

SOLUTIONS FOR THE CAHN-HILLIARD EQUATION WITH MANY BOUNDARY SPIKE LAYERS

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ABSTRACT. In this paper we construct new classes of stationary solutions for the Cahn-Hilliard equation by a novel approach.

One of the results is as follows: Given a positive integer K and a (not necessarily nondegenerate) local minimum point of the mean curvature of the boundary then there are boundary K -spike solutions whose peaks all approach this point. This implies that for any smooth and bounded domain there exist boundary K -spike solutions.

The central ingredient of our analysis is the novel derivation and exploitation of a reduction of the energy to finite dimensions (Lemma 3.5), where the variables are closely related to the peak locations.

1. INTRODUCTION

The Cahn-Hilliard equation [7] was originally derived from the Helmholtz free energy of an isotropic two-component solid and can be written as follows:

$$E(u) = \int_{\Omega} [F(u(x)) + \frac{1}{2}\epsilon^2 |\nabla u(x)|^2] dx.$$

It is a well-accepted and widely studied macroscopic model for phase separation. Here $\Omega \subset R^N$ is the smooth and bounded region occupied by the body, $u(x)$ is an order parameter typically representing the concentration of one of the components. Furthermore, $F(u)$ is the free energy density of a corresponding homogeneous solid which has a double well structure the prototype being $F(u) = (1 - u^2)^2$ since we consider low temperatures. The constant ϵ describes the range of intermolecular forces; the gradient term models spatial fluctuations.

We assume conservation of the order parameter, i.e. there exists \bar{u} with $-1 < \bar{u} < 1$ such that $\bar{u} = \frac{1}{|\Omega|} \int_{\Omega} u dx$. Therefore, a stationary solution of

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$E(u)$ under the conservation constraint $\bar{u} = \frac{1}{|\Omega|} \int_{\Omega} u dx$ satisfies

$$\begin{cases} \epsilon^2 \Delta u - f(u) = \lambda_{\epsilon} & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ \int_{\Omega} u = \bar{u}|\Omega|, \end{cases} \quad (1.1)$$

where $f(u) = F'(u)$ and λ_{ϵ} is a constant.

In this paper we are concerned with solutions of (1.1) which contain spike layers. The one dimensional case was studied by Novick-Cohen and Segal [33], Novick-Cohen and Peletier [32], Bates and Fife [6], Grinfeld and Novick-Cohen [13], [14].

In [44] we constructed a boundary spike layer solution to (1.1) whose peak approaches a given nondegenerate critical point of the mean curvature of the boundary assuming that \bar{u} lies in the metastable region, i.e. $f'(\bar{u}) > 0$, for dimensions $N \geq 2$ and $\epsilon \ll 1$.

Under the same assumptions in [45] we constructed a solution to (1.1) with many boundary spike layers whose peaks are each located near different nondegenerate critical points of the mean curvature of the boundary.

In both [44] and [45] we reduce the problem to finite dimensions and use a fixed-point technique to obtain solutions. In this paper our approach is reducing the energy to finite dimensions and finding extrema for it instead. A new analysis is required. Although many of the estimates required for this analysis are the same as in [17] some major differences are needed to deal with the conservation constraint. These occur in particular in Lemma 3.6 and in Section 5.

The existence of spike layer solutions as well as their profile and the location of the peaks for the semilinear Neumann problem

$$\begin{cases} \epsilon^2 \Delta u - u + u^p = 0 & \text{in } \Omega \\ u > 0 \text{ in } \Omega, \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

for subcritical exponents p which arises as a model in various areas of applied science such as chemotaxis, pattern formation, chemical reactor theory, etc. has been studied by Lin, Ni, Pan, and Takagi [21, 26, 27, 28] and lately by Gui, Wei, and Winter [15], [43], [17], and [20]. For the critical case $p = (N + 2)/(N - 2)$ similar results have obtained for example in [1], [2], [3], [16], [36], [37], [38], [42]. The corresponding Dirichlet problem in the

subcritical case was first investigated by Ni and Wei [30]. for the Dirichlet problem. However, they do not have the conservation constraint and the nonlinearity is simpler than here.

Naturally these stationary solutions are essential for the understanding of the dynamics of the corresponding evolution process.

Other important features of the Cahn-Hilliard equation with physical relevance are spinodal decomposition and pattern formation. In this respect see the recent work of Kielhöfer [18] and Maier-Paape and Wanner [23], [24] and the references therein. The existence of stationary interface solutions has first been proved by Modica [25]. See also the works of Luckhaus and Modica [22] for the geometrical interpretation of the Lagrange multiplier λ_ϵ , Niethammer [31] for the radially symmetric case. See also Kohn and Sternberg [19], and Chen and Kowalczyk [8].

The dynamics of interface solutions has been studied extensively, see for example [39], [5], [4], [9], [10].

The attractor has been investigated for example in [14] and [41].

Henceforth, we assume that $f'(\bar{u}) > 0$.

Before stating our main result we make the following transformation.

$$v = \bar{u} - u,$$

$$g(v) = -f(\bar{u}) + f(\bar{u} - v).$$

Rewrite

$$g'(0) = -m, g(v) = -mv + h(v).$$

Then equation (1.1) becomes

$$\begin{cases} \epsilon^2 \Delta v - mv + h(v) - \frac{1}{|\Omega|} \int_{\Omega} h(v) = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.2)$$

To accommodate more general g we assume that

$$(g1) \quad g(0) = 0, g'(0) = -m < 0.$$

$$(g2) \quad g \in C^2(\mathbb{R}^+), g(v) = -mv + h(v), \text{ where } h \text{ satisfies}$$

$$h(v) = O(|v|^{p_1}), h'(v) = O(|v|^{p_2-1}) \text{ as } |v| \rightarrow \infty$$

for some $1 < p_1, p_2 < \left(\frac{N+4}{N-4}\right)_+$ ($= \infty$ if $N \leq 4$, $\frac{N+4}{N-4}$ if $N \geq 5$);
 there exists $1 < p_3 < \left(\frac{N+4}{N-4}\right)_+$ such that

$$|h'(v + \phi) - h'(v)| \leq \begin{cases} C|\phi|^{p_3-1} & \text{if } p_3 > 2, \\ C(|\phi| + |\phi|^{p_3-1}) & \text{if } p_3 \leq 2. \end{cases}$$

(g3) The equation

$$\begin{cases} \Delta V - mV + h(V) = 0 & \text{in } \mathbb{R}^N, \\ V > 0, V(0) = \max_{z \in \mathbb{R}^n} V(z), \\ V \rightarrow 0 & \text{at } \infty \end{cases} \quad (1.3)$$

has a unique solution $V(y)$ (by the results of [12], V is radial, i.e. $V = V(r)$ and $V' < 0$ for $r = |y| \neq 0$). Furthermore, V is nondegenerate, namely the operator

$$L := \Delta + g'(V) \quad (1.4)$$

is invertible in the space $H_r^2(\mathbb{R}^N) := \{u = u(|y|) \in H^2(\mathbb{R}^N)\}$.

Remark: Assuming $F(u) = (1 - u^2)^2$ (i.e. $f(u) = -4u(1 - u^2)$) and $f'(\bar{u}) > 0$ by changing F at infinity the Cahn-Hilliard equation satisfies conditions (g1) – (g3). See [44]. In [44] it is shown that without loss of generality we can assume that h and its first two derivatives are bounded continuous functions on the real line. For simplicity, we make this assumption for the rest of the paper.

Let $\Gamma \subset \partial\Omega$ be a relatively open set such that

$$\min_{P \in \partial\Gamma} \kappa(P) > \min_{P \in \Gamma} \kappa(P), \quad (1.5)$$

where $\kappa(P)$ is the mean curvature of $\partial\Omega$ at the point P .

Our main result can be stated as follows.

Theorem 1.1. *Assume that condition (1.5) holds. Let g satisfy assumptions (g1)–(g3). Then for ε sufficiently small problem (1.2) has a solution v_ε which possesses exactly K local maximum points $Q_1^\varepsilon, \dots, Q_K^\varepsilon$ with $\mathbf{Q}^\varepsilon = (Q_1^\varepsilon, \dots, Q_K^\varepsilon) \in \Gamma \times \dots \times \Gamma$.*

Moreover $\kappa(Q_i^\varepsilon) \rightarrow \min_{P \in \Gamma} \kappa(P)$, $V\left(\frac{|Q_k^\varepsilon - Q_l^\varepsilon|}{\varepsilon}\right) \rightarrow 0$, $i, k, l = 1, \dots, K$, $k \neq l$ as $\varepsilon \rightarrow 0$. Furthermore, there exists a real constant v_ε^∞ and positive constants

a, b such that $v_\epsilon(x) \rightarrow v_\epsilon^\infty$ as $\epsilon \rightarrow 0$ and

$$|v_\epsilon(x) - v_\epsilon^\infty| \leq a \exp\left(-\frac{b \min_{i=1, \dots, K}(|x - Q_i^\epsilon|)}{\epsilon}\right). \quad (1.6)$$

Theorem 1.1 can be derived from a more general theorem which is as follows.

Theorem 1.2. *Let $\Gamma_i, i = 1, \dots, K$ be relatively open sets in $\partial\Omega$ such that*

$$\min_{P \in \partial\Gamma_i} \kappa(P) > \min_{P \in \Gamma_i} \kappa(P), \quad i = 1, \dots, K.$$

Let g satisfy assumptions (g1)-(g3). Then for ϵ sufficiently small problem (1.2) has a solution v_ϵ which possesses exactly K local maximum points $Q_1^\epsilon, \dots, Q_K^\epsilon$ with $\mathbf{Q}^\epsilon = (Q_1^\epsilon, \dots, Q_K^\epsilon) \in \Gamma_1 \times \dots \times \Gamma_K$. Moreover $\kappa(Q_i^\epsilon) \rightarrow \min_{P \in \Gamma_i} \kappa(P), V(\frac{|Q_k^\epsilon - Q_l^\epsilon|}{\epsilon}) \rightarrow 0, i, k, l = 1, \dots, K, k \neq l$ as $\epsilon \rightarrow 0$. Furthermore, there exists a real constant v_ϵ^∞ and positive constants a, b such that $v_\epsilon(x) \rightarrow v_\epsilon^\infty$ as $\epsilon \rightarrow 0$ and

$$|v_\epsilon(x) - v_\epsilon^\infty| \leq a \exp\left(-\frac{b \min_{i=1, \dots, K}(|x - Q_i^\epsilon|)}{\epsilon}\right). \quad (1.7)$$

More details about the asymptotic behaviour of v_ϵ can be found in the proof of Theorem 1.2.

We have the following interesting corollary.

Corollary 1.3. *Let g satisfy assumptions (g1)-(g3). Then for any smooth and bounded domain and any fixed positive integer $K \in \mathbb{Z}$, there always exists a boundary K -peak solution of (1.1) if ϵ is small enough.*

Theorem 1.1 is the first result about the existence of boundary K -spike solutions for problem (1.2) for any positive integer K in any smooth bounded domain. Note that for a strict local minimum point of $\kappa(P)$ (i.e. there exists a relatively open set $\Gamma \subset \partial\Omega$ with $P \in \Gamma$ such that $\kappa(Q) > \kappa(P)$ for all $Q \in \Gamma$) the boundary K -spike solutions can be chosen such that their peaks approach the same point on the boundary. Intuitively speaking, the boundary spikes attract one another. This is in balance with “forces” coming from the curvature of the boundary which prevent the spikes from moving closer towards one another and towards the strict local minimum point of $\kappa(P)$.

It seems that this new phenomenon cannot occur at a local maximum point of $\kappa(P)$.

In this paper we study local minimum or maximum points of the mean curvature of the boundary *without* assuming their nondegeneracy. Instead we only need the global condition (1.5) which can be genuinely weaker in many cases. To our knowledge, this was not possible in all previous works.

Theorem 1.2 is the main result in this paper. To introduce the most important idea of the proof of Theorem 1.2, we need to give some notations and definitions first.

For our approach it is essential to note that v is a solution of (1.2) if and only if v is a critical point of

$$J_\epsilon(v) = \frac{\epsilon^2}{2} \int_\Omega |\nabla v|^2 + \frac{m}{2} \int_\Omega v^2 - \int_\Omega H(v),$$

where

$$H(v) = \int_0^v h(s) ds, \quad v \in X = \{v \in H^1(\Omega) \mid \int_\Omega v = 0\}.$$

Note that the conservation constraint

$$\int_\Omega v = 0 \tag{1.8}$$

contributes the Lagrange multiplier λ_ϵ in (1.1). Recall on the other hand that for solutions of (1.2) equation (1.8) does not have to be assumed a priori but follows automatically for all solutions in $\{v \in H^2(\Omega) : \frac{\partial v}{\partial \nu} = 0 \text{ at } \partial\Omega\}$.

We start our construction by finding good approximating functions for the solutions. Our approach is by using a projection technique to obtain appropriate functions in the space X . Let V be the unique solution of (1.3). It is known (see [12]) that V is radially symmetric, decreasing and

$$\lim_{|y| \rightarrow \infty} V(y) e^{\sqrt{m}|y|} |y|^{\frac{N-1}{2}} = c_0 > 0.$$

Let $P \in \bar{\Omega}$, $\Omega_{\epsilon, P} := \{y \mid \epsilon y + P \in \Omega\}$ and $\Omega_\epsilon := \{y \mid \epsilon y \in \Omega\}$.

For any smooth domain $U \subset \mathbb{R}^N$ we define a function $u = P_U V$ as the unique solution of

$$\begin{cases} \Delta u - mu + h(V) = 0 \text{ in } U, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial U. \end{cases}$$

Let $\eta > 0$ be a small number. Let Γ_i be as in Theorem 1.2. Set $\Lambda = \{\mathbf{P} = (P_1, \dots, P_K) \in \Gamma_1 \times \dots \times \Gamma_K, V(\frac{|P_k - P_l|}{\epsilon}) < \eta\epsilon, k, l = 1, \dots, K, k \neq l\}$.

Fix $\mathbf{P} = (P_1, P_2, \dots, P_K) \in \bar{\Lambda}$. We set

$$V_i(y) = V(y - \frac{P_i}{\epsilon}), \quad PV_i(y) = P_{\Omega_\epsilon, P_i} V(y - \frac{P_i}{\epsilon}), \quad y \in \Omega_\epsilon,$$

$$P_0 V_i(y) = PV_i(y) - \frac{1}{|\Omega_\epsilon|} \int_{\Omega_\epsilon} PV_i(y) dy,$$

$$w_{\epsilon, \mathbf{P}} = \sum_{i=1}^K P_0 V_i,$$

$$v_\epsilon = w_{\epsilon, \mathbf{P}} + \Phi_{\epsilon, \mathbf{P}} \in H^2(\Omega_\epsilon),$$

where

$$\Phi_{\epsilon, \mathbf{P}} \in \{\Phi \in H^2(\Omega_\epsilon) : \frac{\partial \Phi}{\partial \nu} = 0 \text{ on } \partial \Omega_\epsilon, \int_{\Omega_\epsilon} \Phi dy = 0\}$$

is still unknown. Finally, we introduce

$$\mathcal{K}_{\epsilon, \mathbf{P}} = \mathcal{C}_{\epsilon, \mathbf{P}} = \text{span}\left\{\frac{\partial P_0 V_i}{\partial \tau_{P_i, i_j}}, i = 1, \dots, K, j = 1, \dots, N-1\right\}$$

to denote the approximate kernel and cokernel of the operator obtained from linearizing (1.2) at $w_{\epsilon, \mathbf{P}}$, respectively, where τ_{P_i, i_j} are the $(N-1)$ tangential derivatives at P_i (without loss of generality we may assume that the inward normal vector at P_i is e_N). We denote τ_{P_i, i_j} as $\tau_{P_i, j}$ in the rest of the paper.

We first solve for $\Phi_{\epsilon, \mathbf{P}}$ such that

$$v_\epsilon \in \mathcal{K}_{\epsilon, \mathbf{P}}^\perp,$$

$$\Delta v_\epsilon - m v_\epsilon + h(v_\epsilon) - \frac{1}{|\Omega_\epsilon|} \int_{\Omega_\epsilon} h(v_\epsilon) dy \in \mathcal{C}_{\epsilon, \mathbf{P}}$$

using the Liapunov-Schmidt reduction method. This method evolves from that of [11], [34], and [35] on the semi-classical (i.e. for small parameter \bar{h}) solution of the nonlinear Schrödinger equation

$$\frac{\bar{h}^2}{2} \Delta U - (V - E)U + U^p = 0 \tag{1.9}$$

in R^N , where V is a potential function and E is a real constant. The method of Liapunov-Schmidt reduction was used in [11], [34] and [35] to construct solutions of (1.9) close to nondegenerate critical points of V for \bar{h} sufficiently small.

Then we show that $\Phi_{\epsilon, \mathbf{P}}$ is C^1 in \mathbf{P} . Now we have developed all the tools to introduce the novel function

$$M_\epsilon(\mathbf{P}) = J_\epsilon\left(\sum_{i=1}^K P_0 V_i + \Phi_{\epsilon, \mathbf{P}}\right). \quad (1.10)$$

That means we have reduced the energy J_ϵ to finite dimensions, where the variables are closely related to the location of the peaks. A large part of the paper is devoted to deriving an explicit expression for $M_\epsilon(\mathbf{P})$.

We maximize $M_\epsilon(\mathbf{P})$ over $\bar{\Lambda}$. Condition (1.5) ensures that $M_\epsilon(\mathbf{P})$ attains its maximum inside Λ . We show that the resulting solution has the properties of Theorem 1.2.

Throughout this paper, unless otherwise stated, the letter C will always denote various generic constants which are independent of ϵ , for ϵ sufficiently small; $\delta > 0$ is a very small number; $o(1)$ means $|o(1)| \rightarrow 0$ as $\epsilon \rightarrow 0$.

The paper is organized as follows. Notation, preliminaries and some useful estimates are explained in Section 2. Section 3 contains the setup of our problem and we solve (1.2) up to approximate kernel and cokernel, respectively. We introduce and solve a finite-dimensional optimization problem in Section 4. Finally, in Section 5, we show that the solution to the maximizing problem is indeed a solution of (1.2) and satisfies all the properties of Theorem 1.2.

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2. TECHNICAL ANALYSIS

In this section we introduce a projection and derive some useful estimates. Finally we will prove some lemmas which will be important in deriving an explicit expression for $M_\epsilon(\mathbf{P})$ as defined in (1.10). Propositions 2.1 and 2.2

as well as Lemma 2.3 are from [44] and are presented here for the convenience of the reader.

Throughout the paper we shall use the letter C to denote a generic positive constant which may vary from term to term. We denote $R_+^N = \{(x', x_N) | x_N > 0\}$, where $x' = (x_1, \dots, x_{N-1})$. Let V be the unique solution of (1.3).

Set

$$I(V) = \frac{1}{2} \int_{R^N} (|\nabla V|^2 + mV^2) - \int_{R^N} H(V).$$

Let $P \in \partial\Omega$. Then, since $\partial\Omega$ is smooth, there exists $R_0 > 0$ such that for $|x - P| < R_0$, $\partial\Omega$ can be represented by the graph of a smooth function $\rho(x - P)$, where $\rho(0) = 0, \nabla\rho(0) = 0$. The mean curvature of $\partial\Omega$ at P is $\kappa(P) = \frac{1}{N-1} \sum_{i=1}^{N-1} \rho_{ii}(0)$, where

$$\rho_i = \frac{\partial\rho}{\partial x_i}, \quad i = 1, \dots, N-1.$$

Here we use ρ_α to denote the multiple differentiation $\frac{\partial^{|\alpha|}\rho}{\partial x^\alpha}$ for $\alpha = (\alpha_1, \dots, \alpha_{N-1})$, where $\alpha_i \in \{0, 1, \dots\}$ for $i = 1, \dots, N-1$ and $|\alpha| = \sum_{i=1}^{N-1} \alpha_i$. We denote

$$\|v\|_\epsilon^2 = \epsilon^{-N} \int_\Omega [\epsilon^2 |\nabla v|^2 + mv^2].$$

For $x \in \Omega_0$ set now

$$\begin{cases} \epsilon y_i = x_i - P_i, & i = 1, \dots, N-1, \\ \epsilon y_N = x_N - P_N - \rho(x_1 - P_1, \dots, x_{N-1} - P_{N-1}). \end{cases} \quad (2.1)$$

Furthermore, for $x \in \Omega_0$ we introduce the transformation

$$\begin{cases} T_i(x) = x_i, & i = 1, \dots, N-1, \\ T_N(x) = x_N - P_N - \rho(x_1 - P_1, \dots, x_{N-1} - P_{N-1}). \end{cases} \quad (2.2)$$

Note that then

$$y = \frac{1}{\epsilon} T(x).$$

Then we have

Proposition 2.1. *Let $\chi(x)$ be a smooth cutoff function such that $\chi(x) = 1, x \in B(P, R_0 - \delta)$ and $\chi(x) = 0$ for $x \in B(P, R_0)^C$ (for a positive and sufficiently small number δ .) Then*

$$\begin{aligned} & \left[V - P_{\Omega_{\epsilon,P}} V \right] \left(\frac{x - P}{\epsilon} \right) \\ &= \epsilon v_1(y) \chi(x - P) + \epsilon^2 (v_2(y) \chi(x - P) + v_3(y) \chi(x - P)) + \epsilon^3 \Psi_{\epsilon,P}(x), \end{aligned}$$

where v_1 is the unique solution of

$$\begin{cases} \Delta v - mv = 0 & \text{in } R_+^N, \\ \frac{\partial v}{\partial y_N} = -\frac{V'}{|y|} \frac{1}{2} \sum_{i,j=1}^{N-1} \rho_{ij}(0) y_i y_j & \text{on } \partial R_+^N, \end{cases} \quad (2.3)$$

V' is the radial derivative of V , i.e. $V' = V_r(r)$, $r = \left| \frac{x-P}{\epsilon} \right|$; v_2 is the unique solution of

$$\begin{cases} \Delta v - mv - 2 \sum_{i,j=1}^{N-1} \rho_{ij}(0) y_i \frac{\partial^2 v_1}{\partial y_j \partial y_N} = 0 & \text{in } R_+^N, \\ \frac{\partial v}{\partial y_N} = \sum_{i,j=1}^{N-1} \rho_{ij}(0) y_i \frac{\partial v_1}{\partial y_j} & \text{on } \partial R_+^N; \end{cases} \quad (2.4)$$

v_3 is the unique solution of

$$\begin{cases} \Delta v - mv = 0 & \text{in } R_+^N, \\ \frac{\partial v}{\partial y_N} = -\frac{V'}{|y|} \frac{1}{3} \sum_{i,j,k=1}^{N-1} \rho_{ijk}(0) y_i y_j y_k & \text{on } \partial R_+^N \end{cases} \quad (2.5)$$

and

$$\|\Psi_{\epsilon,P}\|_{\epsilon} \leq C.$$

Proof. A proof can be found in [44]. \square

Note that v_1, v_2 are even functions in $y' = (y_1, \dots, y_{N-1})$ and v_3 is an odd function in $y' = (y_1, \dots, y_{N-1})$ (i.e. $v_1(y', y_N) = v_1(-y', y_N)$, $v_3(y', y_N) = -v_3(-y', y_N)$). Moreover, it is easy to see that $|v_1|, |v_2|, |v_3| \leq C e^{-\mu|y|}$ for some $0 < \mu < \sqrt{m}$.

We next analyze $\partial/\partial\tau_{P_j} P_{\Omega_{\epsilon,P}} V \left(\frac{x-P}{\epsilon} \right)$ for sufficiently small x . Because we choose the coordinate system as explained on page 7, we have $\partial/\partial\tau_{P_j} = \partial/\partial P_j$.

Proposition 2.2.

$$\left[\frac{\partial V}{\partial \tau_{P_j}} - \frac{\partial P_{\Omega_{\epsilon,P}} V}{\partial \tau_{P_j}} \right] \left(\frac{x-P}{\epsilon} \right) = w_1(y) \chi(x-P) + \epsilon w_2^{\epsilon}(x),$$

where w_1 is the unique solution of

$$\begin{cases} \Delta v - mv = 0 & \text{in } R_+^N, \\ \frac{\partial v}{\partial y_N} = -\frac{1}{2} \left(\frac{V''}{|y|^2} - \frac{V'}{|y|^3} \right) \sum_{k,l=1}^{N-1} \rho_{kl}(0) y_k y_l y_j - \frac{V'}{|y|} \sum_{k=1}^{N-1} \rho_{jk}(0) y_k & \text{on } \partial R_+^N. \end{cases} \quad (2.6)$$

and

$$\|w_2^{\epsilon}\|_{\epsilon} \leq C.$$

Proof. A proof can be found in [44]. \square

Note that $|w_1| \leq C \exp(-\mu|y|)$ and $|w_2| \leq C \exp(-\mu|y|)$ for some $\mu < \sqrt{m}$ and w_1 is an odd function in y' .

Finally, let

$$L_0 = \Delta - m + h'(V).$$

We have

Lemma 2.3.

$$\text{Ker}(L_0) \cap H_N^2(R_+^N) = \text{span} \left\{ \frac{\partial V}{\partial y_1}, \dots, \frac{\partial V}{\partial y_{N-1}} \right\},$$

where $H_N^2(R_+^N) = \{u \in H^2(R_+^N), \frac{\partial u}{\partial y_N} = 0 \text{ on } \partial R_+^N\}$.

Proof. See Lemma 4.2 in [28]. \square

The next lemma is the key result in this section. Its proof is similar but differs at a crucial points from the one in [17]. We indicate this difference.

Lemma 2.4. For any $\mathbf{P} = (P_1, \dots, P_K) \in \bar{\Lambda}$ and ϵ sufficiently small

$$\begin{aligned} J_\epsilon \left(\sum_{i=1}^K P_0 V_i \right) &= \epsilon^N \left[\frac{K}{2} I(V) - \epsilon(\beta_1 + o(1)) \sum_{i=1}^K \kappa(P_i) \right. \\ &\quad \left. - \frac{1}{2} \sum_{k,l=1, k \neq l}^K (\gamma_{kl} + o(1)) V \left(\frac{|P_k - P_l|}{\epsilon} \right) + o(\epsilon) \right], \end{aligned} \quad (2.7)$$

where

$$\beta_1 = \frac{1}{N+1} \int_{R^{N-1}} |\nabla V|^2 |y'|^2 dy'$$

and $\gamma_{kl} = \gamma_{lk} \in \Sigma$ for

$$\Sigma = \left\{ \int_{R_+^N} h(V(y)) e^{\sqrt{m} \langle b, y \rangle} dy \mid b \in R^N, |b| = 1 \right\}.$$

Furthermore, if $V \left(\frac{|P_k - P_l|}{\epsilon} \right) = \eta \epsilon$, we have $\gamma_{kl} \in \Sigma_1$, where

$$\Sigma_1 = \left\{ \int_{R_+^N} h(V(y)) e^{\sqrt{m} \langle b, y \rangle} dy \mid b = (b_1, \dots, b_N) \in R^N, b_N = 0, |b| = 1 \right\}.$$

Proof. In [17] we calculated $J_\epsilon(\sum_{i=1}^K P V_i)$. Now we need $J_\epsilon(\sum_{i=1}^K P_0 V_i)$.

Note that

$$J_\epsilon(P_0 V) - J_\epsilon(P V) = \epsilon^N \int_{\Omega_\epsilon} \frac{m}{2} \left(|P_0 V|^2 - |P V|^2 \right) - (H(P_0 V) - H(P V))$$

$$= O(\epsilon^{2N})$$

and

$$\begin{aligned} J_\epsilon\left(\sum_{i=1}^2 P_0 V_i\right) - J\left(\sum_{i=1}^2 P V_i\right) &= \epsilon^N \int_{\Omega_{\epsilon,P}} \left[\frac{m}{2} \left(\left| \sum_{i=1}^2 P_0 V_i \right|^2 - \left| \sum_{i=1}^2 P V_i \right|^2 \right) \right. \\ &\quad \left. - \left(H\left(\sum_{i=1}^2 P_0 V_i\right) - H\left(\sum_{i=1}^2 P V_i\right) \right) \right] = O(\epsilon^{2N}). \end{aligned}$$

Using Lemma 2.8 of [17] the proof is completed. \square

3. LIAPUNOV-SCHMIDT REDUCTION

In this section, we reduce problem (1.2) to finite dimensions by the Liapunov-Schmidt method. We first introduce some notation.

$$\begin{aligned} X &= \{v \in H^2(\Omega_\epsilon) \mid \int_{\Omega_\epsilon} v = 0, \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega_\epsilon\}, \\ Y &= \{v \in L^2(\Omega_\epsilon) \mid \int_{\Omega_\epsilon} v = 0.\} \end{aligned}$$

Define

$$S_\epsilon(v) = \Delta v - mv + h(v) - \frac{1}{|\Omega_\epsilon|} \int_{\Omega_\epsilon} h(v),$$

for $v \in X$. Then solving equation (1.1) is equivalent to

$$S_\epsilon(v) = 0, v \in X.$$

Fix $\mathbf{P} = (P_1, \dots, P_K) \in \bar{\Lambda}$. To study (1.2) we first consider the linearized operator

$$\begin{aligned} L_\epsilon : u &\mapsto \Delta u - mu + h'(w_{\epsilon,\mathbf{P}})u - \frac{1}{|\Omega_\epsilon|} \int_{\Omega_\epsilon} h'(w_{\epsilon,\mathbf{P}})u, \\ X &\rightarrow Y. \end{aligned}$$

Recall that $w_{\epsilon,\mathbf{P}} = \sum_{i=1}^K P_0 V_i$. Choose approximate cokernel and kernel as

$$\begin{aligned} \mathcal{C}_{\epsilon,\mathbf{P}} &= \mathcal{K}_{\epsilon,\mathbf{P}} \\ &= \text{span} \left\{ \frac{\partial P_0 V_i}{\partial \tau_{P_{i,j}}} \mid i = 1, \dots, K, j = 1, \dots, N-1 \right\}, \end{aligned}$$

where (as in the introduction)

$$\mathcal{K}_{\epsilon,\mathbf{P}} \subset X$$

$$\text{and } \mathcal{C}_{\epsilon,\mathbf{P}} \subset Y.$$

Let $\pi_{\epsilon, \mathbf{P}}$ denote the projection from Y onto $\mathcal{C}_{\epsilon, \mathbf{P}}^\perp$. Our goal in this section is to show that the equation

$$\pi_{\epsilon, \mathbf{P}} \circ S_\epsilon(w_{\epsilon, \mathbf{P}} + \Phi_{\epsilon, \mathbf{P}}) = 0$$

has a unique solution $\Phi_{\epsilon, \mathbf{P}} \in \mathcal{K}_{\epsilon, \mathbf{P}}^\perp$ if ϵ is small enough and $\mathbf{P} = (P_1, \dots, P_K) \in \bar{\Lambda}$.

As a preparation in the following two propositions we show the invertibility of the corresponding linearized operator.

Proposition 3.1. *Let $L_{\epsilon, \mathbf{P}} = \pi_{\epsilon, \mathbf{P}} \circ L_\epsilon$. There exist positive constants $\bar{\epsilon}, \bar{\lambda}$ such that for all $\epsilon \in (0, \bar{\epsilon})$ and $\mathbf{P} = (P_1, \dots, P_K) \in \bar{\Lambda}$*

$$\|L_{\epsilon, \mathbf{P}} \Phi\|_{L^2(\Omega_\epsilon)} \geq \bar{\lambda} \|\Phi\|_{H^2(\Omega_\epsilon)} \quad (3.1)$$

for all $\Phi \in \mathcal{K}_{\epsilon, \mathbf{P}}^\perp$.

Proposition 3.2. *There exists a positive constant $\tilde{\epsilon}$ such that for all $\epsilon \in (0, \tilde{\epsilon})$ and $\mathbf{P} = (P_1, \dots, P_K) \in \bar{\Lambda}$ the map*

$$L_{\epsilon, \mathbf{P}} = \pi_{\epsilon, \mathbf{P}} \circ L_\epsilon : \mathcal{K}_{\epsilon, \mathbf{P}}^\perp \rightarrow \mathcal{C}_{\epsilon, \mathbf{P}}^\perp$$

is surjective.

Proof of Propositions 3.1 and 3.2. We refer to [45] for proofs. \square

We are now in a position to solve the equation

$$\pi_{\epsilon, \mathbf{P}} \circ S_\epsilon(w_{\epsilon, \mathbf{P}} + \Phi_{\epsilon, \mathbf{P}}) = 0. \quad (3.2)$$

Since $L_{\epsilon, \mathbf{P}}|_{\mathcal{K}_{\epsilon, \mathbf{P}}^\perp}$ is invertible (call the inverse $L_{\epsilon, \mathbf{P}}^{-1}$) we can rewrite

$$\begin{aligned} \Phi &= -L_{\epsilon, \mathbf{P}}^{-1} \circ \pi_{\epsilon, \mathbf{P}} \circ S_\epsilon(w_{\epsilon, \mathbf{P}}) \\ &\quad - L_{\epsilon, \mathbf{P}}^{-1} \circ \pi_{\epsilon, \mathbf{P}} \circ N_{\epsilon, \mathbf{P}}(\Phi) \\ &\quad + L_{\epsilon, \mathbf{P}}^{-1} \circ \pi_{\epsilon, \mathbf{P}} \circ \bar{H}_{\epsilon, \mathbf{P}}(\Phi) \\ &\equiv G_{\epsilon, \mathbf{P}}(\Phi), \end{aligned} \quad (3.3)$$

where

$$\begin{aligned} N_{\epsilon, \mathbf{P}}(\Phi) &= S_\epsilon(w_{\epsilon, \mathbf{P}} + \Phi) \\ &\quad - [S_\epsilon(w_{\epsilon, \mathbf{P}}) + S'_\epsilon(w_{\epsilon, \mathbf{P}})\Phi], \\ \bar{H}_{\epsilon, \mathbf{P}}(\Phi) &= \frac{1}{|\Omega_\epsilon|} \int_{\Omega_\epsilon} h'(w_{\epsilon, \mathbf{P}})\Phi, \end{aligned}$$

and the operator $G_{\epsilon, \mathbf{P}}$ is defined by (3.3) for $\Phi \in H_N^2(\Omega_\epsilon)$. We are going to show that the operator $G_{\epsilon, \mathbf{P}}$ is a contraction on

$$B_{\epsilon, \delta} \equiv \{\Phi \in H^2(\Omega_\epsilon) \mid \int_{\Omega_\epsilon} \Phi = 0, \|\Phi\|_{H^2(\Omega_\epsilon)} < \delta\}$$

if δ is small enough.

In fact we have the following lemma

Lemma 3.3. *For ϵ sufficiently small, we have*

$$\|N_{\epsilon, \mathbf{P}}(\Phi)\|_{L^2(\Omega_\epsilon)} \leq C\|\Phi_{\epsilon, \mathbf{P}}\|_{L^2(\Omega_\epsilon)}, \quad (3.4)$$

$$\|S_\epsilon(w_{\epsilon, \mathbf{P}})\|_{L^2(\Omega_\epsilon)} \leq C\epsilon, \quad (3.5)$$

$$|\overline{H}_{\epsilon, \mathbf{P}}(\Phi)| \leq C\epsilon^N\|\Phi\|_{L^2(\Omega_\epsilon)}. \quad (3.6)$$

Proof. (3.4) follows from the mean value theorem since $h \in C^2(\mathbb{R})$ and h, h', h'' are bounded real functions.

(3.6) follows since

$$|\overline{H}_{\epsilon, \mathbf{P}}(\Phi)| \leq C \frac{1}{|\Omega_\epsilon|} \|\Phi\|_{L^2(\Omega_\epsilon)} \|h'(w_{\epsilon, \mathbf{P}})\|_{L^2(\Omega_\epsilon)} \leq C\epsilon^N\|\Phi\|_{L^2(\Omega_\epsilon)}.$$

The proof of (3.5) is the same as in [17]. □

Thus

$$\begin{aligned} \|G_{\epsilon, \mathbf{P}}(\Phi)\|_{H^2(\Omega_\epsilon)} &\leq \lambda^{-1}(\|\pi_{\epsilon, \mathbf{P}} \circ N_{\epsilon, \mathbf{P}}(\Phi)\|_{L^2(\Omega_\epsilon)} \\ &+ \|\pi_{\epsilon, \mathbf{P}} \circ S_\epsilon(w_{\epsilon, \mathbf{P}})\|_{L^2(\Omega_\epsilon)} + \|\pi_{\epsilon, \mathbf{P}} \circ \overline{H}_{\epsilon, \mathbf{P}}(\Phi)\|_{L^2(\Omega_\epsilon)}) \\ &\leq \lambda^{-1}C(c(\delta)\delta + \epsilon), \end{aligned}$$

where $\lambda > 0$ is independent of $\delta > 0$ and $c(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. Similarly we show

$$\|G_{\epsilon, \mathbf{P}}(\Phi) - G_{\epsilon, \mathbf{P}}(\Phi')\|_{H^2(\Omega_\epsilon)} \leq \lambda^{-1}C(c(\delta) + O(\epsilon^N))\|\Phi - \Phi'\|_{H^2(\Omega_\epsilon)}$$

if δ, ϵ are small enough and where $c(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. Therefore $G_{\epsilon, \mathbf{P}}$ is a contraction on B_δ . The existence of a fixed point $\Phi_{\epsilon, \mathbf{P}}$ now follows from the Contraction Mapping Principle and $\Phi_{\epsilon, \mathbf{P}}$ is a solution of (3.3).

Because of

$$\begin{aligned} \|\Phi_{\epsilon, \mathbf{P}}\|_{H^2(\Omega_\epsilon)} &\leq \lambda^{-1}(\|N_{\epsilon, \mathbf{P}}(\Phi_{\epsilon, \mathbf{P}})\|_{L^2(\Omega_\epsilon)} \\ &+ \|S_\epsilon(w_{\epsilon, \mathbf{P}})\|_{L^2(\Omega_\epsilon)} + \|\overline{H}_{\epsilon, \mathbf{P}}(\Phi_{\epsilon, \mathbf{P}})\|_{L^2(\Omega_\epsilon)}) \\ &\leq \lambda^{-1}C(\epsilon^1 + c(\delta))\|\Phi_{\epsilon, \mathbf{P}}\|_{H^2(\Omega_\epsilon)} \end{aligned}$$

we have

$$\|\Phi_{\epsilon, \mathbf{P}}\|_{H^2(\Omega_\epsilon)} \leq C\epsilon.$$

We have proved

Lemma 3.4. *There exists $\bar{\epsilon} > 0$ such that for every $(N+1)$ -tuple $\epsilon, P_1, \dots, P_K$ with $0 < \epsilon < \bar{\epsilon}$ and $\mathbf{P} = (P_1, \dots, P_K) \in \bar{\Lambda}$ there is a unique $\Phi_{\epsilon, \mathbf{P}} \in \mathcal{K}_{\epsilon, \mathbf{P}}^\perp$ satisfying $S_\epsilon(w_{\epsilon, \mathbf{P}} + \Phi_{\epsilon, \mathbf{P}}) \in \mathcal{C}_{\epsilon, \mathbf{P}}$ and*

$$\|\Phi_{\epsilon, \mathbf{P}}\|_{H^2(\Omega_\epsilon)} \leq C\epsilon. \quad (3.7)$$

The next lemma is our main estimate.

Lemma 3.5. *Let $\Phi_{\epsilon, \mathbf{P}}$ be defined by Lemma 3.4. Then we have*

$$\begin{aligned} & J_\epsilon(w_{\epsilon, \mathbf{P}} + \Phi_{\epsilon, \mathbf{P}}) \\ &= \epsilon^N \left[\frac{K}{2} I(V) - \beta_1 \epsilon \sum_{i=1}^K \kappa(P_i) \right. \\ & \quad \left. - \frac{1}{2} \sum_{k, l=1, \dots, K, k \neq l} (\gamma_{kl} + o(1)) V\left(\frac{|P_k - P_l|}{\epsilon}\right) + o(\epsilon) \right], \end{aligned} \quad (3.8)$$

where β_1 and γ_{kl} are introduced in Lemma 2.4 and Lemma 2.5, respectively.

Proof.

In fact, for any $\mathbf{P} \in \bar{\Lambda}$, we have

$$\epsilon^{-N} J_\epsilon(w_{\epsilon, \mathbf{P}} + \Phi_{\epsilon, \mathbf{P}}) = \epsilon^{-N} J_\epsilon(w_{\epsilon, \mathbf{P}}) + g_{\epsilon, \mathbf{P}}(\Phi_{\epsilon, \mathbf{P}}) + O(\|\Phi_{\epsilon, \mathbf{P}}\|_{H^2(\Omega_\epsilon)}^2),$$

where

$$\begin{aligned} & g_{\epsilon, \mathbf{P}}(\Phi_{\epsilon, \mathbf{P}}) \\ &= \int_{\Omega_\epsilon} \sum_{i=1}^K (\nabla P_0 V_i \nabla \Phi_{\epsilon, \mathbf{P}} + m P_0 V_i \Phi_{\epsilon, \mathbf{P}}) - \int_{\Omega_\epsilon} h\left(\sum_{i=1}^K P_0 V_i\right) \Phi_{\epsilon, \mathbf{P}} \\ &= \int_{\Omega_\epsilon} \left[\sum_{i=1}^K h(V_i) - h(w_{\epsilon, \mathbf{P}}) \right] \Phi_{\epsilon, \mathbf{P}} + O(\epsilon^{N+1}) \\ &\leq \left\| \sum_{i=1}^K h(V_i) - h(w_{\epsilon, \mathbf{P}}) \right\|_{L^2(\Omega_\epsilon)} \|\Phi_{\epsilon, \mathbf{P}}\|_{L^2(\Omega_\epsilon)} \\ &\leq O(\epsilon^2) \end{aligned}$$

for $N \geq 2$ by Lemma 3.3 and Lemma 3.4.

Estimate (3.8) now follows from Lemma 2.4 and Lemma 3.4. \square

Finally, we show that $\Phi_{\epsilon, \mathbf{P}}$ is actually smooth in \mathbf{P} .

Lemma 3.6. *Let $\Phi_{\epsilon, \mathbf{P}}$ be defined by Lemma 3.4. Then $\Phi_{\epsilon, \mathbf{P}} \in C^1$ in \mathbf{P} .*

Proof. Recall that $\Phi_{\epsilon, \mathbf{P}}$ is a solution of the equation

$$\pi_{\epsilon, \mathbf{P}} \circ S_{\epsilon}(w_{\epsilon, \mathbf{P}} + \Phi_{\epsilon, \mathbf{P}}) = 0 \quad (3.9)$$

such that

$$\Phi_{\epsilon, \mathbf{P}} \in \mathcal{K}_{\epsilon, \mathbf{P}}^{\perp}. \quad (3.10)$$

By definition we easily conclude that the functions $\frac{\partial P_0 V_i}{\partial P_{i,j}}$ and $\frac{\partial^2 P_0 V_i}{\partial \tau_{P_{i,j}} \partial \tau_{P_{i,k}}}$ are C^1 in \mathbf{P} . This implies that the projection $\pi_{\epsilon, \mathbf{P}}$ is C^1 in \mathbf{P} . Applying $\partial/\partial \tau_{P_{i,j}}$ gives

$$\begin{aligned} \pi_{\epsilon, \mathbf{P}} \circ DS_{\epsilon}(w_{\epsilon, \mathbf{P}} + \Phi_{\epsilon, \mathbf{P}}) \left(\sum_{i=1}^K \frac{\partial P_0 V_i}{\partial \tau_{P_{i,j}}} + \frac{\partial \Phi_{\epsilon, \mathbf{P}}}{\partial \tau_{P_{i,j}}} \right) \\ + \frac{\partial \pi_{\epsilon, \mathbf{P}}}{\partial \tau_{P_{i,j}}} \circ S_{\epsilon}(w_{\epsilon, \mathbf{P}} + \Phi_{\epsilon, \mathbf{P}}) = 0, \end{aligned} \quad (3.11)$$

where

$$DS_{\epsilon}(w_{\epsilon, \mathbf{P}} + \Phi_{\epsilon, \mathbf{P}}) = \Delta - m + h'(w_{\epsilon, \mathbf{P}} + \Phi_{\epsilon, \mathbf{P}}) - \frac{1}{|\Omega_{\epsilon}|} \int_{\Omega_{\epsilon}} h'(w_{\epsilon, \mathbf{P}} + \Phi_{\epsilon, \mathbf{P}}).$$

We decompose $\frac{\partial \Phi_{\epsilon, \mathbf{P}}}{\partial \tau_{P_{i,j}}}$ into two parts:

$$\frac{\partial \Phi_{\epsilon, \mathbf{P}}}{\partial \tau_{P_{i,j}}} = \left(\frac{\partial \Phi_{\epsilon, \mathbf{P}}}{\partial \tau_{P_{i,j}}} \right)_1 + \left(\frac{\partial \Phi_{\epsilon, \mathbf{P}}}{\partial \tau_{P_{i,j}}} \right)_2,$$

where $\left(\frac{\partial \Phi_{\epsilon, \mathbf{P}}}{\partial \tau_{P_{i,j}}} \right)_1 \in \mathcal{K}_{\epsilon, \mathbf{P}}$ and $\left(\frac{\partial \Phi_{\epsilon, \mathbf{P}}}{\partial \tau_{P_{i,j}}} \right)_2 \in \mathcal{K}_{\epsilon, \mathbf{P}}^{\perp}$. We can easily conclude that

$\left(\frac{\partial \Phi_{\epsilon, \mathbf{P}}}{\partial \tau_{P_{i,j}}} \right)_1$ is continuous in \mathbf{P} since

$$\int_{\Omega_{\epsilon}} \Phi_{\epsilon, \mathbf{P}} \frac{\partial P_0 V_k}{\partial \tau_{P_{k,l}}} = 0, \quad k = 1, \dots, K, \quad l = 1, \dots, N-1$$

and

$$\begin{aligned} \int_{\Omega_{\epsilon}} \frac{\partial \Phi_{\epsilon, \mathbf{P}}}{\partial \tau_{P_{i,j}}} \frac{\partial P_0 V_k}{\partial \tau_{P_{k,l}}} + \int_{\Omega_{\epsilon}} \Phi_{\epsilon, \mathbf{P}} \frac{\partial^2 P_0 V_k}{\partial \tau_{P_{i,j}} \partial \tau_{P_{k,l}}} = 0 \\ k, i = 1, \dots, K, \quad l, j = 1, \dots, N-1. \end{aligned}$$

We can rewrite equation (3.11) as

$$\begin{aligned} & \pi_{\epsilon, \mathbf{P}} \circ DS_{\epsilon}(w_{\epsilon, \mathbf{P}} + \Phi_{\epsilon, \mathbf{P}}) \left(\left(\frac{\partial \Phi_{\epsilon, \mathbf{P}}}{\partial \tau_{P_{i,j}}} \right)_2 \right) \\ & + \pi_{\epsilon, \mathbf{P}} \circ DS_{\epsilon}(w_{\epsilon, \mathbf{P}} + \Phi_{\epsilon, \mathbf{P}}) \left(\sum_{i=1}^K \frac{\partial P_0 V_i}{\partial \tau_{P_{i,j}}} + \left(\frac{\partial \Phi_{\epsilon, \mathbf{P}}}{\partial \tau_{P_{i,j}}} \right)_1 \right) \\ & + \frac{\partial \pi_{\epsilon, \mathbf{P}}}{\partial \tau_{P_{i,j}}} \circ S_{\epsilon}(w_{\epsilon, \mathbf{P}} + \Phi_{\epsilon, \mathbf{P}}) = 0. \end{aligned} \quad (3.12)$$

As in the proof of Propositions 3.1 and 3.2, we can show that the operator

$$\pi_{\epsilon, \mathbf{P}} \circ DS_{\epsilon}(w_{\epsilon, \mathbf{P}} + \Phi_{\epsilon, \mathbf{P}})$$

is invertible from $\mathcal{K}_{\epsilon, \mathbf{P}}^{\perp}$ to $\mathcal{C}_{\epsilon, \mathbf{P}}^{\perp}$. Then we can take the inverse of $\pi_{\epsilon, \mathbf{P}} \circ DS_{\epsilon}(w_{\epsilon, \mathbf{P}} + \Phi_{\epsilon, \mathbf{P}})$ in the above equation and the inverse is continuous in \mathbf{P} .

Since $\frac{\partial P_0 V_i}{\partial \tau_{P_{i,j}}}, \left(\frac{\partial \Phi_{\epsilon, \mathbf{P}}}{\partial \tau_{P_{i,j}}} \right)_1 \in \mathcal{K}_{\epsilon, \mathbf{P}}$ are continuous in \mathbf{P} and so is $\frac{\partial \pi_{\epsilon, \mathbf{P}}}{\partial \tau_{P_{i,j}}}$, we conclude that $\left(\frac{\partial \Phi_{\epsilon, \mathbf{P}}}{\partial \tau_{P_{i,j}}} \right)_2$ is also continuous in \mathbf{P} . This is equivalent to C^1 -dependence of $\Phi_{\epsilon, \mathbf{P}}$ on \mathbf{P} . The proof is finished. \square

4. THE REDUCED PROBLEM: AN OPTIMIZATION PROCEDURE

In this section, we study an optimization problem.

Fix $\mathbf{P} \in \bar{\Lambda}$. Let $\Phi_{\epsilon, \mathbf{P}}$ be the solution given by Lemma 3.4. We define a new functional

$$M_{\epsilon}(\mathbf{P}) = J_{\epsilon}(w_{\epsilon, \mathbf{P}} + \Phi_{\epsilon, \mathbf{P}}) : \bar{\Lambda} \rightarrow R. \quad (4.1)$$

We shall prove

Proposition 4.1. *For ϵ small, the optimization problem*

$$\max\{M_{\epsilon}(\mathbf{P}) : \mathbf{P} \in \bar{\Lambda}\} \quad (4.2)$$

has a solution $\mathbf{P}^{\epsilon} \in \Lambda$.

Proof. Since $J_{\epsilon}(w_{\epsilon, \mathbf{P}} + \Phi_{\epsilon, \mathbf{P}})$ is continuous in \mathbf{P} , the optimization problem has a solution. Let $M_{\epsilon}(\mathbf{P}^{\epsilon})$ be the maximum where $\mathbf{P}^{\epsilon} \in \bar{\Lambda}$. We claim that $\mathbf{P}^{\epsilon} \in \Lambda$.

In fact, for any $\mathbf{P} \in \bar{\Lambda}$, by Lemma 3.5, we have

$$M_\epsilon(\mathbf{P}) = \epsilon^N \left[\frac{K}{2} I(V) - \epsilon \beta_1 \left(\sum_{i=1}^K \kappa(P_i) \right) - \frac{1}{2} \sum_{k,l=1,\dots,K,k \neq l} (\gamma_{kl} + o(1)) V \left(\frac{|P_k - P_l|}{\epsilon} \right) + o(\epsilon) \right].$$

Since $M_\epsilon(\mathbf{P}^\epsilon)$ is the maximum, we have

$$\begin{aligned} & \beta_1 \sum_{i=1}^K \kappa(P_i^\epsilon) + \frac{1}{\epsilon} \sum_{k \neq l} \left(\frac{1}{2} \gamma_{kl} + o(1) \right) V \left(\frac{|P_k^\epsilon - P_l^\epsilon|}{\epsilon} \right) \\ & \leq \beta_1 \sum_{i=1}^K \kappa(P_i) + \frac{1}{\epsilon} \sum_{k \neq l} \left(\frac{1}{2} \gamma_{kl} + o(1) \right) V \left(\frac{|P_k - P_l|}{\epsilon} \right) + o(1) \end{aligned} \quad (4.3)$$

for any $\mathbf{P} = (P_1, \dots, P_K) \in \bar{\Lambda}$. Choose P_i such that $\kappa(P_i) \rightarrow \min_{P \in \Gamma_i} \kappa(P)$ for $i = 1, 2, \dots, K$ and $V \left(\frac{|P_k - P_l|}{\epsilon} \right) \frac{1}{\epsilon} \rightarrow 0$ for $k \neq l$. This implies that

$$\beta_1 \sum_{i=1}^K \kappa(P_i^\epsilon) + \frac{1}{\epsilon} \sum_{k \neq l} \left(\frac{1}{2} \gamma_{kl} + o(1) \right) V \left(\frac{|P_k^\epsilon - P_l^\epsilon|}{\epsilon} \right) \leq \beta_1 \sum_{i=1}^K \min_{P \in \Gamma_i} \kappa(P) + \delta$$

for any $\delta > 0$.

Note that $\partial\Lambda \subset \{P_i \in \partial\Gamma_i \text{ or } V \left(\frac{|P_k - P_l|}{\epsilon} \right) = \epsilon\eta\}$. Hence if $\mathbf{P}^\epsilon \in \partial\Lambda$, we have that either

$$\kappa(P_i^\epsilon) \geq \min_{P \in \partial\Gamma_i} \kappa(P) \geq \min_{P \in \Gamma_i} \kappa(P) + 2\eta_0$$

for some $i = 1, \dots, K$ and $\eta_0 > 0$ (by condition (1.5)) or

$$\frac{1}{\epsilon} V \left(\frac{|P_k^\epsilon - P_l^\epsilon|}{\epsilon} \right) = \eta$$

for some $k \neq l$.

Therefore, if $\mathbf{P}^\epsilon \in \partial\Lambda$ we have

$$\begin{aligned} & \beta_1 \sum_{i=1}^K \kappa(P_i^\epsilon) + \frac{1}{\epsilon} \sum_{k \neq l} \left(\frac{1}{2} \gamma_{kl} + o(1) \right) V \left(\frac{|P_k^\epsilon - P_l^\epsilon|}{\epsilon} \right) \\ & \geq \beta_1 \sum_{i=1}^K \min_{P \in \Gamma_i} \kappa(P) + \min(\beta_1 \eta_0, \min_{k \neq l, V \left(\frac{|P_k - P_l|}{\epsilon} \right) = \eta\epsilon} \gamma_{kl} \eta). \end{aligned}$$

Note that $\min_{k \neq l, V \left(\frac{|P_k - P_l|}{\epsilon} \right) = \eta\epsilon} \gamma_{kl} \geq \inf_{\tau \in \Sigma_1} \tau \geq \delta_0 > 0$ since for any $\tau \in \Sigma_1$ we have

$$\tau = \int_{R_+^N} h(V) e^{\sqrt{m}\langle b, y \rangle} = \frac{1}{2} \int_{R^N} h(V) e^{\sqrt{m}\langle b, y \rangle} > 0.$$

This is a contradiction to (4.3) if we choose δ small enough.

It follows that $\mathbf{P}^\epsilon \in \Lambda$.

This completes the proof of Proposition 4.1. \square

5. PROOF OF THEOREM 1.2

In this section, we apply results of Sections 3-4 to prove Theorem 1.1, Theorem 1.2 and Corollary 1.3. It remains to prove Theorem 1.2. The other proofs are similar.

Proof of Theorem 1.2. By Lemma 3.4 and Lemma 3.6, there exists ϵ_0 such that for $\epsilon < \epsilon_0$ we have a C^1 -map which, to any $\mathbf{P} \in \bar{\Lambda}$, associates $\Phi_{\epsilon, \mathbf{P}} \in \mathcal{K}_{\epsilon, \mathbf{P}}^\perp$ such that

$$S_\epsilon(w_{\epsilon, \mathbf{P}} + \Phi_{\epsilon, \mathbf{P}}) = \sum_{k=1, \dots, K; l=1, \dots, N-1} \alpha_{kl} \frac{\partial P_0 V_k}{\partial \tau_{P_{k,l}}} \quad (5.1)$$

for some constants $\alpha_{kl} \in R^{K(N-1)}$.

By Proposition 4.1, we have $\mathbf{P}^\epsilon \in \Lambda$, achieving the maximum of the optimization problem in Proposition 4.1. Let $\Phi_\epsilon = \Phi_{\epsilon, \mathbf{P}^\epsilon}$ and $v_\epsilon = w_{\epsilon, \mathbf{P}^\epsilon} + \Phi_{\epsilon, \mathbf{P}^\epsilon}$. Then we have

$$\frac{\partial}{\partial \tau_{P_{i,j}}} \Big|_{\mathbf{P}=\mathbf{P}^\epsilon} M_\epsilon(\mathbf{P}^\epsilon) = 0, \quad i = 1, \dots, K, \quad j = 1, \dots, N-1.$$

Hence we have

$$\begin{aligned} \int_{\Omega_\epsilon} \left[\nabla v_\epsilon \nabla \frac{\partial(w_{\epsilon, \mathbf{P}} + \Phi_{\epsilon, \mathbf{P}})}{\partial \tau_{P_{i,j}}} \Big|_{\mathbf{P}=\mathbf{P}^\epsilon} + m v_\epsilon \frac{\partial(w_{\epsilon, \mathbf{P}} + \Phi_{\epsilon, \mathbf{P}})}{\partial \tau_{P_{i,j}}} \Big|_{\mathbf{P}=\mathbf{P}^\epsilon} \right. \\ \left. - h(v_\epsilon) \frac{\partial(w_{\epsilon, \mathbf{P}} + \Phi_{\epsilon, \mathbf{P}})}{\partial \tau_{P_{i,j}}} \Big|_{\mathbf{P}=\mathbf{P}^\epsilon} \right] = 0. \end{aligned}$$

Thus

$$\begin{aligned} \int_{\Omega_\epsilon} \nabla v_\epsilon \nabla \frac{\partial(P_0 V_i + \Phi_{\epsilon, \mathbf{P}})}{\partial \tau_{P_{i,j}}} \Big|_{\mathbf{P}=\mathbf{P}^\epsilon} \\ + m v_\epsilon \frac{\partial(P_0 V_i + \Phi_{\epsilon, \mathbf{P}})}{\partial \tau_{P_{i,j}}} \Big|_{\mathbf{P}=\mathbf{P}^\epsilon} - h(v_\epsilon) \frac{\partial(P_0 V_i + \Phi_{\epsilon, \mathbf{P}})}{\partial \tau_{P_{i,j}}} \Big|_{\mathbf{P}=\mathbf{P}^\epsilon} = 0 \end{aligned}$$

for $i = 1, \dots, K$ and $j = 1, \dots, N-1$. Because of

$$w_{\epsilon, \mathbf{P}} + \phi_{\epsilon, \mathbf{P}} \in X$$

we have

$$\int_{\Omega_\epsilon} [w_{\epsilon, \mathbf{P}} + \phi_{\epsilon, \mathbf{P}}] = 0.$$

Differentiating both sides, we have

$$\int_{\Omega_\epsilon} \frac{\partial(w_{\epsilon, \mathbf{P}} + \phi_{\epsilon, \mathbf{P}})}{\partial \tau_{P_{i,j}}} = 0.$$

This implies that

$$\int_{\Omega_\epsilon} S_\epsilon(v_\epsilon) \frac{\partial(w_{\epsilon, \mathbf{P}} + \phi_{\epsilon, \mathbf{P}})}{\partial \tau_{P_{i,j}}} = 0.$$

Therefore we have

$$\sum_{k=1, \dots, K; l=1, \dots, N-1} \alpha_{kl} \int_{\Omega_\epsilon} \frac{\partial P_0 V_k}{\partial \tau_{P_{k,l}}} \frac{\partial(P_0 V_i + \Phi_{\epsilon, \mathbf{P}})}{\partial \tau_{P_{i,j}}} = 0. \quad (5.2)$$

Since $\Phi_{\epsilon, \mathbf{P}} \in \mathcal{K}_{\epsilon, \mathbf{P}}^\perp$, we have that

$$\begin{aligned} \left| \int_{\Omega_\epsilon} \frac{\partial P_0 V_k}{\partial \tau_{P_{k,l}}} \frac{\partial \Phi_{\epsilon, \mathbf{P}}}{\partial \tau_{P_{i,j}}} \right| &= \left| - \int_{\Omega_\epsilon} \frac{\partial^2 P_0 V_i}{\partial \tau_{P_{k,l}} \partial \tau_{P_{i,j}}} \Phi_{\epsilon, \mathbf{P}} \right| \\ &\leq \left\| \frac{\partial^2 P_0 V_i}{\partial \tau_{P_{k,l}} \partial \tau_{P_{i,j}}} \right\|_{L^2} \|\Phi_{\epsilon, \mathbf{P}}\|_{L^2} \\ &= O(\epsilon^{-1}). \end{aligned}$$

Note that by Proposition 2.2

$$\int_{\Omega_\epsilon} \frac{\partial P_0 V_k}{\partial \tau_{P_{k,l}}} \frac{\partial P_0 V_i}{\partial \tau_{P_{i,j}}} = \frac{1}{\epsilon^2} \delta_{ik} \delta_{lj} (A + o(1)),$$

where

$$A = \int_{R_+^N} \left(\frac{\partial V}{\partial y_1} \right)^2 > 0.$$

Thus equation (5.2) becomes a system of homogeneous equations for α_{kl} and the matrix of the system is nonsingular since it is diagonally dominant. So $\alpha_{kl} \equiv 0, k = 1, \dots, K, l = 1, \dots, N-1$.

Hence $v_\epsilon = w_{\epsilon, \mathbf{P}} + \Phi_{\epsilon, \mathbf{P}}$ is a solution of (1.2).

By our construction, it is easy to see that $\epsilon^N J_\epsilon(v_\epsilon) \rightarrow \frac{K}{2} I(V)$ and v_ϵ has only K local maximum points $Q_1^\epsilon, \dots, Q_K^\epsilon$ and $Q_i^\epsilon \in \partial \Omega$. By the structure of v_ϵ we see that (up to a permutation) $Q_i^\epsilon - P_i^\epsilon = o(1)$. This proves Theorem 1.2.

Theorem 1.1 follows from Theorem 1.2 by taking $\Gamma_i = \Gamma, i = 1, \dots, K$.

Finally, we prove Corollary 1.3.

If Ω is not a ball, then $\kappa(P)$ has a local minimum on some relatively open set Γ , Theorem 1.1 can be applied.

If Ω is a ball, Corollary 1.3 follows by using perturbation theory in symmetric spaces. See [27] and [29]. \square

REFERENCES

- [1] ADIMURTHI, MANCINI, G., AND YADAVA, S. L., The role of mean curvature in a semilinear Neumann problem involving the critical Sobolev exponent, *Comm. Partial Differential Equations* **20** (1995), 591-631.
- [2] ADIMURTHI, PACELLA, F., AND YADAVA, S. L., Interaction between the geometry of the boundary and positive solutions of a semilinear Neumann problem with critical nonlinearity, *J. Funct. Anal.* **113** (1993), 318-350
- [3] ADIMURTHI, PACELLA, F., AND YADAVA, S. L., Characterization of concentration points and L^∞ -estimates for solutions involving the critical Sobolev exponent, *Diff. Int. Eqn.* **8** (1995), 41-68.
- [4] N. ALIKAKOS, P.W. BATES AND X. CHEN, Convergence of the Cahn-Hilliard equation to the Hele-Shaw model, *Arch. Rat. Mech. Anal.* **128** (1994), 165-205.
- [5] N. ALIKAKOS, P.W. BATES AND G. FUSCO, Slow motion for the Cahn-Hilliard equation in one space dimension, *J. Diff. Eqns.* **90** (1991), 81-134.
- [6] P.W. BATES AND P.C. FIFE, The dynamics of nucleation for the Cahn-Hilliard equation, *SIAM J. Appl. Math.* **53** (1993), 990-1008.
- [7] J.W. CAHN AND J.E. HILLIARD, Free energy of a nonuniform system, I. Interfacial free energy, *J. Chem. Phys.* **28** (1958), 258-267.
- [8] X. CHEN AND M. KOWALCZYK, Existence of equilibria for the Cahn-Hilliard equation via local minimizers of the perimeter, *Comm. Partial Differential Equations* **21** (1996), 1207-1233.
- [9] C.M. ELLIOT AND D.A. FRENCH, Numerical studies of the Cahn-Hilliard equation for phase separation, *IMA J. Appl. Math.* **38** (1987), 97-128.
- [10] C.M. ELLIOT AND S. ZHENG, On the Cahn-Hilliard equation, *Arch. Rat. Mech. Anal.* **96** (1986), 339-357.
- [11] A. FLOER AND A. WEINSTEIN, Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential, *J. Funct. Anal.* **69** (1986), 397-408.
- [12] B. GIDAS, W.-M. NI AND L. NIRENBERG, Symmetry of positive solutions of nonlinear elliptic equations in R^n , in *Mathematical Analysis and Applications* (Part A, Adv. Math. Suppl. Studies **7A**, Academic Press, New York, 1981), 369-402.
- [13] M. GRINFELD AND A. NOVICK-COHEN, Counting stationary solutions of the Cahn-Hilliard equation by transversality arguments, *Proc. Roy. Soc. Edinburgh Sect. A* **125** (1995), 351-370.
- [14] M. GRINFELD AND A. NOVICK-COHN, The viscous Cahn-Hilliard equation: Morse decomposition and structure of the global attractor, *Trans. Amer. Math. Soc.* **351** (1999), 2375-2406.
- [15] C. GUI, Multi-peak solutionis for a semilinear Neumann problem, *Duke Math. J.* **84** (1996), 739-769.
- [16] C. GUI AND N. GHOUSSEUB, Multi-peak solutions for a semilinear Neumann problem involving the critical Sobolev exponent, *Math. Z.* **229** (1998), 443-474.
- [17] C. GUI, J. WEI, AND M. WINTER, Multiple boundary peak solutions for some singularly perturbed Neumann problems, *Ann. Inst. H. Poincaré*, to appear.

- [18] H. KIELHÖFER, Pattern formation of the stationary Cahn-Hilliard model, *Proc. Roy. Soc. Edinburgh Sect. A* **127** (1997), 1219-1243.
- [19] R.V. KOHN AND P. STERNBERG, Local minimisers and singular perturbations, *Proc. Roy. Soc. Edinburgh Sect. A* **111** (1989), 69-84.
- [20] LI, Y. Y., On a singularly perturbed equation with Neumann boundary condition, *Comm. Partial Diff. Equations*, **23** (1998), 487-545.
- [21] C.-S. LIN, W.-M. NI AND I. TAKAGI, Large amplitude stationary solutions to a chemotaxis systems, *J. Diff. Eqns.* **72** (1988), 1-27.
- [22] S. LUCKHAUS AND L. MODICA, The Gibbs-Thompson relation within the gradient theory of phase transitions, *Arch. Rational Mech. Anal.* **107** (1989), 71-83.
- [23] S. MAIER-PAAPE AND T. WANNER, Spinodal decomposition for the Cahn-Hilliard equation in higher dimensions, part I: probability and wavelength estimate, *Comm. Math. Phys.* **195** (1998), 435-464.
- [24] S. MAIER-PAAPE AND T. WANNER, Spinodal decomposition for the Cahn-Hilliard equation in higher dimensions, part II: nonlinear dynamics, Preprint (1997).
- [25] L. MODICA, The gradient theory of phase transitions and the minimal interface criterion, *Arch. Rational Mech. Anal.* **98** (1987), 123-142.
- [26] W.-M. NI, X. PAN AND I. TAKAGI, Singular behavior of least-energy solutions of a semilinear Neumann problem involving critical Sobolev exponents, *Duke Math. J.* **67** (1992), 1-20.
- [27] W.-M. NI AND I. TAKAGI, On the shape of least energy solution to a semilinear Neumann problem, *Comm. Pure Appl. Math.* **41** (1991), 819-851.
- [28] W.-M. NI AND I. TAKAGI, Locating the peaks of least energy solutions to a semilinear Neumann problem, *Duke Math. J.* **70** (1993), 247-281.
- [29] W.-M. NI AND I. TAKAGI, Point-condensation generated by a reaction-diffusion system in axially symmetric domains, *Japan J. Industrial Appl. Math.* **12** (1995), 327-365.
- [30] W.-M. NI AND J. WEI, On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems, *Comm. Pure Appl. Math.* **48** (1995), 731-768.
- [31] B. S. NIETHAMMER, Existence and uniqueness of radially symmetric stationary points within the gradient theory of phase transitions, *European J. Appl. Math.* **6** (1995), 45-67.
- [32] A. NOVICK-COHEN AND L.A. PELETIER, Steady states of the one-dimensional Cahn-Hilliard equation, *Proc. Royal Soc. Edin. A* **123** (1993), 1071-1098.
- [33] A. NOVICK-COHEN AND L.A. SEGAL, Nonlinear aspects of the Cahn-Hilliard equation, *Physica D* **10** (1984), 277-298.
- [34] Y.G. OH, Existence of semi-classical bound states of nonlinear Schrödinger equations with potentials of the class $(V)_a$, *Comm. Partial Differential Equations* **13** (1988), 1499-1519.
- [35] Y.G. OH, On positive multi-bump bound states of nonlinear Schrödinger equations under multiple-well potentials, *Comm. Math. Phys.* **131** (1990), 223-253.
- [36] X. B. PAN, Condensation of least-energy solutions of a semilinear Neumann problem, *J. Partial Diff. Eqns.*, **8** (1995), 1-36.
- [37] X. B. PAN, Condensation of least-energy solutions: the effect of boundary conditions, *Nonlinear Analysis, TMA* **24** (1995), 195-222.
- [38] X. B. PAN, Further study on the effect of boundary conditions, *J. Diff. Eqns.* **117** (1995), 446-468.

- [39] R.L. PEGO, Front migration in the nonlinear Cahn-Hilliard equation, *Proc. Roy. Soc. London A* **422** (1989), 261-278.
- [40] L.A. PELETIER AND J. SERRIN, Uniqueness of positive solutions of semilinear equations in R^n , *Arch. Rational Mech. Anal.* **81** (1983), 181-197.
- [41] P. RYBKA AND K.-H. HOFFMANN, Convergence of solutions to Cahn-Hilliard equation, *Comm. Partial Differential Equations* **24** (1999), 1055-1077.
- [42] Z.-Q. WANG, On the existence of multiple single-peaked solutions for a semilinear Neumann problem, *Arch. Rational Mech. Anal.* **120** (1992), 375-399.
- [43] J. WEI, On the boundary spike layer solutions of singularly perturbed semilinear Neumann problem, *J. Diff. Eqns.* **134** (1997), 104-133.
- [44] J. WEI AND M. WINTER, Stationary solutions for the Cahn-Hilliard equation, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **15** (1998), 459-492.
- [45] J. WEI AND M. WINTER, Multi-peak solutions for a wide class of singular perturbation problems, *J. London Math. Soc.* **59** (1999), 585-606.

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