High frequency Diffraction of an electromagnetic plane wave by an imperfectly conducting rectangular cylinder.

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We shall consider the the problem of determining the scattered far wave field produced when a plane E-polarized wave is incident on an imperfectly conducting rectangular cylinder. By using the the uniform asymptotic solution for the problem of the diffraction of a plane wave by a right-angled impedance wedge, in conjunction with Keller's method, the a high frequency far field solution to the problem is given.

1 Introduction

In dealing with mobile phone propagation in cities the effect of building corners and their surface cladding is of paramount importance for the signal strength of the phones, see references in Nechayev[1]. A building of rectangular cross-section can be modelled by four of these corners. With appropriate polarization this building can be effectively modelled for high frequency diffraction by a rectangular impedance cylinder in two dimensions. To obtain quantitative and qualitative results for the signal strength far from the building when there are multiple diffraction from such corners an effective approach is to use the Keller method of the geometrical theory of diffraction(GTD). This method requires information about the "diffraction coefficient" which are obtained from the solution of canonical impedance wedge problems. These coefficients need to be uniformly valid in the angular variables in order that the method can be used successfully when considering multiple diffractions at different corners. In a previous work Rawlins[2] this aspect was addressed by using a simple exact solution to the specific problem of the diffraction of a plane wave by a right-angled impedance wedge obtained by Rawlins[3]. In this work, Rawlins[2], useful asymptotic results were obtained for the far-field across singular ray directions where the usual diffraction coefficient used in high frequency methods breaks down. In this work we shall use these results to apply to the practical situation of the scattering by an absorbing rectangular building. The determination of the far field when a high-frequency E-polarized electromagnetic plane-wave is obliquely incident on an imperfectly conducting rectangular cylinder is obtained by applying Keller's method of geometrical diffraction. Oblique incidence corresponds to the situation where incident plane wave ray is not running parallel along any of the faces of the rectangular cylinder. To achieve this the uniform results of Rawlins[2] for the diffraction coefficient for a right-angled impedance wedge is used in conjunction with the multiple diffraction that arises from waves travelling from corner to corner of the rectangle.

1.1 Formulation of the boundary value problem

An E_z-polarized plane wave

$$u_i(P) = e^{-i[\omega t + kr sin(\theta+\theta_0)]}, \quad (1)$$

is incident on an imperfectly conducting rectangular cylinder: $|x| \le a$, and $|y| \le b$,$-\infty < z < \infty$; where the polar coordinates (r, θ) are defined by $x = r\cos\theta, y = r\sin\theta$. The permeability, permittivity, and conductivity of the cylinder are μ, ϵ, and σ respectively; and the complex refractive index of the cylinder material is given by

$$N = \sqrt{\frac{\mu}{\mu_0}\left(\frac{\epsilon}{\epsilon_0} + \frac{i\sigma}{\omega\epsilon_0}\right)}.$$

with $k^2 = \epsilon_0\mu_0\omega^2$. For a unique solution u, must satisfy edge conditions at the corners and a radiation condition at infinity. The boundary conditions appropriate to the present problem are given by

$$\frac{\partial u}{\partial y}(b, x) - ik\cos\vartheta u(b, x) = 0, \quad (|x| \le a),$$

$$\frac{\partial u}{\partial x}(y, -a) + ik\cos\vartheta u(y, -a) = 0, \quad (|y| \le b),$$

$$\frac{\partial u}{\partial y}(-b,x)+ik\cos\vartheta u(-b,x)=0, \quad (|x|\leq a),$$

$$\frac{\partial u}{\partial x}(y,a)-ik\cos\vartheta u(y,a)=0, \quad (|y|\leq b).$$

where $\cos\vartheta=-\frac{\mu_0}{\mu}N$, and for absorbing surfaces $\pi\leq\Re\vartheta\leq 3\pi/2$. An exact closed form solution of such a boundary problem is not so far possible. However for practical purposes we may make some realistic practical assumptions that can lead to useful computational results. The sides of the cylinder are assumed to be large compared to the incident wavelength (i.e. $kb\gg 1$). The problem under consideration is that of finding the field at the point $P(r,\theta)$, $0\leq\theta\leq 2\pi$, where r is large compared to the dimensions of the cylinder, $r\gg\sqrt{a^2+b^2}$, i.e. the diffracted far field. Clearly from the symmetry of the problem we need only consider an angle of incidence within the range $\pi/2\leq\theta_0\leq\pi$. To achieve this objective we shall use the results for the uniform, and nonuniform, asymptotics of the solution to the problem of the diffraction of an E_z-polarized plane wave by a right-angled impedance wedge given in Rawlins[2]; in conjunction with Keller's theory of geometrical diffraction (GTD) Keller[4] and by applying a formula due to Zitron[6], to deal with mulitiple diffraction at the corners of the rectangle.

2 Keller's geometrical theory of diffraction and multiple diffraction

According to Keller's GTD the diffracted field $u_d(P)$ at a point P is equal to the sum of the fields on all rays through P:

$$u_d(P)=\sum_{rays}u_j(P). \tag{2}$$

Here $u_j(P)$ is the diffracted field on the j^{th} such ray, and if this is an m-fold diffracted ray then

$$u_d(P)=\frac{e^{iks_j}}{k^{\frac{m}{2}}}\sum_{n=0}^{\infty}\frac{A_{jn}(P)}{(ik)^n}, \tag{3}$$

where $k(=2\pi/\lambda)$ is the propagation constant, s_j the arc length along the ray, and the function A_{jn} depends on the geometry and material of the diffracting object. For a rectangular cylinder all the diffracted rays are produced by wedges of 90^0 angle. Hence the inclusion of higher $A_{jn}(n=1,2,...)$ in the expansion (3) involves the use of more terms in the asymptotic solution of the wedge diffraction problem. The calculation of the diffraction coefficient corresponding to these higher-order terms necessitates the solution of the wedge diffraction problem for non-plane-wave incidence. However, as is shown in the work of Zitron[6], the relevant non-plane waves are expressible in terms of linear combinations of plane waves and their derivatives. Thus the diffraction coefficients are easily found; and therefore it is possible to calculate the off shadow far fields corresponding to wedge excitations which are not shadow boundary fields. In order to calculate diffraction coefficients corresponding to shadow boundary fields we show that these fields too are expressible in terms of plane waves and their derivatives. As far as the solution to the canonical problem of the diffraction of a plane wave by an impedance wedge is concerned the complete solution to this problem has already been derived in detail by Rawlins[2]. In particular the far field expression is given by

$$u_d(r,\theta,\theta_0)=D(\theta,\theta_0)\frac{e^{ikr}}{\sqrt{r}}+O((kr)^{-\frac{3}{2}}), \tag{4}$$

where the "diffraction coefficient" $D(\theta,\theta_0)$ is given by Rawlins[2] as

$$\begin{aligned} &D(\theta,\theta_0) \\ =\ &\frac{2e^{i\frac{\pi}{4}}(\cos\theta-\cos\vartheta)(\sin\theta+\cos\vartheta)(\cos\frac{4\theta_0}{3}-\cos\frac{4(\pi+\vartheta)}{3})}{\sqrt{6\pi k}(\cos\theta_0+\cos\vartheta)(\sin\theta_0-\cos\vartheta)(\cos\frac{4(\theta-\pi-\vartheta)}{3}+\frac{1}{2})} \\ &\frac{(2\cos\frac{2\theta_0}{3}\cos\frac{2\theta}{3}+\frac{1}{2}-\cos\frac{4(\pi+\vartheta)}{3})\sin\frac{2\theta}{3}\sin\frac{2\theta_0}{3}}{(\cos\frac{4(\theta+\pi+\vartheta)}{3}+\frac{1}{2})(\cos\frac{2(\theta-\theta_0)}{3}+\frac{1}{2})(\cos\frac{2(\theta+\theta_0)}{3}+\frac{1}{2})}. \end{aligned} \tag{5}$$

An important property of the diffraction coefficient (5) is that

$$D(\theta,\theta_0)=D(\frac{3\pi}{2}-\theta,\frac{3\pi}{2}-\theta_0), \tag{6}$$

which means that the angle of incidence θ_0 and the angle of observation θ can be measured from either face of the corner wedge provided they are both measured from the same datum face. Other useful properties of $D(\theta,\theta_0)$ which we will require later in an application of the Keller method is the Karp-Karal lemma:

$$D(0,\theta_0)=D(\theta,0)=0; \tag{7}$$

and if we use the notation

$$\lim_{\theta\to 0}\frac{\partial D(\theta,\theta_0)}{\partial\theta}=D_\theta(0,\theta_0),$$

$$\lim_{\theta_0\to 0}\frac{\partial D(\theta,\theta_0)}{\partial\theta_0}=D_{\theta_0}(\theta,0),$$

then

$$D_\theta(0,\theta_0)=\frac{4e^{i\frac{\pi}{4}}(1-\cos\vartheta)(\cos\frac{4\theta_0}{3}-\cos\frac{4(\pi+\vartheta)}{3})}{3\sqrt{6\pi k}(\cos\theta_0+\cos\vartheta)(\sin\theta_0-\cos\vartheta)}\frac{(2\cos\frac{2\theta_0}{3}+\frac{1}{2}-\cos\frac{4(\pi+\vartheta)}{3})\sin\frac{2\theta_0}{3}\cos\vartheta}{(\cos\frac{4(\pi+\vartheta)}{3}+\frac{1}{2})^2(\cos\frac{2\theta_0}{3}+\frac{1}{2})^2},\quad(8)$$

$$D_{\theta_0}(\theta,0)=\frac{-4e^{i\frac{\pi}{4}}(\cos\theta-\cos\vartheta)(\sin\theta+\cos\vartheta)(1-\cos\frac{4(\pi+\vartheta)}{3})}{3\sqrt{6\pi k}(1+\cos\vartheta)\cos\vartheta(\cos\frac{4(\theta-\pi-\vartheta)}{3}+\frac{1}{2})}\frac{(2\cos\frac{2\theta}{3}+\frac{1}{2}-\cos\frac{4(\pi+\vartheta)}{3})\sin\frac{2\theta}{3}}{(\cos\frac{4(\theta+\pi+\vartheta)}{3}+\frac{1}{2})(\cos\frac{2\theta}{3}+\frac{1}{2})^2}.\quad(9)$$

By applying a method due to Zitron[6], to the expression (4) we obtain the field on and near to the ray determined by the point $P_1(r,\theta)$. That is, we obtain the asymptotic expansion of the diffracted field, $u_d(P_1)$, in terms of the coordinates of any point Q_1 which is in the near neighbourhood of the point P_1. By using these techniques it can be shown that the far field in the region around the cylinder is given by:

<u>For $0<\theta<\frac{\pi}{2}$, and $\theta\neq\theta_0-\frac{\pi}{2}$.</u>

$$\begin{aligned}u_d(P)=&\frac{e^{ikr}}{\sqrt{r}}([D(\frac{\pi}{2}-\theta,\theta_0)\\&\times e^{ik[-a(\cos\theta+\sin\theta_0)+b(\cos\theta_0+\sin\theta)]}\\&+D(\frac{\pi}{2}+\theta,\pi-\theta_0)\\&\times e^{ik[-a(\cos\theta+\sin\theta_0)-b(\cos\theta_0+\sin\theta)]}]\\&-\frac{1}{2ik}[\frac{D_\theta(0,\theta_0-\frac{\pi}{2})D_{\theta_0}(\pi+\theta,0)}{(2a)^{\frac{3}{2}}}\\&\times e^{ik[a(2-\cos\theta+\sin\theta_0)+b(\cos\theta_0+\sin\theta)]}\\&+\frac{D_\theta(0,\pi-\theta_0)D_{\theta_0}(\frac{\pi}{2}-\theta,0)}{(2b)^{\frac{3}{2}}}\\&\times e^{ik[b(2+\sin\theta-\cos\theta_0)-a(\sin\theta_0+\cos\theta)]}\\&+\frac{D_\theta(0,\theta_0)D_{\theta_0}(\frac{\pi}{2}+\theta,0)}{(2b)^{\frac{3}{2}}}\\&\times e^{ik[b(2-\sin\theta+\cos\theta_0)-a(\sin\theta_0+\cos\theta)]}\\&+\frac{D_\theta(0,\theta_0+\frac{\pi}{2})D_{\theta_0}(\theta,0)}{(2a)^{\frac{3}{2}}}\\&\times e^{ik[a(2+\cos\theta-\sin\theta_0)-b(\cos\theta_0+\sin\theta)]}\\&+\frac{D_\theta(0,2\pi-\theta_0)D_{\theta_0}(\frac{3\pi}{2}-\theta,0)}{(2b)^{\frac{3}{2}}}\\&\times e^{ik[b(2-\sin\theta+\cos\theta_0)+a(\sin\theta_0+\cos\theta)]}]\\&+O\left[(kd)^{-\frac{5}{2}}\right])+O\left[(kr)^{-\frac{3}{2}}\right].\end{aligned}\quad(10)$$

The expression for the diffraction coefficients in (10) are given by (5) to (9). By an analogous procedure we can obtain $u_d(P)$for the remaining quadrants.

<u>For$\frac{\pi}{2}<\theta<\pi$, and $\theta\neq\frac{3\pi}{2}-\theta_0$.</u>

$$\begin{aligned}u_d(P)=&\frac{e^{ikr}}{\sqrt{r}}([D(\frac{\pi}{2}+\theta,\pi-\theta_0)\\&\times e^{ik[-a(\cos\theta+\sin\theta_0)-b(\cos\theta_0+\sin\theta)]}\\&+D(\theta-\frac{\pi}{2},2\pi-\theta_0)\\&\times e^{ik[a(\cos\theta+\sin\theta_0)+b(\cos\theta_0+\sin\theta)]}]\\&-\frac{1}{2ik}[\frac{D_\theta(0,\theta_0)D_{\theta_0}(\frac{\pi}{2}+\theta,0)}{(2b)^{\frac{3}{2}}}\\&\times e^{ik[b(2-\sin\theta+\cos\theta_0)-a(\sin\theta_0+\cos\theta)]}\\&+\frac{D_\theta(0,\theta_0+\frac{\pi}{2})D_{\theta_0}(\theta,0)}{(2a)^{\frac{3}{2}}}\\&\times e^{ik[a(2+\cos\theta-\sin\theta_0)-b(\cos\theta_0+\sin\theta)]}\\&+\frac{D_\theta(0,2\pi-\theta_0)D_{\theta_0}(\frac{3\pi}{2}-\theta,0)}{(2b)^{\frac{3}{2}}}\\&\times e^{ik[b(2-\sin\theta+\cos\theta_0)+a(\sin\theta_0+\cos\theta)]}\\&+\frac{D_\theta(0,\frac{3\pi}{2}-\theta_0)D_{\theta_0}(2\pi-\theta,0)}{(2a)^{\frac{3}{2}}}\\&\times e^{ik[a(2+\cos\theta-\sin\theta_0)+b(\cos\theta_0+\sin\theta)]}]\\&+O\left[(kd)^{-\frac{5}{2}}\right])+O\left[(kr)^{-\frac{3}{2}}\right].\end{aligned}\quad(11)$$

<u>For $\pi<\theta<\frac{3\pi}{2}$, and $\theta\neq\theta_0+\frac{\pi}{2}$.</u>

$$\begin{aligned}u_d(P)=&\frac{e^{ikr}}{\sqrt{r}}([D(2\pi-\theta,\theta_0-\frac{\pi}{2})\\&\times e^{ik[a(\cos\theta+\sin\theta_0)+b(\cos\theta_0+\sin\theta)]}\\&+D(\theta-\pi,\frac{3\pi}{2}-\theta_0)\\&\times e^{ik[-a(\cos\theta+\sin\theta_0)+b(\cos\theta_0+\sin\theta)]}]\\&-\frac{1}{2ik}[\frac{D_\theta(0,-\theta_0+\frac{3\pi}{2})D_{\theta_0}(2\pi-\theta,0)}{(2a)^{\frac{3}{2}}}\\&\times e^{ik[a(2+\cos\theta-\sin\theta_0)+b(\cos\theta_0+\sin\theta)]}\end{aligned}$$

$$+\frac{D_\theta(0,\theta_0-\frac{\pi}{2})D_{\theta_0}(\theta-\pi,0)}{(2a)^{\frac{3}{2}}}$$
$$\times e^{ik[a(2-\cos\theta+\sin\theta_0)+b(\cos\theta_0+\sin\theta)]}$$
$$+\frac{D_\theta(0,\pi-\theta_0)D_{\theta_0}(\frac{5\pi}{2}-\theta,0)}{(2b)^{\frac{3}{2}}}$$
$$\times e^{ik[b(2+\sin\theta-\cos\theta_0)-a(\sin\theta_0+\cos\theta)]}$$
$$+\frac{D_\theta(0,\theta_0+\frac{\pi}{2})D_{\theta_0}(\theta,0)}{(2a)^{\frac{3}{2}}}$$
$$\times e^{ik[a(2+\cos\theta-\sin\theta_0)-b(\cos\theta_0+\sin\theta)]}$$
$$+\frac{D_\theta(0,2\pi-\theta_0)D_{\theta_0}(\frac{3\pi}{2}-\theta,0)}{(2b)^{\frac{3}{2}}}$$
$$\times e^{ik[b(2-\sin\theta+\cos\theta_0)+a(\sin\theta_0+\cos\theta)]}]$$
$$+O\left[(kd)^{-\frac{5}{2}}\right])+O\left[(kr)^{-\frac{3}{2}}\right]. \quad (12)$$

For $\frac{3\pi}{2}<\theta<2\pi$.

$$u_d(P)=\frac{e^{ikr}}{\sqrt{r}}([D(2\pi-\theta,\theta_0-\frac{\pi}{2})$$
$$\times e^{ik[a(\cos\theta+\sin\theta_0)+b(\cos\theta_0+\sin\theta)]}$$
$$+D(\theta-\pi,\frac{3\pi}{2}-\theta_0)$$
$$\times e^{ik[-a(\cos\theta+\sin\theta_0)+b(\cos\theta_0+\sin\theta)]}$$
$$+D(\theta-\frac{3\pi}{2},\pi-\theta_0)$$
$$\times e^{ik[-a(\cos\theta+\sin\theta_0)-b(\cos\theta_0+\sin\theta)]}]$$
$$-\frac{1}{2ik}[\frac{D_\theta(0,\frac{3\pi}{2}-\theta_0)D_{\theta_0}(2\pi-\theta,0)}{(2a)^{\frac{3}{2}}}$$
$$\times e^{ik[a(2+\cos\theta-\sin\theta_0)+b(\cos\theta_0+\sin\theta)]}$$
$$+\frac{D_\theta(0,\theta_0-\frac{\pi}{2})D_{\theta_0}(\theta-\pi,0)}{(2a)^{\frac{3}{2}}}$$
$$\times e^{ik[a(2-\cos\theta+\sin\theta_0)+b(\cos\theta_0+\sin\theta)]}$$
$$+\frac{D_\theta(0,\pi-\theta_0)D_{\theta_0}(\frac{5\pi}{2}-\theta,0)}{(2b)^{\frac{3}{2}}}$$
$$\times e^{ik[b(2+\sin\theta-\cos\theta_0)-a(\sin\theta_0+\cos\theta)]}$$
$$+\frac{D_\theta(0,\theta_0)D_{\theta_0}(\theta-\frac{3\pi}{2},0)}{(2b)^{\frac{3}{2}}}$$
$$\times e^{ik[b(2-\sin\theta+\cos\theta_0)-a(\sin\theta_0+\cos\theta)]}]$$
$$+O\left[(kd)^{-\frac{5}{2}}\right])+O\left[(kr)^{-\frac{3}{2}}\right]. \quad (13)$$

In the situation, for oblique incidence, where the observation point P is close to the specular or shadow boundaries these expressions are no longer valid. The reason being that the first square brackets of the expressions (10) to (13), become infinite on these boundaries. In this case more refined uniform asymptotics have been carried out to deal with this situation. Combining all these refinements graphs can be drawn a typical graph of the final results is shown below:

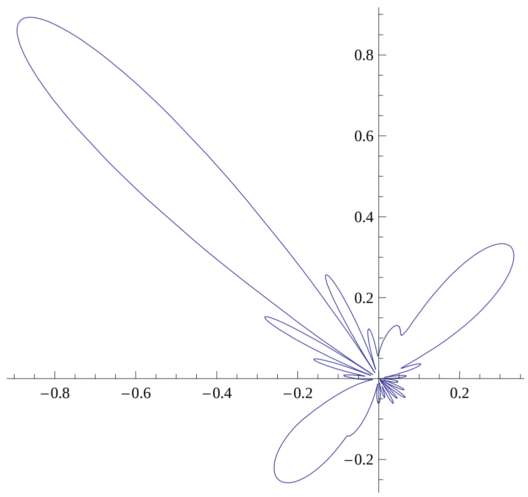

Figure 1: $a=1, b=2, \vartheta=4.14159+I, r=10, k=2\pi, \theta_0=3\pi/4$.

Finally we remark that in the limit as $|N|\to\infty$ the above expressions reduces to far field scattering by a perfectly conducting cylinder which agrees with the results of [5].

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