Robust Synchronization for Two-Dimensional Discrete-Time Coupled Dynamical Networks

Jinling Liang, Zidong Wang, Xiaohui Liu and Panos Louvieris

Abstract— **In this paper, a new synchronization problem is addressed for an array of two-dimensional (2-D) coupled dynamical networks. The class of systems under investigation is described by the 2-D nonlinear state space model which is oriented from the well-known Fornasini-Marchesini second model. For such a new 2-D complex network model, both the network dynamics and the couplings evolve in two independent directions. A new synchronization concept is put forward to account for the phenomenon that the propagations of all 2-D dynamical networks are synchronized in two directions with influence from the coupling strength. The purpose of the problem addressed is to first derive sufficient conditions ensuring the global synchronization and then extend the obtained results to more general cases where the system matrices contain either the norm-bounded or the polytopic parameter uncertainties. An energy-like quadratic function is developed, together with the intensive use of the Kronecker product, to establish the easy-to-verify conditions under which the addressed 2-D complex network model achieves global synchronization. Finally, a numerical example is given to illustrate the theoretical results and the effectiveness of the proposed synchronization scheme.**

Keywords— **Two-dimensional systems, Complex networks, Coupling, Synchronization, Parameter uncertainties.**

I. Introduction

The past ten years have seen a tremendous upsurge in the research efforts towards the discovery of non-trivial topological features of complex networks, see [1,2] for the latest advances in the area. Typical complex networks that have been thoroughly investigated include communication networks, social networks, electrical power grids, cellular and metabolic networks and the internet. Dynamic behaviors of complex networks, such as stability, periodic oscillation, bifurcation, or even chaos, are ubiquitous in the real world and often reconfigurable. Complex networks have been studied in the context of dynamical systems in a range of disciplines which have already become an ideal research area for control engineer, mathematicians, computer scientists, and biologists to manage, analyze, and interpret functional information from real-world networks. Recently, as an emerging phenomenon of a population of dynamically interacting units, the synchronization problem has gained particular research attention for complex networks in various fields [3, 4]. For example, the synchronization problem has been thoroughly investigated for the large-scale networks of chaotic oscillators [5–7], the coupled systems exhibiting spatio-temporal chaos and autowaves [8, 9], the genetic oscillator networks [10, 11], and the array of coupled neural networks with or without delays [12, 13]. More specifically, in [14], it has been shown that the synchronization may help protecting interconnected neurons from the influence of random intrinsic neuronal noise, which affects all neurons in the nervous system. The experimental demonstration of chaotic phase synchronization has been reported in [15] for the unidirectionally coupled time-delay systems using electronic circuits. A variety of synchronous regimes has been found, respectively, in arrays of oscillatory Belousov-Zhabotinsky microdrops, in linear configurations and in arrays of partially stacked drops [16]. Furthermore, it has been experimentally confirmed that the inhibitory coupling is able to produce a rich variety of synchronous patterns.

Most existing results concerning complex networks have been established based on the assumption that the dynamical networks under consideration are accurate that vary continuously over time. Such an assumption, however, is not always true in practice. In nowadays digitalized world, it is quite common that the signal transmission over the network links is conducted in a *discrete-time* manner. Also, the connection weights of the nodes of complex networks are typically dependent on certain resistance and capacitance values that include uncertainties (modeling errors). Therefore, the robust synchronization problem for various kinds of discrete-time complex/neural networks has received considerable research interests in the past few years. For instance, in [17], the robust synchronization problem has been considered for an array of coupled stochastic discrete-time neural networks with time-varying delay, where the individual neural network is subject to parameter uncertainty, stochastic disturbance and time-varying delay. In [18], the synchronization problem has been investigated for an array of coupled complex discrete-time networks with the simultaneous presence of both the discrete and the distributed time delays, where a more general sector-like nonlinear function is employed to describe the nonlinearities existing in the network. The problem of synchronization stability has been further studied in [19] for discrete complex dynamical networks with a time-varying delay, where a new Lyapunov-Krasovskii functional has been constructed by dividing the time-varying delay into a constant part and a variant part.

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J. Liang is with the Department of Mathematics, Southeast University, Nanjing 210096, China.

Z. Wang, X. Liu and P. Louvieris are with the Department of Information Systems and Computing, Brunel University, Uxbridge UB8 3PH, United Kingdom. (Email: Zidong.Wang@brunel.ac.uk).

On another research frontier, the two-dimensional (2-D) systems, where the information propagation occurs in two independent directions, have received considerable research attention in the past few decades. The research on 2-D systems has mainly been inspired by the practical needs to represent continuous- and discrete-time nonlinear dynamic systems by using the Volterra series. 2-D systems have extensive applications in engineering problems such as long-wall coal cutting and metal rolling, the control of sheet-forming processes, image processing, seismographic data processing, thermal processes, and water stream heating, etc., see [20–22]. For example, in [23], a 2-D general learning model has been established for the mulitlayer feedforward neural network (MFNN) since the 2-D model can exhibit two independent dynamics: one index reflects the feedforward process of the network and the other reflects the learning process of the network. Therefore, the learning process of the MFNN in [23] has been treated as a 2-D discrete-time dynamical system. In [24], an improved 2-D Hamming-like neural network model has been developed to process the 2-D images with advantages of direct learning, easy reconfiguration (update or expand), multiple functions, controllable attraction-basin size, high storage capacity, enhanced local input fault tolerances, and suitability for optical implementations. Among a variety of models for 2-D systems, the Fornasini-Marchesini (F-M) first and second models as well as the Roesser model have been the popular ones because of their engineering insights in image processing. It should be noted that the 2-D space Roesser model, which has been proposed firstly in [25] and then extensively studied in the literature [26–29], could be embedded into the F-M model without increasing dimensions. By using the finite-difference method, many fundamental partial differential equations can be transformed into the Roesser model [28]. So far, many important results on 2-D systems have been reported in the literature. For example, the stability analysis problem for 2-D systems has been investigated in [27, 30], the controller and filter design problems have been considered in [26, 31–34], and the model approximation problem for 2-D digital filters has been studied in [35].

It should be pointed out that, up to now, almost all the research efforts on dynamical complex networks have been devoted to the one-dimensional (1-D) case where the network dynamics evolves along one direction only. However, in reality, many kinds of complex networks are best treated in two dimensions. The 2-D phenomenon has been addressed for complex networks as early as in [36] where the 2-D Fourier transform NMR (Nuclear magnetic resonance) was used to study the behaviors of the proton-proton networks with spin-spin couplings. Wireless sensor networks (WSNs) are a typical class of complex networks whose nodes are the sensors linked by the wireless communications. For WSNs, so far, different geometric topologies have been proposed to be the underlying network topologies to achieve the sparseness of the communication networks or to guarantee the package delivery of specific routing methods, and most available topology control algorithms have been based on the *two-dimensional* networks where all sensor nodes are distributed in a 2-D plane [37]. Another type of complex networks that exhibit dynamical behaviors along two dimensions are the genetic regulatory networks, see e.g. [38] for the collective dynamics of coupled 2-D chaotic maps on gene regulatory network of bacterium *Escherichia coli*. To this end, it can be concluded that many phenomena in nature can be modeled as 2-D complex networks whose dynamics (e.g. synchronization) analysis issue has, unfortunately, been largely overlooked in the area due primarily to the mathematical complexity. For example, the time synchronization problem is known to be crucial for 2-D wireless sensor networks, but few results have been available from a dynamics analysis perspective. It is, therefore, the main purpose of this paper to investigate how the synchronization of a 2-D complex network is affected by its topology through the coupling strength in the presence of the parameter uncertainties.

In this paper, we aim to make the one of the first few attempts to address the synchronization problem for an array of 2-D coupled dynamical networks. The class of systems under investigation is described by the 2-D nonlinear state space model which is oriented from the well-known FM second model. The purpose of the problem addressed is to derive sufficient conditions ensuring the global synchronization in terms of linear matrix inequalities. Furthermore, the obtained results are extended to the more general case where the system matrices contain parameter uncertainties in either the polytopic or the norm-bounded form. *The main contribution of this paper is mainly threefold: 1) a new 2-D complex network model is proposed where both the network dynamics and the couplings evolve in two directions; 2) based on this 2-D complex network model, the synchronization concept is put forward to account for the phenomenon that the propagations of all sub-networks are synchronized in both directions with the help from the coupling strength; and 3) an energy-like quadratic function is developed, together with the intensive use of the Kronecker product, to establish the easy-to-verify conditions under which the addressed 2-D complex network model achieves global synchronization.* In summary, this paper deals with a new problem for a new model using comprehensive mathematical analysis tools.

Notations: Throughout this paper, I_n is the $n \times n$ identity matrix and $\mathbb{Z}_+$ is the set of non-negative integers. The notation $X \geq 0$ (respectively, $X > 0$) means that X is real, symmetric and positive semidefinite (respectively, positive definite). '$*$' in a matrix is used to denote the term which is induced by symmetry. $\|\cdot\|$ refers to the Euclidean vector norm; and $\lambda_{\min}(\cdot)$, $\lambda_{\max}(\cdot)$ denote the minimum and the maximum eigenvalues of a real symmetric matrix respectively. The Kronecker product of matrices X and Y is denoted as $X \otimes Y$. Matrices, if they are not explicitly specified, are assumed to have compatible dimensions.

II. Problem formulation and preliminaries

Consider the following dynamical network consisting of N identical nodes with diffusive coupling, in which each node is a 2-D nonlinear state space system generalized from the well-known FM second model:

$$\begin{aligned} x_i(k+1,h+1) =& A_1 x_i(k+1,h) + A_2 x_i(k,h+1) \\ &+ f(x_i(k+1,h), x_i(k,h+1)) \\ &+ \sum_{j=1}^{N} G_{ij}^{(1)} \Gamma x_j(k+1,h) \\ &+ \sum_{j=1}^{N} G_{ij}^{(2)} \Lambda x_j(k,h+1), \end{aligned} \tag{1}$$

where $i = 1,2,\dots,N$; for k and $h \in \mathbb{Z}_+$, $x_i(k,h) \in \mathbb{R}^n$ is the state vector of the ith node; A_1 and A_2 are known real system matrices; Γ and Λ describe the inner couplings of the network in the horizontal and the vertical directions, respectively; $G^{(1)} = (G_{ij}^{(1)})_{N\times N}$ and $G^{(2)} = (G_{ij}^{(2)})_{N\times N}$ are the coupling configuration matrices denoting, respectively, the topological structures of the complex network in the horizontal and the vertical directions that satisfy the following diffusive coupling connections:

$$G_{ij}^{(q)} = G_{ji}^{(q)} \geq 0 \ \ (i \neq j), \quad G_{ii}^{(q)} = -\sum_{j=1, j\neq i}^{N} G_{ij}^{(q)}; \tag{2}$$

where $q = 1,2$; $i,j = 1,2,\dots,N$. The diffusive coupling configuration means that these connected nodes would be decoupled at the synchronized state. The nonlinear vector-valued continuous function $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is known and assumed to satisfy that for all $u, \tilde{u}, v, \tilde{v} \in \mathbb{R}^n$, the following condition holds:

$$\|f(u,v) - f(\tilde{u},\tilde{v})\| \leq \|B_1(u-\tilde{u}) + B_2(v-\tilde{v})\|, \tag{3}$$

where B_1 and B_2 are known real matrices of appropriate dimensions.

Remark 1: As stated in [27,34], the 2-D dynamical systems exist extensively in practical applications, such as those in thermal processes, water stream heating and image data processing and transmission [21,30]. As a field of investigation, 2-D FM second model has received considerable attention, see e.g. [39,40] for stability analysis issue using the Lyapunov approach, [32,34,41] for robust stability and stabilization issues, and [31,33] for the H_∞ filtering issue. It has been shown in [42] that Roesser model could be regarded as a special kind of FM second model without requiring any increasing of dimension, while the converse might not be true. For the representative results concerning Roesser model, we refer the readers to [26,27,29] for the H_∞ control studies, to [28] for application of hyperbolic partial differential equations, to [43] for the positive real control, and to [35] and [44] for the H_∞ model reduction and filtering research, respectively. It should be noted that, in all the references mentioned above, the model under consideration is assumed to be *linear* which would largely limit the application scope. In this paper, the *nonlinear* 2-D dynamical systems are to be studied and our main attention is focused on the synchronization behavior for an array of such systems coupled according to certain topologies.

The boundary condition associated with the dynamical network (1) is given as

$$x_i(0,k) = \varphi_i(k), \quad x_i(k,0) = \psi_i(k); \qquad k \in \mathbb{Z}_+ \tag{4}$$

where $i = 1,2,\dots,N$; φ_i, $\psi_i : \mathbb{R} \to \mathbb{R}^n$ are known nonlinear functions satisfying $\varphi_i(0) = \psi_i(0)$ and the following condition:

$$\lim_{M\to\infty} \sum_{k=0}^{M} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \big(\|\varphi_i(k) - \varphi_j(k)\|^2 + \|\psi_i(k) - \psi_j(k)\|^2 \big) < \infty. \tag{5}$$

For the purpose of simplicity, we introduce the following notations: $x(k,h) := \big(x_1^T(k,h), x_2^T(k,h), \dots, x_N^T(k,h)\big)^T$, $F(x(k+1,h), x(k,h+1)) := \big(f^T(x_1(k+1,h), x_1(k,h+1)), f^T(x_2(k+1,h), x_2(k,h+1)), \dots, f^T(x_N(k+1,h), x_N(k,h+1))\big)^T$. By using the Kronecker product, the coupled 2-D dynamical network (1) can be rewritten in the following compact form:

$$\begin{aligned} x(k+1,h+1) =& (I_N \otimes A_1 + G^{(1)} \otimes \Gamma) x(k+1,h) \\ &+ (I_N \otimes A_2 + G^{(2)} \otimes \Lambda) x(k,h+1) \\ &+ F(x(k+1,h), x(k,h+1)). \end{aligned} \tag{6}$$

To proceed, we need to introduce the following definitions and lemmas.

Definition 1: The synchronization manifold is defined as $\mathbb{S} \triangleq \{(x_1^T(k,h), x_2^T(k,h), \dots, x_N^T(k,h))^T \in \mathbb{R}^{nN} | x_i(k,h) = x_j(k,h)$ for $i,j = 1,2,\dots,N\}$, where $x_i(k,h) = (x_{i1}(k,h), x_{i2}(k,h), \dots, x_{in}(k,h))^T$ $(i = 1,2,\dots,N)$ is the state of node i.

Definition 2: The synchronization manifold $\mathbb{S}$ is said to be globally asymptotically stable for the dynamical network (1) or, in other words, the 2-D two-dimensional network (1) is globally asymptotically synchronized, if

$$\lim_{k+h\to\infty} \|x_i(k,h) - x_j(k,h)\| = 0 \tag{7}$$

holds for all $i,j = 1,2,\dots,N$ and for every boundary condition (4) satisfying (5).

Remark 2: The complex network model (1)-(3) is, to the best of the authors' knowledge, the first 2-D model of this kind that is proposed to reflect both the network dynamics and the couplings propagating in two directions. Also, based on such a 2-D complex network model, the definition of synchronization given in Definition 2 is a new concept that accounts for the phenomenon that the propagations of all sub-networks are synchronized in both directions, thanks to the coupling strength. In the sequel, we aim to develop an energy-like quadratic function and employ the Kronecker product so as to establish easily solvable sufficient conditions under which the addressed 2-D complex network model achieves global synchronization.

Lemma 1: [17] Let e be the N-dimensional vector with all components being 1 and $U = (u_{ij})_{N\times N} \triangleq NI_N - ee^T$. For $i = 1,2,\dots,N$, assume that P is an $n \times n$ matrix, $x =$

$(x_1^T, x_2^T, \ldots, x_N^T)^T$ where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^T \in \mathbb{R}^n$, and $y = (y_1^T, y_2^T, \ldots, y_N^T)^T$ where $y_i = (y_{i1}, y_{i2}, \ldots, y_{in})^T \in \mathbb{R}^n$. Then, we have the following relationships:

(1) $UG^{(j)} = G^{(j)}U = NG^{(j)}, \quad j = 1, 2;$

(2) $x^T(U \otimes P)y = -\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} u_{ij}(x_i - x_j)^T P(y_i - y_j);$

where $G^{(1)}$ and $G^{(2)}$ are matrices defined in model (6).

III. Synchronization for the 2-D dynamical network

In this section, we first investigate the globally asymptotic synchronization problem and derive easy-to-verify criteria for the 2-D dynamical network (1) with couplings. After that, we shall extend the obtained results to the discrete-time coupled networks with parameter uncertainties.

A. The synchronization problem without parameter uncertainties

The following theorem provides a sufficient condition under which the 2-D coupled dynamical network (1) is globally asymptotically synchronized.

Theorem 1: The synchronization manifold $\mathbb{S}$ is globally asymptotically stable for the dynamical network (1) if there exist matrices $P > 0$, $Q > 0$ and a scalar $\varepsilon > 0$ such that the following matrix inequalities hold for all $1 \le i < j \le N$:

$$Q < P,$$

$$\Phi_{ij} \triangleq \begin{bmatrix} \Phi_{ij}^{(1,1)} & \Phi_{ij}^{(1,2)} & A_1^T P - NG_{ij}^{(1)}\Gamma^T P \\ * & \Phi_{ij}^{(2,2)} & A_2^T P - NG_{ij}^{(2)}\Lambda^T P \\ * & * & P - \varepsilon I \end{bmatrix} < 0; \quad (8)$$

where

$$\begin{aligned}\Phi_{ij}^{(1,1)} =& A_1^T P A_1 - NG_{ij}^{(1,1)}\Gamma^T P\Gamma - Q \\ & - NG_{ij}^{(1)}(A_1^T P\Gamma + \Gamma^T P A_1) + \varepsilon B_1^T B_1, \\ \Phi_{ij}^{(1,2)} =& A_1^T P A_2 - NG_{ij}^{(1,2)}\Gamma^T P\Lambda - NG_{ij}^{(2)} A_1^T P\Lambda \\ & - NG_{ij}^{(1)}\Gamma^T P A_2 + \varepsilon B_1^T B_2, \\ \Phi_{ij}^{(2,2)} =& A_2^T P A_2 - NG_{ij}^{(2,2)}\Lambda^T P\Lambda - P + Q \\ & - NG_{ij}^{(2)}(A_2^T P\Lambda + \Lambda^T P A_2) + \varepsilon B_2^T B_2\end{aligned}$$

and $G^{(1,1)} = G^{(1)}G^{(1)} = (G_{ij}^{(1,1)})_{N\times N}$, $G^{(1,2)} = G^{(1)}G^{(2)} = (G_{ij}^{(1,2)})_{N\times N}$, $G^{(2,2)} = G^{(2)}G^{(2)} = (G_{ij}^{(2,2)})_{N\times N}$.

On the synchronization manifold $\mathbb{S}$, the connected nodes are decoupled and each of them would have the same dynamical behavior as the following 2-D nonlinear system:

$$\begin{aligned}y(k+1, h+1) =& A_1 y(k+1, h) + A_2 y(k, h+1) \\ & + f(y(k+1, h), y(k, h+1)). \end{aligned} \quad (9)$$

The dynamical behaviors of system (9) could be various. Among others, the stability is one of the most important features. Similar to the proof of Theorem 1, by considering the Schur complement operation [45], the following corollary is readily accessible.

Corollary 1: The 2-D nonlinear system (9) under the additional assumption of $f(0, 0) = 0$ is globally asymptotically stable if there exist matrices $P > 0$, $Q > 0$ and a scalar $\varepsilon > 0$ such that the following matrix inequality holds:

$$\begin{bmatrix} -Q + \varepsilon B_1^T B_1 & \varepsilon B_1^T B_2 & 0 & A_1^T P \\ * & \mathcal{Q} & 0 & A_2^T P \\ * & * & -\varepsilon I & P \\ * & * & * & -P \end{bmatrix} < 0, \quad (10)$$

where $\mathcal{Q} = -P + Q + \varepsilon B_2^T B_2$.

Remark 3: As stated in Remark 1, most of the 2-D discrete-time models discussed in the literature have been assumed to be linear except, for example, Ref. [40] where the nonlinear local state-space model was firstly studied and an important sufficient asymptotic stability condition was derived. Compared with Theorem 2 in [40], our result in Corollary 1 is more general. For example, the matrix P in Theorem 2 of [40] was required to be positive-definite and diagonal while P in Corollary 1 here is only required to be positive definite. Furthermore, when system (9) reduces to the linear model

$$y(k+1, h+1) = A_1 y(k+1, h) + A_2 y(k, h+1), \quad (11)$$

a sufficient condition could be derived easily from Corollary 1 ensuring the linear model (11) to be globally asymptotically stable, that is, system (11) is globally asymptotically stable if there exist matrices $P > 0$ and $Q > 0$ such that the following matrix inequality

$$\begin{bmatrix} -Q & 0 & A_1^T P \\ * & -P + Q & A_2^T P \\ * & * & -P \end{bmatrix} < 0 \quad (12)$$

holds. Note that (12) has been obtained in [34] and is explicitly stated in Remark 3 of [41].

B. The synchronization problem with parameter uncertainties

Having established the sufficient conditions for the globally asymptotic synchronization for the 2-D dynamical network (1) with couplings, we are now in a position to extend the obtained results to the discrete-time coupled networks with parameter uncertainties.

In reality, modeling errors are unavoidable. For 2-D discrete-time complex networks, the modeling error may result from the fluctuation of the connection weights of the node of complex networks, the inconsistency introduced by the discretization process, and the estimation variance from statistical tests when identifying the network parameters. Parameter uncertainties are a typical class of modeling errors. Generally, according to the way it occurs, the parameter uncertainty can be categorized into norm-bounded uncertainty and polytopic uncertainty, both of which have been extensively studied in the literature on research problems such as robust stability, robust control and robust filtering [17, 26, 32–35]. Needless to say, parameter uncertainties inevitably exist in the modeling process of 2-D complex networks and the purpose of this subsection is therefore to extend our previously obtained results to account for both the norm-bounded and the polytopic uncertainties.

Let us first deal with the 2-D complex networks with norm-bounded parameter uncertainties.

Assumption 1: Matrices A_1 and A_2 in the dynamical network (1) are of the following form

$$A_1 = A_{10} + \Delta A_1, \quad A_2 = A_{20} + \Delta A_2$$

where A_{10} and A_{20} are known constant matrices in $\mathbb{R}^{n\times n}$; ΔA_1 and ΔA_2 are real-valued time-varying matrix functions denoting the perturbations from the environment to the array of dynamical systems which are assumed to satisfy

$$\left[\begin{array}{cc} \Delta A_1 & \Delta A_2 \end{array}\right] = DH \left[\begin{array}{cc} W_1 & W_2 \end{array}\right], \tag{13}$$

where D, W_1 and W_2 are known matrices with appropriate dimensions, and H is a real uncertain matrix function satisfying $H^T H \le I$.

Lemma 2: Given appropriately dimensioned matrices Σ_1, Σ_2 and Σ_3 with $\Sigma_1^T = \Sigma_1$, then

$$\Sigma_1 + \Sigma_2 \Omega \Sigma_3 + \Sigma_3^T \Omega^T \Sigma_2^T < 0 \tag{14}$$

holds for all Ω satisfying $\Omega^T \Omega \le I$ if and only if there exists some $\epsilon > 0$ such that

$$\Sigma_1 + \epsilon^{-1}\Sigma_2\Sigma_2^T + \epsilon\Sigma_3^T\Sigma_3 < 0. \tag{15}$$

Theorem 2: The synchronization manifold $\mathbb{S}$ is globally robustly asymptotically stable for the dynamical network (1) with Assumption 1 if there exist matrices $P > 0$, $Q > 0$ and scalars $\varepsilon > 0$, $\gamma > 0$ such that the matrix inequalities in (16) (shown at the top of the next page) hold for all $1 \le i < j \le N$, where matrices $\Psi_{ij}^{(1,1)}$, $\Psi_{ij}^{(1,2)}$, $\Psi_{ij}^{(2,2)}$ are defined just below (16) and the other symbols are the same as defined in Theorem 1.

Next, let us cope with the robust synchronization problem for 2-D complex networks with polytopic parameter uncertainties.

Assumption 2: Matrices A_1 and A_2 in dynamical network (1) contain partially unknown parameters, that is, $(A_1, A_2) \in \Upsilon$ where

$$\Upsilon \triangleq \{\aleph(\lambda) | \aleph(\lambda) = \sum_{l=1}^{\kappa} \lambda_l \aleph_l; \ \sum_{l=1}^{\kappa} \lambda_l = 1, \ \lambda_l \ge 0\} \tag{17}$$

and $\aleph_l = (A_{1l}, A_{2l})$ denotes the lth vertex of the polytope.

The following theorem is easily accessible and its proof is therefore omitted.

Theorem 3: The synchronization manifold $\mathbb{S}$ is globally robustly asymptotically stable for the dynamical network (1) with Assumption 2 if there exist matrices $P > 0$, $Q > 0$ and scalar $\varepsilon > 0$ such that the following matrix inequalities hold for all $1 \le i < j \le N$ and $l = 1, 2, \dots, \kappa$:

$$Q < P, \left[\begin{array}{cccc} \Theta_{ij}^{(1,1)} & \Theta_{ij}^{(1,2)} & -NG_{ij}^{(1)}\Gamma^T P & A_{1l}^T P \\ * & \Theta_{ij}^{(2,2)} & -NG_{ij}^{(2)}\Lambda^T P & A_{2l}^T P \\ * & * & -\varepsilon I & P \\ * & * & * & -P \end{array}\right] < 0; \tag{18}$$

where

$$\begin{aligned} \Theta_{ij}^{(1,1)} =& -NG_{ij}^{(1,1)}\Gamma^T P\Gamma - Q + \varepsilon B_1^T B_1 \\ & - NG_{ij}^{(1)}(A_{1l}^T P\Gamma + \Gamma^T P A_{1l}), \\ \Theta_{ij}^{(1,2)} =& -NG_{ij}^{(1,2)}\Gamma^T P\Lambda - NG_{ij}^{(2)} A_{1l}^T P\Lambda \\ & - NG_{ij}^{(1)}\Gamma^T P A_{2l} + \varepsilon B_1^T B_2, \\ \Theta_{ij}^{(2,2)} =& -NG_{ij}^{(2,2)}\Lambda^T P\Lambda - P + Q \\ & - NG_{ij}^{(2)}(A_{2l}^T P\Lambda + \Lambda^T P A_{2l}) + \varepsilon B_2^T B_2 \end{aligned}$$

and the other symbols are defined in Theorem 1.

In terms of Theorem 2 and Theorem 3 obtained above, the following corollaries can be readily obtained.

Corollary 2: Under Assumption 1 and an additional assumption that $f(0,0) = 0$, the 2-D nonlinear system (9) is globally robustly asymptotically stable if there exist matrices $P > 0$, $Q > 0$ and scalars $\varepsilon > 0$, $\gamma > 0$ such that the following matrix inequality holds:

$$\left[\begin{array}{ccccc} \mathcal{P}_1 & \varepsilon B_1^T B_2 + \gamma W_1^T W_2 & 0 & A_{10}^T P & 0 \\ * & \mathcal{P}_2 & 0 & A_{20}^T P & 0 \\ * & * & -\varepsilon I & P & 0 \\ * & * & * & -P & PD \\ * & * & * & * & -\gamma I \end{array}\right] < 0, \tag{19}$$

where $\mathcal{P}_1 = -Q + \varepsilon B_1^T B_1 + \gamma W_1^T W_1$ and $\mathcal{P}_2 = -P + Q + \varepsilon B_2^T B_2 + \gamma W_2^T W_2$.

Corollary 3: Under Assumption 1 and an additional assumption that $f(0,0) = 0$, the 2-D nonlinear system (9) is globally robustly asymptotically stable if there exist matrices $P > 0$, $Q > 0$ and scalar $\varepsilon > 0$ such that the following matrix inequality holds for all $l = 1, 2, \dots, \kappa$:

$$\left[\begin{array}{cccc} -Q + \varepsilon B_1^T B_1 & \varepsilon B_1^T B_2 & 0 & A_{1l}^T P \\ * & \mathcal{Q} & 0 & A_{2l}^T P \\ * & * & -\varepsilon I & P \\ * & * & * & -P \end{array}\right] < 0, \tag{20}$$

where $\mathcal{Q}$ is the same as defined in Corollary 1.

Remark 4: The synchronization problem has been extensively investigated in the literature for 1-D complex networks including neural networks and genetic regulatory networks, and numerous results have been reported, see e.g. [7, 10, 17, 46–50]. In this paper, we have taken a major step further to study the robust synchronization problem for a class of new complex network, that is, an array of 2-D coupled dynamical networks. The criteria derived here are in the form of linear matrix inequalities (LMIs) that can be effectively solved and checked by the algorithms such as the interior-point method.

Remark 5: In this paper, the Lipschitz-like nonlinear functions are employed to describe the nonlinearities existing in the coupled network. The main results established contain all the information of the complex networks including the physical parameters, coupling strength, nonlinear parameters and bounds/vertices for the parameter uncertainties. In the next section, a simulation example is provided to show the usefulness of the proposed synchronization conditions.

$$Q < P, \quad \begin{bmatrix} \Psi_{ij}^{(1,1)} & \Psi_{ij}^{(1,2)} & -NG_{ij}^{(1)}\Gamma^T P & A_{10}^T P & -NG_{ij}^{(1)}\Gamma^T PD \\ * & \Psi_{ij}^{(2,2)} & -NG_{ij}^{(2)}\Lambda^T P & A_{20}^T P & -NG_{ij}^{(2)}\Lambda^T PD \\ * & * & -\varepsilon I & P & 0 \\ * & * & * & -P & PD \\ * & * & * & * & -\gamma NI \end{bmatrix} < 0; \tag{16}$$

with

$$\begin{aligned} \Psi_{ij}^{(1,1)} &= -NG_{ij}^{(1,1)}\Gamma^T P\Gamma - Q - NG_{ij}^{(1)}(A_{10}^T P\Gamma + \Gamma^T PA_{10}) + \varepsilon B_1^T B_1 + \gamma NW_1^T W_1, \\ \Psi_{ij}^{(1,2)} &= -NG_{ij}^{(1,2)}\Gamma^T P\Lambda - NG_{ij}^{(2)}A_{10}^T P\Lambda - NG_{ij}^{(1)}\Gamma^T PA_{20} + \varepsilon B_1^T B_2 + \gamma NW_1^T W_2, \\ \Psi_{ij}^{(2,2)} &= -NG_{ij}^{(2,2)}\Lambda^T P\Lambda - P + Q - NG_{ij}^{(2)}(A_{20}^T P\Lambda + \Lambda^T PA_{20}) + \varepsilon B_2^T B_2 + \gamma NW_2^T W_2. \end{aligned}$$

IV. Numerical Examples

In this section, a 2-D network model with five coupled nodes is considered as an example to illustrate the main theoretical results obtained. In a real world, it is known that some dynamical processes in gas absorption, air drying and water stream heating can be described by the Darboux equation [28]:

$$\frac{\partial^2 s(z,t)}{\partial z \partial t} = a_1 \frac{\partial s(z,t)}{\partial t} + a_2 \frac{\partial s(z,t)}{\partial z} + a_0 s(z,t) + bg(z,t), \tag{21}$$

where $s(z,t)$ is a function at z(space) $\in [0, z_{\text{end}}]$ and t(time) $\in [0, \infty)$. a_0, a_1, a_2 and b are real coefficients, and $g(z,t)$ can be regarded as input function or nonlinear perturbation function.

By defining

$$r(z,t) \triangleq \frac{\partial s(z,t)}{\partial t} - a_2 s(z,t)$$

and letting

$$y_1(i,j) \triangleq r(i,j), \quad y_2(i,j) \triangleq s(i,j),$$

where $r(i,j) \triangleq r(i\Delta z, j\Delta t)$ and $s(i,j) \triangleq s(i\Delta z, j\Delta t)$, the partial differential equation (21) can be converted into a second-order nonlinear system described by model (9) with

$$\begin{aligned} A_1 &= \begin{bmatrix} 0.01\delta & 0 \\ \Delta t & 1 + a_2\Delta t \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 1 + a_1\Delta z & (a_0 + a_1 a_2)\Delta z \\ 0 & 0.02\delta \end{bmatrix} \end{aligned} \tag{22}$$

and the nonlinear function $f = \begin{bmatrix} b(\Delta z)g(i,j+1) & 0 \end{bmatrix}^T$. The readers are referred to Refs. [26,28,42] for more details of the transformation process. In this example, we take $a_0 = -1.9773$, $a_1 = -2$, $a_2 = -1$, $b = 0.7171$, $\Delta t = 0.6085$, $\Delta z = 0.2024$ and appropriately choose $g(\cdot,\cdot)$ in (21) such that condition (3) is satisfied with

$$B_1 = \begin{bmatrix} 0.1451 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

First, assume that there are no parameter uncertainties, i.e., the system matrices are completely known with $\delta = 0$. We consider the 2-D complex network with five coupled nodes. For simplicity, the inner coupling matrices and the outer coupling configuration matrices in (6) in the horizontal and the vertical directions are chosen to be

$$\Gamma = \begin{bmatrix} 0.42 & 0.20 \\ 0.36 & -0.52 \end{bmatrix}, \ \Lambda = \begin{bmatrix} 0.30 & 0.22 \\ -0.20 & 0.56 \end{bmatrix},$$

$$G^{(1)} = \begin{bmatrix} -0.3 & 0 & 0.1 & 0.1 & 0.1 \\ 0 & -0.2 & 0.1 & 0.1 & 0 \\ 0.1 & 0.1 & -0.3 & 0 & 0.1 \\ 0.1 & 0.1 & 0 & -0.2 & 0 \\ 0.1 & 0 & 0.1 & 0 & -0.2 \end{bmatrix},$$

$$G^{(2)} = \begin{bmatrix} -0.3 & 0.1 & 0.1 & 0.1 & 0 \\ 0.1 & -0.3 & 0.1 & 0 & 0.1 \\ 0.1 & 0.1 & -0.3 & 0 & 0.1 \\ 0.1 & 0 & 0 & -0.1 & 0 \\ 0 & 0.1 & 0.1 & 0 & -0.2 \end{bmatrix}.$$

Solving the matrix inequality condition (8) in Theorem 1 by using the Matlab LMI Toolbox, a feasible solution can be found as: $\varepsilon = 3.6347$,

$$P = \begin{bmatrix} 1.0192 & 0.0804 \\ 0.0804 & 0.3637 \end{bmatrix}, \ Q = \begin{bmatrix} 0.3669 & 0.1391 \\ 0.1391 & 0.2335 \end{bmatrix}.$$

It follows from Theorem 1 that the synchronization manifold $\mathbb{S}$ is globally asymptotically stable. Let the initial boundary condition be $x_1(0,h) = (0.4, -0.1)^T$, $x_1(k,0) = (0.3, -0.2)^T$; $x_2(0,h) = (0.2, 0.5)^T$, $x_2(k,0) = (0.5, -1.0)^T$; $x_3(0,h) = (1.0, 0.5)^T$, $x_3(k,0) = (0.6, 1.0)^T$; $x_4(0,h) = (-0.5, 0.4)^T$, $x_4(k,0) = (1.0, 0.7)^T$; $x_5(0,h) = (0.4, 1.3)^T$, $x_5(k,0) = (1.9, -0.1)^T$; where k, $h = 1, 2, \ldots, 13$; otherwise, $x_i(0,h) = x_i(k,0) = 0$ $(i = 2,3,4,5)$. It can be seen from Figures 1-4 that the coupled 2-D network is indeed globally asymptotically synchronized under the above conditions.

Next, let us continue to consider the case of norm-bounded uncertainties. Assume that the parameters are

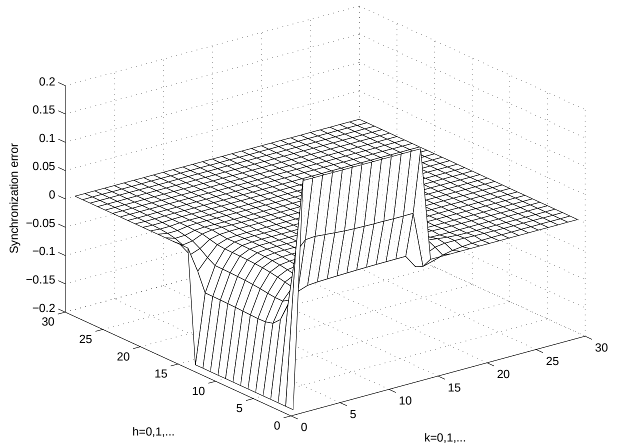

Fig. 1. Synchronization error $x_{11}(k,h) - x_{21}(k,h)$ of network (1)

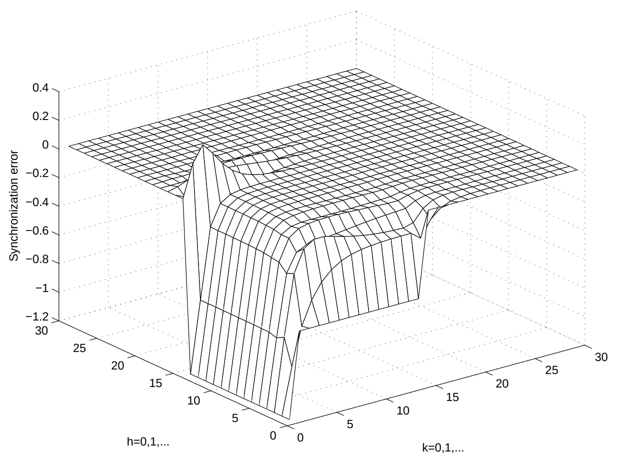

Fig. 2. Synchronization error $x_{12}(k,h) - x_{32}(k,h)$ of network (1)

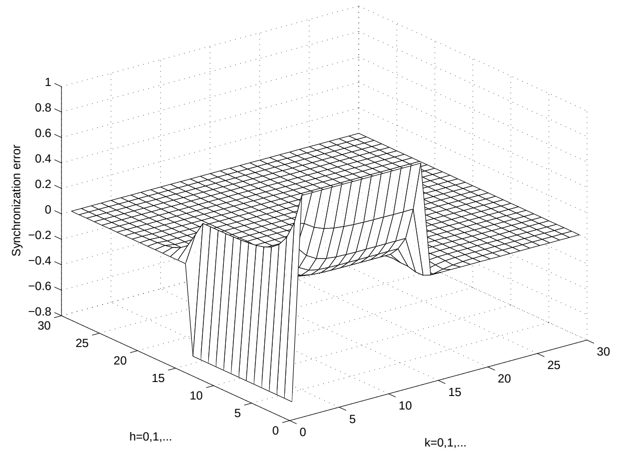

Fig. 3. Synchronization error $x_{11}(k,h) - x_{41}(k,h)$ of network (1)

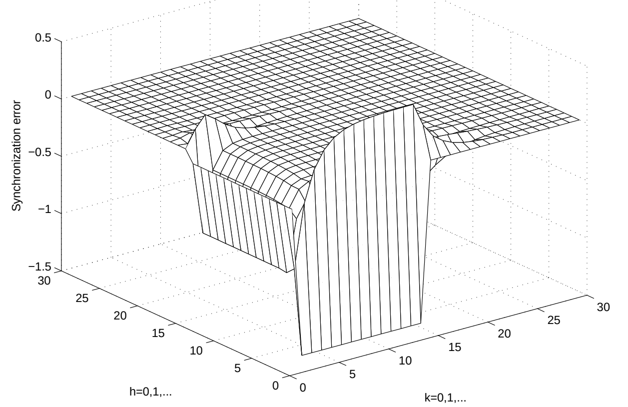

Fig. 4. Synchronization error $x_{12}(k,h) - x_{52}(k,h)$ of network (1)

subject to Assumption 1 with matrices A_{10} and A_{20} defined in (22). Let the condition (13) be satisfied with

$$W_1 = \begin{bmatrix} 0.1 & 0 \end{bmatrix}, W_2 = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}, D^T = \begin{bmatrix} 0.1 & -0.1 \end{bmatrix}.$$

The inequalities in (16) are solvable with $\varepsilon = 0.6638$, $\gamma = 0.0309$,

$$P = \begin{bmatrix} 0.1661 & 0.0130 \\ 0.0130 & 0.0613 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.0612 & 0.0223 \\ 0.0223 & 0.0394 \end{bmatrix}.$$

Using Theorem 2, one knows that the synchronization manifold $\mathbb{S}$ is globally robustly asymptotically stable.

Finally, assume that $|\delta| \leq 1$, i.e., the network considered has polytopic parameter uncertainties. In this case, according to Assumption 2, the parameter uncertainties can be represented by a two-vertex polytope. By resorting to the Matlab Toolbox, the condition (18) is satisfied with $\varepsilon = 14.8932$ and

$$P = \begin{bmatrix} 3.6103 & 0.2433 \\ 0.2433 & 1.2160 \end{bmatrix}, Q = \begin{bmatrix} 1.3298 & 0.4628 \\ 0.4628 & 0.7900 \end{bmatrix}.$$

It follows from Theorem 3 that the synchronization manifold $\mathbb{S}$ is globally robustly asymptotically stable.

V. Conclusions

This paper has been concerned with the problem of robust synchronization for a class of 2-D coupled uncertain dynamical networks. Firstly, some sufficient conditions have been derived which ensure the deterministic coupled complex network to be globally asymptotically synchronized. After that, the obtained results have been extended to the 2-D uncertain complex networks where the parameter uncertainties are assumed to be in either the norm-bounded or the polytopic forms. An illustrative example with numerical simulations has been presented to demonstrate the effectiveness of the obtained criteria.

Appendix A
Proof of Theorem 1

Proof: To establish the synchronization performance, we introduce the following energy-like index

$$\begin{aligned} \mathcal{J} =& x^T(k+1,h+1)(U \otimes P)x(k+1,h+1) \\ &- x^T(k+1,h)(U \otimes Q)x(k+1,h) \\ &- x^T(k,h+1)(U \otimes (P-Q))x(k,h+1), \end{aligned} \tag{23}$$

$$\begin{aligned}\mathcal{J}=&\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\Big\{\big(x_i(k+1,h)-x_j(k+1,h)\big)^T\Big[A_1^TPA_1-NG_{ij}^{(1)}(A_1^TP\Gamma+\Gamma^TPA_1)-NG_{ij}^{(1,1)}\Gamma^TP\Gamma-Q\Big]\\&\times\big(x_i(k+1,h)-x_j(k+1,h)\big)+\big(x_i(k,h+1)-x_j(k,h+1)\big)^T\Big[A_2^TPA_2-NG_{ij}^{(2)}(A_2^TP\Lambda+\Lambda^TPA_2)\\&-NG_{ij}^{(2,2)}\Lambda^TP\Lambda-P+Q\Big]\big(x_i(k,h+1)-x_j(k,h+1)\big)+\big(f(x_i(k+1,h),x_i(k,h+1))\\&-f(x_j(k+1,h),x_j(k,h+1))\big)^TP\big(f(x_i(k+1,h),x_i(k,h+1))-f(x_j(k+1,h),x_j(k,h+1))\big)\\&+2\big(x_i(k+1,h)-x_j(k+1,h)\big)^T\Big[A_1^TPA_2-NG_{ij}^{(1)}\Gamma^TPA_2-NG_{ij}^{(2)}A_1^TP\Lambda-NG_{ij}^{(1,2)}\Gamma^TP\Lambda\Big]\\&\times\big(x_i(k,h+1)-x_j(k,h+1)\big)+2\big(x_i(k+1,h)-x_j(k+1,h)\big)^T\Big[A_1^TP-NG_{ij}^{(1)}\Gamma^TP\Big]\\&\times\big(f(x_i(k+1,h),x_i(k,h+1))-f(x_j(k+1,h),x_j(k,h+1))\big)+2\big(x_i(k,h+1)-x_j(k,h+1)\big)^T\\&\times\Big[A_2^TP-NG_{ij}^{(2)}\Lambda^TP\Big]\big(f(x_i(k+1,h),x_i(k,h+1))-f(x_j(k+1,h),x_j(k,h+1))\big)\Big\}.\end{aligned}\tag{24}$$

---

where $k,\,h\in\mathbb{Z}_+$, $\{P>0,Q>0\}$ is the solution of the matrix inequalities (8), and U is the matrix defined in Lemma 1. Then, we have from Lemma 1 and the properties of Kronecker product that formula (24) holds.

Note that condition (3) on the nonlinear function $f(\cdot,\cdot)$ ensures that, for scalar $\varepsilon>0$, the inequality (25) (shown on the next page) holds for all $1\le i<j\le N$. Substituting (25) into (24) and considering the condition (8), one has

$$\mathcal{J}\le\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\xi_{ij}^T(k,h)\Phi_{ij}\xi_{ij}(k,h)\le 0,\tag{26}$$

where $\xi_{ij}^T(k,h)=\big((x_i(k+1,h)-x_j(k+1,h))^T\quad(x_i(k,h+1)-x_j(k,h+1))^T\quad(f(x_i(k+1,h),x_i(k,h+1))-f(x_j(k+1,h),x_j(k,h+1)))^T\big)$. Hence, for all $x(k,h)\notin\mathbb{S}$, we have the inequality (27) (shown on the next page below (25)).

Defining $\alpha\triangleq 1-\min\limits_{i,j}\lambda_{\min}(-\Phi_{ij})/\max\{\lambda_{\max}(Q),\lambda_{\max}(P-Q)\}$, we have $\alpha<1$ because of the positiveness of $\min\limits_{i,j}\lambda_{\min}(-\Phi_{ij})/\max\{\lambda_{\max}(Q),\lambda_{\max}(P-Q)\}$. On the other hand, inequality (28) (shown on page 9) holds; which means that $\alpha\in(0,1)$ and α is independent of $x(k+1,h)$ and $x(k,h+1)$. Therefore, (28) ensures that the following inequality holds for all $k,\,h\in\mathbb{Z}_+$

$$\begin{aligned}&x^T(k+1,h+1)(U\otimes P)x(k+1,h+1)\\&\quad\le\alpha\big[x^T(k+1,h)(U\otimes Q)x(k+1,h)\\&\qquad+x^T(k,h+1)(U\otimes(P-Q))x(k,h+1)\big].\end{aligned}\tag{29}$$

Upon the relationship (29), it can be established that

$$\begin{aligned}&x^T(k+1,0)(U\otimes P)x(k+1,0)\\&\quad=x^T(k+1,0)(U\otimes P)x(k+1,0),\\&x^T(k,1)(U\otimes P)x(k,1)\\&\quad\le\alpha\big[x^T(k,0)(U\otimes Q)x(k,0)\\&\qquad+x^T(k-1,1)(U\otimes(P-Q))x(k-1,1)\big]\\&\quad\le\alpha\big[x^T(k,0)(U\otimes P)x(k,0)\\&\qquad+x^T(k-1,1)(U\otimes(P-Q))x(k-1,1)\big],\\&x^T(k-1,2)(U\otimes P)x(k-1,2)\\&\quad\le\alpha\big[x^T(k-1,1)(U\otimes Q)x(k-1,1)\\&\qquad+x^T(k-2,2)(U\otimes(P-Q))x(k-2,2)\big],\\&\qquad\vdots\\&x^T(1,k)(U\otimes P)x(1,k)\\&\quad\le\alpha\big[x^T(1,k-1)(U\otimes Q)x(1,k-1)\\&\qquad+x^T(0,k)(U\otimes(P-Q))x(0,k)\big],\\&x^T(0,k+1)(U\otimes P)x(0,k+1)\\&\quad=x^T(0,k+1)(U\otimes P)x(0,k+1).\end{aligned}$$

Adding both sides of the above inequalities and equalities yields

$$\begin{aligned}&\sum_{s=0}^{k+1}x^T(k+1-s,s)(U\otimes P)x(k+1-s,s)\\&\le\alpha\sum_{s=0}^{k}x^T(k-s,s)(U\otimes P)x(k-s,s)\\&\quad+x^T(k+1,0)(U\otimes P)x(k+1,0)\\&\quad+x^T(0,k+1)(U\otimes P)x(0,k+1)\\&\quad-\alpha x^T(0,k)(U\otimes Q)x(0,k)\\&\le\alpha\sum_{s=0}^{k}x^T(k-s,s)(U\otimes P)x(k-s,s)\\&\quad+x^T(k+1,0)(U\otimes P)x(k+1,0)\\&\quad+x^T(0,k+1)(U\otimes P)x(0,k+1).\end{aligned}\tag{30}$$

$$\begin{aligned}&\varepsilon\big(f(x_i(k+1,h),x_i(k,h+1))-f(x_j(k+1,h),x_j(k,h+1))\big)^T\\&\qquad\times\big(f(x_i(k+1,h),x_i(k,h+1))-f(x_j(k+1,h),x_j(k,h+1))\big)\\\leq\;&\big(x_i(k+1,h)-x_j(k+1,h)\big)^T(\varepsilon B_1^TB_1)\big(x_i(k+1,h)-x_j(k+1,h)\big)\\&+2\big(x_i(k+1,h)-x_j(k+1,h)\big)^T(\varepsilon B_1^TB_2)\big(x_i(k,h+1)-x_j(k,h+1)\big)\\&+\big(x_i(k,h+1)-x_j(k,h+1)\big)^T(\varepsilon B_2^TB_2)\big(x_i(k,h+1)-x_j(k,h+1)\big).\end{aligned}\tag{25}$$

$$\begin{aligned}&\frac{\mathcal{J}}{x^T(k+1,h)(U\otimes Q)x(k+1,h)+x^T(k,h+1)(U\otimes(P-Q))x(k,h+1)}\\\leq\;&\frac{-\min\limits_{i,j}\lambda_{\min}(-\Phi_{ij})\sum\limits_{i=1}^{N-1}\sum\limits_{j=i+1}^{N}\|\xi_{ij}^T(k,h)\|^2}{\max\{\lambda_{\max}(Q),\lambda_{\max}(P-Q)\}\sum\limits_{i=1}^{N-1}\sum\limits_{j=i+1}^{N}\big(\|x_i(k+1,h)-x_j(k+1,h)\|^2+\|x_i(k,h+1)-x_j(k,h+1)\|^2\big)}\\\leq\;&-\frac{\min\limits_{i,j}\lambda_{\min}(-\Phi_{ij})}{\max\{\lambda_{\max}(Q),\lambda_{\max}(P-Q)\}}.\end{aligned}\tag{27}$$

$$\alpha\geq\frac{x^T(k+1,h+1)(U\otimes P)x(k+1,h+1)}{x^T(k+1,h)(U\otimes Q)x(k+1,h)+x^T(k,h+1)(U\otimes(P-Q))x(k,h+1)}>0,\tag{28}$$

Using the above relationship iteratively, one obtains

$$\begin{aligned}&\sum_{s=0}^{k+1}x^T(k+1-s,s)(U\otimes P)x(k+1-s,s)\\\leq&\sum_{s=0}^{k}\alpha^s\big[x^T(k+1-s,0)(U\otimes P)x(k+1-s,0)\\&+x^T(0,k+1-s)(U\otimes P)x(0,k+1-s)\big]\\&+\alpha^{k+1}x^T(0,0)(U\otimes P)x(0,0)\\\leq&\sum_{s=0}^{k+1}\alpha^s\big[x^T(k+1-s,0)(U\otimes P)x(k+1-s,0)\\&+x^T(0,k+1-s)(U\otimes P)x(0,k+1-s)\big].\end{aligned}\tag{31}$$

It follows easily from (31) and Lemma 1 that inequality (32) (shown on page 10) holds. Therefore, we can obtain inequality (33), where $\beta=\lambda_{\max}(P)/\lambda_{\min}(P)\geq 1$.

By denoting

$$S_k\triangleq\sum_{s=0}^{k}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\|x_i(k-s,s)-x_j(k-s,s)\|^2,$$

it follows from inequality (33) that the $M+1$ inequalities (shown in the middle of page 10) hold. Summing up both sides of these inequalities yields formula (34) on page 10.

By noting the initial boundary conditions (4)-(5) and the fact that $\alpha\in(0,1)$, the right side of the inequality (34) is bounded, which means that the positive term series $\sum\limits_{l=0}^{\infty}S_l$ is convergent, and hence it can be concluded that $\lim\limits_{l\to\infty}S_l=0$, that is, $\lim\limits_{k+h\to\infty}\|x_i(k,h)-x_j(k,h)\|=0$ holds for all $i,j=1,2,\ldots,N$. From Definition 2, it is known that the synchronization manifold $\mathbb{S}$ is globally asymptotically stable for the 2-D dynamical network (1). The proof is now completed. ■

Appendix B
Proof of Theorem 2

Proof: From the result of Theorem 1, Theorem 2 can be proved by substituting A_1 and A_2 in Assumption 1 (with norm-bounded uncertain parameter matrices) into (8). Then, by the well-known Schur complement operation, we obtain (14) with $\Omega=H$ and

$$\Sigma_1=\begin{bmatrix}\Sigma_1^{(1,1)}&\Sigma_1^{(1,2)}&-NG_{ij}^{(1)}\Gamma^TP&A_{10}^TP\\ *&\Sigma_1^{(2,2)}&-NG_{ij}^{(2)}\Lambda^TP&A_{20}^TP\\ *&*&-\varepsilon I&P\\ *&*&*&-P\end{bmatrix},$$

$$\Sigma_2^T=\sqrt{N}\begin{bmatrix}-G_{ij}^{(1)}D^TP\Gamma&-G_{ij}^{(2)}D^TP\Lambda&0&\frac{1}{N}D^TP\end{bmatrix},$$

$$
\begin{aligned}
&\sum_{s=0}^{k+1}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\big(x_i(k+1-s,s)-x_j(k+1-s,s)\big)^T P\big(x_i(k+1-s,s)-x_j(k+1-s,s)\big)\\
\leq\;&\sum_{s=0}^{k+1}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\alpha^s\Big[\big(x_i(k+1-s,0)-x_j(k+1-s,0)\big)^T P\big(x_i(k+1-s,0)-x_j(k+1-s,0)\big)\\
&+\big(x_i(0,k+1-s)-x_j(0,k+1-s)\big)^T P\big(x_i(0,k+1-s)-x_j(0,k+1-s)\big)\Big]. \qquad (32)
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{s=0}^{k+1}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\|x_i(k+1-s,s)-x_j(k+1-s,s)\|^2\\
\leq\;&\beta\sum_{s=0}^{k+1}\alpha^s\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left[\|x_i(k+1-s,0)-x_j(k+1-s,0)\|^2+\|x_i(0,k+1-s)-x_j(0,k+1-s)\|^2\right] \qquad (33)
\end{aligned}
$$

$$
\begin{aligned}
S_0 &\leq \beta\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left[\|x_i(0,0)-x_j(0,0)\|^2+\|x_i(0,0)-x_j(0,0)\|^2\right],\\
S_1 &\leq \beta\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\big[\alpha\big(\|x_i(0,0)-x_j(0,0)\|^2+\|x_i(0,0)-x_j(0,0)\|^2\big)\\
&\quad+\big(\|x_i(1,0)-x_j(1,0)\|^2+\|x_i(0,1)-x_j(0,1)\|^2\big)\big],\\
S_2 &\leq \beta\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\big[\alpha^2\big(\|x_i(0,0)-x_j(0,0)\|^2+\|x_i(0,0)-x_j(0,0)\|^2\big)\\
&\quad+\alpha\big(\|x_i(1,0)-x_j(1,0)\|^2+\|x_i(0,1)-x_j(0,1)\|^2\big)+\big(\|x_i(2,0)-x_j(2,0)\|^2+\|x_i(0,2)-x_j(0,2)\|^2\big)\big],\\
&\vdots\\
S_M &\leq \beta\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\big[\alpha^M\big(\|x_i(0,0)-x_j(0,0)\|^2+\|x_i(0,0)-x_j(0,0)\|^2\big)\\
&\quad+\alpha^{M-1}\big(\|x_i(1,0)-x_j(1,0)\|^2+\|x_i(0,1)-x_j(0,1)\|^2\big)+\cdots+\alpha\big(\|x_i(M-1,0)-x_j(M-1,0)\|^2\\
&\quad+\|x_i(0,M-1)-x_j(0,M-1)\|^2\big)+\big(\|x_i(M,0)-x_j(M,0)\|^2+\|x_i(0,M)-x_j(0,M)\|^2\big)\big].
\end{aligned}
$$

$$
\begin{aligned}
\sum_{l=0}^{M}S_l \leq&\beta\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\big\{(1+\alpha+\cdots+\alpha^M)\big(\|x_i(0,0)-x_j(0,0)\|^2+\|x_i(0,0)-x_j(0,0)\|^2\big)\\
&+(1+\alpha+\cdots+\alpha^{M-1})\big(\|x_i(1,0)-x_j(1,0)\|^2+\|x_i(0,1)-x_j(0,1)\|^2\big)+\cdots\\
&+(1+\alpha)\big(\|x_i(M-1,0)-x_j(M-1,0)\|^2+\|x_i(0,M-1)-x_j(0,M-1)\|^2\big)\\
&+\big(\|x_i(M,0)-x_j(M,0)\|^2+\|x_i(0,M)-x_j(0,M)\|^2\big)\big\}\\
\leq&\beta\frac{1-\alpha^{M+1}}{1-\alpha}\sum_{l=0}^{M}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\big(\|x_i(l,0)-x_j(l,0)\|^2+\|x_i(0,l)-x_j(0,l)\|^2\big). \qquad (34)
\end{aligned}
$$

$$\Sigma_3 = \sqrt{N} \begin{bmatrix} W_1 & W_2 & 0 & 0 \end{bmatrix};$$

where

$$\begin{aligned}
\Sigma_1^{(1,1)} =& - NG_{ij}^{(1,1)}\Gamma^T P\Gamma - Q + \varepsilon B_1^T B_1 \\
& - NG_{ij}^{(1)}(A_{10}^T P\Gamma + \Gamma^T P A_{10}), \\
\Sigma_1^{(1,2)} =& - NG_{ij}^{(1)}\Gamma^T P A_{20} + \varepsilon B_1^T B_2 - NG_{ij}^{(2)} A_{10}^T P\Lambda \\
& - NG_{ij}^{(1,2)}\Gamma^T P\Lambda, \\
\Sigma_1^{(2,2)} =& - NG_{ij}^{(2,2)}\Lambda^T P\Lambda - P + Q + \varepsilon B_2^T B_2 \\
& - NG_{ij}^{(2)}(A_{20}^T P\Lambda + \Lambda^T P A_{20}).
\end{aligned}$$

By resorting to Lemma 2 and the Schur complement operation, (15) holds if and only if (16) is true, and this ends the proof. ■

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