STABILITY OF MONOTONE SOLUTIONS FOR THE SHADOW GIERER-MEINHARDT SYSTEM WITH FINITE DIFFUSIVITY

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ABSTRACT. We consider the following shadow system of the Gierer-Meinhardt model:

$$\begin{cases} A_t = \epsilon^2 A_{xx} - A + \frac{A^p}{\xi^q},\ 0 < x < 1,\ t > 0, \\ \tau \xi_t = -\xi + \xi^{-s} \int_0^1 A^2\, dx, \\ A > 0,\ A_x(0,t) = A_x(1,t) = 0, \end{cases}$$

where $1 < p < +\infty$, $\frac{2q}{p-1} > s+1$, $s \geq 0$, and $\tau > 0$. It is known that a nontrivial monotone steady-state solution exists if and only if

$$\epsilon < \frac{\sqrt{p-1}}{\pi}.$$

In this paper, we show that for any $\epsilon < \frac{\sqrt{p-1}}{\pi}$, and $p = 2$ or $p = 3$, there exists a unique $\tau_c > 0$ such that for $\tau < \tau_c$ this steady state is linearly stable while for $\tau > \tau_c$ it is linearly unstable. (This result is optimal.) The transversality of this Hopf bifurcation is proven. Other cases for the exponents as well as extensions to higher dimensions are also considered. Our proof makes use of functional analysis and the properties of Weierstrass functions and elliptic integrals.

1. INTRODUCTION

The Gierer-Meinhardt system has been very popular for the theoretical investigation of pattern formation in living organisms. Following the analysis of Turing [17] a lot of work has been established in studying the linear stability of trivial (constant) steady states. Recently there have been many studies on patterns for singularly perturbed systems for which one of the diffusivities is **very small** and the solutions concentrate at finitely many points of the domain, either for the shadow system or for the full system. In

1991 *Mathematics Subject Classification.* Primary 35B45, ; Secondary 35J40.

Key words and phrases. Stability, Gierer-Meinhardt system, Hopf bifurcations, finite diffusivities.

this paper we do not make this smallness assumption. Rather we give a complete picture of the stability behavior of the shadow system in terms of the parameters τ and ϵ but under more restrictive conditions on the exponents in the system.

We study monotone solutions for the following **shadow system** of the generalized Gierer-Meinhardt system ([6], [11]):

$$\begin{cases} A_t = \epsilon^2 \Delta A - A + \frac{A^p}{\xi^q}, \quad x \in \Omega, \quad t > 0, \\ \tau \xi_t = -\xi + \xi^{-s} \frac{1}{|\Omega|} \int_\Omega A^r \, dx, \\ A > 0, \quad \frac{\partial A}{\partial \nu} = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\epsilon > 0, \tau > 0$ are positive constants, $\Delta := \sum_{i=1}^N \frac{\partial^2}{\partial x^2}$ is the usual Laplace operator, $\Omega \subset R^N$ is a bounded and smooth domain, and the exponents (p, q, r, s) satisfy the following condition:

$$(H0) \qquad p > 1, \quad q > 0, \quad r > 0, \quad s \geq 0, \quad \gamma := \frac{qr}{(p-1)(s+1)} > 1.$$

In the original Gierer-Meinhardt system [6], we have $(p, q, r, s) = (2, 1, 2, 0)$ and (H0) holds.

Problem (1.1) can be derived by formally taking $D \to +\infty$ in the following generalized Gierer-Meinhardt system:

$$(GM) \qquad \begin{cases} A_t = \epsilon^2 \Delta A - A + \frac{A^p}{H^q}, \quad \text{in } \Omega, \\ \tau H_t = D \Delta H - H + \frac{A^r}{H^s}, \qquad \text{in } \Omega, \\ A, H > 0, \frac{\partial A}{\partial \nu} = \frac{\partial H}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

The unknowns $A = A(x,t)$ and $H = H(x,t)$ represent the concentrations of the activator and inhibitor, respectively. For the derivation of (1.1) from (GM), we refer the interested reader to [12], [14], [15], [18] for more details.

In this paper, we consider the case $N = 1$. (In the last section, some extensions to higher dimensions are discussed.) Without loss of generality, we may assume that $\Omega = (0, 1)$. That is, we consider

$$\begin{cases} A_t = \epsilon^2 A_{xx} - A + \frac{A^p}{\xi^q}, \quad 0 < x < 1, \quad t > 0, \\ \tau \xi_t = -\xi + \xi^{-s} \int_0^1 A^r \, dx, \\ A > 0, \quad A_x(0, t) = A_x(1, t) = 0. \end{cases} \tag{1.2}$$

The steady-state problem of (1.2) is equivalent to the following problem for the transformed function u_ϵ given by $u_\epsilon(x) = \xi^{-\frac{q}{p-1}} A(x)$:

$$\xi^{1+s-\frac{qr}{p-1}} = \int_0^1 u^r(x)dx$$

and

$$\epsilon^2 u_{xx} - u + u^p = 0,\ u_x(x) < 0,\ 0 < x < 1,\ u_x(0) = u_x(1) = 0. \tag{1.3}$$

Letting

$$L := \frac{1}{\epsilon} \tag{1.4}$$

and rescaling $u(x) = w_L(y)$, where $y = Lx$, we see that w_L satisfies the following ODE:

$$w_L^{''} - w_L + w_L^p = 0,\ w_L^{'}(y) < 0,\ 0 < y < L,\ w_L^{'}(0) = w_L^{'}(L) = 0. \tag{1.5}$$

Since (1.5) is an autonomous ODE, it is easy to see that a nontrivial solution exists if and only if

$$\epsilon < \frac{\sqrt{p-1}}{\pi} \quad (\text{or} \quad L > \frac{\pi}{\sqrt{p-1}}). \tag{1.6}$$

If $\epsilon \geq \frac{\sqrt{p-1}}{\pi}$ (or $L \leq \frac{\pi}{\sqrt{p-1}}$), then $w_L = 1$.

The stability of steady-state solutions to (1.2) has been a subject of study in the last few years. A recent result of [13] (see Theorem 1.1 of [13]) says that a stable solution to (1.2) must be asymptotically monotone. More precisely, if $(A(x,t), \xi(t)), t \geq 0$ is a solution to (1.2) that is linearly neutrally stable, then there is a $t_0 > 0$ such that

$$A_x(x, t_0) \neq 0 \text{ for all } (x,t) \in (0,1) \times [t_0, +\infty). \tag{1.7}$$

Thus all **non-monotone** steady-state solutions are linearly unstable. Therefore we focus our attention on monotone solutions. There are two monotone solutions – the monotone increasing one and the monotone decreasing one. Since these two solutions differ by reflection, we consider the monotone decreasing function only. This solution is then called u_ϵ and it has the least energy among all positive solutions of (1.3), see [15]. If $L \leq \frac{\pi}{\sqrt{p-1}}$, then $w_L = 1$. We also denote the corresponding solutions to (1.2) by

$$A_L(x) = \xi_L^{\frac{q}{p-1}} w_L(Lx), \quad \xi_L^{1+s-\frac{qr}{p-1}} = \int_0^1 w_L^r(Lx)dx. \tag{1.8}$$

In [14] and [15], it is proved that under the assumption that ϵ **is sufficiently small** (or, equivalently, that L is sufficiently large) that $|\frac{qr}{p-1} - s - 1|$ is small, and that either $r = 2$, $1 < p < 5$ or $r = p+1$, then (A_L, ξ_L) is linearly stable for τ small. The authors use the SLEP (singular limit eigenvalue problem) approach. In [18], it is proved that for ϵ **sufficiently small**, and

$$\text{either } r = 2,\ 1 < p \leq 5, \quad \text{or } r = p+1,\ 1 < p < +\infty \tag{1.9}$$

then u_ϵ is linearly stable for τ small. The NLEP (nonlocal eigenvalue problem) approach is used.

An interesting and important question is the following: Are such stability results valid for finite ϵ or finite L, respectively without the above smallness or largeness assumption? This is of practical importance since in real-world experiments one has fixed physical constants and one can not make such a smallness assumption. Thus this present theory is helpful in predicting experimental results. The main purpose of this paper is to investigate this question. It turns out in some cases we are able to study stability for all finite ϵ (or L) and give a complete picture of the stability behavior.

Before stating our results, we first introduce some notation. Let $I = (0, L)$ and $\phi \in H^2(I)$. We define the following operator:

$$\mathcal{L}[\phi] = \phi^{''} - \phi + pw_L^{p-1}\phi. \tag{1.10}$$

In Section 2, we shall prove that $\mathcal{L}$ has the spectrum

$$\lambda_1 > 0, \quad \lambda_j < 0,\ j = 2, 3, \ldots. \tag{1.11}$$

Hence for the map $\mathcal{L}$ from $H^2(I)$ to $L^2(I)$ we know that

$$(H1) \qquad \mathcal{L}^{-1} \quad \text{exists},$$

where $\mathcal{L}^{-1}$ is the inverse of $\mathcal{L}$. This implies that $\mathcal{L}^{-1}w_L$ is well-defined.

Our first result is the following theorem.

Theorem 1.1. *Let (H0) be true. Assume that $L > \frac{\pi}{\sqrt{p-1}}$ and either*

$$r = 2 \tag{1.12}$$

and

$$(H2) \qquad \int_0^L w_L \mathcal{L}^{-1} w_L \, dy > 0$$

or

$$r = p + 1. \tag{1.13}$$

Then (A_L, ξ_L) (given by (1.8) is a linearly stable steady state to (1.2) for τ small.

This theorem reduces the issue of stability to the computation of the integral $\int_0^L w_L \mathcal{L}^{-1} w_L \, dy$. This integral is quite difficult to compute for general L. In the two limiting cases: $L \to +\infty$ or $L \sim \frac{\pi}{\sqrt{p-1}}$, one can use asymptotic analysis to compute this integral (see Lemma 2.2 below). If L is sufficiently large (which is equivalent to ϵ being sufficiently small), one can show that (H2) holds, i.e. $\int_0^L w_L \mathcal{L}^{-1} w_L \, dy > 0$, for $1 < p < 5$ and $\int_0^L w_L \mathcal{L}^{-1} w_L \, dy < 0$ for $p > 5$. Thus Theorem 1.1 recovers results of [14] and [18]. In the second case, it is easy to show that when L is near $\frac{\pi}{\sqrt{p-1}}$, then $w_L \sim 1$, $\mathcal{L}^{-1} w_L \sim \frac{1}{p-1}$, and hence $\int_0^L w_L \mathcal{L}^{-1} w_L \, dy > 0$. This implies that for $r = 2$, and for any $p > 1$, there exists some $L_p > \frac{\pi}{\sqrt{p-1}}$, such that (A_L, ξ_L) is stable for $L < L_p$. This is a new result.

For τ finite, we have the following theorem.

Theorem 1.2. *Let (H0) and (H2) be true. Let $r = 2$ and $L > \frac{\pi}{\sqrt{p-1}}$. Then there exists a unique $\tau_c > 0$ such that for $\tau < \tau_c$, (A_L, ξ_L) is stable and for $\tau > \tau_c$, (A_L, ξ_L) is unstable. At $\tau = \tau_c$, there exists a unique Hopf bifurcation. Furthermore, the Hopf bifurcation is transversal, namely, we have*

$$\frac{d\lambda_R}{d\tau}|_{\tau=\tau_c} > 0, \tag{1.14}$$

where λ_R is the real part of the eigenvalue.

The following theorem follows from Theorem 1.2 by the results in Section 2 where one uses Weierstrass $p(z)$ functions and Jacobi elliptic integrals to show that $\int_0^L w_L \mathcal{L}^{-1} w_L \, dy > 0$ for all $L > \pi$ in the cases $r = 2$, $p = 2, 3$.

The original Gierer-Meinhardt system ($(p,q,r,s)=(2,1,2,0)$)) falls into this class. Thus for the shadow system of the original Gierer-Meinhardt system, we have a complete picture of the stability of (A_L,ξ_L) for any $\tau>0$ and any $L>0$. The result is included in the following theorem.

Theorem 1.3. *Let (H0) be true. Assume that $L>\frac{\pi}{\sqrt{p-1}}$ and $r=2$, $p=2$ or 3. Then there exists a unique $\tau_c>0$ such that for $\tau<\tau_c$, (A_L,ξ_L) is stable and for $\tau>\tau_c$, (A_L,ξ_L) is unstable. At $\tau=\tau_c$, there exists a Hopf bifurcation. Furthermore, the Hopf bifurcation is transversal.*

Theorem 1.3 gives a complete picture of the stability of nontrivial monotone solutions in terms of L since for $L\leq\frac{\pi}{\sqrt{p-1}}$ we necessarily have $w_L\equiv 1$. Combining this with the results of [13], we have completely classified stability and instability of all steady-state solutions for **all** $\epsilon>0$ for the shadow system of the classical Gierer-Meinhardt system.

We remark that standard singular perturbation techniques which work for small ϵ can not be used here since ϵ is finite and not necessarily small. The hypergeometric function approach of [5] does not work here, either. Our rigorous approach is based on functional analysis and PDE estimates. We follow the approaches used in [18], [22], [23]. (Technically speaking, we have to analyze w_L instead of w_∞ which is more difficult).

For the existence and stability of multiple spikes for finite inhibitor diffusivity D and small ϵ, we refer to [7], [16], [20], [21], [22] and the references therein. We recall that in the current work we study the complementary case of finite ϵ and infinite D.

In the last section, we shall discuss some extensions to higher dimensions, in particular about the relevance and validity of the conditions (H0), (H1), and (H2).

The organization of this paper is as follows:

In Section 2, we discuss some properties of w_L. In particular, we calculate the integral $\int_0^L w_L\mathcal{L}^{-1}w_L\,dy$ for $p=2$ and $p=3$.

In Section 3, we derive a nonlocal eigenvalue problem and prove Theorem 1.1 in the case $r=p+1$.

In Section 4, we prove Theorem 1.1 in the case $r=2$.

Theorem 1.2 is proved in Sections 5 and 6: The Hopf bifurcation for finite τ is discussed in Section 5. The transversality of the Hopf bifurcation is proved in Section 6.

Section 7 contains some extensions to higher dimensions.

Acknowledgments.

The research of JW is supported by an Earmarked Grant from RGC of Hong Kong. MW thanks the Department of Mathematics at CUHK for their kind hospitality. We would like to thank Prof. M. J. Ward for useful discussions.

2. Some properties of w_L

In this section, we use Weierstrass functions and elliptic integrals to study the properties of w_L – the unique solution of the following ODE:

$$w_L^{''} - w_L + w_L^p = 0,\ w_L^{'}(0) = w_L^{'}(L) = 0,\ w_L^{'}(y) < 0 \text{ for } 0 < y < L. \tag{2.1}$$

Recall that

$$\mathcal{L}[\phi] = \phi^{''} - \phi + p w_L^{p-1}\phi.$$

We first have

Lemma 2.1. *Consider the following eigenvalue problem:*

$$\begin{cases} \mathcal{L}\phi = \lambda\phi, & 0 < y < L, \\ \phi^{'}(0) = \phi^{'}(L) = 0. \end{cases} \tag{2.2}$$

Then the eigenvalues can be arranged in such a way that

$$\lambda_1 > 0, \quad \lambda_j < 0,\ j = 2, 3, \ldots . \tag{2.3}$$

Moreover, the eigenfunction corresponding to λ_1 (denoted by Φ_1) can be made positive.

Proof: Let the eigenvalues of $\mathcal{L}$ be arranged by $\lambda_1 \geq \lambda_2 \geq \ldots$. It is well-known that $\lambda_1 > \lambda_2$ and that the eigenfunction Φ_1 corresponding to λ_1 is positive. Moreover,

$$-\lambda_1 = \min_{\int_0^L \phi^2\, dy = 1} \left(\int_0^L (|\phi^{'}|^2 + \phi^2 - p w_L^{p-1}\phi^2)\, dy \right) \tag{2.4}$$

$$\leq \left(\int_0^L w_L^2\,dy\right)^{-1}\left(\int_0^L (|w_L^{'}|^2+w_L^2-pw_L^{p-1}w_L^2)\,dy\right)<0.$$

Next we claim that $\lambda_2 \leq 0$. This follows from a classical argument (see Theorem 2.11 of [10]). For the sake of completeness, we include a proof here. By the variational characterization of λ_2, we have

$$-\lambda_2=\sup_{v\in H^1(I)}\inf_{\phi\in H^1(I),\phi\not\equiv 0}\left[\frac{\int_0^L (|\phi^{'}|^2+\phi^2-pw_L^{p-1}\phi^2)\,dy}{\int_0^L\phi^2\,dy}\ :\ v\not\equiv 0,\int_0^L \phi v\,dy=0\right]. \tag{2.5}$$

On the other hand, w_L has least energy, that is

$$E[w_L]=\inf_{u\not\equiv 0,u\in H^1(I)} E[u],$$

where

$$E[u]=\frac{\int_0^L(|u^{'}|^2+u^2)\,dy}{(\int_0^L u^{p+1}\,dy)^{\frac{2}{p+1}}}.$$

Let

$$h(t)=E[w_L+t\phi],\quad \phi\in H^1(I).$$

Then $h(t)$ attains its minimum at $t=1$ and hence

$$h^{''}(0)=2\left[\int_0^L(|\phi^{'}|^2+\phi^2)\,dy-p\int_0^L w_L^{p-1}\phi^2\,dy+2\frac{(\int_0^L w_L^p\phi\,dy)^2}{\int_0^L w_L^{p+1}\,dy}\right]$$
$$\times\frac{1}{\left(\int_0^L w_L^{p+1}\,dy\right)^{2/(p+1)}}\geq 0.$$

By (2.5), we see that

$$-\lambda_2\geq \inf_{\int_0^L \phi w_L^p\,dy=0}\left[\int_0^L(|\phi^{'}|^2+\phi^2)\,dy-p\int_0^L w_L^{p-1}\phi^2\,dy+2\frac{(\int_0^L w_L^p\phi\,dy)^2}{\int_0^L w_L^{p+1}\,dy}\right]$$
$$\times\frac{1}{\left(\int_0^L w_L^{p+1}\,dy\right)^{2/(p+1)}}\geq 0.$$

Finally, we claim that $\lambda_2<0$. But this follows from the proof of uniqueness of w_L, see [9].

□

By Lemma 2.1, $\mathcal{L}^{-1}$ exists and hence $\mathcal{L}^{-1}w_L$ is well-defined. Our next goal in this section is to compute the integral $\int_0^L w_L\mathcal{L}^{-1}w_L\,dy$. We begin with the following simple lemma, whose proof follows from a perturbation argument.

Lemma 2.2. *We have*

$$\lim_{L\to\frac{\pi}{\sqrt{p-1}}} \int_0^L w_L \mathcal{L}^{-1} w_L \, dy = \frac{\pi}{(p-1)^{\frac{3}{2}}}, \tag{2.6}$$

$$\lim_{L\to+\infty} \int_0^L w_L \mathcal{L}^{-1} w_L \, dy = \left(\frac{1}{p-1} - \frac{1}{4}\right) \int_0^\infty w_\infty^2 \, dy, \tag{2.7}$$

where $w_\infty(y)$ *is the unique solution of*

$$w^{''} - w + w^p = 0, \, w^{'}(0) = 0, \, w^{'}(y) < 0, \, w(y) > 0, \, 0 < y < +\infty. \tag{2.8}$$

For general p, it is quite difficult to compute $\int_0^L w_L \mathcal{L}^{-1} w_L \, dy$. However, if $p = 2$ or $p = 3$, this is possible by using elliptic integrals.

We first state the following theorem.

Lemma 2.3. *Let* $p = 2$. *Then we have*

$$\int_0^L w_L \mathcal{L}^{-1} w_L \, dy > 0$$

for all $L > \pi$.

Before we prove Lemma 2.3, let us first write w_L in terms of Weierstrass functions. For the definitions and properties of Weierstrass functions, we refer the reader to [1].

Now we assume that $p = 2$. Let $w_L(0) = M$, $w_L(L) = m$.

¿From (2.1), we have

$$(w_L^{'})^2 = w_L^2 - \frac{2}{3} w_L^3 - M^2 + \frac{2}{3} M^3 \tag{2.9}$$

and

$$-m^2 + \frac{2}{3} m^3 = -M^2 + \frac{2}{3} M^3. \tag{2.10}$$

¿From (2.10), we deduce that

$$\frac{Mm}{M+m} = M + m - \frac{3}{2}. \tag{2.11}$$

Now let

$$\hat{w} = -\frac{1}{6} w_L + \frac{1}{12}. \tag{2.12}$$

Then, by simple computations, $\hat{w}$ satisfies the following equation:

$$(\hat{w}^{'})^2 = 4\hat{w}^3 - g_2\hat{w} - g_3 = 4(\hat{w}-e_1)(\hat{w}-e_2)(\hat{w}-e_3), \tag{2.13}$$

where

$$g_2 = \frac{1}{12}, \quad g_3 = -\frac{1}{216} - \frac{1}{36}\left(-M^2 + \frac{2}{3}M^3\right), \tag{2.14}$$

$$e_1 = \frac{1}{6}(M+m) - \frac{1}{6},\ e_2 = -\frac{1}{6}m + \frac{1}{12},\ e_3 = -\frac{1}{6}M + \frac{1}{12}. \tag{2.15}$$

Recalling the definition of the Weierstrass function $p(z)$ (see [1]), we get

$$\hat{w}(x) = p(x+\alpha; g_2, g_3) \tag{2.16}$$

for some constant α. We omit the dependence of p on g_2 and g_3.

Then we have

$$p(f_i) = e_i,\ p^{'}(f_i) = 0,\ i = 1,2,3,\ f_1 + f_2 + f_3 = 0. \tag{2.17}$$

Thus we obtain that

$$\hat{w}(x) = p(f_3 + x), \quad L = f_1. \tag{2.18}$$

Let us also recall the Weierstrass function $\zeta(z)$:

$$\zeta(z) = \frac{1}{z} - \int_0^z \left(p(u) - \frac{1}{u^2}\right) du,$$

which satisfies

$$\zeta^{'}(u) = -p(u),\ \zeta(f_i) = \eta_i,\ i = 1,2,3,\ \eta_1 + \eta_2 + \eta_3 = 0. \tag{2.19}$$

Now we compute

$$\begin{aligned} \int_0^L \hat{w}(x)dx = \int_0^{f_1} p(f_3+x)dx = -\zeta(u)|_{f_3}^{-f_2} = \zeta(f_3) + \zeta(f_2) \\ = -\zeta(f_1) = -\zeta(L). \end{aligned} \tag{2.20}$$

This implies that

$$\int_0^L w_L^2\, dy = \int_0^L w_L\, dy = \int_0^L \left(-6\hat{w} + \frac{1}{2}\right) dy = 6\zeta(L) + \frac{L}{2}. \tag{2.21}$$

Using the formulas on page 649 of [1], we have

$$\zeta(L) = \frac{K(k)}{3L}[3E(k) + (k-2)K(k)], \tag{2.22}$$

$$e_1 = \frac{(2-k)K^2(k)}{3L^2},$$

$$e_2 = \frac{(2k-1)K^2(k)}{3L^2},$$

$$e_3 = \frac{-(k+1)K^2(k)}{3L^2},$$

where e_1, e_2 and e_3 are defined by (2.15) and satisfy

$$e_1e_2 + e_2e_3 + e_1e_3 = -\frac{1}{4}g_2 = -\frac{1}{48},$$

and $E(k)$ and $K(k)$ are Jacobi elliptic integrals:

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2\sin^2\varphi}\,d\varphi, \quad K(k) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2\sin^2\varphi}}d\varphi.$$

Thus we obtain the following relation between k and L:

$$L = 2(k^2 - k + 1)^{\frac{1}{4}}K(k). \tag{2.23}$$

¿From (2.23), we compute that (dropping the argument k of K):

$$\frac{dL}{dk} = \frac{4K^2((2k-1)K^2 + 4KK^{'}(k^2-k+1))}{L^3}. \tag{2.24}$$

Equation (2.23) determines k as a function of L uniquely if we choose $L > \pi$. Moreover, $\frac{dk}{dL} > 0$ and

$$(2k-1)K + 4K^{'}(k^2-k+1) > 0. \tag{2.25}$$

We are now ready to prove Lemma 2.3.

Proof of Lemma 2.3:

Let us denote $\phi_L = \mathcal{L}^{-1}w_L$. That is, ϕ_L satisfies

$$\phi_L^{''} - \phi_L + 2w_L\phi_L = w_L, \quad \phi_L^{'}(0) = \phi_L^{'}(L) = 0.$$

Set

$$\phi_L = w_L + \frac{1}{2}yw_L^{'}(y) + \Psi. \tag{2.26}$$

Then $\Psi(y)$ satisfies

$$\Psi^{''} - \Psi + 2w_L\Psi = 0,$$

$$\Psi^{'}(0)=0, \quad \Psi^{'}(L)=-\frac{1}{2}Lw_L^{''}(L). \tag{2.27}$$

On the other hand, let $\Psi_0=\frac{\partial w_L}{\partial M}$. Then Ψ_0 satisfies

$$\Psi_0^{''}-\Psi_0+2w_L\Psi_0=0, \tag{2.28}$$

$$\Psi_0(0)=1, \Psi_0^{'}(0)=0.$$

Integrating (2.28), we have

$$\Psi_0^{'}(L)=\int_0^L \frac{\partial w_L}{\partial M}\,dy-2\int_0^L w_L\frac{\partial w_L}{\partial M}\,dy$$

$$=\frac{d}{dM}\left(\int_0^L (w_L-w_L^2)\,dy\right)-\left(w_L(L)-w_L^2(L)\right)\frac{dL}{dM}.$$

Using the equation for w_L, we have $\int_0^L (w_L-w_L^2)\,dy=0$. Thus we obtain

$$\Psi_0^{'}(L)=-(w_L(L)-w_L^2(L))\frac{dL}{dM}. \tag{2.29}$$

Comparing (2.27) and (2.29), we derive the following important relation:

$$\Psi(x)=\frac{1}{2}\frac{L}{\frac{dL}{dM}}\Psi_0(x). \tag{2.30}$$

Hence, we have

$$\int_0^L w_L\phi_L\,dy=\int_0^L \left(w_L+\frac{1}{2}yw_L^{'}+\Psi\right)w_L\,dy$$

$$=\frac{3}{4}\int_0^L w_L^2\,dy+\frac{1}{4}Lw_L^2(L)+\frac{L}{2}\left(\frac{dL}{dM}\right)^{-1}\int_0^L w_L\Psi_0\,dy. \tag{2.31}$$

On the other hand,

$$\int_0^L w_L\Psi_0\,dy=\int_0^L w_L\frac{\partial w_L}{\partial M}\,dy$$

$$=\frac{1}{2}\frac{d}{dM}\int_0^L w_L^2\,dy-\frac{1}{2}w_L^2(L)\frac{dL}{dM}$$

$$=\frac{1}{2}\left[\frac{d}{dL}\int_0^L w_L^2\,dy-w_L^2(L)\right]\frac{dL}{dM}. \tag{2.32}$$

Substituting (2.32) into (2.31), we obtain that

$$\int_0^L w_L\phi_L\,dy=\frac{3}{4}\int_0^L w_L^2\,dy+\frac{1}{4}L\frac{d}{dL}\int_0^L w_L^2\,dy \tag{2.33}$$

$$=\frac{L^{-2}}{4}\frac{d}{dL}\left(L^3\int_0^L w_L^2\,dy\right).$$

We now compute, using the formulas (2.21) and (2.23),

$$L^3\int_0^L w_L^2\,dy = L^3\int_0^L w_L\,dy = 2L^2K[3E+(k-2)K]+\frac{L^4}{2}$$
$$= 8\sqrt{k^2-k+1}K^3[3E+(k-2+\sqrt{k^2-k+1})K]. \tag{2.34}$$

If $2k-1\geq 0$, it is easy to see that

$$\frac{1}{8}\frac{d}{dk}\left(L^3\int_0^L w_L^2\right)>0.$$

If $2k-1<0$, we have to use the inequality (2.25) and the following formulas:

$$\frac{dK}{dk}=\frac{E-(k')^2K}{k(k')^2},\qquad \frac{dE}{dk}=\frac{E-K}{k},$$

where $k'=\sqrt{1-k^2}$, and obtain:

$$\frac{1}{8}\frac{d}{dk}\left(L^3\int_0^L w_L^2\right)=\frac{d}{dk}[\sqrt{k^2-k+1}K^3[3E+\rho_kK]]$$
$$=\sqrt{k^2-k+1}K^2\left[9\frac{dK}{dk}E+3K\frac{dE}{dk}+\frac{d\rho_k}{dk}K^2+4\rho_kK\frac{dK}{dk}+\frac{2k-1}{2(k^2-k+1)}K[3E+\rho_kK]\right]$$
$$=\sqrt{k^2-k+1}K^2\left[3\frac{d(EK)}{dk}+2E\left(\frac{dK}{dk}+\frac{2k-1}{4(k^2-k+1)}K\right)+4\frac{dK}{dk}(E+\rho_kK)\right]$$
$$+\sqrt{k^2-k+1}K^2\left[K\left(\frac{d\rho_k}{dk}K+\frac{2k-1}{2(k^2-k+1)}(2E+\rho_kK)\right)\right],$$

where $\rho_k=k-2+\sqrt{k^2-k+1}$. Each term in the above equality is positive. The proof consists of elementary calculus and thus omitted.

This finishes the proof of the lemma.

$\square$

Our next case is $p=3$. We have

Lemma 2.4. *Assume that $p=3$. Then*

$$\int_0^L w_L\mathcal{L}^{-1}w_L\,dy>0.$$

Proof: By using a similar approach as in Lemma 2.3, we have in the case $p=3$,

$$\int_0^L w_L\mathcal{L}^{-1}w_L\,dy=\frac{1}{4}\int_0^L w_L^2\,dy+\frac{1}{4}L\frac{d}{dL}\left(\int_0^L w_L^2\,dy\right) \tag{2.35}$$

$$= \frac{1}{4}\frac{d}{dL}\left(L\int_0^L w_L^2\,dy\right).$$

In terms of elliptic integrals, we have

$$L = \sqrt{1-\frac{k^2}{2}}K(k), \tag{2.36}$$

$$L\int_0^L w_L^2\,dy = E(k)K(k). \tag{2.37}$$

Then we can compute that

$$\frac{dk}{dL} = \sqrt{1-\frac{k^2}{2}}\left(\frac{(1-\frac{k^2}{2})E^2}{k(k')^2} - \frac{K}{k}\right)^{-1} > 0$$

and

$$\frac{d(EK)}{dk} = \frac{E^2}{k(k')^2} - \frac{K}{k} > 0.$$

This shows that

$$\frac{d}{dL}\left(L\int_0^L w_L^2\,dy\right) = \frac{d(EK)}{dk}\frac{dk}{dL} > 0.$$

$\square$

Remark: In general, let w_L be the unique solution of

$$w_L'' - w_L + w_L^p = 0, w_L'(0) = w_L'(L) = 0, w_L'(y) < 0 \quad \text{for } 0 < y < L, \tag{2.38}$$

then similar computations as in Lemma 2.3 give the following formulas:

$$\int_0^L w_L\mathcal{L}^{-1}w_L\,dy = \left(\frac{1}{p-1}-\frac{1}{4}\right)\int_0^L w_L^2\,dy + \frac{1}{4}L\frac{d}{dL}\left(\int_0^L w_L^2\,dy\right) \tag{2.39}$$

and

$$\frac{d}{dL}\left(\int_0^L w_L\mathcal{L}^{-1}w_L\,dy\right) = \frac{1}{p-1}\frac{d}{dL}\int_0^L w_L^2\,dy + \frac{1}{4}L\frac{d^2}{dL^2}\left(\int_0^L w_L^2\,dy\right). \tag{2.40}$$

The main problem now is that we do not have an explicit formula for $\int_0^L w_L^2\,dy$ for general p. Numerical computation is indispensable.

We put forward the following conjecture which is supported by numerical computations.

Conjecture: The function $\int_0^L w_L\mathcal{L}^{-1}w_L\,dy$ is monotone decreasing in L.

If the conjecture holds, as a consequence there exists a unique L_p (which may be ∞) such that

$$\int_0^L w_L \mathcal{L}^{-1} w_L \, dy > 0 \text{ for } L < L_p \text{ and } \int_0^L w_L \mathcal{L}^{-1} w_L \, dy < 0 \text{ for } L > L_p.$$

3. Nonlocal Eigenvalue Problems

We linearize (1.2) around the solution (A_L, ξ_L), where

$$A_L = \xi^{\frac{q}{p-1}} w_L(Lx), \quad \xi_L^{1+s-\frac{qr}{p-1}} = \int_0^1 w_L^r(Lx) dx. \tag{3.1}$$

It is easy to see that we arrive at the following eigenvalue problem:

$$\epsilon^2 \phi_{xx} - \phi + p w_L^{p-1} \phi - q\eta \xi^{\frac{pq}{p-1}-q-1} w_L^p = \lambda\phi, \tag{3.2}$$

$$-\eta - s\eta \xi_L^{-s-1+\frac{qr}{p-1}} \int_0^1 w_L^r dx + r\xi_L^{-s+\frac{q(r-1)}{p-1}} \int_0^1 w_L^{r-1} \phi \, dx = \tau\lambda\eta.$$

We also rescale:

$$y = Lx. \tag{3.3}$$

Solving the second equation for η and substituting into the equation for ϕ, we arrive at the following nonlocal eigenvalue problem (NLEP):

$$\phi'' - \phi + p w_L^{p-1} \phi - \frac{qr}{s+1+\tau\lambda} \frac{\int_0^L w_L^{r-1} \phi \, dy}{\int_0^L w_L^r \, dy} w_L^p = \lambda\phi, \quad y \in (0, L), \tag{3.4}$$

$$\phi'(0) = \phi'(L) = 0$$

and

$$\lambda = \lambda_R + \sqrt{-1}\lambda_I \in \mathcal{C} \tag{3.5}$$

In the present section, we let $\tau = 0$. Thus (3.4) becomes

$$L_\gamma[\phi] := \mathcal{L}[\phi] - \gamma(p-1) \frac{\int_0^L w_L^{r-1} \phi \, dy}{\int_0^L w_L^r \, dy} w_L^p = \lambda\phi, \quad \phi'(0) = \phi'(L) = 0. \tag{3.6}$$

Let us first show that $\lambda = 0$ is not an eigenvalue of (3.4) if $\gamma \neq 1$.

Lemma 3.1. *Suppose that $\gamma \neq 1$. Then $\lambda = 0$ is not an eigenvalue of (3.4).*

Proof: Suppose $\lambda = 0$. Then we have

$$0 = \mathcal{L}[\phi] - \gamma(p-1)\frac{\int_0^L w_L^{r-1}\phi\,dy}{\int_0^L w_L^r\,dy} w_L^p$$
$$= \mathcal{L}\left(\phi - \gamma\frac{\int_0^L w_L^{r-1}\phi\,dy}{\int_0^L w_L^r\,dy} w_L\right).$$

By Lemma 2.1,

$$\phi - \gamma\frac{\int_0^L w_L^{r-1}\phi\,dy}{\int_0^L w_L^r\,dy} w_L = 0.$$

Multiplying this equation by w_L^{r-1} and integrating, we get

$$(1-\gamma)\int_0^L w_L^{r-1}\phi\,dy = 0.$$

Hence, since $\gamma \neq 1$,

$$\int_0^L w_L^{r-1}\phi\,dy = 0,$$

therefore

$$\mathcal{L}[\phi] = 0$$

and by Lemma 2.1

$$\phi = 0.$$

□

We next show that the unstable eigenvalues are bounded uniformly in τ.

Lemma 3.2. *Let λ be an eigenvalue of (3.4) with $Re(\lambda) \geq 0$. Then there exists a constant C which is independent of $\tau > 0$ such that*

$$|\lambda| \leq C. \tag{3.7}$$

Proof: Multiplying (3.4) by $\bar{\phi}$ – the conjugate of ϕ – and integrating, we obtain that

$$\lambda\int_0^L |\phi|^2\,dy = -\int_0^L (|\phi'|^2 + |\phi|^2 - pw_L^{p-1}|\phi|^2)\,dy$$
$$- \frac{qr}{1+s+\tau\lambda}\,\frac{(\int_0^L w_L^{r-1}\phi\,dy)(\int_0^L w_L^p\bar{\phi}\,dy)}{\int_0^L w_L^r\,dy}. \tag{3.8}$$

Here we have used notation: $|\phi|^2 = \phi\bar{\phi}$. Since

$$\left|\frac{qr}{1+s+\tau\lambda}\right| \leq \frac{qr}{1+s} \quad \text{for } \mathrm{Re}(\lambda) \geq 0, \tag{3.9}$$

we see that

$$\left|\frac{qr}{1+s+\tau\lambda}\frac{(\int_0^L w_L^{r-1}\phi\,dy)(\int_0^L w_L^p\bar{\phi}\,dy)}{\int_0^L w_L^r\,dy}\right| \le C\int_0^L |\phi|^2\,dy, \tag{3.10}$$

where C is independent of τ.

(3.7) follows from (3.8) and (3.10).

□

We first study (3.6) the case $r = p+1$ which is easy since the operator is self-adjoint.

The case $r = 2$ will be studied in the next section. It is more difficult since the operator is not self-adjoint and thus has complex eigenvalues.

Lemma 3.3. *Assume that $r = p+1$ and $L > \frac{\pi}{\sqrt{p-1}}$. Then all eigenvalues of (3.6) are real and*

(a) if $\gamma > 1$, then $\lambda < 0$;

(b) if $\gamma = 1$, then $\lambda \le 0$ and zero is an eigenvalue with eigenfunction w_L:

(c) if $\gamma < 1$, then there exists an eigenvalue $\lambda_0 > 0$ to (3.6).

¿From Lemma 3.3, we see that when $r = p+1$, $\gamma = 1$ is the borderline case between stability and instability.

Proof: Since $r = p+1$, we see that the operator L_γ is selfadjoint and hence the eigenvalues are real. Let $\lambda_0 \ge 0$ be an eigenvalue of (3.6). We first claim that $\lambda_0 \ne \lambda_1$, where λ_1 is the first eigenvalue of $\mathcal{L}$ given by Lemma 2.1. In fact, if $\lambda_0 = \lambda_1$, then we have

$$\gamma\frac{\int_0^L w_L^p\phi\,dy}{\int_0^L w_L^{p+1}\,dy}\int_0^L w_L^p\Phi_1\,dy = 0$$

and hence, since $\Phi_1 > 0$,

$$\int_0^L w_L^p\phi\,dy = 0, \quad \mathcal{L}[\phi] = \lambda_1\phi.$$

Therefore, $\phi = \Phi_1$. This is impossible since $\Phi_1 > 0$. So $\lambda_0 \ne \lambda_1$.

By Lemma 2.1, $(\mathcal{L}-\lambda_0)^{-1}$ exists and hence $\lambda_0 > 0$ is an eigenvalue of (3.6) if and only if it satisfies the following algebraic equation:

$$\int_0^L w_L^{p+1}\,dy = \gamma(p-1)\int_0^L [((\mathcal{L}-\lambda_0)^{-1}w_L^p)w_L^p]\,dy. \tag{3.11}$$

Let

$$\rho(t) = \int_0^L w_L^{p+1}\,dy - \gamma(p-1)\int_0^L [((\mathcal{L}-t)^{-1}w_L^p)w_L^p]\,dy, \quad t \ge 0,\, t \ne \lambda_1.$$

Then $\rho(0) = (1-\gamma)\int_0^L w_L^{p+1}\,dy$ and

$$\rho'(t) = -\gamma(p-1)\int_0^L [((\mathcal{L}-t)^{-2}w_L^p)w_L^p]\,dy < 0.$$

On the other hand,

$$\rho(t) \to -\infty \text{ as } t \to \lambda_1,\, t < \lambda_1,$$

$$\rho(t) \to +\infty \text{ as } t \to \lambda_1,\, t > \lambda_1,$$

$$\rho(t) \to \int_0^L w_L^{p+1}\,dy \text{ as } t \to +\infty.$$

Thus $\rho(t) > 0$ for $t > \lambda_1$ and $\rho(t)$ has a (unique zero) in $(0,\lambda_1)$ if and only if $\rho(0) > 0$.

This shows that for $\gamma > 1$, $\rho(t) \ne 0$ for $t \ge 0$ and for $\gamma < 1$, $\rho(t)$ has a unique root $t = \lambda_0 \in (0,\lambda_1)$.

For $\gamma = 1$, $\rho(0) = 0$ and hence zero is an eigenvalue. Note that $\mathcal{L}w_L = (p-1)w_L^p$. So w_L is the eigenfunction corresponding to the zero eigenvalue.

This proves the lemma.

□

Remark: Combining Lemma 3.2 and Lemma 3.3, we see that for $\gamma > 1$, $r = p+1$ and τ small, the conclusion (a) of Lemma 3.3 still holds.

Thus Theorem 1.1 has been proved in the case $r = p+1$.

4. Nonlocal Eigenvalue Problem: The case $r = 2$

We consider the eigenvalue problem (3.4) and prove Theorem 1.1 in the case $r = 2$. Note that the operator L_γ is not self-adjoint anymore and there are complex eigenvalues.

We first assume that $\tau = 0$. We then have

Lemma 4.1. *Assume that $r = 2, \tau = 0$ and (H2) holds, i.e.*

$$\int_0^L w_L \mathcal{L}^{-1} w_L\,dy > 0. \tag{4.1}$$

Then for any eigenvalue λ of (3.6), we have

$$Re(\lambda) < 0.$$

To prove Lemma 4.1, we need the following key inequality:

Lemma 4.2. *If (H2) holds, i.e.* $\int_0^L w_L \mathcal{L}^{-1} w_L\, dy > 0$*, then there exists a positive constant* $a_1 > 0$ *such that*

$$Q[\phi,\phi] := \int_0^L (|\phi'|^2 + \phi^2 - p w_L^{p-1}\phi^2)\, dy + \frac{2(p-1)\int_0^L w_L^p \phi\, dy \int_0^L w_L \phi\, dy}{\int_0^L w_L^2\, dy} \tag{4.2}$$
$$-(p-1)\frac{\int_0^L w_L^{p+1}\, dy}{\left(\int_0^L w_L^2\, dy\right)^2}\left(\int_0^L w_L\phi\, dy\right)^2 \geq a_1 d_{L^2}^2(\phi, X_1),\ \ \forall \phi \in H^1(0,L),$$

where $X_1 = span\ \{w\}$ *and* d_{L^2} *means the distance in the* L^2*-norm.*

Let us assume that Lemma 4.2 is true. Then we proceed to prove following lemma.

Lemma 4.3. *Let* (λ,ϕ) *satisfy (3.4) with* $Re(\lambda) \geq 0$*. Assume that* $r = 2$ *and (H2) holds, i.e.* $\int_0^L w_L \mathcal{L}^{-1} w_L\, dy > 0$*. Then we have*

$$Re[\bar{\lambda}\chi(\tau\lambda) - \lambda] + (p-1)|\chi(\tau\lambda) - 1|^2 \left(\frac{\int_0^L w_L^{p+1}\, dy}{\int_0^L w_L^2\, dy}\right) \leq 0, \tag{4.3}$$

where

$$\chi(\tau\lambda) = \frac{\gamma}{1 + \frac{\tau\lambda}{s+1}}, \quad \gamma = \frac{qr}{s+1}. \tag{4.4}$$

and $\bar{\lambda}$ *is the conjugate of* λ*.*

Proof of Lemma 4.3: Let (λ,ϕ) be a solution of (3.4). Set $\lambda = \lambda_R + \sqrt{-1}\lambda_I$ and $\phi = \phi_R + \sqrt{-1}\phi_I$. Let $\chi(\tau\lambda)$ be given in (4.4). Then, by taking (3.4) and its conjugate, we obtain the following two equations:

$$\mathcal{L}\phi - (p-1)\chi(\tau\lambda)\frac{\int_0^L w_L\phi\, dy}{\int_0^L w_L^2\, dy} w_L^p = \lambda\phi, \tag{4.5}$$

$$\mathcal{L}\bar{\phi} - (p-1)\bar{\chi}(\tau\lambda)\frac{\int_0^L w_L\bar{\phi}\, dy}{\int_0^L w_L^2\, dy} w_L^p = \bar{\lambda}\bar{\phi}. \tag{4.6}$$

Multiplying (4.5) by $\bar{\phi}$ and integrating by parts, we obtain

$$-\lambda\int_0^L |\phi|^2 - (p-1)\chi(\tau\lambda)\frac{(\int_0^L w_L\phi\, dy)(\int_0^L w_L^p\bar{\phi}\, dy)}{\int_0^L w_L^2\, dy} \tag{4.7}$$
$$= \int_0^L (|\phi'|^2 + |\phi|^2)\, dy - p\int_0^L w_L^{p-1}|\phi|^2\, dy.$$

Multiplying (4.6) by w_L, we obtain

$$(p-1)\int_0^L w_L^p\bar{\phi}\,dy-(p-1)\bar{\chi}(\tau\lambda)\frac{\int_0^L w_L\bar{\phi}\,dy}{\int_0^L w_L^2\,dy}\int_0^L w_L^{p+1}\,dy=\bar{\lambda}\int_0^L w_L\bar{\phi}\,dy. \tag{4.8}$$

Multiplying (4.8) by $\int_0^L w_L\phi\,dy$ and substituting the resulting expression into (4.7), we arrive at

$$\int_0^L(|\phi'|^2+|\phi|^2-pw_L^{p-1}|\phi|^2)\,dy+\lambda\int_0^L|\phi|^2\,dy \tag{4.9}$$
$$=-\chi(\tau\lambda)\left[\bar{\lambda}+\overline{\chi}(\tau\lambda)(p-1)\left(\frac{\int_0^L w_L^{p+1}\,dy}{\int_0^L w_L^2\,dy}\right)\right]\frac{|\int_0^L w_L\phi\,dy|^2}{\int_0^L w_L^2\,dy}.$$

We write (4.9) in terms of the quadratic functional Q defined in Lemma 4.2 and deduce, using (4.8) again, that

$$\left[\mathrm{Re}[\bar{\lambda}\chi(\tau\lambda)-\lambda]+(p-1)|\chi(\tau\lambda)-1|^2\left(\frac{\int_0^L w_L^{p+1}\,dy}{\int_0^L w_L^2\,dy}\right)\right]\frac{|\int_0^L w_L\phi\,dy|^2}{\int_0^L w_L^2\,dy} \tag{4.10}$$
$$=-Q[\phi_R,\phi_R]-Q[\phi_I,\phi_I]-\mathrm{Re}(\lambda)\left[\int_0^L|\phi|^2\,dy-\frac{|\int_0^L w_L\phi\,dy|^2}{\int_0^L w_L^2\,dy}\right]\le 0,$$

which proves the lemma.

□

Lemma 4.1 follows from Lemma 4.3:

Proof of Lemma 4.1:

In fact, let $\tau=0$. Then from (4.3), we have

$$\mathrm{Re}[\bar{\lambda}\chi(\tau\lambda)-\lambda]+(p-1)|\chi(\tau\lambda)-1|^2\left(\frac{\int_0^L w_L^{p+1}\,dy}{\int_0^L w_L^2\,dy}\right)$$
$$=(\gamma-1)\mathrm{Re}(\lambda)+(p-1)|\gamma-1|^2\left(\frac{\int_0^L w_L^{p+1}\,dy}{\int_0^L w_L^2\,dy}\right)\le 0$$

and hence

$$\mathrm{Re}(\lambda)\le-(p-1)(\gamma-1)\left(\frac{\int_0^L w_L^{p+1}\,dy}{\int_0^L w_L^2\,dy}\right)<0$$

since $\gamma>1$.

□

Remark: Combining Lemma 3.2 and Lemma 4.1, we see that for $\gamma>1$, $r=2$, $\int_0^L w_L\mathcal{L}^{-1}w_L\,dy>0$ and τ small, the conclusion of Lemma 4.1 still holds.

We are now ready to prove Lemma 4.2.

Proof of Lemma 4.2:

We consider the following self-adjoint operator:

$$\mathcal{L}_1\phi := \mathcal{L}\phi - (p-1)\frac{\int_0^L w_L\phi\,dy}{\int_0^L w_L^2\,dy}w_L^p$$
$$-(p-1)\frac{\int_0^L w_L^p\phi\,dy}{\int_0^L w_L^2\,dy}w_L + (p-1)\frac{\int_0^L w_L^{p+1}\,dy\int_0^L w_L\phi\,dy}{(\int_0^L w_L^2\,dy)^2}w_L. \tag{4.11}$$

Clearly, $\mathcal{L}_1$ is self-adjoint and

$$Q[\phi,\phi]\geq 0 \iff \quad \mathcal{L}_1 \text{ has no positive eigenvalues.}$$

By simple computations, we get

$$\mathcal{L}_1 w_L = 0.$$

On the other hand, if $\mathcal{L}_1\phi = 0$, then

$$\mathcal{L}\phi = c_1(\phi)w_L + c_2(\phi)w_L^p,$$

where

$$c_1(\phi) = (p-1)\frac{\int_0^L w_L^p\phi\,dy}{\int_0^L w_L^2\,dy} - (p-1)\frac{\int_0^L w_L^{p+1}\,dy\int_0^L w_L\phi\,dy}{(\int_0^L w_L^2\,dy)^2}, \tag{4.12}$$

$$c_2(\phi) = (p-1)\frac{\int_0^L w_L\phi\,dy}{\int_0^L w_L^2\,dy}. \tag{4.13}$$

Hence

$$\phi - c_1(\phi)(\mathcal{L}^{-1}w_L) - c_2(\phi)\frac{1}{p-1}w_L = 0. \tag{4.14}$$

Substituting (4.14) into (4.12), we have

$$c_1(\phi) = (p-1)c_1(\phi)\frac{\int_0^L w_L^p\mathcal{L}^{-1}w_L\,dy}{\int_0^L w_L^2\,dy}$$
$$-(p-1)c_1(\phi)\frac{\int_0^L w_L^{p+1}\,dy\int_0^L w_L\mathcal{L}^{-1}w_L\,dy}{(\int_0^L w_L^2\,dy)^2}$$
$$= c_1(\phi) - (p-1)c_1(\phi)\frac{\int_0^L w_L^{p+1}\,dy\int_0^L w_L\mathcal{L}^{-1}w_L\,dy}{(\int_0^L w_L^2\,dy)^2}$$

after integration by parts. By (H2), we have $c_1(\phi) = 0$. Hence $\phi = c_2(\phi)\frac{1}{p-1}w_L$. This shows that w_L is the only eigenfunction of $\mathcal{L}_1$ corresponding to the eigenvalue zero.

Now suppose $\mathcal{L}_1$ has a positive eigenvalue $\lambda_0 > 0$ with ϕ_0 as eigenfunction. Since $\mathcal{L}_1$ is self-adjoint and w_L is an eigenfunction, we may assume that

$$\int_0^L w_L \phi_0 \, dy = 0. \tag{4.15}$$

Using (4.15), we see that ϕ_0 satisfies

$$(\mathcal{L} - \lambda_0)\phi_0 = (p-1)\frac{\int_0^L w_L^p \phi_0}{\int_0^L w_L^2} w_L. \tag{4.16}$$

Note that $\int_0^L w_L^p \phi_0 \, dy \neq 0$. In fact, if $\int_0^L w_L^p \phi_0 \, dy = 0$, then $\lambda_0 > 0$ is an eigenvalue of $\mathcal{L}$. By Lemma 2.1, $\lambda_0 = \lambda_1$ and ϕ_0 has constant sign. This contradicts with the fact that $\phi_0 \perp w_L$. Therefore $\lambda_0 \neq \lambda_1$. Hence $\mathcal{L} - \lambda_0$ is invertible. So (4.16) implies

$$\phi_0 = (p-1)\frac{\int_0^L w_L^p \phi_0 \, dy}{\int_0^L w_L^2 \, dy} (\mathcal{L} - \lambda_0)^{-1} w_L.$$

Thus

$$\int_0^L w_L^p \phi_0 \, dy = (p-1)\frac{\int_0^L w_L^p \phi_0 \, dy}{\int_0^L w_L^2 \, dy} \int_0^L ((\mathcal{L} - \lambda_0)^{-1} w_L) w_L^p \, dy.$$

Since $\int_0^L w_L^p \phi_0 \, dy \neq 0$, we have

$$\int_0^L w_L^2 \, dy = (p-1) \int_0^L ((\mathcal{L} - \lambda_0)^{-1} w_L) w_L^p \, dy$$

and therefore

$$\int_0^L w_L^2 \, dy = \int_0^L ((\mathcal{L} - \lambda_0)^{-1} w_L)((\mathcal{L} - \lambda_0) w_L + \lambda_0 w_L) \, dy.$$

Since $\lambda_0 > 0$ this gives

$$0 = \int_0^L ((\mathcal{L} - \lambda_0)^{-1} w_L) w_L \, dy. \tag{4.17}$$

Let $\beta(t) = \int_0^L ((\mathcal{L} - t)^{-1} w_L) w_L \, dy$ for $t > 0, t \neq \lambda_1$, then

$$\beta(0) = \int_0^L (\mathcal{L}^{-1} w_L) w_L \, dy > 0$$

by assumption (H2) and

$$\beta'(t) = \int_0^L ((\mathcal{L} - t)^{-2} w_L) w_L \, dy > 0.$$

This implies $\beta(t) > 0$ for all $t \in (0, \lambda_1)$.

On the other hand,

$$\beta(t) \to 0 \text{ as } t \to +\infty$$

and hence $\beta(t) < 0$ for $t > \lambda_1$.

In conclusion, $\beta(t) \neq 0$ for $t > 0, t \neq \lambda_1$. This shows that (4.17) is impossible. So $\mathcal{L}_1$ has no positive eigenvalue.

Since

$$Q[\phi, \phi] = -\int_0^L (\mathcal{L}_1 \phi)\phi \, dy,$$

we see that $Q[\phi, \phi] \geq 0$ for all ϕ and equality holds if and only if $\phi = cw_L$ for some constant c.

The proof is completed.

□

5. Uniqueness of the Hopf Bifurcation when $r = 2$

In the previous two sections, we have assumed that $\tau = 0$. In this section, we study the case $\tau > 0$. In general, it is quite difficult to analyze the corresponding Hopf bifurcation. However, when $r = 2$, we have a good picture.

We begin with a perturbation result whose proof is the same as in [2], where the Hopf bifurcation for $L >> 1$ is studied.

Lemma 5.1. *For τ large, there exists a real and positive eigenvalue λ_0 to (3.4). Moreover, as $\tau \to +\infty$,*

$$\lambda_0 = \lambda_1 + O\left(\frac{1}{\tau}\right), \tag{5.1}$$

where λ_1 is given in Lemma 2.1.

For $r = 2$, the following lemma shows the existence and uniqueness of the Hopf bifurcation.

Lemma 5.2. *Let $r = 2$ and assume that (H2) holds, i.e. $\int_0^L w_L \mathcal{L}^{-1} w_L \, dy > 0$. Then there exists a unique $\tau = \tau_c(L, p)$ such that problem (3.4) has two conjugate imaginary eigenvalues*

$$\lambda = \pm\sqrt{-1}\lambda_I, \quad \lambda_I > 0.$$

Proof:

Let $\lambda_0 = \sqrt{-1}\lambda_I$ be an eigenvalue of (3.4). We shall derive the equation for λ_I and τ.

Without loss of generality, we may assume that $\lambda_I > 0$. (Note that $-\sqrt{-1}\lambda_I$ is also an eigenvalue of (3.4).) Then $\phi_0 = (\mathcal{L} - \sqrt{-1}\lambda_I)^{-1} w_L^2$ up to a real constant factor. Then (3.4) becomes

$$\frac{\int_{R^2} w_L \phi_0 \, dy}{\int_{R^2} w_L^2 \, dy} = \frac{s+1+\sqrt{-1}\tau\lambda_I}{qr}. \tag{5.2}$$

Let $\phi_0 = \phi_0^R + \sqrt{-1}\phi_0^I$. Then from (5.2), by taking the real and imaginary parts, respectively, we obtain the following two equations:

$$\frac{\int_0^L w_L \phi_0^R \, dy}{\int_0^L w_L^2 \, dy} = \frac{s+1}{qr}, \tag{5.3}$$

$$\frac{\int_0^L w_L \phi_0^I \, dy}{\int_0^L w_L^2 \, dy} = \frac{\tau\lambda_I}{qr}. \tag{5.4}$$

Note that (5.3) is independent of τ.

Let us now compute $\int_0^L w_L \phi_0^R \, dy$. Observe that (ϕ_0^R, ϕ_0^I) satisfies

$$\mathcal{L}\phi_0^R = w_L^p - \lambda_I \phi_0^I, \quad \mathcal{L}\phi_0^I = \lambda_I \phi_0^R.$$

So $\phi_0^R = \lambda_I^{-1}\mathcal{L}\phi_0^I$ and

$$\phi_0^I = \lambda_I (\mathcal{L}^2 + \lambda_I^2)^{-1} w_L^p, \quad \phi_0^R = \mathcal{L}(\mathcal{L}^2 + \lambda_I^2)^{-1} w_L^p. \tag{5.5}$$

Substituting (5.5) into (5.3) and (5.4), we obtain

$$\frac{\int_0^L [w_L \mathcal{L}(\mathcal{L}^2 + \lambda_I^2)^{-1} w_L^p] \, dy}{\int_0^L w_L^2 \, dy} = \frac{s+1}{qr}, \tag{5.6}$$

$$\frac{\int_0^L [w_L (\mathcal{L}^2 + \lambda_I^2)^{-1} w_L^p] \, dy}{\int_0^L w_L^2 \, dy} = \frac{\tau}{qr}. \tag{5.7}$$

Let $\alpha(\lambda_I) = \frac{\int_0^L w_L \mathcal{L}(\mathcal{L}^2+\lambda_I^2)^{-1} w_L^p \, dy}{\int_0^L w_L^2 \, dy}$. Then integration by parts gives $\alpha(\lambda_I) = (p-1)\frac{\int_0^L w_L^p (\mathcal{L}^2+\lambda_I^2)^{-1} w_L^p \, dy}{\int_0^L w_L^2 \, dy}$. Note that

$$\alpha'(\lambda_I) = -2\lambda_I \frac{\int_0^L w_L^p (\mathcal{L}^2 + \lambda_I^2)^{-2} w_L^p \, dy}{\int_0^L w_L^2 \, dy} < 0,$$

$$\alpha(0) = \frac{\int_0^L w_L (\mathcal{L}^{-1} w_L^p) \, dy}{\int_0^L w_L^2 \, dy} = \frac{1}{p-1} > \frac{s+1}{qr},$$

and

$$\alpha(\lambda_I) \to 0 \quad \text{as} \quad \lambda_I \to \infty.$$

So there exists a unique $\lambda_I > 0$ such that (5.6) holds. Substituting this unique λ_I into (5.7), we obtain a unique $\tau = \tau_c$.

Finally, we show that $\tau_c > 0$. To this end, we make use of the inequality (4.3). Substituting $\chi(\tau) = \frac{\gamma}{1+\frac{\tau\sqrt{-1}\lambda_I}{s+1}}$, $\lambda = \sqrt{-1}\lambda_I$ into (4.3), we see that

$$\mathrm{Re}(\bar{\lambda}\chi(\tau\lambda)) + |\chi(\tau\lambda) - 1|^2 \left(\frac{\int_0^L w_L^{p+1}\, dy}{\int_0^L w_L^2\, dy} \right) \leq 0. \tag{5.8}$$

Since

$$\mathrm{Re}(\bar{\lambda}\chi(\tau\lambda)) = -\frac{\tau\lambda_I^2\gamma}{s+1+\frac{(\tau\lambda_I)^2}{s+1}} < 0,$$

we see immediately that $\tau = \tau_c > 0$. (In fact, (5.8) also gives an explicit bound for τ_c.)

Lemma 5.2 is thus proved.

□

6. Transversality of the Hopf Bifurcation for $r = 2$

In Section 5, we have shown that for $r = 2$, there exists a unique $\tau = \tau_c > 0$ such that the eigenvalue problem (3.4) has a Hopf bifurcation. In this section, we show the transversality of this Hopf bifurcation and thus finish the proof of Theorem 1.2. Namely, we prove the following lemma.

Lemma 6.1. *Suppose $r = 2$ and (H2) holds, i.e. $\int_0^L w_L \mathcal{L}^{-1} w_L\, dy > 0$. Let τ_c be the unique point, where a Hopf bifurcation for (3.4) occurs. Then we have*

$$\frac{d\lambda_R}{d\tau}|_{\tau=\tau_c} > 0. \tag{6.1}$$

Thus the eigenvalues cross through the imaginary axis from the left to the right as τ crosses τ_c.

Proof: Let $\lambda = \lambda_R + \sqrt{-1}\lambda_I$ be an eigenvalue of (3.4) with eigenfunction Ψ. Then similar to the proof of Lemma 5.2, (3.4) is equivalent to

$$\frac{s+1+\tau\lambda_R+\sqrt{-1}\tau\lambda_I}{qr} = \frac{\int_0^L w_L\Psi\, dy}{\int_0^L w_L^2\, dy}, \tag{6.2}$$

where Ψ satisfies

$$\mathcal{L}\Psi = \lambda\Psi + w_L^p, \quad \Psi'(0) = \Psi'(L) = 0. \tag{6.3}$$

Let us assume that

$$\Psi=\Psi_R+\sqrt{-1}\Psi_I, \quad \lambda=\lambda_R+\sqrt{-1}\lambda_I.$$

Then (6.3) is equivalent to

$$(\mathcal{L}-\lambda_R)\Psi_R=-\lambda_I\Psi_I+w_L^p, \tag{6.4}$$

$$(\mathcal{L}-\lambda_R)\Psi_I=\lambda_I\Psi_R. \tag{6.5}$$

¿From (6.5) we can express Ψ_R in terms of Ψ_I and substitute it into (6.4). Thus we obtain

$$\Psi_I=\lambda_I[(\mathcal{L}-\lambda_R)^2+\lambda_I^2]^{-1}w_L^p, \tag{6.6}$$

$$\Psi_R=(\mathcal{L}-\lambda_R)[(\mathcal{L}-\lambda_R)^2+\lambda_I^2]^{-1}w_L^p. \tag{6.7}$$

Substituting (6.6) and (6.7) into (6.2), we obtain the two equations

$$\frac{s+1+\tau\lambda_R}{qr}\int_0^L w_L^2\,dy=\int_0^L\left(w_L(\mathcal{L}-\lambda_R)[(\mathcal{L}-\lambda_R)^2+\alpha_I]^{-1}w_L^p\right)\,dy \tag{6.8}$$

and

$$\frac{\tau}{qr}\int_0^L w_L^2\,dy=\int_0^L\left(w_L[(\mathcal{L}-\lambda_R)^2+\alpha_I]^{-1}w_L^p\right)\,dy, \tag{6.9}$$

where

$$\alpha_I=\lambda_I^2>0.$$

Substituting (6.9) into (6.8), we deduce that

$$\frac{s+1+2\tau\lambda_R}{qr}\int_0^L w_L^2\,dy=\int_0^L\left(w_L\mathcal{L}[(\mathcal{L}-\lambda_R)^2+\alpha_I]^{-1}w_L^p\right)\,dy. \tag{6.10}$$

Now, differentiating (6.10) and (6.9) with respect to $\tau=\tau_c$ and recalling that

$$\lambda_R(\tau_c)=0,$$

we have

$$\frac{1}{qr}\int_0^L w_L^2\,dy=$$

$$=2\int_0^L(\mathcal{L}w_L[\mathcal{L}^2+\alpha_I]^{-2}w_L^p)\,dy\frac{d\lambda_R}{d\tau}|_{\tau=\tau_c}-\int_0^L(w_L[\mathcal{L}^2+\alpha_I]^{-2}w_L^p)\,dy\frac{d\alpha_I}{d\tau}|_{\tau=\tau_c} \tag{6.11}$$

and

$$\frac{2\tau_c}{qr}\int_0^L w_L^2\,dy\frac{d\lambda_R}{d\tau}|_{\tau=\tau_c} =$$

$$= 2\int_0^L (\mathcal{L}w_L[\mathcal{L}^2+\alpha_I]^{-2}\mathcal{L}w_L^p)\,dy\frac{d\lambda_R}{d\tau}|_{\tau=\tau_c} - \int_0^L (\mathcal{L}w_L[\mathcal{L}^2+\alpha_I]^{-2}w_L^p)\,dy. \tag{6.12}$$

Multiplying (6.11) by $\int_0^L(\mathcal{L}w_L[\mathcal{L}^2+\alpha_I]^{-2}w_L^p)\,dy$ and (6.12) by $\int_0^L(w_L[\mathcal{L}^2+\alpha_I]^{-2}w_L^p)\,dy$ and subtracting the resulting equations, we arrive at

$$\Bigg[[\frac{2\tau_c}{qr}\int_0^L w_L^2\,dy - 2\int_0^L(\mathcal{L}^2 w_L[\mathcal{L}^2+\alpha_I]^{-2}w_L^p)\,dy]\int_0^L(w_L[\mathcal{L}^2+\alpha_I]^{-2}w_L^p)\,dy$$

$$+2\left(\int_0^L \mathcal{L}w_L[\mathcal{L}^2+\alpha_I]^{-2}w_L^p\,dy\right)^2\Bigg]\frac{d\lambda_R}{d\tau}|_{\tau=\tau_c} \tag{6.13}$$

$$=\frac{1}{qr}\int_0^L w_L^2\,dy\int_0^L(\mathcal{L}w_L[\mathcal{L}^2+\alpha_I]^{-2}w_L^p)\,dy$$

$$=\frac{p-1}{qr}\int_0^L w_L^2\,dy\int_0^L(w_L^p[\mathcal{L}^2+\alpha_I]^{-2}w_L^p)\,dy>0.$$

On the other hand, by (6.9) we have at $\tau=\tau_c$,

$$\frac{\tau_c}{qr}=\frac{\int_0^L(w_L[\mathcal{L}^2+\alpha_I]^{-1}w_L^p)\,dy}{\int_0^L w_L^2\,dy}.$$

Thus we see that

$$[\frac{2\tau_c}{qr}\int_0^L w_L^2\,dy - 2\int_0^L(\mathcal{L}^2 w_L[\mathcal{L}^2+\alpha_I]^{-2}w_L^p)\,dy]\int_0^L(w_L[\mathcal{L}^2+\alpha_I]^{-2}w_L^p)\,dy$$

$$=2\alpha_I\left(\int_0^L(w_L[\mathcal{L}^2+\alpha_I]^{-2}w_L^p)\,dy\right)^2>0. \tag{6.14}$$

Substituting (6.14) into (6.13), we conclude that

$$\frac{d\lambda_R}{d\tau}|_{\tau=\tau_c}>0.$$

□

Now Theorem 1.2 follows from Lemma 4.1, Lemma 5.1, Lemma 5.2 and Lemma 6.1.

□

7. Extensions to Higher Dimensions

In the previous sections, we have studied the one-dimensional case. We observe that two key ingredients are needed in our proofs: first (H1) that the operator $\mathcal{L}$ is invertible and second (H2) that the integral $\int_0^L w_L \mathcal{L}^{-1} w_L \, dy$ is positive.

Now let us extend this idea to general domains in $R^N, N \geq 2$. Namely, we consider

$$\begin{cases} A_t = \Delta A - A + \frac{A^p}{\xi^q}, \ x \in \Omega_L, \ t > 0, \\ \tau \xi_t = -\xi + \xi^{-s} \frac{1}{|\Omega_L|} \int_{\Omega_L} A^r \, dx, \\ A > 0, \frac{\partial A}{\partial \nu} = 0 \ \text{ on } \partial \Omega_L, \end{cases} \tag{7.1}$$

where we have rescaled the domain by $\Omega_L = \frac{1}{\epsilon} \Omega$ $(L = \frac{1}{\epsilon})$ and therefore the factor ϵ^2 in the equation vanishes. In this case, let us assume that $\Omega_L \subset R^N$ is a smooth and bounded domain, and the exponents (p, q, r, s) satisfy the following condition:

$$(H0) \qquad p > 1, \quad q > 0, \quad r > 0, \quad s \geq 0, \quad \gamma := \frac{qr}{(p-1)(s+1)} > 1,$$

where p is subcritical:

$$1 < p < \frac{N+2}{N-2} \ \text{ if } \ N \geq 3; \quad 1 < p < +\infty \ \text{ if } \ N = 2.$$

The steady state solution of (7.1) is given by

$$A = \xi^{\frac{q}{p-1}} u, \quad \xi^{1+s-\frac{qr}{p-1}} = \frac{1}{|\Omega_L|} \int_{\Omega_L} u^r \, dx, \tag{7.2}$$

where u is a solution of the following problem:

$$\begin{cases} \Delta u - u + u^p = 0, \quad u > 0 \ \text{ in } \ \Omega_L, \\ \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \partial \Omega_L. \end{cases} \tag{7.3}$$

We again consider the minimizing solution $w_L(x)$ which satisfies (7.3) and

$$E[w_L] = \inf_{u \in H^1(\Omega_L), u \not\equiv 0} E[u], \tag{7.4}$$

where

$$E[u] = \frac{\int_{\Omega_L} (|\nabla u|^2 + u^2) \, dy}{(\int_{\Omega_L} u^{p+1} \, dy)^{\frac{2}{p+1}}}.$$

The corresponding steady-state solution to the shadow system (7.1) is denoted by

$$A_L = \xi_L^{\frac{q}{p-1}} w_L, \quad \xi_L^{1+s-\frac{qr}{p-1}} = \frac{1}{|\Omega_L|} \int_{\Omega_L} w_L^r \, dx. \tag{7.5}$$

Let

$$\mathcal{L}[\phi] = \Delta\phi - \phi + p w_L^{p-1} \phi.$$

Then we have the following lemma, whose proof is similar to Lemma 2.1.

Lemma 7.1. *Consider the following eigenvalue problem:*

$$\begin{cases} \mathcal{L}\phi = \lambda\phi, \ \textit{in } \Omega_L, \\ \frac{\partial \phi}{\partial \nu} = 0 \ \textit{on } \partial\Omega_L. \end{cases} \tag{7.6}$$

Then $\lambda_1 > 0$ *and* $\lambda_2 \leq 0$.

We now make two important assumptions:

We first assume that

$$(H1) \qquad \mathcal{L}^{-1} \ \text{ exists.}$$

Under (H1), we sometimes assume that

$$(H2) \qquad \int_{\Omega_L} w_L (\mathcal{L}^{-1} w_L) \, dy > 0.$$

We can now state the following theorem.

Theorem 7.2. *Assume that either*

$$r = p + 1, \ \textit{and (H1) holds,}$$

or

$$r = 2, \ \textit{and (H1) and (H2) hold.}$$

Then (A_L, ξ_L) *is linearly stable for* τ *small.*

In the case $r = 2$*, there exists a unique* $\tau = \tau_c$ *such that* (A_L, ξ_L) *is stable for* $\tau < \tau_c$*, unstable for* $\tau > \tau_c$*, and there is a Hopf bifurcation at* $\tau = \tau_c$*. Furthermore, the Hopf bifurcation is transversal.*

The proof of Theorem 7.2 is similar to the one-dimensional case. We omit the details here.

It remains an interesting and difficult question as to verify (H1) and (H2) analytically. If L is large, it is shown in [3] and[19] that assumption (H1) is true and assumption (H2) holds if

$$1 < p < 1 + \frac{4}{N}. \tag{7.7}$$

This recovers the results of [18].

It is difficult to verify (H1) and (H2) for general ϵ. One may ask: Does (H1) hold true for **generic** domains? In summary, the stability issue for higher-dimensional systems still holds many challenging open problems.

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