the estimated output of the filter, and $\hat{f}$, $\hat{G}$ and $\hat{m}$ are filter parameters of appropriate dimensions that are smooth functions to be scheduled.

Defining

$$\eta_k := \begin{bmatrix} x_k^T & \hat{x}_k^T \end{bmatrix}^T, E := \begin{bmatrix} -I_p & I_p \end{bmatrix}, C_\beta := \begin{bmatrix} (1-\beta)I_p & \beta I_p \end{bmatrix},$$

we can get the following augmented system

$$
\begin{cases} \eta_{k+1} = \tilde{f}(\eta_k) + \tilde{g}(\eta_k)v_k + \tilde{h}(\eta_k)v_k w_k^2 + \tilde{s}(\eta_k)w_k^1 \\ \qquad + (\gamma_k - \beta)(\bar{f}(\eta_k) + \bar{g}(\eta_k)v_k) \\ \tilde{z}_k := z_k - \hat{z}_k = m(x_k) - \hat{m}(\hat{x}_k), \end{cases} \tag{9}
$$

where

$$
\begin{aligned}
\tilde{f}(\eta_k) &= \begin{bmatrix} f(x_k) \\ \hat{f}(\hat{x}_k) + \hat{G}(\hat{x}_k)C_\beta l(x_k) \end{bmatrix}, \\
\tilde{g}(\eta_k) &= \begin{bmatrix} g(x_k) \\ \hat{G}(\hat{x}_k)C_\beta k(x_k) \end{bmatrix}, \\
\bar{f}(\eta_k) &= \begin{bmatrix} 0 \\ \hat{G}(\hat{x}_k)El(x_k) \end{bmatrix}, \quad \tilde{s}(\eta_k) = \begin{bmatrix} s(x_k) \\ 0 \end{bmatrix}, \\
\bar{g}(\eta_k) &= \begin{bmatrix} 0 \\ \hat{G}(\hat{x}_k)Ek(x_k) \end{bmatrix}, \quad \tilde{h}(\eta_k) = \begin{bmatrix} h(x_k) \\ 0 \end{bmatrix}.
\end{aligned} \tag{10}
$$

In this paper, we aim to design the filter gain matrices $\hat{f}(\hat{x}_k)$, $\hat{G}(\hat{x}_k)$ and $\hat{m}(\hat{x}_k)$ in (8) such that the following requirements are simultaneously satisfied:

a) The zero-solution of the augmented system (9) with $v_k = 0$ is robustly asymptotically stable in the mean square.
b) Under the zero-initial condition, the filtering error $\tilde{z}_k$ satisfies

$$
\sum_{k=0}^{\infty} \mathbb{E}\{\|\tilde{z}_k\|^2\} < \gamma^2 \sum_{k=0}^{\infty} \mathbb{E}\{\|v_k\|^2\} \tag{11}
$$

for all nonzero v_k where $\gamma > 0$ is a given disturbance attenuation level.

3 Main Results

We first consider the following general stochastic system

$$
\begin{cases} \eta_{k+1} = H_k(\eta_k, v_k, \gamma_k, w_k) \\ \tilde{z}_k = \tilde{m}_k(\eta_k). \end{cases} \tag{12}
$$

where $H_k : \mathbb{R}^{4n} \times \mathbb{R}^{2q} \times \mathbb{R} \times \mathbb{R}^{l+1} \to \mathbb{R}^{4n}$ and $\tilde{m}_k : \mathbb{R}^{4n} \to \mathbb{R}^m$ are smooth, time-variant nonlinear matrix-valued functions, and $\eta_k, v_k, \gamma_k, w_k$ are defined previously.

Lemma 1 *Consider the general stochastic system (12). For the given scalar $\gamma > 0$, if there exist a family of positive real valued functions: $V_k : \mathbb{R}^{4n} \times \mathbb{I}^+ \to \mathbb{R}^+$ ($V_k(0) = 0$ for all $k \in \mathbb{I}^+$) satisfying the following HJI inequality*

$$
V_k(\eta) > \sup_{v \in \mathbb{R}^{2q}} \Big\{ \|\tilde{z}_k\|^2 - \gamma^2 \|v\|^2 + \mathbb{E}_{w_k, \gamma_k}\{V_{k+1}[H_k(\eta, v, \gamma_k, w_k)]\} \Big\} \tag{13}
$$

for all nonzero $\eta \in \mathbb{R}^{4n}$ where $\mathbb{E}_y\{\cdot\}$ is defined as in [1], then the system (12) has l_2-gain less than γ, i.e., the following H_∞ criterion is satisfied:

$$
\sum_{i=j}^{k-1} \mathbb{E}\{\|\tilde{z}_i\|^2\} < \mathbb{E}\{V_j(\eta_j)\} + \gamma^2 \sum_{i=j}^{k-1} \mathbb{E}\{\|v_i\|^2\} \tag{14}
$$

for all $0 \le j < k$ and for all $v \in \mathbb{R}^{2q}$.

Proof: This proof can be accomplished readily along the same line of the proof of Theorem 2 in [1] by paying attention to the random variable γ_k.

Remark 2 *Note that the augmented system (9) under consideration in this paper is time-invariant, and is therefore a special case of the general stochastic system (12). For the nonlinear time-invariant system $\eta_{k+1} = H(\eta_k, v_k, \gamma_k, w_k)$ and the time-invariant filtering error $\tilde{z}_k = z_k - \hat{z}_k = m(x_k) - \hat{m}(\hat{x}_k)$, we can easily obtain the following HJI inequality from (13):*

$$
V(\eta) \ge \sup_{v \in \mathbb{R}^{2q}} \Big\{ \|\tilde{z}\|^2 - \gamma^2 \|v\|^2 + \mathbb{E}_{w_k, \gamma_k}\{V[H(\eta, v, \gamma_k, w_k)]\} \Big\} \tag{15}
$$

which can ensure the time-invariant system to have l_2-gain less than γ.

Remark 3 *Under the zero-initial condition, the requirement (11) for the filter error $\tilde{z}_k$ to satisfy can be obtained from the H_∞ criterion (14) as long as one takes $j = 0$ and then lets $k \to +\infty$.*

Corollary 1 *Consider the augmented system (9) with a given disturbance attenuation level $\gamma > 0$. If there exists a positive definite matrix $Q = Q^T > 0$ satisfying*

$$
A := \gamma^2 I - \tilde{g}^T(\eta)Q\tilde{g}(\eta) - \beta(1-\beta)\bar{g}^T(\eta)Q\bar{g}(\eta) - \theta\tilde{h}^T(\eta)Q\tilde{h}(\eta) > 0, \tag{16}
$$

for all $\eta \in \mathbb{R}^{4n}$, and

$$
\eta^T Q \eta > BA^{-1}B^T + \beta(1-\beta)\bar{f}^T(\eta)Q\bar{f}(\eta) + \|\tilde{z}\|^2 + \tilde{f}^T(\eta)Q\tilde{f}(\eta) + \mathrm{trace}[\Theta^{\frac{1}{2}}\tilde{s}^T(\eta)Q\tilde{s}(\eta)\Theta^{\frac{1}{2}}], \tag{17}
$$

for all nonzero $\eta \in \mathbb{R}^{4n}$, where

$$
B := \tilde{f}^T(\eta)Q\tilde{g}(\eta) + \beta(1-\beta)\bar{f}^T(\eta)Q\bar{g}(\eta), \tag{18}
$$

then the error $\tilde{z}_k$ satisfies the H_∞ performance constraint (11) for all nonzero exogenous disturbances under the zero-initial condition.

Proof: We define the positive real-valued function $V(\eta)$ as $V(\eta) = \eta^T Q \eta$ for all $\eta \in \mathbb{R}^{4n}$. Then, we have

$$
\begin{aligned}
&\sup_{v \in \mathbb{R}^{2q}} \Big\{ \|\tilde{z}\|^2 - \gamma^2 \|v\|^2 + \mathbb{E}_{w_k, \gamma_k}\{V[H(\eta, v, \gamma_k, w_k)]\} \Big\} \\
&= \sup_{v \in \mathbb{R}^{2q}} \Big\{ -v^T A v + 2Bv + \beta(1-\beta)\bar{f}^T(\eta)Q\bar{f}(\eta) \\
&\quad + \tilde{f}^T(\eta)Q\tilde{f}(\eta) + \mathrm{trace}[\Theta^{\frac{1}{2}}\tilde{s}^T(\eta)Q\tilde{s}(\eta)\Theta^{\frac{1}{2}}] + \|\tilde{z}\|^2 \Big\} \\
&= BA^{-1}B^T + \tilde{f}^T(\eta)Q\tilde{f}(\eta) + \beta(1-\beta)\bar{f}^T(\eta)Q\bar{f}(\eta) \\
&\quad + \mathrm{trace}[\Theta^{\frac{1}{2}}\tilde{s}^T(\eta)Q\tilde{s}(\eta)\Theta^{\frac{1}{2}}] + \|\tilde{z}\|^2 \\
&< \eta^T Q \eta (= V(\eta))
\end{aligned} \tag{19}
$$

where the second equation holds when maximizing $v^* = A^{-1}B^T$ by applying the "completing the square" rule. Therefore, by Lemma 1 and Remark 3, the proof of Corollary 1 is complete.

Theorem 1 *Given the disturbance attenuation level $\gamma > 0$ and the filter parameters $\hat{f}$, $\hat{G}$ and $\hat{m}$. If there exist two positive definite matrices $Q_1 = Q_1^T > 0$ and $Q_2 = Q_2^T > 0$ satisfying*

$$\begin{aligned}A = &\gamma^2 I - \beta(1-\beta)k^T(x)E^T\hat{G}^T(\hat{x})Q_2\hat{G}(\hat{x})Ek(x)\\ &-k^T(x)C_\beta^T\hat{G}^T(\hat{x})Q_2\hat{G}(\hat{x})C_\beta k(x) - g^T(x)Q_1 g(x)\\ &-\theta h^T(x)Q_1 h(x) > 0, \end{aligned} \tag{20}$$

for all $x \in \mathbb{R}^{2n}, \hat{x} \in \mathbb{R}^{2n}$, and

$$\begin{aligned}\mathbb{H}(x,\hat{x}) := &BA^{-1}B^T + 2\hat{f}^T(\hat{x})Q_2\hat{G}(\hat{x})C_\beta l(x) + \|\tilde{z}\|^2\\ &+l^T(x)C_\beta^T\hat{G}^T(\hat{x})Q_2\hat{G}(\hat{x})C_\beta l(x) - x^TQ_1x\\ &+f^T(x)Q_1f(x) + \hat{f}^T(\hat{x})Q_2\hat{f}(\hat{x}) - \hat{x}^TQ_2\hat{x}\\ &+\beta(1-\beta)l^T(x)E^T\hat{G}^T(\hat{x})Q_2\hat{G}(\hat{x})El(x)\\ &+\mathrm{trace}[\Theta^{\frac{1}{2}}s^T(x)Q_1s(x)\Theta^{\frac{1}{2}}] < 0, \end{aligned} \tag{21}$$

for all nonzero $x \in \mathbb{R}^{2n}, \hat{x} \in \mathbb{R}^{2n}$, where

$$\begin{aligned}B = &f^T(x)Q_1g(x) + l^T(x)C_\beta^T\hat{G}^T(\hat{x})Q_2\hat{G}(\hat{x})C_\beta k(x)\\ &+\beta(1-\beta)l^T(x)E^T\hat{G}^T(\hat{x})Q_2\hat{G}(\hat{x})Ek(x)\\ &+\hat{f}^T(\hat{x})Q_2\hat{G}(\hat{x})C_\beta k(x), \end{aligned} \tag{22}$$

then the zero-solution of the augmented system (9) with $v_k = 0$ is asymptotically stable in the mean square and the filtering error $\tilde{z}_k$ satisfies the H_∞ performance constraint (11) for all nonzero exogenous disturbances under the zero-initial condition.

Proof: By taking $Q = \mathrm{diag}\{Q_1, Q_2\}$ and noticing the definition of $\tilde{f}(.)$, $\tilde{g}(.)$, $\tilde{h}(.)$, $\tilde{s}(.)$, $\bar{f}(.)$ and $\bar{g}(.)$ in (10), it follows immediately (16), (17) and (18) reduce to (20), (21) and (22), respectively. Therefore, we known easily from Corollary 1 that the filtering error $\tilde{z}_k$ satisfies (11) for all nonzero exogenous disturbances under the zero-initial condition.

Next, define the difference of the Lyapunov function as

$$\Delta V(\eta_k) = \mathbb{E}\{V(\eta_{k+1})|\eta_k\} - V(\eta_k), \tag{23}$$

where $V(\eta) = \eta^TQ\eta = x^TQ_1x + \hat{x}^TQ_2\hat{x}$ and $\eta = [x^T \;\; \hat{x}^T]^T$. Calculating the difference of $V(\eta_k)$ along the system (9) with $v_k = 0$ and taking the mathematical expectation, it follows from (20) and (21) that

$$\begin{aligned}&\mathbb{E}\{\Delta V(\eta_k)\} = \mathbb{E}\{V(\eta_{k+1}) - V(\eta_k)\}\\ =&\mathbb{E}\Big\{f^T(x_k)Q_1f(x_k) + \hat{f}^T(\hat{x}_k)Q_2\hat{f}(\hat{x}_k) - x_k^TQ_1x_k\\ &-\hat{x}_k^TQ_2\hat{x}_k + l^T(x_k)C_\beta^T\hat{G}^T(\hat{x}_k)Q_2\hat{G}(\hat{x}_k)C_\beta l(x_k)\\ &+\beta(1-\beta)l^T(x_k)E^T\hat{G}^T(\hat{x}_k)Q_2\hat{G}(\hat{x}_k)El(x_k)\\ &+2\hat{f}^T(\hat{x}_k)Q_2\hat{G}(\hat{x}_k)C_\beta l(x_k)\\ &+\mathrm{trace}[\Theta^{\frac{1}{2}}s^T(x_k)Q_1s(x_k)\Theta^{\frac{1}{2}}]\Big\}.\\ \le&\mathbb{E}\{\mathbb{H}(x_k,\hat{x}_k)\} < 0,\end{aligned}$$

which, by the Lyapunov stability theory, shows the zero-solution of the augmented system (9) with $v_k = 0$ is asymptotically stable in the mean square, and the proof of Theorem 1 is then complete.

Remark 4 *Theorem 1 is proved mainly by the "completing the square" technique which results in very little conservatism. In fact, Theorem 1 can be specialized to the existing results for systems with either Lipschitz or sector-bounded nonlinearities and for linear systems, which means that Theorem 1 serves as a theoretic basis for the H_∞ filtering problems of nonlinear stochastic systems.*

Obviously, it is generally difficult to solve the inequalities (20) and (21). In the following corollary, we aim to decouple the conditions of Theorem 1 into four inequalities that can be solved independently and more easily.

Under the standard assumption of $k^T(x)k(x) \equiv I$ (see e.g. [5]), the following corollary is obtained.

Corollary 2 *Given the disturbance attenuation level $\gamma > 0$ and the filter parameters $\hat{f}$, $\hat{G}$ and $\hat{m}$. The H_∞ filtering problem for the system (7) is solved by filter (8) if there exist three positive constants λ, μ and ε and two positive definite matrices $Q_1 = Q_1^T > 0$ and $Q_2 = Q_2^T > 0$ satisfying:*

$$\begin{aligned}&C_\beta^T\hat{G}^T(\hat{x})Q_2\hat{G}(\hat{x})C_\beta\\ &\qquad+\beta(1-\beta)E^T\hat{G}^T(\hat{x})Q_2\hat{G}(\hat{x})E \le \lambda I, \end{aligned} \tag{24}$$

$$\gamma^2 I - g^T(x)Q_1g(x) - \theta h^T(x)Q_1h(x) \ge (\mu+\lambda)I, \tag{25}$$

for all $x \in \mathbb{R}^{2n}, \hat{x} \in \mathbb{R}^{2n}$, and

$$\begin{aligned}\mathbb{H}_1(x) := &\frac{3}{\mu}\|f^T(x)Q_1g(x)\|^2 + f^T(x)Q_1f(x) + (\frac{3\lambda^2}{\mu}\\ &+\lambda + \varepsilon^{-1})\|l(x)\|^2 - x^TQ_1x + 2\|m(x)\|^2\\ &+\mathrm{trace}[\Theta^{\frac{1}{2}}s^T(x)Q_1s(x)\Theta^{\frac{1}{2}}] < 0, \end{aligned} \tag{26}$$

$$\begin{aligned}\mathbb{H}_2(\hat{x}) := &(\frac{3}{\mu} + \varepsilon)\|\hat{f}^T(\hat{x})Q_2\hat{G}(\hat{x})C_\beta\|^2 - \hat{x}^TQ_2\hat{x}\\ &+\hat{f}^T(\hat{x})Q_2\hat{f}(\hat{x}) + 2\|\hat{m}(\hat{x})\|^2 < 0, \end{aligned} \tag{27}$$

for all nonzero $x \in \mathbb{R}^{2n}, \hat{x} \in \mathbb{R}^{2n}$.

Proof: Using the elementary inequality $\|a+b\|^2 \le 2(\|a\|^2 + \|b\|^2)$, we can get

$$\|\tilde{z}\|^2 = \|m(x) - \hat{m}(\hat{x})\|^2 \le 2\|m(x)\|^2 + 2\|\hat{m}(\hat{x})\|^2. \tag{28}$$

Similarly, it follows easily from (22) and (24)-(25) that

$$\begin{aligned}BA^{-1}B^T \le &\frac{3}{\mu}\Big(\|f^T(x)Q_1g(x)\|^2 + \lambda^2\|l(x)\|^2\\ &+\|\hat{f}^T(\hat{x})Q_2\hat{G}(\hat{x})C_\beta\|^2\Big). \end{aligned} \tag{29}$$

Applying the well-known fact: $2x^Ty \le \varepsilon x^Tx + \varepsilon^{-1}y^Ty$, $\forall\, \varepsilon > 0$, we have

$$\begin{aligned}&2\hat{f}^T(\hat{x})Q_2\hat{G}(\hat{x})C_\beta l(x)\\ &\qquad\le \varepsilon\|\hat{f}^T(\hat{x})Q_2\hat{G}(\hat{x})C_\beta\|^2 + \varepsilon^{-1}\|l(x)\|^2. \end{aligned} \tag{30}$$

Then, we can obtain from (24), (28)-(30) that

$$\mathbb{H}(x,\hat{x}) \le \mathbb{H}_1(x) + \mathbb{H}_2(\hat{x}) < 0. \tag{31}$$

Therefore, the proof of this corollary follows directly from that of Theorem 1.

For the purpose of practical applications, we are now in a position to study the problem of H_∞ filtering for the nonlinear system (7) but with a linear filter. In what follows, we adopt a linear filter of the following structure

$$\begin{cases} \hat{x}_{k+1} = F_f \hat{x}_k + G_f y_k \\ \hat{z}_k = M_f \hat{x}, \quad \hat{x}_0 = 0. \end{cases} \tag{32}$$

Corollary 3 *Given the disturbance attenuation level $\gamma > 0$ and the filter parameters F_f, G_f and M_f. If there exist two positive constants μ and ε and two positive definite matrices $Q_1 = Q_1^T > 0$ and $Q_2 = Q_2^T > 0$ satisfying*

$$\gamma^2 I - g^T(x)Q_1 g(x) - \theta h^T(x)Q_1 h(x) \geq (\mu + \lambda_{\max}(\bar{w}))I, \quad \text{for all } x \in \mathbb{R}^{2n}, \tag{33}$$

$$\begin{aligned} &\frac{3}{\mu}\|f^T(x)Q_1 g(x)\|^2 + f^T(x)Q_1 f(x) - x^T Q_1 x \\ &+ (\frac{3\lambda^2_{\max}(\bar{w})}{\mu} + \lambda_{\max}(\bar{w}) + \varepsilon^{-1})\|l(x)\|^2 + 2\|m(x)\|^2 \\ &+ \mathrm{trace}[\Theta^{\frac{1}{2}} s^T(x) Q_1 s(x) \Theta^{\frac{1}{2}}] \\ &< 0, \quad \text{for all nonzero } x \in \mathbb{R}^{2n}, \end{aligned} \tag{34}$$

$$(\frac{3}{\mu} + \varepsilon)F_f^T Q_2 G_f C_\beta C_\beta^T G_f^T Q_2 F_f + F_f^T Q_2 F_f - Q_2 + 2M_f^T M_f < 0, \tag{35}$$

where

$$\bar{w} := C_\beta^T G_f^T Q_2 G_f C_\beta + \beta(1-\beta)E^T G_f^T Q_2 G_f E, \tag{36}$$

then the H_∞ filtering problem for the system (7) is solved by linear filter (32).

Proof: Noting the fact that $\bar{w} \leq \lambda_{\max}(\bar{w})I$, this corollary follows immediately from Corollary 2.

Now, let us take a look at the linear system. As we expect, the filter parameters can be characterized by the solution to a set of LMIs that can be easily solved by utilizing available software packages on the condition that the system (1) is degenerated to a linear system.

Let $\vec{f}(\vec{x}_k) = \vec{F}\vec{x}_k$, $\vec{g}(\vec{x}_k) = \vec{G}$, $\vec{h}(\vec{x}_k) = \vec{H}$, $\vec{S}(\vec{x}_k) = \vec{S}\vec{x}_k$, $\vec{l}(\vec{x}_k) = \vec{L}\vec{x}_k$, $\vec{m}(\vec{x}_k) = \vec{M}\vec{x}_k$ and $\vec{k}(\vec{x}_k) = \vec{K}$, where $\vec{K}$ satisfies the hypothesis that $\vec{K}^T\vec{K} = I$. Here, $\Theta = \mathbb{E}\{w_k^1 {w_k^1}^T\}$ is reduced to a scalar θ_1. Similar to what we have done previously, we can obtain a linear stochastic system as follows

$$\begin{cases} x_{k+1} = Fx_k + Gv_k + Hv_k w_k^2 + Sx_k w_k^1 \\ y_k = C_{\gamma_k}(Lx_k + Kv_k) \\ z_k = Mx_k, \end{cases} \tag{37}$$

where

$$F = \begin{bmatrix} \vec{F} & 0 \\ I_n & 0 \end{bmatrix}, G = \begin{bmatrix} \vec{G} & 0 \\ 0 & 0 \end{bmatrix}, H = \begin{bmatrix} \vec{H} & 0 \\ 0 & 0 \end{bmatrix}, S = \begin{bmatrix} \vec{S} & 0 \\ 0 & 0 \end{bmatrix},$$

$$L = \begin{bmatrix} \vec{L} & 0 \\ 0 & \vec{L} \end{bmatrix}, K = \begin{bmatrix} \vec{K} & 0 \\ 0 & \vec{K} \end{bmatrix}, M = \begin{bmatrix} \vec{M} & 0 \end{bmatrix}.$$

In the case when the linear filter (32) with $F_f = F$ and $M_f = M$ is still employed, then it can be seen from the following corollary that the filter parameter G_f can be designed by solving certain LMIs.

Corollary 4 *Given the disturbance attenuation level $\gamma > 0$. The H_∞ filtering problem for the system (7) is solved by the linear filter (32) with $F_f = F$ and $M_f = M$ if there exist two positive definite matrices $Q_1 = Q_1^T > 0$ and $Q_2 = Q_2^T > 0$, one real matrix X, and two positive constants λ and μ such that the following LMIs hold for a given positive scalar $\varepsilon > 0$:*

$$\begin{bmatrix} -\lambda I & C_\beta^T X^T & \beta(1-\beta)E^T X^T \\ XC_\beta & -Q_2 & 0 \\ \beta(1-\beta)XE & 0 & -\beta(1-\beta)Q_2 \end{bmatrix} < 0, \tag{38}$$

$$(\mu + \lambda - \gamma^2)I + G^T Q_1 G + \theta H^T Q_1 H < 0, \tag{39}$$

$$\begin{bmatrix} \Gamma & F^T Q_1 G & \lambda L^T \\ G^T Q_1 F & -\frac{\mu}{3}I & 0 \\ \lambda L & 0 & -\frac{\mu}{3}I \end{bmatrix} < 0, \tag{40}$$

$$\begin{bmatrix} -Q_2 + 2M^T M + F^T Q_2 F & F^T X C_\beta & F^T X C_\beta \\ C_\beta^T X^T F & -\frac{\mu}{3}I & 0 \\ C_\beta^T X^T F & 0 & -\varepsilon^{-1}I \end{bmatrix} < 0, \tag{41}$$

where

$$\Gamma = -Q_1 + (\lambda + \varepsilon^{-1})L^T L + 2M^T M + F^T Q_1 F + \theta_1 S^T Q_1 S. \tag{42}$$

Moreover, if the LMIs (38)-(41) are feasible, the desired filter parameter is given by

$$G_f = Q_2^{-1}X. \tag{43}$$

Proof: Noting that $f(x) = Fx$, $g(x) = G$, $h(x) = H$, $s(x) = Sx$, $l(x) = Lx$, $k(x) = K$, $f(\hat{x}) = F\hat{x}$, $\hat{G}(\hat{x}) = G_f$, $m(x) = Mx$, $\hat{m}(\hat{x}) = M\hat{x}$, and together with (43), it can be seen that (38)-(41) imply (24)-(27), respectively, in virtue of Schur's complement formula. The rest of the proof follows directly from Corollary 2.

Remark 5 *The parameter $\varepsilon > 0$ is fixed so that (40) and (41) are LMIs. In implementation, a linear search algorithm can be used to find a suitable scalar $\varepsilon > 0$. Similar strategy has been adopted in [6].*

4 An Illustrative Example

Consider the following nonlinear discrete-time stochastic system

$$
\begin{cases}
\vec{x}_{1,k+1} = -\dfrac{\vec{x}_{1,k}^3}{1+5\vec{x}_{1,k}^2} - \dfrac{1}{5}\vec{x}_{2,k} + \dfrac{1}{2}\vec{v}_k \\
\qquad\qquad + (\dfrac{\sqrt{20}}{5}\vec{x}_{1,k} - \dfrac{\sqrt{20}\vec{x}_{1,k}^2\vec{x}_{2,k}}{1+5\vec{x}_{1,k}^2})w_k^1 \\
\vec{x}_{2,k+1} = -\dfrac{\vec{x}_{2,k}}{4+\vec{x}_{1,k}^2+\vec{x}_{2,k}^2} + \dfrac{1}{2}\vec{v}_k w_k^2 + \dfrac{1}{4}\vec{x}_{2,k}w_k^1 \\
\qquad z_k = \dfrac{1}{6}\vec{x}_{1,k} + \dfrac{1}{5}\vec{x}_{2,k}
\end{cases} \tag{44}
$$

the delayed sensor measurement

$$
\vec{y}_k = -\frac{\vec{x}_{1,k}\vec{x}_{2,k}}{4+3\vec{x}_{2,k}^2} + \vec{v}_k \tag{45}
$$

$$
y_k = (1-\gamma_k)\vec{y}_k + \gamma_k\vec{y}_{k-1}. \tag{46}
$$

Let the disturbance attenuation be $\gamma = 1.65$, the variance be $\theta = \theta_1 = 0.5$ and the probability be $\mathrm{Prob}\{r_k = 1\} = \beta = 0.8$. We adopt a linear filter as follows

$$
\begin{cases}
\hat{x}_{k+1} = \begin{bmatrix} 0.25 & -0.05 & 0 & 0.02 \\ -0.05 & 0.3333 & 0 & 0 \\ 0 & 0 & 0.25 & 0 \\ 0.02 & 0 & 0 & 0.3333 \end{bmatrix} \hat{x}_k + \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} y_k \\
\quad \hat{z}_k = \begin{bmatrix} 0.2 & 0.1667 & 0 & 0 \end{bmatrix} \hat{x}_k, \quad \hat{x}_0 = 0.
\end{cases} \tag{47}
$$

It is not difficult to verify that $\hat{f}$, $\hat{G}$ and $\hat{m}$ satisfy the conditions of Theorem 1 with $Q_1 = \mathrm{diag}\{4, 4, 0.32, 0.05\}$ and $Q_2 = \mathrm{diag}\{1, 1, 1, 1\}$. Therefore, it follows from Theorem 1 that the filter of the form (47) is a desired state estimator. Simulation results are shown in Fig. 1 and Fig. 2, where the trajectory and estimation of the output z_k are given in Fig. 1 and the estimation error $\tilde{z}_k$ is depicted in Fig. 2, which coincide with our theoretical analysis.

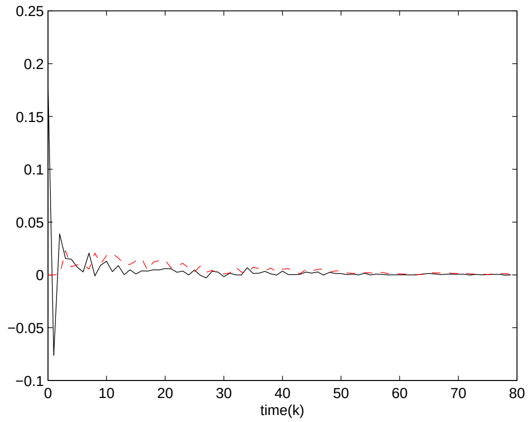

Fig. 1. z (solid) and $\hat{z}$ (dashed)

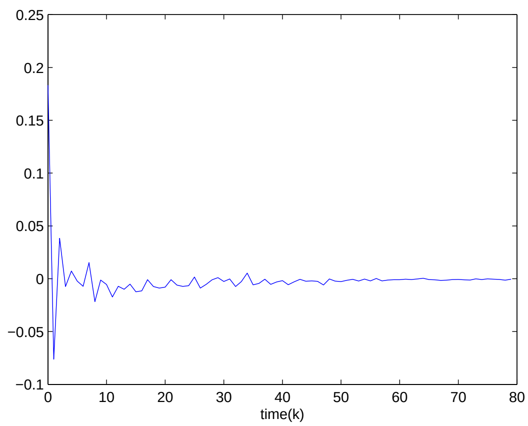

Fig. 2. The estimation error $\tilde{z}$ (solid)

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