State Estimation for Jumping Recurrent Neural Networks with Discrete and Distributed Delays

Zidong Wang, Yurong Liu and Xiaohui Liu

Abstract

This paper is concerned with the state estimation problem for a class of Markovian neural networks with discrete and distributed time-delays. The neural networks have a finite number of modes, and the modes may jump from one to another according to a Markov chain. The main purpose is to estimate the neuron states, through available output measurements, such that for all admissible time-delays, the dynamics of the estimation error is globally asymptotically stable in the mean square. An effective linear matrix inequality approach is developed to solve the neuron state estimation problem. Both the existence conditions and the explicit characterization of the desired estimator are derived. Furthermore, it is shown that the traditional stability analysis issue for delayed neural networks with Markovian jumping parameters can be included as a special case of our main results. Finally, numerical examples are given to illustrate the applicability of the proposed design method.

Keywords

Neural networks; Markovian jumping systems; State estimation; Time-delays; Asymptotic stability; Linear matrix inequalities.

I. Introduction

In the past decade, the dynamic behavior of the delayed recurrent neural networks (RNNs) has become a popular subject of research that attracts increasing interests. There are basically two reasons. First, RNNs have been successfully applied in many areas, including image processing, pattern recognition, associative memory, and optimization problems. Second, it has been widely known that many biological and artificial neural networks contain inherent time delays, which may cause oscillation and instability (see e.g. [1], [2], [6], [21], [28]).

Recently, many important results have been published on various analysis aspects for RNNs with time delays. In particular, the existence of equilibrium point, global asymptotic stability, global exponential stability, and the existence of periodic solutions have been intensively investigated, see [7], [8], [9], [18], [23], [30], [31], [32] for some recent publications. Generally speaking, the time delays considered can be categorized as constant delays, time-varying delays, and distributed delays, and the methods used include the linear matrix inequality (LMI) approach, Lyapunov functional method, M-matrix theory, topological degree theory, and techniques of inequality analysis. For example, most recently, in [7], the global robust stability has been studied for a class of delayed interval recurrent neural networks which contain time-invariant uncertain parameters whose

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values are unknown but bounded in given compact sets, and in [30], the global asymptotic stability analysis problem has been dealt with for a class of neural networks with discrete *and* distributed time-delays by using an effective LMI approach.

In practice, it is sometimes the case that a neural network has *finite state* representations (also called modes, patterns, or clusters), and the modes may switch (or jump) from one to another at different times [4], [10], [11], [13], [19], [22], [27]. An ideal assumption with the conventional RNNs is that the *continuous* variables propagate from one processing unit to the next. Such an assumption, unfortunately, does not hold for the case when an RNN switches within several modes, and therefore RNNs sometimes suffer from the problems in catching long-term dependencies in the input stream. Such a phenomenon is referred to as the problem of information latching [3]. Recently, it has been revealed in [27] that, the switching (or jumping) between different RNN modes can be governed by a *Markovian chain*. Specifically, the class of RNNs with Markovian jump parameters has two components in the state vector. The first one which varies continuously is referred to be the continuous state of the RNN, and the second one which varies discretely is referred to be the mode of the RNN. It should be pointed out that, the control and filtering problems for dynamical systems with Markovian jumping parameters have already been widely studied, see e.g. [20], [29]. However, up to now, the dynamical behavior of Markovian jumping RNNs has received very little research attention, despite its practical importance.

On the other hand, in relatively large-scale neural networks, normally only the partial information about the neuron states is available in the network outputs. Therefore, in order to utilize the neural networks, one would need to estimate the neuron state through available measurements. Recently, the state estimation problem for neural networks has received some research interests, see [12], [16], [25], [28]. In [25], an adaptive state estimator has been described by using techniques of optimization theory, the calculus of variations and gradient descent dynamics. In [28], an LMI approach has been developed to solve the neuron state estimation problem for recurrent neural networks with time-varying delays. So far, to the best of the authors' knowledge, neither the state estimation problem nor the stability analysis problem has been studied in the literature for Markovian jumping RNNs with both discrete and distributed time-delays. This situation motivates our present investigation.

This paper is concerned with the state estimation problem for a class of Markovian neural networks with discrete and distributed time-delays. The neural networks have a finite number of modes, and the modes may jump from one to another according to a Markov chain. The problem addressed is to estimate the neuron states, through available output measurements, such that for all admissible time-delays, the dynamics of the estimation error is globally asymptotically stable in the mean square. An effective LMI approach is developed to solve the neuron state estimation problem. In particular, we derive the conditions for the existence of the desired estimators for the delayed neural networks. We also parameterize the explicit expression of the set of desired estimators, and show that the main results can be used to establish the stability criterion for a general class of delayed neural networks with Markovian jumping parameters. Two numerical examples are used to demonstrate the usefulness of the proposed design methods.

The rest of this paper is arranged as follows. The state estimation problem is formulated in Section II for Markovian jumping delayed neural networks. In Section III, we give the main results that comprise the existence conditions and the explicit expression of the desired estimators. In Section IV, the results are specialized to the stability analysis problem of delayed neural networks with Markovian jumping parameters. Illustrative examples are provided in Section V, and some remarks are concluded in Section VI.

Notation. The notations in this paper are quite standard. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n\times m$ real matrices. The superscript "T" denotes the transpose and the notation $X \geq Y$ (respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). I is the identity matrix with compatible dimension. We let $h > 0$ and $C([-h, 0]; \mathbb{R}^n)$ denote the family of continuous functions φ from $[-h, 0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{-h\leq\theta\leq 0} |\varphi(\theta)|$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. If A is a matrix, denote by $\|A\|$ its operator norm, i.e., $\|A\| = \sup\{|Ax| : |x| = 1\} = \sqrt{\lambda_{\max}(A^TA)}$ where $\lambda_{\max}(\cdot)$ (respectively, $\lambda_{\min}(\cdot)$) means the largest (respectively, smallest) eigenvalue of A. $l_2[0,\infty]$ is the space of square integrable vector. Moreover, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., the filtration contains all P-null sets and is right continuous). Denote by $L^p_{\mathcal{F}_0}([-h, 0]; \mathbb{R}^n)$ the family of all $\mathcal{F}_0$-measurable $C([-h, 0]; \mathbb{R}^n)$-valued random variables $\xi = \{\xi(\theta) : -h \leq \theta \leq 0\}$ such that $\sup_{-h\leq\theta\leq 0} \mathbb{E}|\xi(\theta)|^p < \infty$ where $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure P. Sometimes, the arguments of a function will be omitted in the analysis when no confusion can arise.

II. Problem formulation

Consider the following delayed neural network with n neurons:

$$\dot{u}(t) = -Au(t) + W_0 g_0(u(t)) + W_1 g_1(u(t-h)) + W_2 \int_{t-\tau}^{t} g_2(u(s))ds + V \tag{1}$$

where $u(t) = [u_1(t), u_2(t), \cdots, u_n(t)]^T \in \mathbb{R}^n$ is the state vector of the neural network, $A = \text{diag}(a_1, a_2, \cdots, a_n)$ is a diagonal matrix with positive entries $a_i > 0$. $W_0 = (w^0_{ij})_{n\times n}$, $W_1 = (w^1_{ij})_{n\times n}$, and $W_2 = (w^2_{ij})_{n\times n}$ are, respectively, the connection weight matrix, the discretely delayed connection weight matrix, and the distributively delayed connection weight matrix. $g_i(u(t)) = [g_{i1}(u_1(t)), g_{i2}(u_2(t)), \cdots, g_{in}(u_n(t))]^T$ denotes the neuron activation function with $g_i(0) = 0$, and $V = [V_1, V_2, \cdots, V_n]^T$ is a constant vector. The scalar $h > 0$, which may be unknown, denotes the discrete time delay, whereas the scalar $\tau > 0$ is the known distributed time-delay.

In the literature, the researchers used to assume that the activation functions are continuous, differentiable, monotonically increasing and bounded, such as the sigmoid-type functions. However, in many electronic circuits, the input-output functions of amplifiers may be neither monotonically increasing nor continuously differentiable, hence nonmonotonic functions can be more appropriate to describe the neuron activation in designing and implementing an artificial neural network. In this paper, we assume that the neuron activation functions in (1), $g_i(\cdot)$, satisfy the following Lipschitz condition:

$$|g_k(x) - g_k(y)| \leq |G_k(x-y)|, \quad (k = 0, 1, 2) \tag{2}$$

where $G_i \in \mathbb{R}^{n\times n}$ are known constant matrices. The type of activation functions in (2) is not necessarily monotonic and smooth, and have been used in numerous papers, see e.g. [7], [9], [30] and references therein.

As is well known, the information about the neuron states are often incomplete from the network measurements (outputs) in applications, and the network measurements are subject to nonlinear disturbances. Similar to [28], we assume that the network measurements satisfy

$$y(t) = Cu(t) + f(t, u(t)), \tag{3}$$

where $y(t) \in \mathbb{R}^m$ is the measurement output, C is a known constant matrix with appropriate dimension. $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ is the neuron-dependent nonlinear disturbances on the network outputs, and satisfies

$$|f(t, x) - f(t, y)| \leq |F(x - y)|, \tag{4}$$

where the constant matrix $F \in \mathbb{R}^{n\times n}$ is known.

As discussed in the previous section, delayed RNNs with Markovian jumping parameters are more appropriate to describe a class of RNNs with finite state representation, where the network dynamics can switch from one to another with the switch law being a Markov law. Based on the model (1)-(4), we now introduce the Markovian jumping RNNs with time-delays.

Let $\{r(t),\ t \geq 0\}$ be a right-continuous Markov process on the probability space which takes values in the finite space $\mathcal{S} = \{1, 2, \ldots, N\}$ with generator $\Gamma = (\gamma_{ij})$ $(i, j \in \mathcal{S})$ given by

$$P\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j \end{cases} \tag{5}$$

where $\Delta > 0$ and $\lim_{\Delta\to 0} o(\Delta)/\Delta = 0$, $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ and $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$.

In this paper, we will focus on the following delayed recurrent neural network with Markovian jumping parameters, which can be seen as a variation of the model (1)-(4):

$$\begin{aligned} \dot{u}(t) &= -A(r(t))u(t) + W_0(r(t))g_0(u(t)) + W_1(r(t))g_1(u(t-h)) \\ &\quad + W_2(r(t)) \int_{t-\tau}^{t} g_2(u(s))ds + V(r(t)), \qquad (6) \\ y(t) &= C(r(t))u(t) + f(t, u(t)), \qquad (7) \end{aligned}$$

where $u(t)$ and $y(t)$ have the same meanings as those in (1) and (3), $g_k(\cdot)$ and $f(t, u(t))$ satisfy (2) and (4), respectively. For fixed system mode, $A(r(t))$, $W_k(r(t))$ $(k = 0, 1, 2)$, $V(r(t))$ and $C(r(t))$ are known constant matrices with appropriate dimensions.

Notice that the Markov process $\{r(t),\ t \geq 0\}$ takes values in the finite space $\mathcal{S} = \{1, 2, \ldots, N\}$. For notation simplicity, we denote

$$A(i) := A_i, \quad W_0(i) := W_{0i}, \quad W_1(i) := W_{1i}, \quad W_2(i) := W_{2i}, \quad V(i) := V_i, \quad C(i) := C_i. \tag{8}$$

The main objective of this paper is to develop an efficient algorithm to estimate the neuron states $u(t)$ in (6) from the available network outputs in (7). From now on we shall work on the network mode $r(t) = i,\ \ \forall i \in \mathcal{S}$. The full-order state estimator is of the form

$$\begin{aligned} \dot{\hat{u}}(t) &= -A_i\hat{u}(t) + W_{0i}g_0(\hat{u}(t)) + W_{1i}g_1(\hat{u}(t-h)) + W_{2i}\int_{t-\tau}^{t} g_2(\hat{u}(s))ds + V(r(t)) \\ &\quad + K_i[y(t) - C_i\hat{u}(t) - f(t, \hat{u}(t))], \qquad (9) \end{aligned}$$

where $\hat{u}(t)$ is the estimation of the neuron state, and $K_i \in \mathbb{R}^{n\times m}$ is the estimator gain matrix to be designed.

Let the error state be

$$e(t) = u(t) - \hat{u}(t), \tag{10}$$

then it follows from (1), (3) and (9) that

$$\begin{aligned} \dot{e}(t) &= (-A_i - K_iC_i)e(t) + W_{0i}[g_0(u(t)) - g(\hat{u}(t))] + W_{1i}[g_1(u(t-h)) - g_1(\hat{u}(t-h))] \\ &\quad + W_{2i}\int_{t-\tau}^{t}[g_2(u(s)) - g_2(\hat{u}(s))]ds - K_i[f(t,u(t)) - f(t,\hat{u}(t))]. \end{aligned} \tag{11}$$

Now, let $e(t;\xi)$ denote the state trajectory of the error-state system (11) from the initial data $e(\theta) = \xi(\theta)$ on $-h \le \theta \le 0$ in $L^2_{\mathcal{F}_0}([-h,0];\mathbb{R}^n)$. It can be easily seen that the system (11) admits a trivial solution (equilibrium point) $e(t;0) \equiv 0$ corresponding to the initial data $\xi = 0$.

Definition 1: For the system (11) and every $\xi \in L^2_{\mathcal{F}_0}([-h,0];\mathbb{R}^n)$, the equilibrium point is asymptotically stable in the mean square if, for every network mode,

$$\lim_{t\to\infty} \mathbb{E}|e(t;\xi)|^2 = 0. \tag{12}$$

We shall design a state estimator for the delayed neural network described by (1) and (3) such that, for every mode, the dynamics of the system (11) is globally asymptotically stable in the mean square, for the nonlinear activation function $g_k(\cdot)$ and the nonlinear disturbance $f(\cdot,\cdot)$.

III. Main results and proofs

The following lemmas will be essential in establishing our results in terms of LMIs.

Lemma 1: Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ and $\varepsilon > 0$. Then we have $2x^Ty \le \varepsilon x^Tx + \varepsilon^{-1}y^Ty$.

Proof: The proof follows from the inequality $(\varepsilon^{1/2}x - \varepsilon^{-1/2}y)^T(\varepsilon^{1/2}x - \varepsilon^{-1/2}y) \ge 0$ immediately. ■

Lemma 2: [5] Given constant matrices Σ_1, Σ_2, Σ_3 where $\Sigma_1 = \Sigma_1^T$ and $0 < \Sigma_2 = \Sigma_2^T$, then

$$\Sigma_1 + \Sigma_3^T\Sigma_2^{-1}\Sigma_3 < 0$$

if and only if

$$\begin{bmatrix} \Sigma_1 & \Sigma_3^T \\ \Sigma_3 & -\Sigma_2 \end{bmatrix} < 0$$

or equivalently

$$\begin{bmatrix} -\Sigma_2 & \Sigma_3 \\ \Sigma_3^T & \Sigma_1 \end{bmatrix} < 0.$$

Lemma 3: [15] For any positive definite matrix $M > 0$, scalar $\gamma > 0$, vector function $\omega : [0,\gamma] \to \mathbb{R}^n$ such that the integrations concerned are well defined, the following inequality holds:

$$\left(\int_0^{\gamma}\omega(s)ds\right)^T M\left(\int_0^{\gamma}\omega(s)ds\right) \le \gamma\left(\int_0^{\gamma}\omega^T(s)M\omega(s)ds\right) \tag{13}$$

We are now ready to deal with the analysis problem, that is, deriving the conditions under which the error dynamics of the estimation process (11) is globally asymptotically stable in the mean square. The following theorem shows that such conditions can be obtained if a quadratic matrix inequality involving several scalar parameters is feasible.

Theorem 1: Let the estimator gain K_i be given. If there exist four sequences of positive scalars $\{\varepsilon_{0i} > 0,\ \varepsilon_{1i} > 0,\ \varepsilon_{2i} > 0,\ \varepsilon_{3i} > 0,\ i \in \mathcal{S}\}$ and a sequence of positive definite matrices $P_i = P_i^T > 0$ $(i \in \mathcal{S})$ such that the following quadratic matrix inequalities

$$\begin{aligned} &(-A_i - K_iC_i)^T P_i + P_i(-A_i - K_iC_i) + \sum_{j=1}^{N} \gamma_{ij} P_j + \sum_{k=0}^{2} \varepsilon_{ki}^{-1} P_i W_{ki} W_{ki}^T P_i \\ &+\varepsilon_{0i} G_0^T G_0 + \varepsilon_{1i} G_1^T G_1 + \varepsilon_{2i} \tau^2 G_2^T G_2 + \varepsilon_{3i}^{-1} P_i K_i K_i^T P_i + \varepsilon_{3i} F^T F < 0, \quad \forall i \in \mathcal{S} \end{aligned} \tag{14}$$

hold, then the error-state system (11) of the neural network (1)-(3) is globally asymptotically stable in the mean square.

Proof: For presentation convenience, we define:

$$A_{Ki} := -A_i - K_iC_i, \quad \psi_k(t) := g_k(u(t)) - g_k(\hat{u}(t)) \ \ (k = 0, 1, 2), \quad \phi(t) := f(t, u(t)) - f(t, \hat{u}(t)), \tag{15}$$

and then the system (11) becomes

$$\dot{e}(t) = A_{Ki} e(t) + W_{0i} \psi_0(t) + W_{1i} \psi_1(t-h) + W_{2i} \int_{t-\tau}^{t} \psi_2(s) ds - K_i \phi(t). \tag{16}$$

It follows immediately from (2) and (4) that

$$\begin{aligned} \psi_k^T(\cdot)\psi_k(\cdot) &= |g_k(u(\cdot)) - g_k(\hat{u}(\cdot))|^2 \leq |G_k e(\cdot)|^2 = e^T(\cdot) G_k^T G_k e(\cdot), \quad (k = 0, 1, 2) & (17) \\ \phi^T(t)\phi(t) &= |f(t, u(t)) - f(t, \hat{u}(t))|^2 \leq |Fe(t)|^2 = e^T(t) F^T F e(t). & (18) \end{aligned}$$

Let $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{S}; \mathbb{R}_+)$ denote the family of all nonnegative functions $\Phi(e, t, i)$ on $\mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{S}$ which are continuously twice differentiable in e and differentiable in t. Fix $\xi \in L^2_{\mathcal{F}_0}([-h, 0]; \mathbb{R}^n)$ arbitrarily and write $e(t; \xi) = e(t)$. Define a Lyapunov functional candidate $\Phi(e, t, i) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{S}; \mathbb{R}_+)$ by

$$\Phi(e(t), r(t) = i) := \Phi(e(t), t, i) = e^T(t) P_i e(t) + \int_{t-h}^{t} e^T(s) Q_1 e(s) ds + \int_{-\tau}^{0} \int_{t+s}^{t} e^T(\eta) Q_2 e(\eta) d\eta ds, \tag{19}$$

where $P_i > 0$ is the positive definite solution to (14), and $Q_1 \geq 0$ and $Q_2 \geq 0$ are defined by

$$Q_1 = \varepsilon_{1i} G_1^T G_1, \quad Q_2 = \varepsilon_{2i} \tau G_2^T G_2. \tag{20}$$

We know from [26] that $\{e(t), r(t)\}$ $(t \geq 0)$ is a $C([-h, 0]; \mathbb{R}^n) \times \mathcal{S}$-valued Markov process. Along the trajectory of (6), the weak infinitesimal operator $\mathcal{L}$ (see [20]) of the stochastic process $\{r(t), x(t)\}$ $(t \geq 0)$ is given by:

$$\begin{aligned} \mathcal{L}\Phi(e(t), r(t)) &:= \lim_{\Delta \to 0^+} \frac{1}{\Delta} \Big[\mathbb{E}\big\{ \Phi(e(t+\Delta), r(t+\Delta)) | e(t), r(t) = i \big\} - \Phi(e(t), r(t) = i) \Big] \\ &= e^T(t) \Big[A_{Ki}^T P_i + P_i A_{Ki} + \sum_{j=1}^{N} \gamma_{ij} P_j + Q_1 + \tau Q_2 \Big] e(t) \\ &\quad + 2\psi_0^T(t) W_{0i}^T P_i e(t) + 2\psi_1^T(t-h) W_{1i}^T P_i e(t) + 2 \left(\int_{t-\tau}^{t} \psi_2(e(s)) ds \right)^T W_{2i}^T P_i e(t) \\ &\quad - 2\phi^T(t) K_i^T P_i e(t) - e^T(t-h) Q_1 e(t-h) - \int_{t-\tau}^{t} e^T(s) Q_2 e(s) ds \\ &\quad + \sum_{j=1}^{N} \gamma_{ij} \int_{t-h}^{t} e^T(s) Q_1 e(s) ds + \sum_{j=1}^{N} \gamma_{ij} \int_{-\tau}^{0} \int_{t+s}^{t} e^T(\eta) Q_2 e(\eta) d\eta ds. \end{aligned} \tag{21}$$

Since ε_{ki} $(k=0,1,2,3)$ are positive scalars, it follows from Lemma 1 and (17)-(18) that

$$\begin{aligned}
2\psi_0^T(t)W_{0i}^TP_ie(t) &\leq \varepsilon_{0i}\psi_0^T(t)\psi_0(t)+\varepsilon_{0i}^{-1}e^T(t)P_iW_{0i}W_{0i}^TP_ie(t)\\
&\leq e^T(t)\left[\varepsilon_{0i}G_0^TG_0+\varepsilon_{0i}^{-1}P_iW_{0i}W_{0i}^TP_i\right]e(t), && (22)\\
2\psi_1^T(t-h)W_{1i}^TP_ie(t) &\leq \varepsilon_{1i}\psi_1^T(t-h)\psi_1(t-h)+\varepsilon_{1i}^{-1}e^T(t)P_iW_{1i}W_{1i}^TP_ie(t)\\
&\leq \varepsilon_{1i}e^T(t-h)G_1^TG_1e(t-h)+\varepsilon_{1i}^{-1}e^T(t)P_iW_{1i}W_{1i}^TP_ie(t), && (23)\\
2\left(\int_{t-\tau}^t\psi_2(e(s))ds\right)^TW_{2i}^TP_ie(t) &\leq \varepsilon_{2i}\left(\int_{t-\tau}^t\psi_2(e(s))ds\right)^T\left(\int_{t-\tau}^t\psi_2(e(s))ds\right)\\
&\quad+\varepsilon_{2i}^{-1}e^T(t)P_iW_{2i}W_{2i}^TP_ie(t), && (24)\\
-2\phi^T(t)K_i^TP_ie(t) &\leq \varepsilon_{3i}\phi^T(t)\phi(t)+\varepsilon_{3i}^{-1}e^T(t)P_iK_iK_i^TP_ie(t)\\
&\leq e^T(t)\left[\varepsilon_{3i}F^TF+\varepsilon_{3i}^{-1}P_iK_iK_i^TP_i\right]e(t). && (25)
\end{aligned}$$

By resorting to Lemma 3 and the definition of Q_2 in (20), we have

$$\begin{aligned}
\varepsilon_{2i}\left(\int_{t-\tau}^t\psi_2(e(s))ds\right)^T\left(\int_{t-\tau}^t\psi_2(e(s))ds\right) &\leq \varepsilon_{2i}\tau\int_{t-\tau}^t\psi_2^T(e(s))\psi_2(x(s))ds\\
&\leq \varepsilon_{2i}\tau\int_{t-\tau}^te^T(s)G_2^TG_2e(s)ds=\int_{t-\tau}^te^T(s)Q_2e(s)ds, \quad (26)
\end{aligned}$$

and hence

$$2\left(\int_{t-\tau}^t\psi_2(e(s))ds\right)^TW_{2i}^TP_ie(t)\leq\int_{t-\tau}^te^T(s)Q_2e(s)ds+\varepsilon_{2i}^{-1}e^T(t)P_iW_{2i}W_{2i}^TP_ie(t). \quad (27)$$

Furthermore, it follows from $\sum_{j=1}^N\gamma_{ij}=0$ that

$$\sum_{j=1}^N\gamma_{ij}\int_{t-h}^tx^T(s)Qx(s)ds=\left(\sum_{j=1}^N\gamma_{ij}\right)\left(\int_{t-h}^tx^T(s)Qx(s)ds\right)=0, \quad (28)$$

and

$$\sum_{j=1}^N\gamma_{ij}\int_{-\tau}^0\int_{t+s}^te^T(\eta)Q_2e(\eta)d\eta ds=\left(\sum_{j=1}^N\gamma_{ij}\right)\left(\int_{-\tau}^0\int_{t+s}^te^T(\eta)Q_2e(\eta)d\eta ds\right)=0. \quad (29)$$

Define

$$\begin{aligned}
\Pi_i \quad := \quad & A_{Ki}^TP_i+P_iA_{Ki}+\sum_{j=1}^N\gamma_{ij}P_j+\sum_{k=0}^2\varepsilon_{ki}^{-1}P_iW_{ki}W_{ki}^TP_i\\
&+\varepsilon_{0i}G_0^TG_0+\varepsilon_{1i}G_1^TG_1+\varepsilon_{2i}\tau^2G_2^TG_2+\varepsilon_{3i}^{-1}P_iK_iK_i^TP_i+\varepsilon_{3i}F^TF, \quad (30)
\end{aligned}$$

and (14) implies $\Pi_i<0$.

In the light of (22)-(29), and considering the definitions in (20), we obtain from (21) that

$$\begin{aligned}
\mathcal{L}\Phi(e(t),i) \quad \leq \quad & e^T(t)\Big[A_{Ki}^TP_i+P_iA_{Ki}+\sum_{j=1}^N\gamma_{ij}P_j+\sum_{k=0}^2\varepsilon_{ki}^{-1}P_iW_{ki}W_{ki}^TP_i+\varepsilon_{0i}G_0^TG_0\\
&+\varepsilon_{1i}G_1^TG_1+\varepsilon_{2i}\tau^2G_2^TG_2+\varepsilon_{3i}^{-1}P_iK_iK_i^TP_i+\varepsilon_{3i}F^TF\Big]e(t)=e^T(t)\Pi_ie(t). \quad (31)
\end{aligned}$$

Taking the mathematical expectation of both sides of (31), we have

$$\mathcal{L}\mathbb{E}\Phi(e(t),i) \le \mathbb{E}\left(e^T(t)\Pi_i e(t)\right) \le -\lambda_{\min}(-\Pi_i)\mathbb{E}|e(t)|^2. \tag{32}$$

To this end, from Lyapunov stability theory, we arrive at the conclusion that the error-state system (11) of the neural network (1)-(3) is globally asymptotically stable in the mean square. This completes the proof of Theorem 1. ■

Remark 1: In Theorem 1, when an estimator is given, the analysis result (i.e., the stability criterion for the error dynamics) is established in terms of the quadratic matrix inequalities in (14), which contain several scalar parameters. It is worth pointing out that, following the similar line of the proof of Theorem 1 in [24], it is not difficult to prove the exponential stability (in the mean square) of the error-state system (11).

Our next goal is to deal with the design problem, that is, giving a *practical* design procedure for the estimator gain, K_i, such that the set of inequalities (14) in Theorem 1 are satisfied. Obviously, the inequalities in (14) are difficult to solve, since they are nonlinear and coupled, each involving many parameters. A meaningful approach to tackling such a problem is to convert the nonlinearly coupled matrix inequalities into linear matrix inequalities (LMIs), while the estimator gain is designed simultaneously. It should be mentioned that, in the past decade, LMIs have gained much attention for their computational tractability and usefulness in many areas, including the stability testing for neural networks [9], because the so-called interior point method has been proven to be numerically very efficient for solving the LMIs [14].

In the following theorem, it is shown that the desired estimator gain can be designed if a set of LMIs are feasible.

Theorem 2: If there exist four sequences of positive scalars $\{\varepsilon_{0i},\ \varepsilon_{1i},\ \varepsilon_{2i},\ \varepsilon_{3i},\ i \in \mathcal{S}\}$, a sequence of positive definite matrices $P_i = P_i^T \in \mathbb{R}^{n\times n}$ and a sequence of matrices $R_i \in \mathbb{R}^{n\times n}$ $(i \in \mathcal{S})$ such that the following linear matrix inequalities

$$\begin{bmatrix} -A_iP_i - P_iA_i - R_iC_i - C_i^TR_i^T + \sum_{j=1}^N \gamma_{ij}P_j & P_iW_{0i} & \varepsilon_{0i}G_0^T & P_iW_{1i} & \varepsilon_{1i}G_1^T & P_iW_{2i} & \varepsilon_{2i}\tau G_2^T & R_i & \varepsilon_{3i}F^T \\ W_{0i}^TP_i & -\varepsilon_{0i}I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \varepsilon_{0i}G_0 & 0 & -\varepsilon_{0i}I & 0 & 0 & 0 & 0 & 0 & 0 \\ W_{1i}^TP_i & 0 & 0 & -\varepsilon_{1i}I & 0 & 0 & 0 & 0 & 0 \\ \varepsilon_{1i}G_1 & 0 & 0 & 0 & -\varepsilon_{1i}I & 0 & 0 & 0 & 0 \\ W_{2i}^TP_i & 0 & 0 & 0 & 0 & -\varepsilon_{2i}I & 0 & 0 & 0 \\ \varepsilon_{2i}\tau G_2 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_{2i}I & 0 & 0 \\ R_i^T & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_{3i}I & 0 \\ \varepsilon_{3i}F & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_{3i}I \end{bmatrix} < 0, \tag{33}$$

hold, then with the estimator gain

$$K_i = P_i^{-1}R_i, \tag{34}$$

the error-state system (11) of the neural network (1)-(3) is globally asymptotically stable in the mean square.

Proof: First, let us define

$$\Sigma_{1i} := -A_iP_i - P_iA_i - R_iC_i - C_i^TR_i^T + \sum_{j=1}^N \gamma_{ij}P_j. \tag{35}$$

Pre- and post-multiplying the inequality (33) by the block-diagonal matrix

$$\mathrm{diag}\{I, \varepsilon_{0i}^{-1/2} I, \varepsilon_{0i}^{-1/2} I, \varepsilon_{1i}^{-1/2} I, \varepsilon_{1i}^{-1/2} I, \varepsilon_{2i}^{-1/2} I, \varepsilon_{2i}^{-1/2} I, \varepsilon_{3i}^{-1/2} I, \varepsilon_{3i}^{-1/2} I\}$$

yield

$$\begin{bmatrix} \Sigma_{1i} & \varepsilon_{0i}^{-1/2} P_i W_{0i} & \varepsilon_{0i}^{1/2} G_0^T & \varepsilon_{1i}^{-1/2} P_i W_{1i} & \varepsilon_{1i}^{1/2} G_1^T & \varepsilon_{2i}^{-1/2} P_i W_{2i} & \varepsilon_{2i}^{1/2} \tau G_2^T & \varepsilon_{3i}^{-1/2} R_i & \varepsilon_{3i}^{1/2} F^T \\ \varepsilon_{0i}^{-1/2} W_{0i}^T P_i & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \varepsilon_{0i}^{1/2} G_0 & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ \varepsilon_{1i}^{-1/2} W_{1i}^T P_i & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 \\ \varepsilon_{1i}^{1/2} G_1 & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 \\ \varepsilon_{2i}^{-1/2} W_{2i}^T P_i & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 \\ \varepsilon_{2i}^{1/2} \tau G_2 & 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 \\ \varepsilon_{3i}^{-1/2} R_i^T & 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 \\ \varepsilon_{3i}^{1/2} F & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix} < 0, \tag{36}$$

or

$$\begin{bmatrix} \Sigma_{1i} & \Sigma_{3i}^T \\ \Sigma_{3i} & -\Sigma_{2i} \end{bmatrix} < 0, \tag{37}$$

where Σ_{1i} is defined in (35), and

$$\Sigma_{2i} := I,\ \Sigma_{3i} := \begin{bmatrix} \varepsilon_{0i}^{-1/2} P_i W_{0i} & \varepsilon_{0i}^{1/2} G_0^T & \varepsilon_{1i}^{-1/2} P_i W_{1i} & \varepsilon_{1i}^{1/2} G_1^T & \varepsilon_{2i}^{-1/2} P_i W_{2i} & \varepsilon_{2i}^{1/2} \tau G_2^T & \varepsilon_{3i}^{-1/2} R_i & \varepsilon_{3i}^{1/2} F^T \end{bmatrix}^T.$$

It follows from the Schur Complement Lemma (Lemma 2) that (37) holds if and only if

$$\Sigma_{1i} + \Sigma_{3i}^T \Sigma_{2i}^{-1} \Sigma_{3i} < 0,$$

or

$$\begin{aligned} &-A_i P_i - P_i A_i - R_i C_i - C_i^T R_i^T + \sum_{j=1}^{N} \gamma_{ij} P_j + \sum_{k=0}^{2} \varepsilon_{ki}^{-1} P_i W_{ki} W_{ki}^T P_i \\ &+\varepsilon_{0i} G_0^T G_0 + \varepsilon_{1i} G_1^T G_1 + \varepsilon_{2i} \tau^2 G_2^T G_2 + \varepsilon_{3i}^{-1} R_i R_i^T + \varepsilon_{3i} F^T F < 0. \end{aligned} \tag{38}$$

Noticing that $R_i = P_i K_i$, it can be easily seen that (38) is the same as (14). Hence, it follows from Theorem 1 that, with the estimator gain given by (34), the error-state system (11) of the neural network (1)-(3) is globally asymptotically stable in the mean square. The proof of Theorem 2 is now complete. ■

Remark 2: In Theorem 2, the matrix inequalities in (33) are linear on the parameters $\varepsilon_{0i1} > 0$, $\varepsilon_{1i} > 0$, $\varepsilon_{2i} > 0$, $\varepsilon_{3i} > 0$, $P_i > 0$, and R_i. Therefore, the global asymptotic convergence of the error dynamics can be readily checked by solving the set of LMIs (33).

IV. Stability of Markovian jumping RNNs

In the past few years, the stability analysis issue for RNNs with time delays has received considerable research interests, and various sufficient conditions have been proposed to guarantee the global asymptotic or exponential stability for the RNNs with time-delays, see e.g. [7], [8], [9], [18], [23], [31], [32] for some recent publications.

As discussed in the introduction, RNNs with Markovian jumping parameters can represent an important class of neural networks that have *finite state* representations (also called modes, patterns, or clusters), and the modes may switch (or jump) from one to another according to a Markov chain. However, up to now, the stability analysis issue for Markovian jumping RNNs has not been studied yet. The purpose of this section is to point out that, as a by-product, the main results developed in the previous section can be easily specialized to solve the conventional stability analysis problem for Markovian jumping RNNs.

Consider the delayed neural network (1), and let u^* be its the equilibrium. For presentation convenience, we can shift the intended equilibrium u^* to the origin by letting $x = u - u^*$, and then the system (1) can be transformed into:

$$\dot{x}(t) = -Ax(t) + W_0 l_0(x(t)) + W_1 l_1(x(t-h)) + W_2 \int_{t-\tau}^{t} l_2(x(s))ds \tag{39}$$

where $x(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^T \in \mathbb{R}^n$ is the state vector of the transformed system, and the transformed neuron activation function $l_k(x) = g_k(x + u^*) - g_k(u^*)$ satisfies

$$|l_k(x)| \leq |G_k x|, \quad k = 0, 1, 2 \tag{40}$$

where $G_k \in \mathbb{R}^{n \times n}$ is a known constant matrix.

The stability analysis problem for the delayed neural network of the type (39) has been studied recently by some researchers, see [24], [30] and references therein. We now consider the corresponding delayed neural networks with Markovian jumping parameters as follows:

$$\dot{x}(t) = -A(r(t))x(t) + W_0(r(t))l_0(x(t)) + W_1(r(t))l_1(x(t-h)) + W_2(r(t)) \int_{t-\tau}^{t} l_2(x(s))ds \tag{41}$$

where the Markov process $\{r(t),\ t \geq 0\}$ is defined in (5). Again, we denote $A(r(t)) = A_i$, $W_0(r(t)) = W_{0i}$, $W_1(r(t)) = W_{1i}$, and $W_2(r(t)) = W_{2i}$.

Our aim is to establish LMI-based criteria to test whether the Markovian jumping delayed RNN (41) is globally asymptotically stable in the mean square. This has been done in the following theorem.

Theorem 3: If there exist three sequences of positive scalars $\{\varepsilon_{0i},\ \varepsilon_{1i},\ \varepsilon_{2i},\ i \in \mathcal{S}\}$ and a sequence of positive definite matrices $P_i = P_i^T \in \mathbb{R}^{n \times n}$ $(i \in \mathcal{S})$ such that the following linear matrix inequalities

$$\begin{bmatrix} -A_iP_i - P_iA_i + \sum_{j=1}^{N}\gamma_{ij}P_j & P_iW_{0i} & \varepsilon_{0i}G_0^T & P_iW_{1i} & \varepsilon_{1i}G_1^T & P_iW_{2i} & \varepsilon_{2i}\tau G_2^T \\ W_{0i}^TP_i & -\varepsilon_{0i}I & 0 & 0 & 0 & 0 & 0 \\ \varepsilon_{0i}G_0 & 0 & -\varepsilon_{0i}I & 0 & 0 & 0 & 0 \\ W_{1i}^TP_i & 0 & 0 & -\varepsilon_{1i}I & 0 & 0 & 0 \\ \varepsilon_{1i}G_1 & 0 & 0 & 0 & -\varepsilon_{1i}I & 0 & 0 \\ W_{2i}^TP_i & 0 & 0 & 0 & 0 & -\varepsilon_{2i}I & 0 \\ \varepsilon_{2i}\tau G_2 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_{2i}I \end{bmatrix} < 0, \tag{42}$$

hold, then the Markovian jumping neural network (41) with discrete and distributed delays is globally asymptotically stable in the mean square.

Proof: The proof follows the same line of the proofs of Theorem 1 and Theorem 2. ■

Remark 3: It is shown in Theorem 3 that the asymptotic stability of the delayed network (41) can be checked by examining the solvability of the LMIs (42), which can be readily conducted by utilizing the Matlab LMI toolbox.

If we consider the following Markovian jumping RNN with discrete delays only:

$$\dot{x}(t) = -A(r(t))x(t) + W_0(r(t))l_0(x(t)) + W_1(r(t))l_1(x(t-h)), \tag{43}$$

we will have the following corollaries that are still believed to be new, since there have been very few results on the stability analysis problems for delayed neural networks with Markovian switching.

Corollary 1: If there exist two sequences of positive scalars $\{\varepsilon_{0i},\ \varepsilon_{1i},\ i \in \mathcal{S}\}$ and a sequence of positive definite matrices $P_i = P_i^T \in \mathbb{R}^{n\times n}$ $(i \in \mathcal{S})$ such that the following linear matrix inequalities

$$\begin{bmatrix} -A_iP_i - P_iA_i + \sum_{j=1}^{N}\gamma_{ij}P_j & P_iW_{0i} & \varepsilon_{0i}G_0^T & P_iW_{1i} & \varepsilon_{1i}G_1^T \\ W_{0i}^TP_i & -\varepsilon_{0i}I & 0 & 0 & 0 \\ \varepsilon_{0i}G_0 & 0 & -\varepsilon_{0i}I & 0 & 0 \\ W_{1i}^TP_i & 0 & 0 & -\varepsilon_{1i}I & 0 \\ \varepsilon_{1i}G_1 & 0 & 0 & 0 & -\varepsilon_{1i}I \end{bmatrix} < 0, \tag{44}$$

hold, then Markovian jumping neural network (43) with discrete delays is globally asymptotically stable in the mean square.

Furthermore, if we consider the following Markovian jumping RNN with distributed delays only:

$$\dot{x}(t) = -A(r(t))x(t) + W_0(r(t))l_0(x(t)) + W_2(r(t))\int_{t-\tau}^{t} l_2(x(s))ds \tag{45}$$

we will have the following corollary.

Corollary 2: If there exist two sequences of positive scalars $\{\varepsilon_{0i},\ \varepsilon_{2i},\ i \in \mathcal{S}\}$ and a sequence of positive definite matrices $P_i = P_i^T \in \mathbb{R}^{n\times n}$ $(i \in \mathcal{S})$ such that the following linear matrix inequalities

$$\begin{bmatrix} -A_iP_i - P_iA_i + \sum_{j=1}^{N}\gamma_{ij}P_j & P_iW_{0i} & \varepsilon_{0i}G_0^T & P_iW_{2i} & \varepsilon_{2i}\tau G_2^T \\ W_{0i}^TP_i & -\varepsilon_{0i}I & 0 & 0 & 0 \\ \varepsilon_{0i}G_0 & 0 & -\varepsilon_{0i}I & 0 & 0 \\ W_{2i}^TP_i & 0 & 0 & -\varepsilon_{2i}I & 0 \\ \varepsilon_{2i}\tau G_2 & 0 & 0 & 0 & -\varepsilon_{2i}I \end{bmatrix} < 0, \tag{46}$$

hold, then Markovian jumping neural network (45) with distributed delays is globally asymptotically stable in the mean square.

V. Numerical examples

Two simple examples are presented here so as to illustrate the usefulness of our main results.

Example 1: In this example, we examine the asymptotic stability (in the mean square) of the RNN (39) with both discrete and distributed delays.

Consider a two-neuron two-mode Markovian delayed neural network (39) with parameters given as follows:

$$A_1 = \begin{bmatrix} 1.2 & 0 \\ 0 & 1.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.3 & 0 \\ 0 & 1.6 \end{bmatrix}, \quad G_0 = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.03 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} 0.04 & 0 \\ 0 & 0.06 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.07 \end{bmatrix}, \quad W_{01} = \begin{bmatrix} 0.06 & -0.075 \end{bmatrix},$$

$$W_{02} = \begin{bmatrix} 0.03 & 0.005 \\ 0.005 & 0.01 \end{bmatrix}, \quad W_{11} = \begin{bmatrix} 0.055 & 0.025 \\ 0.025 & 0.04 \end{bmatrix}, \quad W_{12} = \begin{bmatrix} 0.04 & 0.01 \\ 0.01 & 0.015 \end{bmatrix},$$

$$W_{21} = \begin{bmatrix} 0.065 & 0.02 \\ 0.02 & 0.045 \end{bmatrix}, \quad W_{22} = \begin{bmatrix} 0.03 & 0.005 \\ 0.005 & 0.035 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} -7 & 7 \\ 6 & -6 \end{bmatrix}.$$

By using the Matlab LMI toolbox, we solve the LMIs (42) and obtain

$$\varepsilon_{01} = 2.0395, \quad \varepsilon_{02} = 2.1804, \quad \varepsilon_{11} = 3.3257, \quad \varepsilon_{12} = 1.5882, \quad \varepsilon_{21} = 1.5228e-005, \quad \varepsilon_{22} = 1.0204e-005,$$

$$P_1 = \begin{bmatrix} 0.0094 & -0.0077 \\ -0.0077 & 0.0188 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.0087 & -0.0079 \\ -0.0079 & 0.0184 \end{bmatrix}.$$

Therefore, it follows from Theorem 3 that the two-neuron neural network (39) is globally asymptotically stable in the mean square.

Example 2: Now, we demonstrate how to design an estimator for the delayed neural network. Assume that the delayed Markovian neural network in (6)(7) is described by the following data

$$A_1 = \begin{bmatrix} 1.6 & 0 \\ 0 & 1.8 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.2 & 0 \\ 0 & 1.5 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0.8 & 0 \\ 0.9 & 0 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0 & 0.6 \\ 0 & 5 \end{bmatrix}, \quad G_0 = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.03 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0.04 & 0 \\ 0 & 0.06 \end{bmatrix},$$

$$G_2 = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.07 \end{bmatrix}, \quad F = \begin{bmatrix} 0.08 & 0 \\ 0 & 0.08 \end{bmatrix}, \quad W_{01} = \begin{bmatrix} 0.06 & -0.075 \end{bmatrix},$$

$$W_{02} = \begin{bmatrix} 0.03 & 0.005 \\ 0.005 & 0.01 \end{bmatrix}, \quad W_{11} = \begin{bmatrix} 0.055 & 0.025 \\ 0.025 & 0.04 \end{bmatrix}, \quad W_{12} = \begin{bmatrix} 0.04 & 0.01 \\ 0.01 & 0.015 \end{bmatrix},$$

$$W_{21} = \begin{bmatrix} 0.065 & 0.02 \\ 0.02 & 0.045 \end{bmatrix}, \quad W_{22} = \begin{bmatrix} 0.03 & 0.005 \\ 0.005 & 0.035 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} -7 & 7 \\ 6 & -6 \end{bmatrix}, \quad V_1 = V_2 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}.$$

Solving the LMIs (33) gives

$$\varepsilon_{01} = 61.0213, \quad \varepsilon_{02} = 61.0213, \quad \varepsilon_{11} = 60.9263, \quad \varepsilon_{12} = 60.9263,$$

$$\varepsilon_{21} = 60.9376, \quad \varepsilon_{22} = 60.9376, \quad \varepsilon_{31} = 33.7265, \quad \varepsilon_{32} = 33.7265,$$

$$P_1 = \begin{bmatrix} 64.1525 & 0.3833 \\ 0.3833 & 65.7707 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 53.1974 & 0.2594 \\ 0.2594 & 50.7558 \end{bmatrix},$$

$$R_1 = \begin{bmatrix} -207.2495 & 61.4881 \\ 224.9221 & 1.0919 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 207.2495 & -61.4881 \\ -224.9221 & -1.0919 \end{bmatrix},$$

and hence we have

$$K_1 = P_1^{-1} R_1 = \begin{bmatrix} -3.2511 & 0.9584 \\ 3.4387 & 0.0110 \end{bmatrix}, \quad K_2 = P_2^{-1} R_2 = \begin{bmatrix} 3.9176 & -1.1558 \\ -4.4515 & -0.0156 \end{bmatrix}.$$

It is not difficult to verify that, with the obtained estimator gain K_i $(i = 1, 2)$, the error dynamics for the delayed neural network converges to zero asymptotically in the mean square.

VI. Conclusions

In this paper, we have dealt with the problem of state estimation for a class of delayed neural networks with Markovian jumping parameters. We have removed the traditional monotonicity and smoothness assumptions on the activation function. A linear matrix inequality (LMI) approach has been developed to solve the problem addressed. Specifically, the conditions for the existence of the expected estimators have been derived in terms of the positive definite solution to an LMI involving several scalar parameters, and the analytical expression characterizing the desired estimators has been obtained. We have also shown that the main results can be easily extended to cope with the stability analysis problem for delayed Markovian jumping neural networks. Finally, two numerical examples have been used to demonstrate the usefulness of the main results.

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