Global Exponential Stability of Generalized Recurrent Neural Networks with Discrete and Distributed Delays

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Abstract

This paper is concerned with analysis problem for the global exponential stability of a class of recurrent neural networks (RNNs) with mixed discrete and distributed delays. We first prove the existence and uniqueness of the equilibrium point under mild conditions, assuming neither differentiability nor strict monotonicity for the activation function. Then, by employing a new Lyapunov-Krasovskii functional, a linear matrix inequality (LMI) approach is developed to establish sufficient conditions for the RNNs to be globally exponentially stable. Therefore, the global exponential stability of the delayed RNNs can be easily checked by utilizing the numerically efficient Matlab LMI toolbox, and no tuning of parameters is required. A simulation example is exploited to show the usefulness of the derived LMI-based stability conditions.

Keywords

I. Introduction

Recurrent neural networks (RNNs), especially Hopfield neural networks and cellular neural networks, have found successful applications in many areas, such as image processing, pattern recognition, associative memory, and optimization problems. Hence, there has been a rapidly growing research interest on the mathematical properties of RNNs, including the stability, the attractivity and the oscillation, and a great number of results have been available in the literature. For example, in [8], by using the comparison principle, the theory of monotone flow and monotone operator, some sufficient criteria have been given to ensure the existence, uniqueness and global exponential stability of the periodic solution of a class of RNNs. In [12], the absolute exponential stability has been studied for a class of continuous-time recurrent neural networks with locally Lipschitz continuous and monotone nondecreasing activation functions. Recently, in [25], a linear matrix inequality (LMI) approach has been developed to deal with the analysis problem of robust global exponential stability for interval RNNs.

It has been recognized that the time delay, which is an inherent feature of signal transmission between neurons, is one of the main sources for causing instability and poor performances of neural networks [1], [2], [4], [16], [22]. Therefore, stability analysis for RNNs with constant or time-varying delays has been an attractive subject of research in the past few years. Various sufficient conditions, either delay-dependent or delay-independent, have been proposed to guarantee the global asymptotic or exponential stability for the RNNs, see e.g. [5], [7], [13], [14], [26] for some recent publications, where only the *discrete* time-delays have been handled.

Since a neural network usually has a spatial nature due to the presence of an amount of parallel pathways of a variety of axon sizes and lengths, it is desired to model them by introducing continuously *distributed delays* over a certain duration of time, such that the distant past has less influence compared to the recent behavior of the state [19], [21]. For example, in [21], a neural circuit has been designed with distributed delays, which solves a general problem of recognizing patterns in a time-dependent signal. Therefore, when modeling neural networks, both the discrete and distributed time delays should be taken into account [20]. Very recently, many results have been reported on the stability analysis issue for various neural networks with distributed time-delays, such as generalized neural networks [6], [17], bi-directional associative memory networks [15], Hopfield neural networks [27], [29], cellular neural networks [30]. However, despite its significance in modeling neural networks, so far, the global exponential stability analysis problem for general RNNs with *both discrete and distributed delays* has received little research attention, mainly due to the mathematical difficulties in dealing with discrete and distributed delays simultaneously. Hence, it is our intention in this paper to tackle such an important yet challenging problem.

In this paper, we are concerned with the analysis issue for the *global exponential stability* of RNNs with mixed discrete *and* distributed delays. Different from most of the existing results, we develop a unified

framework to cope with the discrete and distributed time-delays by using the numerically efficient linear matrix inequality (LMI) approach, under more general assumptions on the activation functions. Therefore, the global exponential stability of the delayed RNNs can be easily checked by utilizing the numerically efficient Matlab LMI toolbox, and no tuning of parameters is required [3], [25]. A simulation example is exploited to show the usefulness of the derived LMI-based stability conditions.

Notations: The notations are quite standard. Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n \times m}$ denote, respectively, the n-dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript "T" denotes matrix transposition. The notation $X \geq Y$ (respectively, $X > Y$) means that X and Y are symmetric matrices, and that $X - Y$ is positive semidefinite (respectively, positive definite). $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. If A is a matrix, denote by $\|A\|$ its operator norm, i.e., $\|A\| = \sup\{|Ax| : |x| = 1\} = \sqrt{\lambda_{\max}(A^T A)}$ where $\lambda_{\max}(\cdot)$ (respectively, $\lambda_{\min}(\cdot)$) means the largest (respectively, smallest) eigenvalue of A. Sometimes, the arguments of a function or a matrix will be omitted in the analysis when no confusion can arise.

II. PROBLEM FORMULATION

Consider the following recurrent network with discrete and distributed time-delays:

$$\frac{du_i(t)}{dt} = -d_i u_i(t) + \sum_{j=1}^{n} a_{ij} f_j(u_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(u_j(t - \tau_1)) + \int_{t-\tau_2}^{t} \sum_{j=1}^{n} c_{ij} h_j(u_j(s)) ds + I_i, \quad i = 1, ..., n, \quad (1)$$

where n is the number of the neurons in the neural network, $u_i(t)$ denotes the state of the ith neural neuron at time t, $f_j(u_j(t))$, $g_j(u_j(t))$ and $h_j(u_j(t))$ are the activation functions of jth neuron at time t. The constants a_{ij}, b_{ij} and c_{ij} denote, respectively, the connection weights, the discretely delayed connection weights, and the distributively delayed connection weights, of the jth neuron on the i neuron. I_i is the external bias on the ith neuron, d_i denotes the rate with which the ith neuron will reset its potential to the resting state in isolation when disconnected from the network and external inputs. τ_1 is the constant discrete time delay, while τ_2 describes the distributed time delay.

The neural network (1) can be rewritten in the following matrix-vector form:

$$\frac{du(t)}{dt} = -Du(t) + AF(u(t)) + BG(u(t - \tau_1)) + C \int_{t-\tau_2}^{t} H(u(s)) ds + I, \quad (2)$$

where $u(t) = [u_1(t), u_2(t), \cdots, u_n(t)]^T$, $D = \text{diag}(d_1, ..., d_n)$, $A = (a_{ij})_{n \times n}$, $B = (b_{ij})_{n \times n}$, $C = (c_{ij})_{n \times n}$, $I = [I_1, ..., I_n]^T$, and $F(u(t)) = (f_1(u_1(t)), ..., f_n(u_n(t)))$, $G(u(t - \tau_1)) = (g_1(u_1(t - \tau_1)), ..., g_n(u_n(t - \tau_1)))$, $H(u(s)) = (h_1(u_1(s)), ..., h_n(u_n(s)))$.

Traditionally, the activation functions are assumed to be continuous, differentiable, monotonically increasing and bounded, such as the sigmoid-type of function. However, as discussed in [16], in many electronic circuits, the input-output functions of amplifiers may be neither monotonically increasing nor continuously differentiable, hence nonmonotonic functions can be more appropriate to describe the neuron activation in

designing and implementing an artificial neural network. In this paper, we make following assumptions for the neuron activation functions.

Assumption 1: For $i \in \{1, 2, ..., n\}$, the neuron activation functions in (2) satisfy

$$l_i^- \leq \frac{f_i(s_1) - f_i(s_2)}{s_1 - s_2} \leq l_i^+, \tag{3}$$

$$\sigma_i^- \leq \frac{g_i(s_1) - g_i(s_2)}{s_1 - s_2} \leq \sigma_i^+, \tag{4}$$

$$v_i^- \leq \frac{h_i(s_1) - h_i(s_2)}{s_1 - s_2} \leq v_i^+, \tag{5}$$

where l_i^-, l_i^+, σ_i^-, σ_i^+, v_i^-, v_i^+ are some constants.

Assumption 2: The neuron activation functions in (2) are bounded.

Remark 1: The constants l_i^-, l_i^+, σ_i^-, σ_i^+, v_i^-, v_i^+ in Assumption 1 are allowed to be positive, negative or zero. Hence, the resulting activation functions could be non-monotonic, and more general than the usual sigmoid functions.

Remark 2: Usually, various fixed point theorems such as Brouwer's fixed point theorem, Schauder fixed point theorem and contraction mapping principle can be exploited to prove the existence of equilibrium points of neural networks. For example, under Assumption 2, it is not difficult to ensure the existence of equilibrium point of the system (2) by using Brouwer's fixed point theorem. In the sequel we shall analyze the *global exponential stability* of the equilibrium point, which in turn implies the uniqueness of the equilibrium point.

We are now in a position to introduce the notion of the global exponential stability for the system (2). Now, let the initial conditions associated with (2) be of the form

$$u(s) = \phi(s), \quad s \in [-\tau^*, \ 0], \quad \tau^* = \max\{\tau_1, \ \tau_2\}, \tag{6}$$

where ϕ is a continuous real-valued function defined on its domain. Then, under Assumption 1, the solution of (2) exists for all $t \geq 0$ and is unique (see [11]).

Definition 1: The equilibrium point u^* of (2) associated with a given I is said to be globally exponentially stable, if there exist positive constants $k > 0$ and $\mu > 0$ such that every solution $u(t)$ of (2) satisfies

$$|u(t) - u^*| \leq \mu \mathrm{e}^{-kt} \sup_{-\tau^* \leq s \leq 0} |\phi(s) - u^*|, \quad \forall t > 0.$$

The main purpose of this paper is to establish LMI-based sufficient conditions under which the global exponential stability is guaranteed for the neural network (2) with both discrete and distributed time delays.

III. MAIN RESULTS AND PROOFS

The following lemmas are essential in establishing our main results.

Lemma 1: Let X, Y be any n-dimensional real column vectors, and let P be an $n \times n$ symmetric positive definite matrix. Then, the following matrix inequality holds:

$$2X^T P Y \leq X^T P X + Y^T P Y.$$

Proof: The proof follows from the matrix inequality

$$(P^{1/2} X - P^{1/2} Y)^T (P^{1/2} X - P^{1/2} Y) \geq 0$$

directly. ∎

Lemma 2: [10] For any symmetric positive definite matrix $M > 0$, scalar $\gamma > 0$, vector function $\omega : [0, \gamma] \to \mathbb{R}^n$ such that the integrations concerned are well defined, the following inequality holds:

$$\left(\int_0^\gamma \omega(s) ds \right)^T M \left(\int_0^\gamma \omega(s) ds \right) \leq \gamma \left(\int_0^\gamma \omega^T(s) M \omega(s) ds \right) \tag{7}$$

For presentation convenience, in the following, we denote

$$L_1 = \text{diag}(l_1^+ l_1^-, ..., l_n^+ l_n^-), \quad L_2 = \text{diag}(\frac{l_1^+ + l_1^-}{2}, ..., \frac{l_n^+ + l_n^-}{2}), \tag{8}$$

$$\Sigma_1 = \text{diag}(\sigma_1^+ \sigma_1^-, ..., \sigma_n^+ \sigma_n^-), \quad \Sigma_2 = \text{diag}(\frac{\sigma_1^+ + \sigma_1^-}{2}, ..., \frac{\sigma_n^+ + \sigma_n^-}{2}), \tag{9}$$

$$\Upsilon_1 = \text{diag}(v_1^+ v_1^-, ..., v_n^+ v_n^-), \quad \Upsilon_2 = \text{diag}(\frac{v_1^+ + v_1^-}{2}, ..., \frac{v_n^+ + v_n^-}{2}). \tag{10}$$

The main results of this paper are given in the following theorem.

Theorem 1: Let u^* be the equilibrium point of the system (2). Suppose that τ_1 is the discrete time delay, τ_2 describes the distributed delay, and ϵ_0 ($0 < \epsilon_0 < 1$) is a fixed constant. Then, under Assumption 1, the equilibrium point is globally exponentially stable if there exist three symmetric positive definite matrices P_1, P_2, P_3, three diagonal matrices $\Lambda = \text{diag}(\lambda_1, ..., \lambda_n) > 0$, $\Gamma = \text{diag}(\gamma_1, ..., \gamma_n) > 0$ and $\Delta = \text{diag}(\delta_1, ..., \delta_n) > 0$ such that the following LMI holds:

$$\Psi = \begin{bmatrix} \Pi & P_1 A + \Lambda L_2 & \Gamma \Sigma_2 & P_1 B & \Delta \Upsilon_2 & P_1 C \\ A^T P_1 + \Lambda L_2 & -\Lambda & 0 & 0 & 0 & 0 \\ \Gamma \Sigma_2 & 0 & (1 + \epsilon_0 \tau_1) P_2 - \Gamma & 0 & 0 & 0 \\ B^T P_1 & 0 & 0 & -P_2 & 0 & 0 \\ \Delta \Upsilon_2 & 0 & 0 & 0 & \tau_2 P_3 - \Delta & 0 \\ C^T P_1 & 0 & 0 & 0 & 0 & -\frac{1-\epsilon_0}{\tau_2} P_3 \end{bmatrix} < 0, \tag{11}$$

where

$$\Pi = -P_1 D - D P_1 - \Lambda L_1 - \Gamma \Sigma_1 - \Delta \Upsilon_1, \tag{12}$$

and all the matrices here are constant.

Proof: To simplify the exponential stability analysis of (2), we shift the equilibrium point u^* of (2) to the origin by letting $x(t) = u(t) - u^*$, and then the system (2) can be transformed into:

$$\frac{dx(t)}{dt} = -Dx(t) + A\hat{F}(x(t)) + B\hat{G}(x(t-\tau_1)) + C\int_{t-\tau_2}^{t} \hat{H}(x(s))ds, \tag{13}$$

where $x(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^T$ is the state vector of the transformed system, and the transformed neuron activation functions are

$$\hat{F}(x(t)) := (\hat{f}_1(x_1(t)), ..., \hat{f}_n(x_n(t))) = \hat{F}(u(t)) - \hat{F}(u^*), \tag{14}$$

$$\hat{G}(x(t)) := (\hat{g}_1(x_1(t)), ..., \hat{g}_n(x_n(t))) = \hat{G}(u(t)) - \hat{G}(u^*), \tag{15}$$

$$\hat{H}(x(t)) := (\hat{h}_1(x_1(t)), ..., \hat{h}_n(x_n(t))) = \hat{H}(u(t)) - \hat{H}(u^*). \tag{16}$$

According to (3)-(5), it can be easily checked that the transformed neuron activation functions satisfy

$$l_i^- \leq \frac{\hat{f}_i(s_1) - \hat{f}_i(s_2)}{s_1 - s_2} \leq l_i^+, \quad (i = 1, ..., n) \tag{17}$$

$$\sigma_i^- \leq \frac{\hat{g}_i(s_1) - \hat{g}_i(s_2)}{s_1 - s_2} \leq \sigma_i^+, \quad (i = 1, ..., n) \tag{18}$$

$$v_i^- \leq \frac{\hat{h}_i(s_1) - \hat{h}_i(s_2)}{s_1 - s_2} \leq v_i^+. \quad (i = 1, ..., n) \tag{19}$$

In order to establish the stability conditions, we introduce the following Lyapunov-Krasovskii functional

$$\Xi(t) := e^{2kt}V(t) := e^{2kt}\sum_{i=1}^{4} V_i(t), \tag{20}$$

where

$$V_1(t) = x^T(t)P_1x(t), \tag{21}$$

$$V_2(t) = \int_{t-\tau_1}^{t} \hat{G}^T(x(s))P_2\hat{G}(x(s))ds, \tag{22}$$

$$V_3(t) = \epsilon_0 \int_0^{\tau_1} \int_{t-s}^{t} \hat{G}^T(x(\eta))P_2\hat{G}(x(\eta))d\eta ds, \tag{23}$$

$$V_4(t) = \int_0^{\tau_2} \int_{t-s}^{t} \hat{H}^T(x(\eta))P_3\hat{H}(x(\eta))d\eta ds. \tag{24}$$

To facilitate the exponential stability analysis, we first calculate the time derivative of $V_i(t)$ along a given trajectory of the system (13) as follows:

$$\dot{V}_1(t) = 2x^T(t)P_1\left(-Dx(t) + A\hat{F}(x(t)) + B\hat{G}(x(t-\tau_1)) + C\int_{t-\tau_2}^t \hat{H}(x(s))ds\right), \tag{25}$$

$$\dot{V}_2(t) = \hat{G}^T(x(t))P_2\hat{G}(x(t)) - \hat{G}^T(x(t-\tau_1))P_2\hat{G}(x(t-\tau_1)), \tag{26}$$

$$\dot{V}_3(t) = \epsilon_0\tau_1\hat{G}^T(x(t))P_2\hat{G}(x(t)) - \epsilon_0\int_{t-\tau_1}^t \hat{G}^T(x(s))P_2\hat{G}(x(s))ds, \tag{27}$$

$$\begin{aligned}
\dot{V}_4(t) &= \tau_2\hat{H}^T(x(t))P_3\hat{H}(x(t)) - \int_{t-\tau_2}^t \hat{H}^T(x(s))P_3\hat{H}(x(s))ds \\
&= \tau_2\hat{H}^T(x(t))P_3\hat{H}(x(t)) - (1-\epsilon_0)\int_{t-\tau_2}^t \hat{H}^T(x(s))P_3\hat{H}(x(s))ds - \epsilon_0\int_{t-\tau_2}^t \hat{H}^T(x(s))P_3\hat{H}(x(s))ds \\
&\leq \tau_2\hat{H}^T(x(t))P_3\hat{H}(x(t)) - \frac{1-\epsilon_0}{\tau_2}\left(\int_{t-\tau_2}^t \hat{H}(x(s))ds\right)^T P_3\left(\int_{t-\tau_2}^t \hat{H}(x(s))ds\right) \\
&\quad - \epsilon_0\int_{t-\tau_2}^t \hat{H}^T(x(s))P_3\hat{H}(x(s))ds. \tag{28}
\end{aligned}$$

Note that Lemma 2 has been used in deriving (28).

With Lemma 1, the relations (25)-(28) lead to

$$\begin{aligned}
\dot{V}(t) &= \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) + \dot{V}_4(t) \\
&\leq 2x^T(t)P_1\left(-Dx(t) + A\hat{F}(x(t)) + B\hat{G}(x(t-\tau_1)) + C\int_{t-\tau_2}^t \hat{H}(x(s))ds\right) \\
&\quad + (1+\epsilon_0\tau_1)\hat{G}^T(x(t))P_2\hat{G}(x(t)) - \hat{G}^T(x(t-\tau_1))P_2\hat{G}(x(t-\tau_1)) \\
&\quad + \tau_2\hat{H}^T(x(t))P_3\hat{H}(x(t)) - \frac{1-\epsilon_0}{\tau_2}\left(\int_{t-\tau_2}^t \hat{H}(x(s))ds\right)^T P_3\left(\int_{t-\tau_2}^t \hat{H}(x(s))ds\right) \\
&\quad - \epsilon_0\int_{t-\tau_1}^t \hat{G}^T(x(s))P_2\hat{G}(x(s))ds - \epsilon_0\int_{t-\tau_2}^t \hat{H}^T(x(s))P_3\hat{H}(x(s))ds \\
&= \eta^T(t)\Psi_1\eta(t) - \epsilon_0\int_{t-\tau_1}^t \hat{G}^T(x(s))P_2\hat{G}(x(s))ds - \epsilon_0\int_{t-\tau_2}^t \hat{H}^T(x(s))P_3\hat{H}(x(s))ds, \tag{29}
\end{aligned}$$

where

$$\eta(t) = \begin{bmatrix} x^T(t) & \hat{F}^T(x(t)) & \hat{G}^T(x(t)) & \hat{G}^T(x(t-\tau_1)) & \hat{H}^T(x(t)) & \int_{t-\tau_2}^t \hat{H}^T(x(s))ds \end{bmatrix}^T, \tag{30}$$

$$\Psi_1 = \begin{bmatrix} -P_1D - DP_1 & P_1A & 0 & P_1B & 0 & P_1C \\ A^TP_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (1+\epsilon_0\tau_1)P_2 & 0 & 0 & 0 \\ B^TP_1 & 0 & 0 & -P_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tau_2P_3 & 0 \\ C^TP_1 & 0 & 0 & 0 & 0 & -\frac{1-\epsilon_0}{\tau_2}P_3 \end{bmatrix}. \tag{31}$$

By (17)–(19), we have

$$\left(\frac{\hat{f}_i(x_i(t))}{x_i(t)} - l_i^+\right)\left(\frac{\hat{f}_i(x_i(t))}{x_i(t)} - l_i^-\right) \leq 0, \quad i = 1,...,n,$$

$$\left(\frac{\hat{g}_i(x(t))}{x_i(t)} - \sigma_i^+\right)\left(\frac{\hat{g}_i(x(t))}{x_i(t)} - \sigma_i^-\right) \leq 0, \quad i = 1,...,n,$$

$$\left(\frac{\hat{h}_i(x(t))}{x_i(t)} - v_i^+\right)\left(\frac{\hat{h}_i(x(t))}{x_i(t)} - v_i^-\right) \leq 0. \quad i = 1,...,n,$$

Hence we get

$$(\hat{f}_i(x_i(t)) - l_i^+ x_i(t))(\hat{f}_i(x_i(t)) - l_i^- x_i(t)) \leq 0, \quad i = 1,...,n, \tag{32}$$

$$(\hat{g}_i(x(t)) - \sigma_i^+ x_i(t))(\hat{g}_i(x(t)) - \sigma_i^- x_i(t)) \leq 0, \quad i = 1,...,n, \tag{33}$$

$$(\hat{h}_i(x(t)) - v_i^+ x_i(t))(\hat{h}_i(x(t)) - v_i^- x_i(t)) \leq 0. \quad i = 1,...,n, \tag{34}$$

which are equivalent to

$$\begin{bmatrix} x(t) \\ \hat{F}(x(t)) \end{bmatrix}^T \begin{bmatrix} l_i^+ l_i^- e_i e_i^T & -\frac{l_i^+ + l_i^-}{2} e_i e_i^T \\ -\frac{l_i^+ + l_i^-}{2} e_i e_i^T & e_i e_i^T \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{F}(x(t)) \end{bmatrix} \leq 0, \quad i = 1,...,n, \tag{35}$$

$$\begin{bmatrix} x(t) \\ \hat{G}(x(t)) \end{bmatrix}^T \begin{bmatrix} \sigma_i^+ \sigma_i^- e_i e_i^T & -\frac{\sigma_i^+ + \sigma_i^-}{2} e_i e_i^T \\ -\frac{\sigma_i^+ + \sigma_i^-}{2} e_i e_i^T & e_i e_i^T \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{G}(x(t)) \end{bmatrix} \leq 0, \quad i = 1,...,n, \tag{36}$$

$$\begin{bmatrix} x(t) \\ \hat{H}(x(t)) \end{bmatrix}^T \begin{bmatrix} v_i^+ v_i^- e_i e_i^T & -\frac{v_i^+ + v_i^-}{2} e_i e_i^T \\ -\frac{v_i^+ + v_i^-}{2} e_i e_i^T & e_i e_i^T \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{H}(x(t)) \end{bmatrix} \leq 0, \quad i = 1,...,n, \tag{37}$$

where e_i denotes the unit column vector having "1" element on its ith row and zeros elsewhere.

Consequently, we have

$$\eta^T(t)\Psi_1\eta(t) - \sum_{i=1}^n \lambda_i \begin{bmatrix} x(t) \\ \hat{F}(x(t)) \end{bmatrix}^T \begin{bmatrix} l_i^+ l_i^- e_i e_i^T & -\frac{l_i^+ + l_i^-}{2} e_i e_i^T \\ -\frac{l_i^+ + l_i^-}{2} e_i e_i^T & e_i e_i^T \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{F}(x(t)) \end{bmatrix}$$

$$- \sum_{i=1}^n \gamma_i \begin{bmatrix} x(t) \\ \hat{G}(x(t)) \end{bmatrix}^T \begin{bmatrix} \sigma_i^+ \sigma_i^- e_i e_i^T & -\frac{\sigma_i^+ + \sigma_i^-}{2} e_i e_i^T \\ -\frac{\sigma_i^+ + \sigma_i^-}{2} e_i e_i^T & e_i e_i^T \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{G}(x(t)) \end{bmatrix}$$

$$- \sum_{i=1}^n \delta_i \begin{bmatrix} x(t) \\ \hat{H}(x(t)) \end{bmatrix}^T \begin{bmatrix} v_i^+ v_i^- e_i e_i^T & -\frac{v_i^+ + v_i^-}{2} e_i e_i^T \\ -\frac{v_i^+ + v_i^-}{2} e_i e_i^T & e_i e_i^T \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{H}(x(t)) \end{bmatrix}$$

$$= \eta^T(t)\Psi_1\eta(t) + \begin{bmatrix} x(t) \\ \hat{F}(x(t)) \end{bmatrix}^T \begin{bmatrix} -\Lambda L_1 & \Lambda L_2 \\ \Lambda L_2 & -\Lambda \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{F}(x(t)) \end{bmatrix}$$

$$+ \begin{bmatrix} x(t) \\ \hat{G}(x(t)) \end{bmatrix}^T \begin{bmatrix} -\Gamma\Sigma_1 & \Gamma\Sigma_2 \\ \Gamma\Sigma_2 & -\Gamma \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{G}(x(t)) \end{bmatrix} + \begin{bmatrix} x(t) \\ \hat{H}(x(t)) \end{bmatrix}^T \begin{bmatrix} -\Delta\Upsilon_1 & \Delta\Upsilon_2 \\ \Delta\Upsilon_2 & -\Delta \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{H}(x(t)) \end{bmatrix}$$

$$
= \eta^T(t)
\begin{bmatrix}
\Pi & P_1 A + \Lambda L_2 & \Gamma \Sigma_2 & P_1 B & \Delta \Upsilon_2 & P_1 C \\
A^T P_1 + \Lambda L_2 & -\Lambda & 0 & 0 & 0 & 0 \\
\Gamma \Sigma_2 & 0 & (1 + \epsilon_0 \tau_1) P_2 - \Gamma & 0 & 0 & 0 \\
B^T P_1 & 0 & 0 & -P_2 & 0 & 0 \\
\Delta \Upsilon_2 & 0 & 0 & 0 & \tau_2 P_3 - \Delta & 0 \\
C^T P_1 & 0 & 0 & 0 & 0 & -\frac{1-\epsilon_0}{\tau_2} P_3
\end{bmatrix}
\eta(t)
$$

$$
= \eta^T(t) \Psi \eta(t)
$$

$$
\leq \lambda_{\max}(\Psi) |\eta(t)|^2, \tag{38}
$$

where $\lambda_{\max}(\Psi) < 0$ by (11), and Ψ, Π and Ψ_1 are defined in (11), (12) and (31), respectively.

It follows from (35)-(38) that

$$
\eta^T(t) \Psi_1 \eta(t) \leq \eta^T(t) \Psi \eta(t) \leq \lambda_{\max}(\Psi) |\eta(t)|^2. \tag{39}
$$

Therefore, from (29) and (39), we obtain

$$
\begin{aligned}
\dot{V}(t) & \leq \lambda_{\max}(\Psi) |\eta(t)|^2 - \epsilon_0 \int_{t-\tau_1}^{t} \hat{G}^T(x(s)) P_2 \hat{G}(x(s)) ds - \epsilon_0 \int_{t-\tau_2}^{t} \hat{H}^T(x(s)) P_3 \hat{H}(x(s)) ds \\
& \leq \lambda_{\max}(\Psi) |x(t)|^2 - \epsilon_0 \int_{t-\tau_1}^{t} \hat{G}^T(x(s)) P_2 \hat{G}(x(s)) ds - \epsilon_0 \int_{t-\tau_2}^{t} \hat{H}^T(x(s)) P_3 \hat{H}(x(s)) ds.
\end{aligned} \tag{40}
$$

Also, from the definitions of $V_i(t)$, it is not difficult to obtain the following inequalities

$$
\begin{aligned}
V_1(t) & \leq \lambda_{\max}(P_1) |x(t)|^2, \tag{41} \\
V_3(t) & \leq \epsilon_0 \int_0^{\tau_1} \int_{t-\tau_1}^{t} \hat{G}^T(x(\eta)) P_2 \hat{G}(x(\eta)) d\eta ds \\
& = \epsilon_0 \tau_1 \int_{t-\tau_1}^{t} \hat{G}^T(x(s)) P_2 \hat{G}(x(s)) ds, \tag{42} \\
V_4(t) & \leq \int_0^{\tau_2} \int_{t-\tau_2}^{t} \hat{H}^T(x(\eta)) P_3 \hat{H}(x(\eta)) d\eta ds \\
& = \tau_2 \int_{t-\tau_2}^{t} \hat{H}^T(x(s)) P_3 \hat{H}(x(s)) ds. \tag{43}
\end{aligned}
$$

We are now ready to deal with the exponential stability of (2). Consider the Lyapunov-Krasovskii functional $\Xi(t)$ in (20), where k is a constant to be determined. Using (20), (22) and (40)-(43), we have

$$
\begin{aligned}
\frac{d}{dt}\Xi(t) \ =\ & 2k\mathrm{e}^{2kt}V(t) + \mathrm{e}^{2kt}\dot{V}(t)\\
\leq\ & 2k\mathrm{e}^{2kt}\left[\lambda_{\max}(P_1)|x(t)|^2 + (1+\epsilon_0\tau_1)\int_{t-\tau_1}^{t}\hat{G}^T(x(s))P_2\hat{G}(x(s))ds\right.\\
& \left. + \tau_2\int_{t-\tau_2}^{t}\hat{H}^T(x(s))P_3\hat{H}(x(s))ds\right] + \mathrm{e}^{2kt}\left[\lambda_{\max}(\Psi)|x(t)|^2\right.\\
& \left. - \epsilon_0\int_{t-\tau_1}^{t}\hat{G}^T(x(s))P_2\hat{G}(x(s))ds - \epsilon_0\int_{t-\tau_2}^{t}\hat{H}^T(x(s))P_3\hat{H}(x(s))ds\right]\\
\leq\ & \mathrm{e}^{2kt}\left[(2k\lambda_{\max}(P_1) + \lambda_{\max}(\Psi))\,|x(t)|^2 + (2k(1+\epsilon_0\tau_1)-\epsilon_0)\int_{t-\tau_1}^{t}\hat{G}^T(x(s))P_2\hat{G}(x(s))ds\right.\\
& \left. + (2k\tau_2-\epsilon_0)\int_{t-\tau_2}^{t}\hat{H}^T(x(s))P_3\hat{H}(x(s))ds\right]. \tag{44}
\end{aligned}
$$

Set

$$
k_0 = \min\left\{-\frac{\lambda_{\max}(\Psi)}{2\lambda_{\max}(P_1)},\ \frac{\epsilon_0}{2(1+\epsilon_0\tau_1)},\ \frac{\epsilon_0}{2\tau_2}\right\}.
$$

From now on, we take k to be a constant satisfying

$$
k \leq k_0, \tag{45}
$$

and then obtain from (44) that

$$
\frac{d}{dt}(\mathrm{e}^{2kt}V(t)) \leq 0, \tag{46}
$$

which, together with (22) and (40)-(43), imply that

$$
\begin{aligned}
\mathrm{e}^{2kt}V(t) \ \leq\ & V(0)\\
=\ & V_1(0) + V_2(0) + V_3(0) + V_4(0)\\
\leq\ & \lambda_{\max}(P_1)|x(0)|^2 + (1+\epsilon_0\tau_1)\int_{-\tau_1}^{0}\hat{G}^T(x(s))P_2\hat{G}(x(s))ds\\
& + \tau_2\int_{-\tau_2}^{0}\hat{H}^T(x(s))P_3\hat{H}(x(s))ds\\
\leq\ & \lambda_{\max}(P_1)|x(0)|^2 + (1+\epsilon_0\tau_1)\lambda_{\max}(P_2)\int_{-\tau_1}^{0}|\hat{G}(x(s))|^2ds\\
& + \tau_2\lambda_{\max}(P_3)\int_{-\tau_2}^{0}|\hat{H}(x(s))|^2ds. \tag{47}
\end{aligned}
$$

Let

$$
\sigma = \max_{1\leq i\leq n}\{|\sigma_i^-|,\ |\sigma_i^+|\},\ \ \upsilon = \max_{1\leq i\leq n}\{|\upsilon_i^-|,\ |\upsilon_i^+|\}, \tag{48}
$$

$$
\mu_0 = \lambda_{\max}(P_1) + (1+\epsilon_0\tau_1)\tau_1\sigma^2\lambda_{\max}(P_2) + \tau_2^2\upsilon^2\lambda_{\max}(P_3). \tag{49}
$$

Then, it is indicated from (47) that

$$
\begin{aligned}
\mathrm{e}^{2kt}V(t) &\leq \lambda_{\max}(P_1)|x(0)|^2 + (1 + \epsilon_0\tau_1)\tau_1\sigma^2\lambda_{\max}(P_2) \sup_{-\tau_1 \leq s \leq 0}|x(s)|^2 + \tau_2^2 v^2\lambda_{\max}(P_3) \sup_{-\tau_2 \leq s \leq 0}|x(s)|^2 \\
&\leq \left(\lambda_{\max}(P_1) + (1 + \epsilon_0\tau_1)\tau_1\sigma^2\lambda_{\max}(P_2) + \tau_2^2 v^2\lambda_{\max}(P_3)\right) \sup_{-\tau^* \leq s \leq 0}|x(s)|^2 \\
&= \mu_0 \sup_{-\tau^* \leq s \leq 0}|x(s)|^2 \\
&= \mu_0 \sup_{-\tau^* \leq s \leq 0}|\phi(s) - u^*|^2,
\end{aligned}
\tag{50}
$$

and therefore

$$
V(t) \leq \mu_0 \mathrm{e}^{-2kt}|\phi(s) - u^*|^2.
\tag{51}
$$

Noticing $V(t) \geq V_1(t) \geq \lambda_{\max}(P_1)|x(t)|^2$, we obtain

$$
|x(t)|^2 \leq \frac{\mu_0}{\lambda_{\max}(P_1)}\mathrm{e}^{-2kt} \sup_{-\tau^* \leq s \leq 0}|\phi(s) - u^*|^2.
\tag{52}
$$

Letting $\mu = \sqrt{\frac{\mu_0}{\lambda_{\max}(P_1)}}$, we can rewrite (52) as

$$
|x(t)| \leq \mu\mathrm{e}^{-kt} \sup_{-\tau^* \leq s \leq 0}|\phi(s) - u^*|,
\tag{53}
$$

or

$$
|u(t) - u^*| \leq \mu\mathrm{e}^{-kt} \sup_{-\tau^* \leq s \leq 0}|\phi(s) - u^*|.
\tag{54}
$$

Hence, the equilibrium point u^* of (2) is globally exponentially stable. The proof of this theorem is now complete. ∎

Remark 3: In Theorem 1, sufficient conditions are provided for the system (2) to be globally asymptotically stable. Such conditions are expressed in the form of LMIs, which could be easily checked by utilizing the LMI Matlab toolbox [9], and no turning of parameters will be needed. It should be pointed out that, in the past few years, linear matrix inequalities (LMIs) have gained much attention for their computational tractability and usefulness in system engineering (see *e.g.* [3]) as the so-called interior point method (see [18]) has been proved to be numerically very efficient for solving the LMIs. The number of analysis and design problems that can be formulated as LMI problems is large and continues to grow.

Remark 4: The LMI Control Toolbox implements state-of-the-art interior-point LMI solvers. While these solvers are significantly faster than classical convex optimization algorithms, it should be kept in mind that the complexity of LMI computations remains higher than that of solving, say, a Riccati equation. For instance, problems with a thousand design variables typically take over an hour on today's workstations [9]. However, research on LMI optimization is a very active area in the applied math, optimization and the operations research community, and substantial speed-ups can be expected in the future.

If we are only interested in the global *asymptotic* stability of the RNN (2), the following theorem is easily accessible.

Theorem 2: Let u^* be the equilibrium point of the system (2). Suppose that τ_1 is the discrete time delay, and τ_2 describes the distributed delay. Then, under Assumption 1, the equilibrium point is globally asymptotically stable if there exist three symmetric positive definite matrices P_1, P_2, P_3, three diagonal matrices $\Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_n) > 0$, $\Gamma = \mathrm{diag}(\gamma_1, ..., \gamma_n) > 0$ and $\Delta = \mathrm{diag}(\delta_1, ..., \delta_n) > 0$ such that the following LMI holds:

$$\Phi = \begin{bmatrix} \Pi & P_1 A + \Lambda L_2 & \Gamma\Sigma_2 & P_1 B & \Delta\Upsilon_2 & P_1 C \\ A^T P_1 + \Lambda L_2 & -\Lambda & 0 & 0 & 0 & 0 \\ \Gamma\Sigma_2 & 0 & P_2 - \Gamma & 0 & 0 & 0 \\ B^T P_1 & 0 & 0 & -P_2 & 0 & 0 \\ \Delta\Upsilon_2 & 0 & 0 & 0 & \tau_2 P_3 - \Delta & 0 \\ C^T P_1 & 0 & 0 & 0 & 0 & -\frac{1}{\tau_2} P_3 \end{bmatrix} < 0, \tag{55}$$

where

$$\Pi = -P_1 D - D P_1 - \Lambda L_1 - \Gamma\Sigma_1 - \Delta\Upsilon_1. \tag{56}$$

Proof: To avoid unnecessary duplication, here we only give the sketch of the proof, and omit the details. As in the proof of Theorem 1, we shift the equilibrium point u^* of (2) to the origin by letting $x(t) = u(t) - u^*$, and then transform the system (2) into the system (13).

Construct the following Lyapunov-Krasovskii functional:

$$\bar{V}(t) = x^T(t) P_1 x(t) + \int_{t-\tau_1}^{t} \hat{G}^T(x(s)) P_2 \hat{G}(x(s)) ds + \int_{0}^{\tau_2} \int_{t-s}^{t} \hat{H}^T(x(\eta)) P_3 \hat{H}(x(\eta)) d\eta ds. \tag{57}$$

Then, following the similar line in calculating $\dot{V}(t)$ (from (20) to (40)), we can have

$$\dot{\bar{V}}(t) \leq \eta^T(t) \Phi \eta(t) \leq \lambda_{\max}(\Phi) |\eta(t)|^2 \leq \lambda_{\max}(\Phi) |x(t)|^2, \tag{58}$$

which implies that the equilibrium point of (2) is globally asymptotically stable. ∎

IV. NUMERICAL EXAMPLE

In this section, we present a simulation example so as to illustrate the usefulness of our main results. Our aim is to examine the global exponential stability of the delayed RNN (2) with network parameters given as follows:

$$D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \quad A = \begin{bmatrix} 1.2 & -0.8 & 0.6 \\ 0.5 & -1.5 & 0.7 \\ -0.8 & -1.2 & -1.4 \end{bmatrix}, \quad B = \begin{bmatrix} -1.4 & 0.9 & 0.5 \\ -0.6 & 1.2 & 0.8 \\ 0.5 & -0.7 & 1.1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1.8 & 0.7 & -0.8 \\ 0.6 & 1.4 & 1 \\ -0.4 & -0.6 & 1.2 \end{bmatrix}, \quad I = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \tau_1 = 0.1, \quad \tau_2 = 0.2.$$

Fix $\epsilon_0 = 0.01$ and take the activation function as follows:

$$f_1(x) = g_1(x) = h_1(x) = \tanh(-1.2x),$$

$$f_2(x) = g_2(x) = h_2(x) = \tanh(1.4x),$$

$$f_3(x) = g_3(x) = h_3(x) = \tanh(-2.4x).$$

From the facts that

$$\frac{d}{dx}\tanh(x) = \frac{4e^{2x}}{(e^{2x}+1)^2}, \quad 0 < \frac{d}{dx}\tanh(x) \le 1,$$

one has

$$L_1 = \Sigma_1 = \Upsilon_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad L_2 = \Sigma_2 = \Upsilon_2 = \begin{bmatrix} -0.6 & 0 & 0 \\ 0 & 0.7 & 0 \\ 0 & 0 & -1.2 \end{bmatrix}.$$

With the parameters given above, it is obvious that $u^* = [0, \ 0, \ 0]^T$ is an equilibrium point of (2). By using the Matlab LMI toolbox, we solve the LMI (11) and obtain

$$P_1 = \begin{bmatrix} 3.3622 & 0.0540 & 0.6000 \\ 0.0540 & 4.9174 & -0.4745 \\ 0.6000 & -0.4745 & 5.5093 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 9.1332 & -0.6157 & 0.7295 \\ -0.6157 & 8.8077 & -0.4198 \\ 0.7295 & -0.4198 & 2.5549 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 5.8857 & 1.8836 & -0.2181 \\ 1.8836 & 6.4192 & -0.6144 \\ -0.2181 & -0.6144 & 3.0254 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 18.6721 & 0 & 0 \\ 0 & 19.4351 & 0 \\ 0 & 0 & 6.8673 \end{bmatrix},$$

$$\Gamma = \begin{bmatrix} 19.4997 & 0 & 0 \\ 0 & 19.3297 & 0 \\ 0 & 0 & 5.3101 \end{bmatrix}, \quad \Delta = \begin{bmatrix} 9.3159 & 0 & 0 \\ 0 & 7.5492 & 0 \\ 0 & 0 & 1.4863 \end{bmatrix}.$$

Therefore, it follows from Theorem 1 that the RNN (2) with given parameters is globally exponentially stable, which is further verified by the simulation result given in Fig. 1.

V. Conclusions

In this paper, we have dealt with the problem of global exponential stability analysis for a class of general recurrent neural networks, which involve both the discrete and distributed time delays. We have removed the traditional monotonicity and smoothness assumptions on the activation function. A linear matrix inequality (LMI) approach has been developed to solve the problem addressed. The conditions for the global exponential stability have been derived in terms of the symmetric positive definite solution to the LMIs, and a simulation example has been used to demonstrate the usefulness of the main results.

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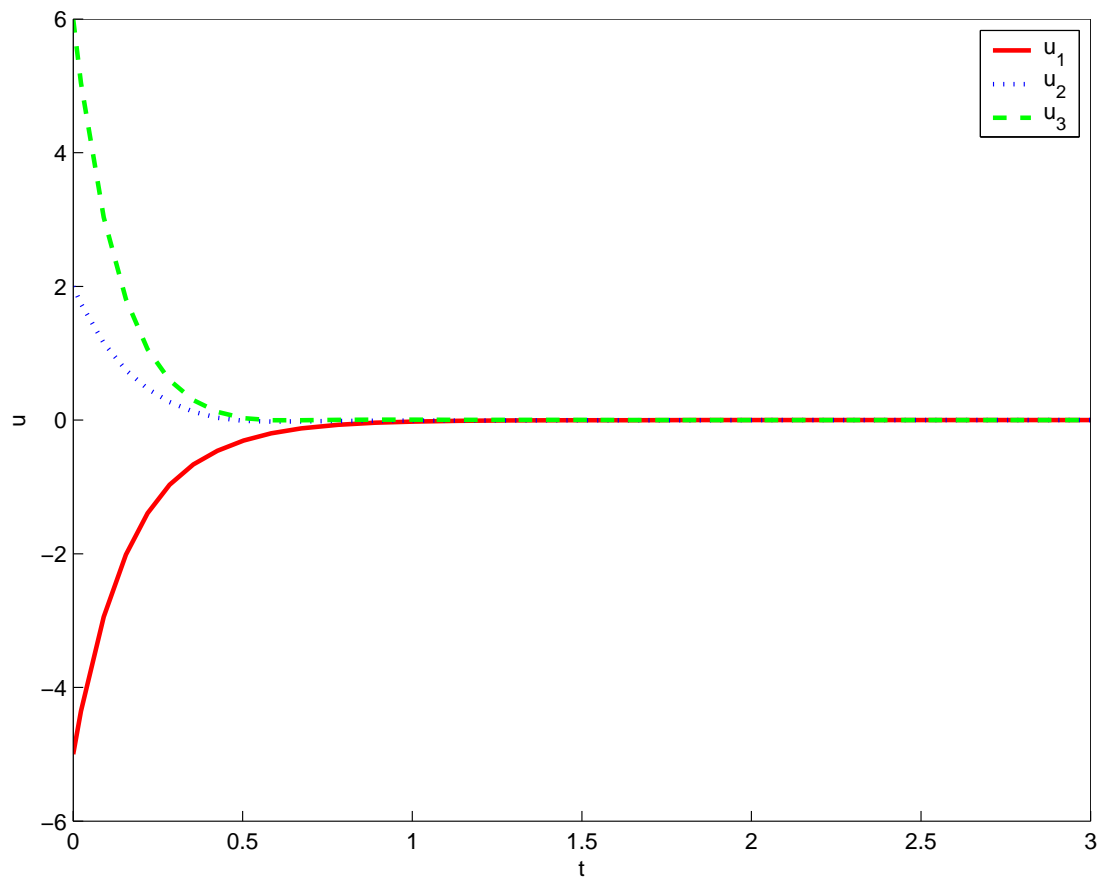
Fig. 1. State trajectory of the RNN in the example.