Compact support probability distributions in random matrix theory

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Abstract

We consider a generalization of the fixed and bounded trace ensembles introduced by Bronk and Rosenzweig up to an arbitrary polynomial potential. In the large-n limit we prove that the two are equivalent and that their eigenvalue distribution coincides with that of the "canonical" ensemble with measure $\exp[-n\text{Tr } V(M)]$. The mapping of the corresponding phase boundaries is illuminated in an explicit example. In the case of a Gaussian potential we are able to derive exact expressions for the one- and two-point correlator for finite n, having finite support.

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1 Introduction

Random matrix ensembles have been extensively studied since the early works of Wigner and Dyson, as effective mathematical reference models for the description of statistical properties in the spectra of complex physical systems, ranging from such diverse areas as nuclear resonances or quantum billiards to mesoscopic transport or quenched QCD. Even a cursory glance at some recent review monographs [1], [2], [3], [4], shows the impressive development of analytical tools and the variety of applications to physical systems reached in the past decade and a combined bibliography, although very incomplete, of over a thousand papers.

Historically the matrices of the ensemble belong to one of three classes, they are real symmetric or complex-Hermitian or with quaternionic entries but in recent years other ensembles , like complex non-Hermitian or real non-symmetric matrices have been studied. To keep our paper as simple as possible, we restrict ourselves to complex-Hermitian matrices, although the results of this paper apply also to the other two traditional ensembles with minimal changes.

A random matrix ensemble is defined by the joint probability density for the independent entries of the matrix. In a large number of papers, particularly those related to two-dimensional quantum gravity, the probability density has the form

$$\mathcal{P}(M) \equiv \frac{1}{\mathcal{Z}} e^{-\beta \operatorname{Tr} V(M)} \tag{1.1}$$

where $V(x)$ is a polynomial. Since this probability density is invariant under the similarity transformation $M = U\Lambda U^\dagger$ which diagonalizes the matrix M, most problems may be formulated in terms of the joint probability density for the eigenvalues

$$\mathcal{P}(\lambda_1, .., \lambda_n) \equiv \frac{1}{z} \Delta_n^2(\lambda) e^{-\beta \sum_1^n V(\lambda_i)} \tag{1.2}$$

and may be called eigenvalue models. We shall call the probability density (1.1) or (1.2) the "canonical" density. In the analysis of the "large-n " limit of observables, evaluated with the "canonical" probability density, the method of orthogonal polynomials [5] proved to be most effective. In the present paper we study matrix ensembles defined by the probability density

$$\mathcal{P}_\delta(M) \equiv \frac{1}{\mathcal{Z}_\delta} \delta\left(A^2 - \frac{1}{n} \operatorname{Tr} V(M)\right) \tag{1.3}$$

and the closely related probability density where the step function replaces the Dirac delta function. We follow the classic book by Mehta [6] and call collectively these models **(generalized) restricted trace ensembles**. They are a generalization of ensembles studied long ago by Rosenzweig and Bronk [7] where only the case $V(x) = x^2$ was considered. While the ensemble is still invariant under the unitary transformation which diagonalizes the random matrix, the method of orthogonal polynomials cannot be directly applied because the constraint of the delta function introduces an additional interaction among the eigenvalues. Restricted trace ensembles seem to us interesting for several features : the interaction among eigenvalues is introduced through a constraint very similar to the non linear sigma model in quantum field theory, the spectral density has compact support both for finite n and in the "large-n " limit (unlike the usual Gaussian random model), and they relate to "canonical" probability densities (1.1) or (1.2) just in the same way as the microcanonical ensemble is related to the canonical ensemble in statistical mechanics.

The effectiveness of random matrix theory is related mainly to "universal" properties of some observables, that is independence, in the "large-n " limit, of some observables from the chosen probability density. The ensemble averaged density of eigenvalues $\rho(\lambda) = \frac{1}{n} \operatorname{Tr} < \delta(\lambda - M) >$, in "canonical"

eigenvalue models (1.1) or (1.2) is known to depend from the chosen function $V(x)$, yet a number of critical exponents deduced from the spectral density were shown to be independent from the details of the chosen function $V(x)$. Much earlier the density-density connected correlator $\rho_c(\lambda, \mu)$

$$
\begin{aligned}
\rho_c(\lambda, \mu) &\equiv \ < \frac{1}{n} \operatorname{Tr}\delta(\lambda - M) \frac{1}{n} \operatorname{Tr}\delta(\mu - M) > - < \frac{1}{n} \operatorname{Tr}\delta(\lambda - M) > < \frac{1}{n} \operatorname{Tr}\delta(\mu - M) > = \\
&= \ \rho(\lambda, \mu) - \rho(\lambda)\,\rho(\mu)
\end{aligned} \tag{1.4}
$$

was shown to have "local universality" properties , that is for $|\lambda - \mu| \sim O(1/n)$ and far from the extrema of the support of the spectral density, in the "large-n" limit. This was the basis for the use of random matrix theory for statistical fluctuations of observables around their mean values. Other forms of universality were derived more recently by several authors, including a form of "wide correlator" which depends on the "canonical" potential $V(x)$ only through the extrema of the spectral density. The proof by Beenakker and a list of other authors is recalled in sect.1 D of ref. [3].

In sect.2 we exploit a scale transformation, already used by Rosenzweig in a more limited extent, to relate observables in restricted trace ensembles where $V(x) = x^2$ with the corresponding ones in the random Gaussian model. This allows explicit evaluations for the spectral density and the two point correlators for finite n.

We then consider a generalization of the restricted trace ensembles to a generic $V(x)$ in sect.3. There a very general proof of the equivalence, in the large-n limit, of the generalized restricted trace ensembles with the corresponding "canonical" ones is presented. This proof is a wide generalization of the old result of the equivalence, in the large-n limit, of the restricted trace ensembles with the random Gaussian model .

Unlike the original restricted trace ensembles, the generalized ensembles have a non trivial phase diagram in the large-n limit. Despite the equivalence shown in sect.3 with "canonical" probability distributions, the mapping of parameters in "equivalent" models is one-to-one only in the "perturbative phase". We show this in detail in one example of phase diagram in sect.4.

Let us stress that the present paper is concerned with derivation of exact analytic results for the probability distributions we consider. Applications of physical interest are deferred to a future paper. While this paper was being written, we were informed of a poster presented by T.Nagao at StatPhys 20, discussing generalized fixed trace ensembles of random matrices. There the old model by Rosenzweig is generalized by considering a joint probability density of eigenvalues of the form

$$
\mathcal{P}(\lambda_1, .., \lambda_n) \ \equiv \ \frac{1}{z} \Delta_n^\beta(\lambda) \prod_1^n \lambda_i^\alpha \, \delta(A^2 - \sum_1^n \lambda_i^2) \ \ . \tag{1.5}
$$

This study has very little overlap with the present paper.

2 Restricted trace ensembles at finite n.

Let us begin by describing the most relevant features of two closely related ensembles : the fixed trace and the bounded trace ensembles. Let M be a n x n Hermitian matrix. The **fixed trace ensemble** corresponding to the Gaussian model is defined by the probability distribution

$$
\begin{aligned}
\mathcal{P}_\delta(M) &\equiv \ \frac{1}{\mathcal{Z}_\delta} \delta(A^2 - \frac{1}{n}\operatorname{Tr} M^2) \ \ , \\
\mathcal{Z}_\delta &\equiv \ \int \mathcal{D}M \, \delta(A^2 - \frac{1}{n}\operatorname{Tr} M^2) = \left(\frac{1}{2}\right)^{(n^2-n)/2} \omega_{n^2} \frac{(A\sqrt{n})^{n^2}}{2A^2} \ \ ,
\end{aligned} \tag{2.1}
$$

where $\mathcal{D}M \equiv \prod_{i=1,..n} dM_{ii} \prod_{i>j} \operatorname{Re} dM_{ij} \operatorname{Im} dM_{ij}$, $\omega_{n^2} = \frac{2\pi^{n^2/2}}{\Gamma(\frac{n^2}{2})}$ is the surface area of the unit sphere in n^2 dimensions, and the factor $\frac{1}{n}$ has been introduced in view of the large-n limit. Expectation values of $O(n^2)$ invariant amplitudes are trivially evaluated for every n as for instance

$$< (\operatorname{Tr} M^2)^k >_\delta \equiv \int \mathcal{D}M \; (\operatorname{Tr} M^2)^k \; \mathcal{P}_\delta(M) = (nA^2)^k \quad . \tag{2.2}$$

However we are interested in more general expectation values, which are functions of the distribution of eigenvalues. They may be evaluated from the joint probability distribution $\mathcal{P}_\delta(\lambda_1,..,\lambda_n)$ which is obtained from eq.(2.1) after integration of the unitary degrees of freedom

$$\mathcal{P}_\delta(\lambda_1,..,\lambda_n) = \frac{1}{z_\delta} \Delta_n^2(\lambda)\, \delta(A^2 - \frac{1}{n}\sum_{i=1}^n \lambda_i^2) , \quad \Delta_n(\lambda) \equiv \prod_{1 \le r < s \le n} (\lambda_r - \lambda_s) = \det[\lambda_i^{j-1}] ,$$

$$z_\delta \equiv \int_{-\infty}^\infty \prod_{i=1}^n d\lambda_i \, \Delta_n^2(\lambda)\, \delta(A^2 - \frac{1}{n}\sum_{i=1}^n \lambda_i^2) = \left(\frac{A^2}{2}\right)^{\frac{n^2}{2}-1} \frac{n^{n^2/2}}{2} \frac{(2\pi)^{n/2}}{\Gamma(\frac{n^2}{2})} \prod_{j=1}^n j! \quad . \tag{2.3}$$

Closely related to this matrix ensemble is the **bounded trace ensemble**. It is defined by the probability distribution

$$\mathcal{P}_\theta(M) \equiv \frac{1}{\mathcal{Z}_\theta} \theta(A^2 - \frac{1}{n}\operatorname{Tr} M^2) \quad ,$$

$$\mathcal{Z}_\theta \equiv \int \mathcal{D}M\, \theta(A^2 - \frac{1}{n}\operatorname{Tr} M^2) = \left(\frac{1}{2}\right)^{(n^2-n)/2} \omega_{n^2} \frac{(A\sqrt{n})^{n^2}}{n^2} \quad . \tag{2.4}$$

In the same way of eq.(2.2), one easily finds

$$< (\operatorname{Tr} M^2)^k >_\theta = \left(\frac{n^{2+k}}{n^2 + 2k}\right) A^{2k} \quad . \tag{2.5}$$

which exhibits the usual factorization of $O(n^2)$ invariant amplitudes only in the large-n limit. In order to evaluate expectations which only depend on the distribution probability of the eigenvalues, one may use the joint probability distribution, analogous to eq.(2.3) :

$$\mathcal{P}_\theta(\lambda_1,..,\lambda_n) = \frac{1}{z_\theta} \Delta_n^2(\lambda)\, \theta(A^2 - \frac{1}{n}\sum_1^n \lambda_i^2) \quad ,$$

$$z_\theta \equiv \int_{-\infty}^\infty \prod_{i=1}^n d\lambda_i \, \Delta_n^2(\lambda)\, \theta(A^2 - \frac{1}{n}\sum_1^n \lambda_i^2) = \left(\frac{A^2}{2}\right)^{\frac{n^2}{2}} n^{n^2/2} \frac{(2\pi)^{n/2}}{\Gamma(\frac{n^2}{2}+1)} \prod_{j=1}^n j! \quad . \tag{2.6}$$

Of course the two ensembles are related by a simple differential equation. Since

$$\frac{\partial}{\partial A^2} \mathcal{P}_\theta(M) = \frac{\mathcal{Z}_\delta}{\mathcal{Z}_\theta}\left[\mathcal{P}_\delta(M) - \mathcal{P}_\theta(M)\right] = \frac{n^2}{2A^2}\left[\mathcal{P}_\delta(M) - \mathcal{P}_\theta(M)\right]$$

one easily obtains a simple relation between the two expectations for any generic observable

$$< O(M) >_\delta = \left(1 + \frac{2A^2}{n^2}\frac{\partial}{\partial A^2}\right) < O(M) >_\theta \quad . \tag{2.7}$$

3

A remarkable feature of both the "fixed trace ensemble" and the "bounded trace ensemble" is that the density of states $\rho(\lambda)$ has compact support for any n, finite or infinite. We here obtain the exact expression of the eigenvalue distribution of the fixed trace ensemble for any value of n, based on the known results for the Gaussian model.

Let us first recall a few useful formulas of the Gaussian model. The partition function and the eigenvalue density are

$$z_G = \int d\lambda_1 \ldots d\lambda_n \, \Delta_n^2 \, e^{-a(\lambda_1^2 + \ldots + \lambda_n^2)} \quad , \tag{2.8}$$

$$\rho_G(\lambda) = e^{-a\lambda^2} \frac{1}{z_G} \int d\lambda_1 \ldots d\lambda_{n-1} \Delta_{n-1}^2 e^{-a(\lambda_1^2 + \ldots + \lambda_{n-1}^2)} \prod_{i=1}^{n-1} (\lambda - \lambda_i)^2 \quad , \tag{2.9}$$

where the positive parameter a is arbitrary, and for shortness we set $\Delta_n^2 \equiv \Delta^2(\lambda_1, \ldots, \lambda_n)$. Both integrals may be computed for finite n by means of orthogonal polynomials, which in this case are the Hermite ones:

$$z_G = \frac{(2\pi)^{n/2}}{(2a)^{n^2/2}} \prod_{k=1}^{n} k! \quad , \qquad \rho_G(\lambda) = \sqrt{\frac{a}{\pi}} e^{-a\lambda^2} \frac{1}{n} \sum_{k=0}^{n-1} \frac{H_k^2(\lambda\sqrt{a})}{2^k k!} \quad . \tag{2.10}$$

Instead of evaluating the sum by means of the Christoffel-Darboux formula, it is useful for our discussion to use the expansion

$$[H_k(x)]^2 = \sum_{\ell=0}^{k} \frac{(k!)^2 2^{k-\ell}}{(\ell!)^2 (k-\ell)!} H_{2\ell}(x) \tag{2.11}$$

to obtain, with some simple algebra:

$$\rho_G(\lambda) = \sqrt{\frac{a}{\pi}} e^{-a\lambda^2} \frac{1}{n} \sum_{k=0}^{n-1} \binom{n}{k+1} \frac{H_{2k}(\lambda\sqrt{a})}{2^k k!} \quad . \tag{2.12}$$

To study the integrals for the fixed trace ensemble, it is convenient to adopt the following notation. Let us denote by $\omega(n, R)$ the surface of the sphere in R^n of radius R, and by da_n the element of surface integration. The partition function and the eigenvalue density, for $|\lambda| \leq R$, are:

$$z_\delta = \int_{-\infty}^{\infty} \prod_{i=1}^{n} d\lambda_i \, \Delta_n^2(\lambda) \, \delta(A^2 - \frac{1}{n} \sum_{i=1}^{n} \lambda_i^2) = \frac{n}{2R} \int_{\omega(n,R)} da_n \, \Delta_n^2 \quad , \quad R^2 \equiv nA^2 \quad , \tag{2.13}$$

$$\rho_\delta(\lambda) = \frac{1}{z_\delta} \int \prod_{i=1}^{n} d\lambda_i \, \Delta_n^2 \, \delta(\lambda - \lambda_n) \, \delta(A^2 - \frac{1}{n} \sum_{i=1}^{n} \lambda_i^2) =$$

$$= \frac{n}{2z_\delta \sqrt{R^2 - \lambda^2}} \int_{\omega(n-1, \sqrt{R^2 - \lambda^2})} da_{n-1} \Delta_{n-1}^2 \prod_{i=1}^{n-1} (\lambda - \lambda_i)^2 \quad . \tag{2.14}$$

After a change of scale, to restrict both integrals to the surface of unit radius:

$$z_\delta = \frac{n}{2} R^{n^2-2} \int_{\omega(n,1)} da_n \, \Delta_n^2 \quad , \tag{2.15}$$

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$$\rho_\delta(\lambda) = \frac{n}{2z_\delta}(R^2 - \lambda^2)^{\frac{1}{2}(n^2-3)} \int_{\omega(n-1,1)} da_{n-1}\Delta_{n-1}^2 \prod_{i=1}^{n-1}(\frac{\lambda}{\sqrt{R^2-\lambda^2}} - \lambda_i)^2 \quad . \tag{2.16}$$

Let us first evaluate the partition function. We start from the integral expression (2.8) for z_G, and change to spherical variables with radial component r. The volume element is $r^{n-1}drda_n$, and $\Delta^2(\lambda_1,\ldots,\lambda_n) = r^{n(n-1)}\Delta^2(\lambda_1/r,\ldots,\lambda_n/r)$. Therefore we have:

$$z_G = \int_0^\infty dr r^{n^2-1}e^{-ar^2} \int_{\omega(n,1)} da_n\Delta_n^2 \quad . \tag{2.17}$$

The surface integral is the same appearing in (2.15), and we conclude:

$$z_\delta = z_G \frac{n(R\sqrt{a})^{n^2}}{R^2\,\Gamma(\frac{n^2}{2})} \quad . \tag{2.18}$$

The same procedure is used in the evaluation of the eigenvalue density. In radial coordinates, the integral for the Gaussian density is

$$\rho_G(\lambda) = e^{-a\lambda^2}\frac{1}{z_G}\int_0^\infty dr r^{n^2-2}e^{-ar^2} \int_{\omega(n-1,1)} da_{n-1}\Delta_{n-1}^2 \prod_{i=1}^{n-1}(\frac{\lambda}{r} - \lambda_i)^2 \quad . \tag{2.19}$$

The surface contribution is much alike the one in the expression (2.16) for $\rho_\delta(\lambda)$. To implement this similarity, we introduce the expansion

$$\int_{\omega(n-1,1)} da_{n-1}\Delta_{n-1}^2 \prod_{i=1}^{n-1}(x - \lambda_i)^2 = \sum_{k=0}^{2n-2} c_k x^k \quad . \tag{2.20}$$

Since $\rho_G(\lambda)$ is even in λ, only the even coefficients are different from zero. The expressions for the densities in the two ensembles are:

$$\rho_G(\lambda) = e^{-a\lambda^2}\frac{1}{z_G}a^{-\frac{1}{2}(n^2-1)}\frac{1}{2}\sum_{k=0}^{n-1} c_{2k}(\lambda\sqrt{a})^{2k}\Gamma\left(\frac{n^2-1}{2} - k\right) \quad , \tag{2.21}$$

$$\rho_\delta(\lambda) = \frac{n}{2z_\delta}(R^2 - \lambda^2)^{\frac{(n-1)^2}{2}-1}\sum_{k=0}^{n-1} c_{2k}\lambda^{2k}(R^2-\lambda^2)^{n-1-k} \quad , \quad R^2 \equiv nA^2 \quad . \tag{2.22}$$

The coefficients c_{2k} are obtained by comparing the polynomial expression in (2.21) and the exactly known result (2.12):

$$c_{2k} = 2^{1-\frac{n^2}{2}}\frac{(2\pi)^{\frac{n}{2}}(-4)^k}{\sqrt{\pi}\,(2k)!}\frac{\prod_{j=1}^n j!}{\Gamma\left(\frac{n^2-1}{2} - k\right)}\frac{1}{n}\sum_{\ell=k}^{n-1}(-1)^\ell\frac{(2\ell)!}{2^\ell \ell!(\ell-k)!}\binom{n}{\ell+1} \quad . \tag{2.23}$$

More explicitly, the spectral densities for the lowest order random matrices are

$$\begin{aligned}
\rho_\delta(\lambda) &= \frac{1}{\pi\sqrt{2A^2 - \lambda^2}} \quad , \quad \text{for } n = 2 \,, \\
\rho_\delta(\lambda) &= \frac{35\sqrt{3}}{576A^7}(A^2 - \frac{\lambda^2}{3})\Big[3A^4 - 2\lambda^2 A^2 + 3\lambda^4\Big], \quad \text{for } n = 3 \,, \\
\rho_\delta(\lambda) &= \frac{32}{429\pi A^{14}}\left(A^2 - \frac{1}{4}\lambda^2\right)^{\frac{7}{2}}\Big[12A^6 + 30\lambda^2 A^4 - 53\lambda^4 A^2 + 38\lambda^6\Big], \quad \text{for } n = 4 \,, \\
\rho_\delta(\lambda) &= \frac{2028117\sqrt{5}}{5^4\,(2A)^{23}}(A^2 - \frac{\lambda^2}{5})^7\Big[375A^8 - 300\lambda^2 A^6 + 4490\lambda^4 A^4 - 5996\lambda^6 A^2 + 2711\lambda^8\Big], \quad \text{for } n = 5.
\end{aligned}$$
$$\tag{2.24}$$

To evaluate the spectral density for the bounded trace ensemble , for finite n , one may proceed in a similar way as in the Gaussian case, to obtain

$$\rho_\theta(\lambda) = \frac{1}{z_\theta} \int_0^{\sqrt{R^2-\lambda^2}} dr \, r^{n^2-2} \int_{\omega(n-1,1)} da_{n-1} \Delta_{n-1}^2 \prod_{i=1}^{n-1} (\frac{\lambda}{r} - \lambda_i)^2 \tag{2.25}$$

and therefore

$$\rho_\theta(\lambda) = \frac{1}{z_\theta}(R^2 - \lambda^2)^{\frac{n^2-1}{2}} \sum_{k=0}^{n-1} \frac{c_{2k}}{n^2 - 2k - 1} \left(\frac{\lambda^2}{R^2 - \lambda^2}\right)^k \quad , \quad R^2 \equiv nA^2 \ . \tag{2.26}$$

The same result may be obtained by inverting the differential equation (2.7). In a similar way , it is possible to write the explicit expressions of the two-point correlator of restricted trace ensembles in terms of the known two-point correlator of the Gaussian ensemble at finite n. We obtain

$$\rho_G(\lambda, \mu) = \frac{1}{2z_G} e^{-a(\lambda^2+\mu^2)} a^{-\frac{1}{2}(n^2-4)} \sum_{r,s=0}^{2n-2} c_{r,s} (\lambda\sqrt{a})^r (\mu\sqrt{a})^s \Gamma\left(\frac{n^2 - r - s}{2} - 1\right) \quad , \tag{2.27}$$

$$\rho_\delta(\lambda, \mu) = \frac{1}{z_\delta}(R^2 - \lambda^2 - \mu^2)^{\frac{1}{2}(n^2-3)} \sum_{r,s=0}^{2n-2} c_{r,s} \lambda^r \mu^s (R^2 - \lambda^2 - \mu^2)^{-\frac{(r+s)}{2}} \quad , \tag{2.28}$$

where the coefficients $c_{r,s} = c_{s,r}$ are defined by

$$(x - y)^2 \int_{\omega(n-2,1)} da_{n-2} \Delta_{n-2}^2 \prod_{k=1}^{n-2} (x - \lambda_i)^2 (y - \lambda_i)^2 = \sum_{r,s=0}^{2n-2} c_{r,s} x^r y^s \ . \tag{2.29}$$

3   Generalized restricted trace ensembles at large n

With some generality, for an arbitrary polynomial potential $V(M) = \sum g_k M^k$, where M is an Hermitian $n \times n$ matrix, we define the generalized **fixed trace ensemble** and the generalized **bounded trace ensemble** by the two probability densities :

$$\mathcal{P}_\delta(M) \equiv \frac{1}{\mathcal{Z}_\delta} \delta\left(A^2 - \frac{1}{n} \operatorname{Tr} V(M)\right) \quad , \tag{3.1}$$

$$\mathcal{P}_\theta(M) \equiv \frac{1}{\mathcal{Z}_\theta} \theta\left(A^2 - \frac{1}{n} \operatorname{Tr} V(M)\right) \quad , \tag{3.2}$$

where $\mathcal{Z}_\delta$ and $\mathcal{Z}_\theta$ are the normalization factors, and we used the same notation of the previous section, where they correspond to the simplest case $V(x) = x^2$.

Both ensembles are invariant under the action of the unitary group. Therefore, when changing matrix the parameterization from n^2 independent matrix elements to the n real eigenvalues and the parameters for eigenvectors, the measures factorize into a part given by the Haar measure of $SU(n)$ and a part involving only the eigenvalues. The latter provides the joint probability density of the eigenvalues,

the starting point for all spectral statistics. Letting ϕ stand for the *delta* or the *theta* function, the expression for the joint probability density is:

$$P_\phi(\lambda_1, \ldots, \lambda_n) = \frac{1}{z_\phi} \phi \left(A^2 - \frac{1}{n} \sum_{i=1}^n V(\lambda_i) \right) \Delta^2(\lambda_1, \ldots, \lambda_n) \quad , \tag{3.3}$$

$$z_\phi = \int \prod_{i=1}^n d\lambda_i \, \phi(A^2 - \frac{1}{n} \sum_{i=1}^n V(\lambda_i)) \, \Delta^2(\lambda_1, \ldots, \lambda_n) \quad . \tag{3.4}$$

The two ensembles are obviously related by the differential equation analogous to eq.(2.7):

$$\mathcal{P}_\delta(\lambda_1, \ldots, \lambda_n) = \left(1 + \frac{z_\theta}{z_\delta} \frac{\partial}{\partial A^2} \right) \mathcal{P}_\theta(\lambda_1, \ldots, \lambda_n) \quad , \tag{3.5}$$

which will be used to study the properties of the bounded trace ensemble from a knowledge of the fixed trace one. Indeed, in this general setting, the latter is easier to evaluate in the large-n limit. Besides the two restricted trace ensembles, it is useful to consider also the "canonical" ensemble, with same potential $V(M)$ and a parameter K:

$$\mathcal{P}(\lambda_1, \ldots, \lambda_n) = \frac{1}{z} \, e^{-Kn \sum_{i=1}^n V(\lambda_i)} \Delta^2(\lambda_1, \ldots, \lambda_n) \quad , \tag{3.6}$$

$$z = \int \prod_{i=1}^n d\lambda_i \, e^{-Kn \sum_{i=1}^n V(\lambda_i)} \Delta^2(\lambda_1, \ldots, \lambda_n) \quad . \tag{3.7}$$

As it is well known, the partition function for the eigenvalues may be given the interpretation as the partition function of a one dimensional gas of n particles with pairwise repulsive interaction and, in the canonical case, subject to the external potential $V(\lambda)$. In the restricted trace ensembles the potential enters as a constraint depending on the positions of *all* particles. This main difference makes the analysis of these models difficult and interesting, especially for the issue of the universality properties of correlators.

While for "canonical" models the powerful technique of orthogonal polynomials applies, giving at least formally and for any value of n the explicit expressions of all correlators, for the restricted trace ensembles we must content ourselves with the analysis of the eigenvalue density in the large n limit. This is easily done for the fixed trace ensemble, whose δ constraint can be taken into account in the energy functional through a Lagrange multiplier. In the large-n limit, the eigenvalue configuration is described by a normalized density $\rho(\lambda)$, and the energy functional associated to it is

$$H[\rho] = - \int d\lambda d\mu \rho(\lambda) \rho(\mu) \log |\lambda - \mu| + \alpha \left(A^2 - \int d\lambda \rho(\lambda) V(\lambda) \right) + \beta \left(1 - \int d\lambda \rho(\lambda) \right) \quad . \tag{3.8}$$

The saddle point configuration is the one that minimizes the energy, and is precisely the sought limit density ρ_δ. It solves the following equation, valid for any λ inside the unknown support L of ρ_δ:

$$0 = \frac{\delta H[\rho]}{\delta \rho(\lambda)} = -2 \int d\mu \rho_\delta(\mu) \log |\lambda - \mu| - \alpha V(\lambda) - \beta \quad . \tag{3.9}$$

A derivative in λ eliminates the parameter β associated to the constraint of normalization, and yields a Cauchy-Hilbert integral equation for the limit density:

$$\fint_L d\mu \frac{\rho_\delta(\mu)}{\lambda - \mu} = \frac{\alpha}{2} V'(\lambda) \quad , \qquad \lambda \in L \quad . \tag{3.10}$$

For any α, which is still unknown, and after having fixed a geometry for the support L (an interval, for example) the equation (3.10) is solved using analyticity arguments, and the extrema of L are fixed by the normalization condition [15]. Inside the family of pairs $L(\alpha)$ and $\rho_\delta(\lambda; \alpha)$ parameterized by α, the pair that describes the large-n limit of the fixed trace ensemble is determined by the value $\alpha = \overline{\alpha}$, solution of the equation

$$A^2 = \int_{L(\overline{\alpha})} d\lambda \rho_\delta(\lambda; \overline{\alpha}) V(\lambda) \quad . \tag{3.11}$$

The number $\overline{\beta}$ of the extremal solution may be evaluated from eq. (3.9) by choosing a convenient value of λ in L.

The density ρ_δ so far obtained, coincides with the limit density of the canonical model (3.7), with parameter $K = \overline{\alpha}$. In the particularly simple case $V(M) = M^2$, one obtains also for the restricted trace ensemble a limit density described by Wigner's semicircle law, with radius $2A$. The energy functional (3.8) evaluated at the extremum, is

$$H[\rho_\delta] = -\int d\lambda d\mu \rho_\delta(\lambda) \rho_\delta(\mu) \log|\lambda - \mu| \quad , \tag{3.12}$$

where the double integral may be simplified by using the equation (3.9) and the constraints:

$$\int d\lambda d\mu \rho_\delta(\lambda) \rho_\delta(\mu) \log|\lambda - \mu| = -\frac{1}{2}\overline{\alpha}A^2 - \frac{1}{2}\overline{\beta} \quad . \tag{3.13}$$

We then obtain the large-n expression of the partition function

$$z_\delta \to e^{-\frac{1}{2}n^2(\overline{\alpha}A^2 + \overline{\beta})} \equiv e^{-n^2 f(A^2)} \quad . \tag{3.14}$$

Since $z_\delta = \frac{\partial}{\partial A^2} z_\theta$, eq.(3.14) implies

$$\frac{z_\theta}{z_\delta} = \frac{\mathcal{Z}_\theta}{\mathcal{Z}_\delta} \to -\frac{1}{n^2 \frac{\partial}{\partial A^2} f(A^2)} \quad . \tag{3.15}$$

A simple check is provided by the monomial potentials $V(x) = x^{2k}$. In this simple case, the normalization constants z_θ and z_δ may be evaluated by a rescaling of the eigenvalues with the result $\frac{z_\theta}{z_\delta} = \frac{2kA^2}{n^2}$. The result (3.15) is most useful and it implies the generalization of eq.(2.7)

$$< O(M) >_\delta = \left(1 + c_n \frac{\partial}{\partial A^2}\right) < O(M) >_\theta \quad ; \quad c_n \to -\frac{1}{n^2 \frac{\partial}{\partial A^2} f(A^2)} \quad . \tag{3.16}$$

By using this equation both for the spectral density and for the density-density correlator (1.4), we obtain an exact equation, for any n :

$$
\begin{aligned}
\rho_{\delta,c}(\lambda, \mu) &= \rho_\delta(\lambda, \mu) - \rho_\delta(\lambda)\rho_\delta(\mu) = \\
&= \left(1 + c_n \frac{\partial}{\partial A^2}\right)\rho_\theta(\lambda, \mu) - \left(1 + c_n \frac{\partial}{\partial A^2}\right)\rho_\theta(\lambda)\left(1 + c_n \frac{\partial}{\partial A^2}\right)\rho_\theta(\mu) = \\
&= \rho_{\theta,c}(\lambda, \mu) + c_n \frac{\partial}{\partial A^2}\rho_{\theta,c}(\lambda, \mu) - (c_n)^2\left(\frac{\partial}{\partial A^2}\rho_\theta(\lambda)\right)\left(\frac{\partial}{\partial A^2}\rho_\theta(\mu)\right) \quad .
\end{aligned} \tag{3.17}
$$

We have not proven that the generalized restricted trace matrix ensembles have a topological expansion in the "large-n" limit and the factorization of invariant operators, analogous to matrix ensembles defined by "canonical" probability densities. The analysis of next section, where the fixed trace

constraint is reached as a limit of the probability density $\mathcal{P}_l(M)$ indicates that such properties are very likely. Therefore it seems reasonable to assume, as for the "canonical" probability densities,

$$\rho_\phi(\lambda, \mu) \to \rho_\phi(\lambda)\rho_\phi(\mu) + \frac{1}{n^2}\overline{\rho}_\phi(\lambda, \mu) + O(\frac{1}{n^3}) \quad , \tag{3.18}$$

where ϕ stands for the δ or the θ functions. This assumption (3.18), as well as more general assumptions, together with eq.(3.17) and eq.(3.16) imply , in the "large-n" limit $\rho_{\delta,c}(\lambda, \mu) = \rho_{\theta,c}(\lambda, \mu)$. The results of this section are rather general and formal. The determination of the Lagrange multiplier $\overline{\alpha}$ in eq.(3.11) of course depends on the model potential $V(M)$ in a non trivial way and on the various phases of the model. We provide a specific example in the next section, by the study of the potential $V(M) = g_2 M^2 + g_4 M^4$.

4   Phase transitions

In the previous section it was shown that, in the large-n limit, the spectral density of restricted trace ensembles with polynomial potential $V(\lambda, g_i)$, where g_i are the couplings, coincides with the spectral density of the "canonical" ensemble with potential $\bar{\alpha}V(\lambda, g_i)$. The scaling factor $\bar{\alpha}$, solution of eq. (3.11), is actually a nonlinear function of the couplings g_i. The correspondence between the two sets of parameters, namely g_i and $\bar{\alpha}g_i$, is one-to-one only in the perturbative phase.
In this section we show in detail the case of the even quartic potential

$$V(M) = g_2\,M^2 + g_4\,M^4 \quad . \tag{4.1}$$

where the nonlinear relation originates different phase diagrams. To this end, we find it useful to consider the **squared trace ensemble** $\mathcal{P}_l(M)$

$$
\begin{aligned}
\mathcal{P}_l(M) &= \frac{1}{\mathcal{Z}_l}\exp\left[-l\left(-2nA^2\mathrm{Tr}V(M) \;+\; (\mathrm{Tr}V(M))^2\right)\right] \quad , \\
\mathcal{Z}_l &= \int \mathcal{D}M \exp\left[-l\left(-2nA^2\mathrm{Tr}V(M) \;+\; (\mathrm{Tr}V(M))^2\right)\right] \quad .
\end{aligned}
\tag{4.2}
$$

The large-n limit of the model described by the probability distribution $\mathcal{P}_l(M)$ is easily found by the saddle point approximation. These type of models, where the exponent of the Boltzmann weight is a sum of different powers of traces of even powers of the random matrix was analyzed in several matrix models in zero and one dimension [8] - [14]. The additional "trace-squared" terms were interpreted to provide touching interactions to the dynamical triangulated surfaces defined by the matrix potential $\mathrm{Tr}\,V(M)$.
For any fixed l, the model in eq.(4.2)-(4.1) is equivalent in the large-n limit to a random matrix ensemble with the well studied "canonical" probability distribution

$$
\begin{aligned}
\mathcal{P}(M) &= \frac{1}{\mathcal{Z}}\exp(-n\mathrm{Tr}V(M)) \quad , \\
\mathcal{Z} &= \int \mathcal{D}M \exp(-n\mathrm{Tr}V(M)) \quad , \quad V(M) = g_2'\,M^2 + g_4'\,M^4 \quad ,
\end{aligned}
\tag{4.3}
$$

provided the parameters g_2' and g_4' are suitable functions of the parameters of the model in eqs.(4.2)-(4.1). This may be accomplished by two equations, such as the requirement that the expectations of

$< \text{Tr}\, M^2 >$ and $< \text{Tr}\, M^4 >$ should be the same for the two probability distributions.

On the other hand, for fixed n, in the large-l limit, $\mathcal{P}_l(M)$ reproduces precisely the generalization of the fixed trace ensemble $\mathcal{P}_\delta(M)$, as one sees from the following representation of the δ-function $\delta(x) = \lim_{l\to\infty} \sqrt{\frac{l}{\pi}} \exp(-lx^2)$. Of course, when choosing $g_4 = 0$, we merely reobtain the results of the analysis by Bronk and Rosenzweig.

Let us now recall the saddle point analysis for the large-n limit of the ensemble $\mathcal{P}_l(M)$, eqs.(4.2)-(4.1). Since it proceeds along well known analysis, we include, for more generality the cases of the random matrix M belonging to the orthogonal, unitary, or symplectic ensembles, corresponding to the parameter $\beta = 1, 2$ or 4. It is important to notice that, unlike the familiar quartic probability distribution (4.3), the probability distribution (4.2)-(4.1) is well defined for any real value of the two parameters g_2, g_4. Let us begin by assuming $g_2 > 0$, $g_4 > 0$, which corresponds to the perturbative (or one-cut) phase ; later in the section the complete phase diagram will be described. For any finite positive value of the parameter l, the density of eigenvalues $\rho_l(\lambda)$ is the solution of the singular integral equation

$$\beta \int d\mu \frac{\rho_l(\mu)}{\lambda - \mu} \;=\; 2l \left(g_2 c_2 + g_4 c_4 - A^2 \right) V'(\lambda) \;=\; 2g_2'\lambda \;+\; 4g_4'\lambda^3 \tag{4.4}$$

where the moments c_k are defined by

$$c_k \;\equiv\; \int d\lambda\, \lambda^k \rho_l(\lambda) \tag{4.5}$$

and g_k' are the effective couplings :

$$g_k' \;=\; 2l \left(g_2 c_2 + g_4 c_4 - A^2 \right) g_k \;\;. \tag{4.6}$$

From the symmetry of the potential the support of $\rho_l(\lambda)$ is expected to be one segment or two segments, in either case symmetric with respect to the origin. The solution of the saddle-point equation (4.4) in the one segment phase reads

$$\rho_l(\lambda) \;=\; \frac{2}{\beta\pi}(g_2' \;+\; g_4'b^2 \;+\; 2g_4'\lambda^2)\sqrt{b^2 - \lambda^2} \;\;, \tag{4.7}$$

where the endpoint of the support $[-b, b]$ is given by the normalization condition on the eigenvalue density

$$1 \;=\; \int_{-b}^{b} d\lambda \rho_l(\lambda) \;=\; 2l \left(g_2 c_2 + g_4 c_4 - A^2 \right) \frac{b^2}{2\beta}(2g_2 + 3b^2 g_4) \;\;, \tag{4.8}$$

where we have used again the g_k's.

The moments c_2 and c_4 can be obtained when requiring self consistency by inserting the solution eq. (4.7) back into the definitions (4.5), which yields the linear system of equations

$$c_2 \;=\; \frac{2}{\beta} 2l \left(g_2 c_2 + g_4 c_4 - A^2 \right) \frac{b^4}{8}(g_2 + 2b^2 g_4) \;\;,$$

$$c_4 \;=\; \frac{2}{\beta} 2l \left(g_2 c_2 + g_4 c_4 - A^2 \right) \frac{b^6}{64}(4g_2 + 9b^2 g_4) \;\;. \tag{4.9}$$

For a potential of higher degree we will again get a linear system of equations for the corresponding moments c_k, $k = 1, .., m$, which is due to the fact that the solution of the saddle-point equation will

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again depend linearly on the coupling constants g'_k as in eq. (4.7). Instead of solving the eqs. (4.9) for c_2 and c_4 we can also express them entirely in terms of the couplings with the help of eq. (4.8)

$$c_2 = \frac{b^2(g_2 + 2b^2 g_4)}{2(2g_2 + 3b^2 g_4)} \ ,$$

$$c_4 = \frac{b^4(4g_2 + 9b^2 g_4)}{16(2g_2 + 3b^2 g_4)} \ . \tag{4.10}$$

The same trick can be used to express the eigenvalue density eq. (4.7) only in terms of the g_k, which reads

$$\rho_l(\lambda) = \frac{4}{\pi b^2(2g_2 + 3b^2 g_4)} \left(g_2 \ + \ g_4 b^2 \ + \ 2g_4 \lambda^2 \right) \sqrt{b^2 - \lambda^2} \ . \tag{4.11}$$

where the endpoint of the support b is the root of the fourth order equation in b^2

$$l\left(9(g_4)^2 b^8 \ + \ 20 g_4 g_2 b^6 \ + \ 8((g_2)^2 - 6A^2 g_4) b^4 \ - \ 32 A^2 g_2 b^2 \right) = 16\beta \tag{4.12}$$

which, for vanishing g_4 and positive g_2 is asymptotic to $b^2 \sim 2(A^2 + \sqrt{A^4 + \beta/(2l)})/g_2$. By comparing eq(4.4) with the analogous saddle point equation for the "canonical" quartic probability distribution (4.3) it is obvious that they have the same eigenvalue density, in the large-n limit, for both phases of the model, provided the effective coupling eq. (4.6) are precisely identified with those of the "canonical" distribution

$$g'_2 = 2l \left(g_2 c_2 + g_4 c_4 - A^2 \right) g_2 = \frac{2\beta}{b^2(2g_2 + 3b^2 g_4)} g_2 \ ,$$

$$g'_4 = 2l \left(g_2 c_2 + g_4 c_4 - A^2 \right) g_4 = \frac{2\beta}{b^2(2g_2 + 3b^2 g_4)} g_4 \ . \tag{4.13}$$

Of course, the last equality on the right sides of previous equations only holds in the one cut phase. For simplicity, let us now proceed with $\beta = 2$. In terms of g'_2 and g'_4, the equation for the support (4.12) is the more familiar equation $3g'_4 b^4 + 2g'_2 b^2 - 4 = 0$. The phase diagram of the "canonical" quartic model $\mathcal{P}(M)$, eq.(4.3), is well known: if g'_2 is fixed positive , the one-cut solution (4.7)-(4.8) holds for any real g'_4 such that

$$g'_4 \geq -\frac{1}{12}(g'_2)^2 \ , \tag{4.14}$$

which is a border of existence for the model. If g'_4 is fixed positive, the one-cut solution holds for any real g'_2 such that

$$g'_2 \geq -2\sqrt{g'_4} \ , \tag{4.15}$$

which is the line of phase transition to the symmetric two-cut solution :

$$\rho_l(\lambda) = \frac{2g'_4 |\lambda|}{\pi} \sqrt{(D^2 - \lambda^2)(\lambda^2 - C^2)} \ , \tag{4.16}$$

with ends of support $[-D, -C] \cup [C, D]$ being solutions of

$$g'_2 + g'_4(C^2 + D^2) = 0 \quad , \quad g'_4(D^2 - C^2)^2 = 4 \tag{4.17}$$

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The map between $\{g_2, g_4\}$ and $\{g'_2, g'_4\}$ in this phase, may be found after the evaluation of $\{c_2, c_4\}$ and the requirement of self-consistency , just as before.

It is straightforward to see that the phase transition line (4.15) becomes, in the parameters of the model (4.2)-(4.1), the couple of lines

$$g_4 = \frac{l}{4} (g_2)^2 \left(-A^2 \pm \sqrt{A^4 - 3/(2l)} \right) \quad , \quad g_2 > 0 \quad , \quad g_4 < 0 \ . \tag{4.18}$$

Therefore if $A^4 - 3/(2l) < 0$ the model (4.2)-(4.1) has the one-cut solution for every real value of g_2 , g_4. In the other case $A^4 - 3/(2l) > 0$ the two-cut solution holds in the region of parameters bounded by the two curves (4.18), while the one-cut solution holds everywhere else in the plane of real values of g_2 , g_4.

The image of the existence line (4.14) , in the space of parameters g_2 , g_4 is a couple of curves :

$$\begin{aligned} g_4 &= \frac{l}{12} (g_2)^2 (A^2 - \sqrt{A^4 + 7/(6l)}) \quad , \quad g_2 > 0 \quad , \quad g_4 < 0 \ , \\ g_4 &= \frac{l}{12} (g_2)^2 (A^2 + \sqrt{A^4 + 7/(6l)}) \quad , \quad g_2 < 0 \quad , \quad g_4 > 0 \ . \end{aligned} \tag{4.19}$$

There are two regions of the plane of real variables g_2 , g_4 : the first one bounded by the first line (4.19) and the positive axis g_2, and the second one bounded by the second line (4.19) and the negative axis g_2, where the equation of the support (4.12) of the one-cut solution has three possible values. The one-cut solution (4.11) as function of the parameters g_2 , g_4 has a first order discontinuity in these regions due to the cubic type instability of the solution of the eq.(4.12) with respect to the parameters g_2 , g_4 . As usual, the lines of the first order transition are determined by comparing the evaluation of the free energy of the model, as functions of the different possible values of the endpoint of the support b.

In the remaining part of this section, we consider the limit $l \to \infty$ where we obtain the distribution $\mathcal{P}_\delta(M)$ with the potential (4.1) explicitly. We shall denote $\overline{c}_k \equiv \lim_{l \to \infty} c_k$ and $\overline{b} = \lim_{l \to \infty} b$. Because of the δ-function in the distribution it will hold

$$A^2 = g_2 \overline{c}_2 + g_4 \overline{c}_4 \tag{4.20}$$

whereas the quantity $l(g_2 c_2 + g_4 c_4 - A^2)$ will stay finite, as one can see from eq. (4.8). Eq. (4.20) is actually eq. (3.11) for the quartic potential considered in this section. Eq. (4.4) shows that the model with $\mathcal{P}_\delta(M)$ has the same eigenvalue density of the "canonical" quartic model (4.3), provided $g'_2 = \overline{\alpha} g_2$, and $g'_4 = \overline{\alpha} g_4$, where

$$\begin{aligned} \overline{\alpha} &= \lim_{l \to \infty} 2l \ (g_2 c_2 + g_4 c_4 - A^2) = \\ &= \left(\frac{\overline{b}^2}{4} (2g_2 + 3\overline{b}^2 g_4) \right)^{-1} . \end{aligned} \tag{4.21}$$

The results for the moments eqs. (4.10) and the density eq. (4.11) carry over when replacing everything by barred quantities. Eq. (4.21) gives the solution to eq. (3.11) and shows its dependence on the coupling constants of the quartic potential eq. (4.1).

The phase diagram for $l = \infty$ is similar to the one previously described for finite l, with some simplifications. The couple of lines (4.18) which are boundaries of the two-cut phase become the line

$$g_4 = -\frac{3}{16} \frac{(g_2)^2}{A^2} \quad , \quad g_2 > 0 \quad , \quad g_4 < 0 \ , \tag{4.22}$$

and the negative g_4 axis. Similarly there are two regions of multiple solution for the one-cut support, where a first order discontinuity will occur. One is bounded by the positive part of the g_2 axis and the line

$$g_4 = -\frac{7}{144}\frac{(g_2)^2}{A^2} \quad , \quad g_2 > 0 \quad , \quad g_4 < 0 \quad . \tag{4.23}$$

The second region is the entire region $g_4 > 0$ and $g_2 < 0$. Eq.(4.20) for the endpoint $\overline{b}$ of the one-cut solution turns into

$$0 \;=\; \overline{b}^2\left[9(g_4)^2\overline{b}^6 \;+\; 20g_2g_4\overline{b}^4 \;+\; 8((g_2)^2 - 6g_4A^2)\overline{b}^2 \;-\; 32g_2A^2\right] \quad . \tag{4.24}$$

The vanishing support $\overline{b} = 0$ actually provides the limiting solution $\rho_\delta(\lambda) = \delta(\lambda)$ in the sector $g_2 < 0$, $g_4 < 0$. In other regions of parameter space the support is determined by the solution of the third order equation in $\overline{b}^2$ above.

Let us finally extract the result for the Gaussian distribution $\mathcal{P}_\delta(M)$ with potential $V(M) = g_2M^2$ from the above formula by setting $g_4 = 0$. Eq. (4.24) leads to

$$\overline{b}^2 \;=\; \frac{4A^2}{g_2} \tag{4.25}$$

with the corresponding eigenvalue density from eq. (4.11)

$$\rho_\delta(\lambda) = \frac{2}{\pi\overline{b}^2}\sqrt{\overline{b}^2 - \lambda^2} \quad . \tag{4.26}$$

this is the well known semi-circle spectral density and together with eq.(2.7) it reproduces the old result [7] that the spectral density of the restricted trace ensembles is equal, in the "large-n" limit, to the spectral density of the Gaussian ensemble.

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