Fault Detection for Markovian Jump Systems with Sensor Saturations and Randomly Varying Nonlinearities

Hongli Dong, Zidong Wang and Huijun Gao

Abstract— **This paper addresses the fault detection problem for discrete-time Markovian jump systems with incomplete knowledge of transition probabilities, randomly varying nonlinearities and sensor saturations. For the Markovian mode jumping, the transition probability matrix is allowed to have partially unknown entries, while the cases with completely known or completely unknown transition probabilities are also investigated as two special cases. The randomly varying nonlinearities and the sensor saturations are introduced to reflect the limited capacity of the communication networks resulting from the noisy environment, probabilistic communication failures, measurements of limited amplitudes, etc. Two energy norm indices are used for the fault detection problem in order to account for, respectively, the restraint of disturbance and the sensitivity of faults. The purpose of the problem addressed is to design an optimized fault detection filter such that 1) the fault detection dynamics is stochastically stable; 2) the effect from the exogenous disturbance on the residual is attenuated with respect to a minimized $\mathcal{H}_\infty$-norm; and 3) the sensitivity of the residual to the fault is enhanced by means of a maximized $\mathcal{H}_\infty$-norm. The characterization of the gains of the desired fault detection filters is derived in terms of the solution to a convex optimization problem that can be easily solved by using the semi-definite programme method. Finally, a simulation example is employed to show the effectiveness of the fault detection filtering scheme proposed in this paper.**

Keywords— **Fault detection; Markovian jumping systems; randomly varying nonlinearities; sensor saturation; incomplete knowledge of transition probabilities; optimized filter.**

I. Introduction

In the past decade, the FDI problem has received considerable research attention and a rich body of literature has appeared on both the theoretical research and practical applications, see e.g. [1,7–10,18,19,22]. FDI techniques are essentially employed in modern manufacturing processes to minimize downtime, increase the safety of plant operations and reduce costs. A practically motivated way of handling the FDI problems is to introduce two performance indices for the robustness and sensitivity, one is the $\mathcal{H}_\infty$ norm of transfer function from the unknown input to the residual that is made to be small, and the other is the $\mathcal{H}_\infty$ norm of the transfer function from the fault to the residual that is designed to be large [4]. Another recently popular model-based way for tackling the FDI issues is to construct the residual that is as close to the fault (or weighted fault) as possible within an as small as possible, see e.g. [8, 19]. Also, in [6], a reference model has been introduced so as to transfer the robust fault detection problem into an equivalent $\mathcal{H}_\infty$ model match problem.

Markovian jump systems (MJSs) have gained particular research interests in the past two decades because of their practical applications in a variety of areas [2, 3, 5, 11–15]. So far, existing results about MJSs have covered a wide range of research problems including those for stability analysis, filter design and controller design. Nevertheless, compared to the fruitful results for filtering and control problems of MJSs, the corresponding fault detection problem of MJSs has received much less attention [8] due primarily to the difficulty in accommodating the multiple fault detection performances. On the other hand, much of the effort has been devoted to deal with the phenomena of sensor/actuator/state saturations in the literature, see e.g. [16, 20]. However, the sensor saturation issue has seldom been taken into account in designing fault detection filters due probably to the mathematical complexities.

Recently, the MJSs with *partially* unknown transition probabilities have been brought to the attention of researchers in the area of control engineering [21]. On the other hand, it is well known that nonlinearities exist universally in practice and it is quite common to describe them as additive nonlinear disturbances that are caused by environmental circumstances. As discussed in [17], in nowadays prevalent networked control system, the nonlinear disturbances themselves may experience random abrupt changes due to random changes and failures arising from networked-induced phenomenon, which give rise to the so-called randomly varying nonlinearities. In other words, the type and intensity of the so-called randomly varying nonlinearities could be changeable in a probabilistic way. Unfortunately, up to now, the fault detection problem for discrete-time

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H. Dong is with the Research Institute of Intelligent Control and Systems, Harbin Institute of Technology, Harbin 150001, China, and also with the College of Electrical and Information Engineering, Northeast Petroleum University, Daqing 163318, China. (Email: shiningdhl@gmail.com)

Z. Wang is with the Department of Information Systems and Computing, Brunel University, Uxbridge, Middlesex, UB8 3PH, United Kingdom. (Email: Zidong.Wang@brunel.ac.uk)

H. Gao is with the Research Institute of Intelligent Control and Systems, Harbin Institute of Technology, Harbin 150001, China. (Email: hjgao@hit.edu.cn)

Markovian jump systems with randomly varying nonlinearities has not been investigated yet, not to mention the case when the sensor saturation occurs as well.

In this paper, we are motivated to deal with the FDI problem for MJSs where the transition probability matrix has partially unknown entries, the nonlinearities vary randomly and the sensor saturations occur with given amplitudes. *The main contributions of this paper can be highlighted as follows. 1) The randomly varying nonlinearities, which are modeled by a Bernoulli random binary distributed white sequence with a known conditional probability, are introduced to describe the binary switch between two kinds of nonlinear disturbances. 2) In the plant under consideration, both the incomplete knowledge of mode transition probabilities and the sensor saturations are present, which render more practical significance of system model. 3) Two energy norm indices are used for the fault detection problem in order to account for, respectively, the restraint of disturbance and the sensitivity of faults. 4) Intensive stochastic analysis is carried out to enforce multiple performance requirements against the uncertainties, nonlinearities and saturations.*

II. Problem Formulation

Let $\theta(k)$ $(k \geq 0)$ be a Markov chain on the probability space which takes values in the finite space $S = \{1, 2, ..., s\}$ with transition probability matrix $\hat{\Psi} = [\lambda_{ij}]$ given by

$$\text{Prob}\left\{\theta(k+1) = j|\theta(k) = i\right\} = \lambda_{ij},\ \forall i, j \in S$$

where $\lambda_{ij} \geq 0$ $(i, j \in S)$ is the transition probability from i to j and $\sum_{j=1}^{s} \lambda_{ij} = 1$, $\forall i \in S$.

We assume that some elements in the transition probability matrix $\hat{\Psi}$ are unknown. For notation clarity, for any $i \in S$, we denote that

$$S^i_{\mathcal{K}} := \{j : \lambda_{ij} \text{ is known}\},\ S^i_{\mathcal{UK}} := \{j : \lambda_{ij} \text{ is unknown}\}. \quad (1)$$

Also, we define $\lambda^i_{\mathcal{K}} := \sum_{j \in S^i_{\mathcal{K}}} \lambda_{ij}$ throughout the paper.

Remark 1: Note that $S = S^i_{\mathcal{K}} + S^i_{\mathcal{UK}}$ $(i \in S)$. Moreover, when $S^i_{\mathcal{K}} \neq \emptyset$, it can be further described as $S^i_{\mathcal{K}} = \{\mathcal{K}^i_1, \mathcal{K}^i_2, \ldots, \mathcal{K}^i_m\}$, $\forall\ 1 \leq m \leq s$, where $\mathcal{K}^i_m \in \mathbb{N}^+$ ($\mathbb{N}^+$ represents the sets of positive integers) denote the mth known element with the index $\mathcal{K}^i_m$ in the ith row of the matrix $\hat{\Psi}$.

Consider, on a probability space (Ω, $\mathscr{F}$, Prob), the following class of Markovian jump discrete systems:

$$\begin{cases} x(k+1) = A(\theta(k))x(k) + \alpha(k)g(\theta(k), x(k)) \\ \qquad\qquad + (1-\alpha(k))h(\theta(k), x(k)) \\ \qquad\qquad + D_1(\theta(k))w(k) + G(\theta(k))f(k) \\ y(k) = \sigma(C(\theta(k))x(k)) + D_2(\theta(k))w(k) \\ \qquad\qquad + E(\theta(k))f(k) \end{cases} \quad (2)$$

where $x(k) \in \mathbb{R}^{n_x}$ represents the state vector; $y(k) \in \mathbb{R}^{n_y}$ is the process output; $w(k) \in \mathbb{R}^{n_w}$ is the disturbance input which belongs to $l_2[0, \infty)$; $g(\cdot)$ and $h(\cdot)$ are nonlinear vector functions. $f(k) \in \mathbb{R}^l$ is the fault to be detected. For fixed system mode, $A(\theta(k)), D_1(\theta(k)), G(\theta(k)), C(\theta(k)), D_2(\theta(k))$ and $E(\theta(k))$ are constant matrices with appropriate dimensions.

The stochastic variable $\alpha(k)$ is a Bernoulli distributed white noise sequences taking values on 0 and 1 with

$$\text{Prob}\{\alpha(k) = 1\} = \bar{\alpha}, \quad \text{Prob}\{\alpha(k) = 0\} = 1 - \bar{\alpha}.$$

In this paper, we assume that Markov chain $\theta(k)$ is independent of the stochastic variable $\alpha(k)$.

The nonlinear functions $g(\theta(k), x(k))$ and $h(\theta(k), x(k))$ are assumed to satisfy $g(\theta(k), 0) = 0$, $h(\theta(k), 0) = 0$, and

$$\begin{aligned} & \|g(\theta(k), x(k)+\delta(k)) - g(\theta(k), x(k))\| \\ \leq\ & \|B_1(\theta(k))\delta(k)\| \\ & \|h(\theta(k), x(k)+\delta(k)) - h(\theta(k), x(k))\| \\ \leq\ & \|B_2(\theta(k))\delta(k)\| \end{aligned} \quad (3)$$

where, for fixed system mode, $B_1(\theta(k))$ and $B_2(\theta(k))$ are known matrices, and $\delta(k)$ is a vector.

The saturation function σ: $\mathbb{R}^{n_y} \to \mathbb{R}^{n_y}$ is defined as

$$\sigma(v) = \left[\begin{array}{cccc} \sigma_1^T(v_1) & \sigma_2^T(v_2) & \cdots & \sigma_{n_y}^T(v_{n_y}) \end{array}\right]^T \quad (4)$$

with $\sigma_i(v_i) = \text{sign}(v_i)\min\{v_{i,\max}, |v_i|\}$, where $v_{i,\max}$ is the ith element of the vector $v_{\max}$, the saturation level.

Definition 1: [20] A nonlinearity $\Psi : \mathbb{R}^m \mapsto \mathbb{R}^m$ is said to satisfy a sector condition if

$$\left(\Psi(v) - \bar{H}_1 v\right)^T \left(\Psi(v) - \bar{H}_2 v\right) \leq 0, \forall v \in \mathbb{R}^r \quad (5)$$

for some real matrices $\bar{H}_1, \bar{H}_2 \in \mathbb{R}^{r\times r}$, where $\bar{H} = \bar{H}_2 - \bar{H}_1$ is a positive-definite symmetric matrix. In this case, we say that Ψ belongs to the sector $\left[\bar{H}_1\ \ \bar{H}_2\right]$.

Assuming that there exist two diagonal matrices L_1 and L_2 such that $0 \leq L_1 < I \leq L_2$, then the saturation function $\sigma(C(\theta(k))x(k))$ in (2) can be decomposed into a linear and a nonlinear part as

$$\sigma(C(\theta(k))x(k)) = L_1 C(\theta(k))x(k) + \Psi(C(\theta(k))x(k)) \quad (6)$$

where $\Psi(C(\theta(k))x(k))$ is a nonlinear vector-valued function satisfying the sector condition with $\bar{H}_1 = 0$, $\bar{H}_2 = L$, and can be described as follows:

$$\Psi^T(C(\theta(k))x(k))\left(\Psi(C(\theta(k))x(k)) - LC(\theta(k))x(k)\right) \leq 0 \quad (7)$$

where $L = L_2 - L_1$.

For presentation convenience, for each possible $\theta(k) = i$ $(i \in S)$, a matrix $N(\theta(k))$ and a function $l(\theta(k))$ are denoted by N_i and l_i, respectively.

We consider a fault detection filter of the following form:

$$\begin{cases} \hat{x}(k+1) = A_i\hat{x}(k) + \bar{\alpha}g_i(\hat{x}(k)) + (1-\bar{\alpha})h_i(\hat{x}(k)) \\ \qquad\qquad + K_i[y(k) - C_i\hat{x}(k)] \\ \tilde{r}(k) = M[y(k) - C_i\hat{x}(k)] \end{cases} \quad (8)$$

where $\hat{x}(k) \in \mathbb{R}^{n_x}$ is the state estimate, $\tilde{r}(k) \in \mathbb{R}^l$ is the so-called residual, and K_i and M are appropriately dimensioned filter matrices to be determined.

Letting $e(k) = x(k) - \hat{x}(k)$, by augmenting $\eta(k) = \begin{bmatrix} x^T(k) & e^T(k) \end{bmatrix}^T$, the overall fault detection dynamics is governed by the following augmented system:

$$\begin{cases} \eta(k+1) = \mathcal{Y}_i(\eta(k)) + (\alpha(k) - \bar{\alpha})\Lambda_2 \mathcal{G}_i(\eta(k)) + \mathcal{D}_{di} w(k) \\ \qquad\qquad + \mathcal{D}_{fi} f(k) + \mathcal{K}_{\sigma i} \sigma(C_i H_1 \eta(k)) \\ \tilde{r}(k) = M[\sigma(C_i H_1 \eta(k)) + \hat{C}_i \eta(k) + D_{2i} w(k) \\ \qquad\qquad + E_i f(k)] \end{cases} \tag{9}$$

where

$$\begin{aligned} \mathcal{Y}_i(\eta(k)) &= \mathcal{A}_i \eta(k) + \Lambda_1 \mathcal{G}_i(\eta(k)), \mathcal{K}_{\sigma i} = \begin{bmatrix} 0 & -K_i^T \end{bmatrix}^T, \\ H_1 &= \begin{bmatrix} I & 0 \end{bmatrix}, \quad \hat{C}_i = \begin{bmatrix} -C_i & C_i \end{bmatrix}, \\ \mathcal{A}_i &= \begin{bmatrix} A_i & 0 \\ K_i C_i & A_i - K_i C_i \end{bmatrix}, \mathcal{D}_{fi} = \begin{bmatrix} G_i \\ G_i - K_i E_i \end{bmatrix}, \\ \Lambda_1 &= \begin{bmatrix} \bar{\alpha} I & (1-\bar{\alpha})I & 0 & 0 \\ 0 & 0 & \bar{\alpha} I & (1-\bar{\alpha})I \end{bmatrix}, \\ \Lambda_2 &= \begin{bmatrix} I & -I & 0 & 0 \\ I & -I & 0 & 0 \end{bmatrix}, \quad \mathcal{D}_{di} = \begin{bmatrix} D_{1i} \\ D_{1i} - K_i D_{2i} \end{bmatrix}, \\ \mathcal{G}_i(\eta(k)) &= \begin{bmatrix} \mathcal{H}_i^T(x(k)) & \tilde{\mathcal{H}}_i^T(e(k)) \end{bmatrix}^T, \\ \mathcal{H}_i(x(k)) &= \begin{bmatrix} g_i^T(x(k)) & h_i^T(x(k)) \end{bmatrix}^T, \\ \tilde{\mathcal{H}}_i(e(k)) &= \begin{bmatrix} \tilde{g}_i^T(e(k)) & \tilde{h}_i^T(e(k)) \end{bmatrix}^T, \\ \tilde{g}_i(e(k)) &:= g_i(x(k)) - g_i(\hat{x}(k)), \\ \tilde{h}_i(e(k)) &:= h_i(x(k)) - h_i(\hat{x}(k)). \end{aligned} \tag{10}$$

Moreover, it follows from (3), (6) and (7) that

$$\|\mathcal{G}_i(\eta(k))\| \le \|\tilde{B}_i \eta(k)\|, \tag{11}$$
$$\sigma(C_i H_1 \eta(k)) = \tilde{L}_{1i} \eta(k) + \Psi(C_i H_1 \eta(k)), \tag{12}$$
$$\Psi^T(C_i H_1 \eta(k)) \left(\Psi(C_i H_1 \eta(k)) - \tilde{L}_{2i} \eta(k) \right) \le 0 \tag{13}$$

where

$$\begin{aligned} &\tilde{B}_i := \begin{bmatrix} B_{1i}^T & B_{2i}^T & 0 & 0 \\ 0 & 0 & B_{1i}^T & B_{2i}^T \end{bmatrix}^T, \quad \tilde{L}_{1i} := \begin{bmatrix} L_1 C_i & 0 \end{bmatrix}, \\ &\tilde{L}_{2i} := \begin{bmatrix} L C_i & 0 \end{bmatrix}. \end{aligned} \tag{14}$$

Definition 2: The fault detection dynamics in (10) is said to be stochastically stable in the mean square for any initial conditions $\eta(0)$ and $\theta(0) \in S$ if, when $w(k) = 0$ and $f(k) = 0$, there exists a finite $W(\theta(0)) > 0$ such that

$$\mathbb{E}\left\{ \sum_{k=0}^{\infty} \|\eta(k)\|^2 \Big| \eta(0), \theta(0) \right\} < \eta^T(0) W(\theta(0)) \eta(0).$$

The main purpose of this paper is to design a fault detection filter of the form (8) such that the following requirements are met simultaneously:

a) The fault detection dynamics (9) is stochastically stable.

b) Under the zero-initial condition, the following inequality holds for any nonzero $w(k)$

$$\sum_{k=0}^{\infty} \mathbb{E}\{\|\tilde{r}(k)\|^2\} \le \gamma^2 \sum_{k=0}^{\infty} \|w(k)\|^2 \Big|_{f(k)=0} \tag{15}$$

where $\gamma > 0$ is made as small as possible in the feasibility of (15) so as to minimize the effect from the exogenous disturbance on the residual.

c) Under the zero-initial condition, the following inequality holds for any nonzero $f(k)$

$$\sum_{k=0}^{\infty} \mathbb{E}\{\|\tilde{r}(k)\|^2\} \ge \beta^2 \sum_{k=0}^{\infty} \|f(k)\|^2 \Big|_{w(k)=0} \tag{16}$$

where $\beta > 0$ is made as large as possible in the feasibility of (16) so as to enhance the sensitivity of faults on the residual.

Remark 2: It should be noted that the performance index γ reflects the robustness of residuals against the disturbance in the fault-free case, and the performance index β quantifies the sensitivity of the residuals with respect to the fault in the disturbance-free case. Therefore, in order to achieve a satisfactory trade-off between the robustness against the disturbances and the sensitivity to the faults, the fault detection dynamics (9) should be made stochastically stable where the index

$$J = \gamma / \beta, \tag{17}$$

is used to evaluate the overall performance of the designed fault detection filter.

We further adopt a residual evaluation stage including an evaluation function $\bar{J}(\tilde{r})$ and a threshold $\bar{J}_{th}$ of the following form:

$$\bar{J}(\tilde{r}) = \left\{ \sum_{s=k-\mathcal{L}}^{s=k} \tilde{r}^T(s) \tilde{r}(s) \right\}^{\frac{1}{2}}, \quad \bar{J}_{th} = \sup_{w \in l_2, f=0} \mathbb{E}\{\bar{J}(\tilde{r})\} \tag{18}$$

Based on (18), the occurrence of faults can be detected by comparing $\bar{J}(\tilde{r})$ with $\bar{J}_{th}$ according to the following rule:

$$\bar{J}(\tilde{r}) > \bar{J}_{th} \Longrightarrow \text{with faults} \Longrightarrow \text{alarm},$$
$$\bar{J}(\tilde{r}) \le \bar{J}_{th} \Longrightarrow \text{no faults}.$$

III. Main Results

Lemma 1: Consider the discrete-time Markovian jump system (2) with known transition probability matrix $\hat{\Psi}$. Let the filter parameters K_i $(i \in S)$, M and the index $\gamma > 0$ be given. The system (9) is stochastically stable and satisfies the constraint (15) if there exist a set of matrices $P_i > 0$ $(i \in S)$ and positive scalars ε_1, ε_2 satisfying

$$\hat{\Phi}_i = \begin{bmatrix} \hat{\Phi}_{11} & * \\ \hat{\Phi}_{21} & \hat{\Phi}_{22} \end{bmatrix} \le 0 \tag{19}$$

where

$$\begin{aligned} \hat{\Phi}_{11} &= \begin{bmatrix} \Phi_{11} + \mathcal{M}_i^T \mathcal{M}_i + \varepsilon_1 \tilde{B}_i^T \tilde{B}_i & * \\ \Lambda_1^T \bar{P}_i \bar{\mathcal{A}}_i & \Phi_{22} - \varepsilon_1 I \end{bmatrix}, \\ \hat{\Phi}_{21} &= \begin{bmatrix} M^T \mathcal{M}_i + \mathcal{K}_{\sigma i}^T \bar{P}_i \bar{\mathcal{A}}_i + \varepsilon_2 \tilde{L}_{2i} & \mathcal{K}_{\sigma i}^T \bar{P}_i \Lambda_1 \\ \mathcal{D}_{di}^T \bar{P}_i \bar{\mathcal{A}}_i + \mathcal{D}_{2i}^T \mathcal{M}_i & \mathcal{D}_{di}^T \bar{P}_i \Lambda_1 \end{bmatrix}, \\ \hat{\Phi}_{22} &= \begin{bmatrix} \mathcal{K}_{\sigma i}^T \bar{P}_i \mathcal{K}_{\sigma i} + M^T M - \varepsilon_2 I & * \\ \mathcal{D}_{di}^T \bar{P}_i \mathcal{K}_{\sigma i} + \mathcal{D}_{2i}^T M & \Xi_i \end{bmatrix}, \\ \bar{P}_i &= \sum_{j \in S} \lambda_{ij} P_j, \quad \Phi_{11} = \bar{\mathcal{A}}_i^T \bar{P}_i \bar{\mathcal{A}}_i - P_i, \end{aligned}$$

$$\bar{\mathcal{A}}_i = \mathcal{A}_i + \mathcal{K}_{\sigma i}\tilde{L}_{1i}, \quad \Xi_i = \mathcal{D}_{di}^T\bar{P}_i\mathcal{D}_{di} + \mathcal{D}_{2i}^T\mathcal{D}_{2i} - \gamma^2 I,$$
$$\mathcal{M}_i = M(\tilde{L}_{1i} + \hat{C}_i), \quad \mathcal{D}_{2i} = MD_{2i},$$
$$\Phi_{22} = \Lambda_1^T\bar{P}_i\Lambda_1 + \bar{\alpha}(1-\bar{\alpha})\Lambda_2^T\bar{P}_i\Lambda_2.$$

Proof: Consider (9) with $w(k) = 0$ and $f(k) = 0$, and define the following Lyapunov function:

$$V(\eta(k), \theta(k)) = \eta^T(k)P(\theta(k))\eta(k) \tag{20}$$

We can obtain that

$$\begin{aligned} &\mathbb{E}\{V(\eta(k+1), \theta(k+1)) \mid \eta(k), \theta(k)\} - V(\eta(k), \theta(k)) \\ < \ & -\lambda_{\min}(-\Gamma_i)\|\xi(k)\|^2 < -\lambda_{\min}(-\Gamma_i)\|\eta(k)\|^2, \end{aligned} \tag{21}$$

where

$$\Gamma_i = \begin{bmatrix} \Phi_{11} + \varepsilon_1\tilde{B}_i^T\tilde{B}_i & * & * \\ \Lambda_1^T\bar{P}_i\bar{\mathcal{A}}_i & \Phi_{22} - \varepsilon_1 I & * \\ \mathcal{K}_{\sigma i}^T\bar{P}_i\bar{\mathcal{A}}_i + \varepsilon_2\tilde{L}_{2i} & \mathcal{K}_{\sigma i}^T\bar{P}_i\Lambda_1 & \mathcal{K}_{\sigma i}^T\bar{P}_i\mathcal{K}_{\sigma i} - \varepsilon_2 I \end{bmatrix}.$$

which implies

$$\mathbb{E}\left\{\sum_{k=0}^{\infty}\|\eta(k)\|^2\Big|\eta(0), \theta(0)\right\} < \eta^T(0)\mathcal{W}(\theta(0))\eta(0)$$

where $\mathcal{W}(\theta(0)) := (\lambda_{\min}(-\Gamma_i))^{-1}P(\theta(0)) > 0$. Hence the fault detection dynamics (9) is stochastically stable.

Next, consider system (9) with $f(k) = 0$. We introduce the following index:

$$\begin{aligned} J_1 \ := \ & \mathbb{E}\{V(\eta(k+1), \theta(k+1)) \mid \eta(k), \theta(k)\} \\ & -V(\eta(k), \theta(k)) + \mathbb{E}\{\|\tilde{r}(k)\|^2\} - \gamma^2\|w(k)\|^2 \end{aligned}$$

we have $\mathbb{E}\{J_1\} \leq 0$. By considering the zero initial conditions, we can obtain(15), and then the proof is complete. ■

Lemma 2: Consider the discrete-time Markovian jump system (2) with known transition probability matrix $\hat{\Psi}$. Let the filter parameters K_i $(i \in S)$, M and the index $\beta > 0$ be given. For the system (9), the constraint (16) is met if there exist a set of matrices $P_i > 0$ $(i \in S)$ and positive constant scalars ε_1, ε_2 satisfying

$$\Omega_i = \begin{bmatrix} \hat{\Omega}_{11} & * \\ \hat{\Omega}_{21} & \hat{\Omega}_{22} \end{bmatrix} \leq 0 \tag{22}$$

where

$$\hat{\Omega}_{11} = \begin{bmatrix} \Phi_{11} - \mathcal{M}_i^T\mathcal{M}_i + \varepsilon_1\tilde{B}_i^T\tilde{B}_i & * \\ \Lambda_1^T\bar{P}_i\bar{\mathcal{A}}_i & \Phi_{22} - \varepsilon_1 I \end{bmatrix},$$
$$\hat{\Omega}_{21} = \begin{bmatrix} -M^T\mathcal{M}_i + \mathcal{K}_{\sigma i}^T\bar{P}_i\bar{\mathcal{A}}_i + \varepsilon_2\tilde{L}_{2i} & \mathcal{K}_{\sigma i}^T\bar{P}_i\Lambda_1 \\ \mathcal{D}_{fi}^T\bar{P}_i\bar{\mathcal{A}}_i - E_i^TM^T\mathcal{M}_i & \mathcal{D}_{fi}^T\bar{P}_i\Lambda_1 \end{bmatrix},$$
$$\hat{\Omega}_{22} = \begin{bmatrix} \mathcal{K}_{\sigma i}^T\bar{P}_i\mathcal{K}_{\sigma i} - M^TM - \varepsilon_2 I & * \\ \mathcal{D}_{fi}^T\bar{P}_i\mathcal{K}_{\sigma i} - E_i^TM^TM & \Omega_{33} \end{bmatrix},$$
$$\Omega_{33} = \mathcal{D}_{fi}^T\bar{P}_i\mathcal{D}_{fi} + \beta^2 I - E_i^TM^TME_i,$$

and the other symbols are the same as defined in Lemma 1.

Proof: Consider the system (9) with $w(k) = 0$ and define

$$\begin{aligned} J_2 \ := \ & \mathbb{E}\{V(\eta(k+1), \theta(k+1)) \mid \eta(k), \theta(k)\} \\ & -V(\eta(k), \theta(k)) - \mathbb{E}\{\|\tilde{r}(k)\|^2\} + \beta^2\|f(k)\|^2, \end{aligned}$$

we have

$$\mathbb{E}\{J_2\} \ \leq \ \mathbb{E}\{\bar{\xi}^T(k)\Omega_i\bar{\xi}(k)\},$$

where

$$\bar{\xi}(k) := \begin{bmatrix} \xi^T(k) & f^T(k) \end{bmatrix}^T.$$

Furthermore, it follows from (22) in Lemma 2 that

$$\begin{aligned} &\mathbb{E}\{V(\eta(k+1), \theta(k+1)) - V(\eta(k), \theta(k))\} \\ &-\mathbb{E}\{\|\tilde{r}(k)\|^2\} + \beta^2\|f(k)\|^2 \leq 0 \end{aligned}$$

for all nonzero $f(k)$. Considering the zero initial conditions, it is easy to see that

$$\sum_{k=0}^{\infty}\mathbb{E}\{\|\tilde{r}(k)\|^2\} \geq \beta^2\sum_{k=0}^{\infty}\|f(k)\|^2$$

which is equivalent to (16). The proof is now complete. ■

The following lemma is easily accessible from Lemma 1 and Lemma 2, and therefore its proof is omitted.

Lemma 3: Consider the discrete-time Markovian jump system (2) with known transition probability matrix $\hat{\Psi}$. Let the filter parameters K_i $(i \in S)$, M and the indices $\beta > 0$, $\gamma > 0$ be given. The system (9) is stochastically stable while satisfying the constraints (15)-(16) if there exist a set of matrices $P_i > 0$ $(i \in S)$ and positive constant scalars ε_1, ε_2 such that inequalities (19) and (22) hold simultaneously.

Next, given the unknown transition probability matrix described in (1), we first propose the following performance analysis results with a given fault detection filter (8), and then deal with the design problem of the fault detection filter for system (2).

Theorem 1: Consider the discrete-time Markovian jump system (2) subject to randomly varying nonlinearities, sensor saturation and incomplete knowledge of transition probabilities. Let the indices $\beta > 0$, $\gamma > 0$ and the fault detection filter parameters K_i $(i \in S)$, M be given. The fault detection dynamics (9) is stochastically stable while achieving the performance constraints (15)-(16) if there exist matrices $P_i > 0$ $(i \in S)$ and positive constant scalars ε_1, ε_2 such that the following inequalities hold:

$$\Pi_{ij} = \begin{bmatrix} \Pi_{11} & * \\ \Pi_{21} & \Pi_{22} \end{bmatrix} \leq 0 \tag{23}$$

$$\bar{\Pi}_{ij} = \begin{bmatrix} \bar{\Pi}_{11} & * \\ \bar{\Pi}_{21} & \bar{\Pi}_{22} \end{bmatrix} \leq 0 \tag{24}$$

where, if $\lambda_{\mathcal{K}}^i = 0$, Q_j is defined to be $Q_j = P_j$ $(j \in S_{\mathcal{UK}}^i)$, otherwise

$$\begin{cases} Q_j = \dfrac{1}{\lambda_{\mathcal{K}}^i}P_{\mathcal{K}}^i = \dfrac{1}{\lambda_{\mathcal{K}}^i}\displaystyle\sum_{j\in S_{\mathcal{K}}^i}\lambda_{ij}P_j, & \forall j \in S_{\mathcal{K}}^i \\ Q_j = P_j, & \forall j \in S_{\mathcal{UK}}^i \end{cases}$$

and

$$\Pi_{11} = \begin{bmatrix} \bar{\Phi}_{11} + \mathcal{M}_i^T \mathcal{M}_i + \varepsilon_1 \tilde{B}_i^T \tilde{B}_i & * \\ \Lambda_1^T Q_j \bar{\mathcal{A}}_i & \bar{\Phi}_{22} - \varepsilon_1 I \end{bmatrix},$$
$$\Pi_{21} = \begin{bmatrix} M^T \mathcal{M}_i + \mathcal{K}_{\sigma i}^T Q_j \bar{\mathcal{A}}_i + \varepsilon_2 \tilde{L}_{2i} & \mathcal{K}_{\sigma i}^T Q_j \Lambda_1 \\ \mathcal{D}_{di}^T Q_j \bar{\mathcal{A}}_i + \mathcal{D}_{2i}^T \mathcal{M}_i & \mathcal{D}_{di}^T Q_j \Lambda_1 \end{bmatrix},$$
$$\Pi_{22} = \begin{bmatrix} \mathcal{K}_{\sigma i}^T Q_j \mathcal{K}_{\sigma i} + M^T M - \varepsilon_2 I & * \\ \mathcal{D}_{di}^T Q_j \mathcal{K}_{\sigma i} + \mathcal{D}_{2i}^T M & \bar{\Phi}_{33} \end{bmatrix},$$
$$\bar{\Pi}_{11} = \begin{bmatrix} \bar{\Phi}_{11} - \mathcal{M}_i^T \mathcal{M}_i + \varepsilon_1 \tilde{B}_i^T \tilde{B}_i & * \\ \Lambda_1^T Q_j \bar{\mathcal{A}}_i & \bar{\Phi}_{22} - \varepsilon_1 I \end{bmatrix},$$
$$\bar{\Pi}_{21} = \begin{bmatrix} -M^T \mathcal{M}_i + \mathcal{K}_{\sigma i}^T Q_j \bar{\mathcal{A}}_i + \varepsilon_2 \tilde{L}_{2i} & \mathcal{K}_{\sigma i}^T Q_j \Lambda_1 \\ \mathcal{D}_{fi}^T Q_j \bar{\mathcal{A}}_i - E_i^T M^T \mathcal{M}_i & \mathcal{D}_{fi}^T Q_j \Lambda_1 \end{bmatrix},$$
$$\bar{\Pi}_{22} = \begin{bmatrix} \mathcal{K}_{\sigma i}^T Q_j \mathcal{K}_{\sigma i} - M^T M - \varepsilon_2 I & * \\ \mathcal{D}_{fi}^T Q_j \mathcal{K}_{\sigma i} - E_i^T M^T M & \bar{\Omega}_{33} \end{bmatrix},$$
$$\bar{\Phi}_{11} = \bar{\mathcal{A}}_i^T Q_j \bar{\mathcal{A}}_i - P_i,$$
$$\bar{\Phi}_{22} = \Lambda_1^T Q_j \Lambda_1 + \bar{\alpha}(1-\bar{\alpha}) \Lambda_2^T Q_j \Lambda_2,$$
$$\bar{\Phi}_{33} = \mathcal{D}_{di}^T Q_j \mathcal{D}_{di} + \mathcal{D}_{2i}^T \mathcal{D}_{2i} - \gamma^2 I,$$
$$\bar{\Omega}_{33} = \mathcal{D}_{fi}^T Q_j \mathcal{D}_{fi} + \beta^2 I - E_i^T M^T M E_i.$$

Proof: Note that $\hat{\Phi}_i$ in (19) can be rewritten as

$$\begin{aligned} \hat{\Phi}_i &= \begin{bmatrix} \widetilde{\Phi}_{11} & * \\ \widetilde{\Phi}_{21} & \widetilde{\Phi}_{22} \end{bmatrix} \\ &\quad + \sum_{j \in S^i_{\mathcal{UK}}} \lambda_{ij} \begin{bmatrix} \underline{\Phi}_{11} & * \\ \underline{\Phi}_{21} & \underline{\Phi}_{22} \end{bmatrix} \\ &= \lambda^i_{\mathcal{K}} \Pi_{ij}_{j \in S^i_{\mathcal{K}}} + \sum_{j \in S^i_{\mathcal{UK}}} \lambda_{ij} \Pi_{ij} \end{aligned}$$

where

$$\widetilde{\Phi}_{11} = \begin{bmatrix} \vec{\Phi}_{11} + \lambda^i_{\mathcal{K}} \bar{\mathcal{M}}_i & * \\ \Lambda_1^T P^i_{\mathcal{K}} \bar{\mathcal{A}}_i & \vec{\Phi}_{22} - \lambda^i_{\mathcal{K}} \varepsilon_1 I \end{bmatrix},$$
$$\widetilde{\Phi}_{21} = \begin{bmatrix} \mathcal{K}_{\sigma i}^T P^i_{\mathcal{K}} \bar{\mathcal{A}}_i + \lambda^i_{\mathcal{K}} \tilde{\mathcal{M}}_i & \mathcal{K}_{\sigma i}^T P^i_{\mathcal{K}} \Lambda_1 \\ \mathcal{D}_{di}^T P^i_{\mathcal{K}} \bar{\mathcal{A}}_i + \lambda^i_{\mathcal{K}} \mathcal{D}_{2i}^T \mathcal{M}_i & \mathcal{D}_{di}^T P^i_{\mathcal{K}} \Lambda_1 \end{bmatrix},$$
$$\widetilde{\Phi}_{22} = \begin{bmatrix} \widetilde{\Xi}_{11} & * \\ \mathcal{D}_{di}^T P^i_{\mathcal{K}} \mathcal{K}_{\sigma i} + \lambda^i_{\mathcal{K}} \mathcal{D}_{2i}^T M & \mathcal{D}_{di}^T P^i_{\mathcal{K}} \mathcal{D}_{di} + \lambda^i_{\mathcal{K}} \bar{\mathcal{D}}_{2i} \end{bmatrix},$$
$$\underline{\Phi}_{11} = \begin{bmatrix} \check{\Phi}_{11} + \bar{\mathcal{M}}_i & * \\ \Lambda_1^T P_j \bar{\mathcal{A}}_i & \check{\Phi}_{22} - \varepsilon_1 I \end{bmatrix},$$
$$\underline{\Phi}_{21} = \begin{bmatrix} \mathcal{K}_{\sigma i}^T P_j \bar{\mathcal{A}}_i + \tilde{\mathcal{M}}_i & \mathcal{K}_{\sigma i}^T P_j \Lambda_1 \\ \mathcal{D}_{di}^T P_j \bar{\mathcal{A}}_i + \mathcal{D}_{2i}^T \mathcal{M}_i & \mathcal{D}_{di}^T P_j \Lambda_1 \end{bmatrix},$$
$$\underline{\Phi}_{22} = \begin{bmatrix} \mathcal{K}_{\sigma i}^T P_j \mathcal{K}_{\sigma i} + M^T M - \varepsilon_2 I & * \\ \mathcal{D}_{di}^T P_j \mathcal{K}_{\sigma i} + \mathcal{D}_{2i}^T M & \mathcal{D}_{di}^T P_j \mathcal{D}_{di} + \bar{\mathcal{D}}_{2i} \end{bmatrix},$$
$$\widetilde{\Xi}_{11} = \mathcal{K}_{\sigma i}^T P^i_{\mathcal{K}} \mathcal{K}_{\sigma i} - \lambda^i_{\mathcal{K}} (\varepsilon_2 I - M^T M),$$
$$\lambda^i_{\mathcal{K}} := \sum_{j \in S^i_{\mathcal{K}}} \lambda_{ij}, \quad P^i_{\mathcal{K}} = \sum_{j \in S^i_{\mathcal{K}}} \lambda_{ij} P_j,$$
$$\bar{\mathcal{D}}_{2i} = \mathcal{D}_{2i}^T \mathcal{D}_{2i} - \gamma^2 I, \quad \vec{\Phi}_{11} = \bar{\mathcal{A}}_i^T P^i_{\mathcal{K}} \bar{\mathcal{A}}_i,$$
$$\vec{\Phi}_{22} = \Lambda_1^T P^i_{\mathcal{K}} \Lambda_1 + \bar{\alpha}(1-\bar{\alpha}) \Lambda_2^T P^i_{\mathcal{K}} \Lambda_2,$$
$$\check{\Phi}_{11} = \bar{\mathcal{A}}_i^T P_j \bar{\mathcal{A}}_i, \quad \check{\Phi}_{22} = \Lambda_1^T P_j \Lambda_1 + \bar{\alpha}(1-\bar{\alpha}) \Lambda_2^T P_j \Lambda_2,$$
$$\bar{\mathcal{M}}_i = \mathcal{M}_i^T \mathcal{M}_i + \varepsilon_1 \tilde{B}_i^T \tilde{B}_i - P_i,$$
$$\tilde{\mathcal{M}}_i = \varepsilon_2 \tilde{L}_{2i} + M^T \mathcal{M}_i.$$

Therefore, inequality (23) guarantees that (19) holds. Similarly, it is not difficult to see from (24) that the inequality (22) is true. The proof of this theorem is complete. ■

Based on the analysis results with a given fault detection filter, we are now ready to solve the filter design problem for system (9) in the following theorem with the incomplete knowledge of transition probabilities.

Theorem 2: Consider system (2) with the unknown transition probability matrix described in (1). Let $\beta > 0$, $\gamma > 0$ be given indices. The fault detection dynamics (9) is stochastically stable while achieving the performance constraints (15)-(16) if there exist matrices $P_i > 0$, N_{ij} $(i, j \in S)$, $\bar{M}$ and positive constant scalars ε_1, ε_2 such that the following linear matrix inequalities (LMIs) hold:

$$\begin{bmatrix} \bar{\Upsilon}_{11} & * & * \\ \bar{\Upsilon}_{21} & \bar{\Upsilon}_{22} & * \\ \bar{\Upsilon}_{31} & \bar{\Upsilon}_{32} & \bar{\Upsilon}_{33} \end{bmatrix} \leq 0, \tag{25}$$

$$\begin{bmatrix} \hat{\Upsilon}_{11} & * & * \\ \hat{\Upsilon}_{21} & \hat{\Upsilon}_{22} & * \\ \bar{\Upsilon}_{31} & \hat{\Upsilon}_{32} & \bar{\Upsilon}_{33} \end{bmatrix} \leq 0, \tag{26}$$

where

$$\bar{\Upsilon}_{11} = \mathrm{diag}\{\Upsilon_{11} + \tilde{\Upsilon}_{11}, -\varepsilon_1 I\},$$
$$\bar{\Upsilon}_{21} = \begin{bmatrix} \bar{M}(\tilde{L}_{1i} + \hat{C}_i) + \varepsilon_2 \tilde{L}_{2i} & 0 \\ D_{2i}^T \bar{M}(\tilde{L}_{1i} + \hat{C}_i) & 0 \end{bmatrix},$$
$$\bar{\Upsilon}_{22} = \begin{bmatrix} -\varepsilon_2 I + \bar{M} & * \\ D_{2i}^T \bar{M} & D_{2i}^T \bar{M} D_{2i} - \gamma^2 I \end{bmatrix},$$
$$\bar{\Upsilon}_{31} = \begin{bmatrix} Q_j \mathcal{A}_{0i} + N_{ij}(\hat{C}_i + \tilde{L}_{1i}) & Q_j \Lambda_1 \\ 0 & \sqrt{\bar{\alpha}(1-\bar{\alpha})} Q_j \Lambda_2 \end{bmatrix},$$
$$\bar{\Upsilon}_{32} = \begin{bmatrix} N_{ij} & Q_j \hat{D}_{1i} + N_{ij} D_{2i} \\ 0 & 0 \end{bmatrix},$$
$$\bar{\Upsilon}_{33} = \mathrm{diag}\{-Q_j, -Q_j\},$$
$$\hat{\Upsilon}_{11} = \mathrm{diag}\{\Upsilon_{11} - \tilde{\Upsilon}_{11}, -\varepsilon_1 I\},$$
$$\hat{\Upsilon}_{21} = \begin{bmatrix} -\bar{M}(\tilde{L}_{1i} + \hat{C}_i) + \varepsilon_2 \tilde{L}_{2i} & 0 \\ -E_i^T \bar{M}(\tilde{L}_{1i} + \hat{C}_i) & 0 \end{bmatrix},$$
$$\hat{\Upsilon}_{22} = \begin{bmatrix} -\varepsilon_2 I - \bar{M} & * \\ -E_i^T \bar{M} & -E_i^T \bar{M} E_i + \beta^2 I \end{bmatrix},$$
$$\hat{\Upsilon}_{32} = \begin{bmatrix} N_{ij} & Q_j \hat{G}_i + N_{ij} E_i \\ 0 & 0 \end{bmatrix}, \Upsilon_{11} = \varepsilon_1 \tilde{B}_i^T \tilde{B}_i - P_i,$$
$$\tilde{\Upsilon}_{11} = (\tilde{L}_{1i} + \hat{C}_i)^T \bar{M} (\tilde{L}_{1i} + \hat{C}_i), \mathcal{A}_{0i} = \mathrm{diag}\{A_i, A_i\},$$
$$\bar{H} = \begin{bmatrix} 0 & -I \end{bmatrix}^T, \hat{D}_{1i} = \mathbf{1}_2 \otimes D_{1i}, \hat{G}_i = \mathbf{1}_2 \otimes G_i. \tag{27}$$

Furthermore, $K_i = (\bar{H}^T Q_j \bar{H})^{-1} \bar{H}^T N_{ij}$, and M can be obtained by means of the matrix $\bar{M}$, where M is a factorization of $\bar{M}$ (i.e., $\bar{M} = M^T M$).

Proof: We rewrite the parameters in Theorem 1 in the following form

$$\begin{aligned} \mathcal{A}_i &= \mathcal{A}_{0i} + \bar{H} K_i \hat{C}_i, \quad \mathcal{D}_{di} = \hat{D}_{1i} + \bar{H} K_i D_{2i}, \\ K_{\sigma i} &= \bar{H} K_i, \quad \mathcal{D}_{fi} = \hat{G}_i + \bar{H} K_i E_i. \end{aligned} \tag{28}$$

Noticing (28) and applying the Schur complement equivalence, together with some straightforward algebraic manipulations, (25) and (26) can be obtained from (23) and (24), respectively. The proof is now complete. ■

Remark 3: Theorem 2 provides a solution to the fault detection filter design problem for the discrete Markovian jump system (2) under partially unknown transition probabilities. Obviously, in the spirit of fault detection, the index $\gamma > 0$ should be made as small as possible subject to (25) so as to minimize the effect from the exogenous disturbance on the residual, while the index $\beta > 0$ should be made as large as possible subject to (26) in order to maximize the sensitivity of faults on the residual. Based on such a principle, we will propose an algorithm that locally optimizes the gains of the fault detection filters.

To achieve both the satisfactory robustness against disturbances and the satisfactory sensitivity to faults, we suggest the following locally Optimized Fault Detection Filter Design ($OFDFD$) algorithm.

Algorithm $OFDFD$:

Step 1. Obtain $\gamma_{\min}$ (the minimum of γ) and $\beta_{\max}$ (the maximum of β) by solving (25) and (26) in Theorem 2, respectively.

Step 2. If, with γ and β replaced by $\gamma_{\min}$ and $\beta_{\max}$ respectively, (25) and (26) are feasible for Theorem 2, we can obtain the locally optimized parameters K_i and M for the desired fault detection filter and exit. Otherwise, go to Step 3.

Step 3. Increase $\gamma_{\min}$ by μ and decrease $\beta_{\max}$ by μ where $\mu > 0$ is a sufficiently small scalar, and then solve (25) and (26) with the updated $\gamma_{\min}$ and $\beta_{\max}$. Repeat such a procedure until (25) and (26) are feasible, and therefore obtain the locally optimized filter parameters $\{K_i, M\}$ and the index $J_{\min} = \gamma_{\min}/\beta_{\max}$.

Step 4. Stop.

Remark 4: Based on the proposed Algorithm $OFDFD$, the main results in Theorem 2 can be applied to solve the fault detection problem for a wide class of Markovian jump systems involving sensor saturations and randomly varying nonlinearities that result typically from networked environments. The Algorithm $OFDFD$ is developed to check the existence of the desired fault detection filter gains, and the explicit expression of such filter gains is characterized in terms of the solution to a set of LMIs that can be effectively solved by the algorithms such as the interior-point method.

Remark 5: The system (2) under consideration is quite comprehensive that reflects partially known mode transition probabilities, randomly varying nonlinearities as well as the sensor saturations. Furthermore, two energy norm indices are used for the fault detection problem in order to account for, respectively, the restraint of disturbance and the sensitivity of faults. Note that the main results established contain all the information of the addressed general systems including the physical parameters, the transition probabilities, occurrence probabilities of the randomly varying nonlinearities, and the amplitudes of the sensor saturations. In the next section, a simulation example is

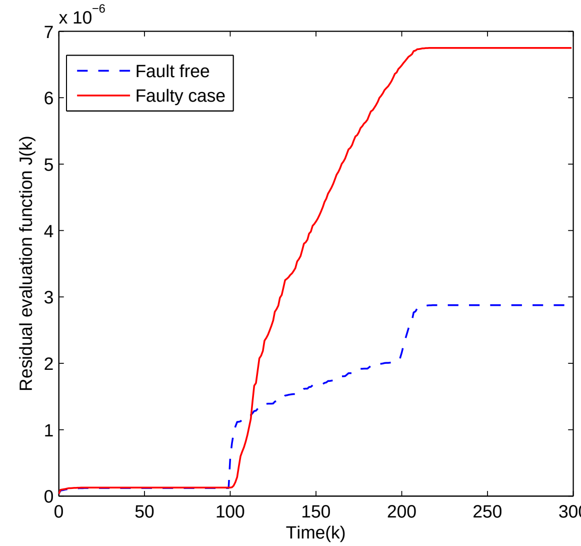

Fig. 1. Eveolution of $\bar{J}(\tilde{r}) = \{\sum_{l=0}^{k} \tilde{r}^T(l)\tilde{r}(l)\}^{\frac{1}{2}}$ in the case $\hat{\Psi}_1$

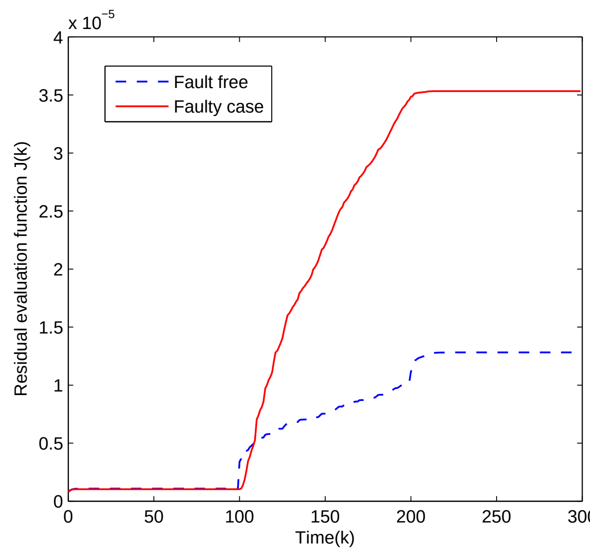

Fig. 2. Eveolution of $\bar{J}(\tilde{r}) = \{\sum_{l=0}^{k} \tilde{r}^T(l)\tilde{r}(l)\}^{\frac{1}{2}}$ in the case $\hat{\Psi}_2$

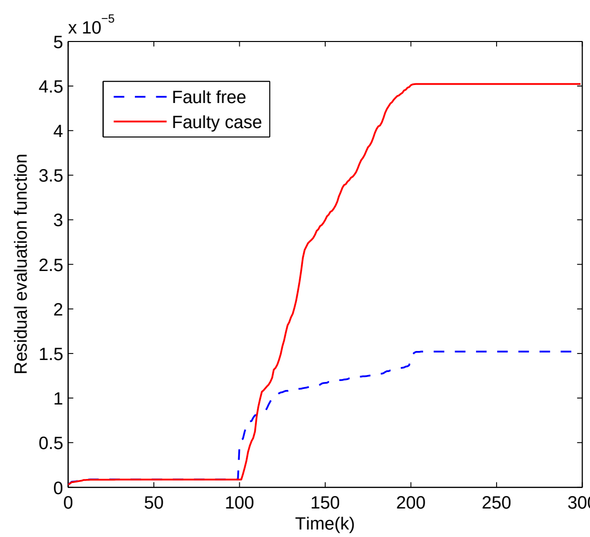

Fig. 3. Eveolution of $\bar{J}(\tilde{r}) = \{\sum_{l=0}^{k} \tilde{r}^T(l)\tilde{r}(l)\}^{\frac{1}{2}}$ in the case $\hat{\Psi}_3$

TABLE I
THE OPTIMAL INDICES AND FILTER GAINS FOR DIFFERENT CASES

Transition probability matrix	$J_{\min}$	K_1	K_2	M
$\hat{\Psi}_1$ (Completely known)	0.8992	$\begin{bmatrix} 0.6387 \\ 0.1695 \end{bmatrix}$	$\begin{bmatrix} 0.0058 \\ -0.1199 \end{bmatrix}$	0.0547
$\hat{\Psi}_2$ (Partially known)	1.2983	$\begin{bmatrix} 0.5643 \\ 0.3226 \end{bmatrix}$	$\begin{bmatrix} 0.1708 \\ -0.5382 \end{bmatrix}$	0.4362
$\hat{\Psi}_3$ (Completely unknown)	1.6180	$\begin{bmatrix} 0.1628 \\ 0.0608 \end{bmatrix}$	$\begin{bmatrix} 0.1166 \\ -0.0127 \end{bmatrix}$	0.3308

TABLE II
THRESHOLDS AND TIME STEPS OF FAULT DETECTION FOR DIFFERENT CASES

Transition probability matrix	$\hat{\Psi}_1$ (Completely known)	$\hat{\Psi}_2$ (Partially known)	$\hat{\Psi}_3$ (Completely unknown)
Thresholds	$1.2113 * 10^{-6}$	$0.4823 * 10^{-5}$	$0.8116 * 10^{-5}$
Time steps	111	112	117

provided to show the usefulness of the proposed fault detection technique.

IV. An Illustrative Example

Consider the following three cases for the transition probability matrix $\hat{\Psi}$ of the Markov process:

$$\hat{\Psi}_1 = \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix}, \hat{\Psi}_2 = \begin{bmatrix} ? & ? \\ 0.4 & 0.6 \end{bmatrix}, \hat{\Psi}_3 = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}.$$

Apparently, the matrix $\hat{\Psi}_1$ (respectively, $\hat{\Psi}_2, \hat{\Psi}_3$) means that the transition probabilities are completely known (respectively, partially known and completely unknown).

Assume that the system involves two modes and the other system data are given as follows:

$$\begin{aligned} A_1 &= \begin{bmatrix} -0.6 & 0.4 \\ 0.3 & 0.5 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0.3 & 0.5 \\ 0.4 & 0.5 \end{bmatrix}, \\ D_{11} &= \begin{bmatrix} -0.1 \\ 0.7 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0.1 \\ -0.5 \end{bmatrix}, \\ G_1 &= G_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ C_1 = \begin{bmatrix} 0 & 0.5 \end{bmatrix}, \\ C_2 &= \begin{bmatrix} 0.2 & 0.2 \end{bmatrix}, \ D_{21} = D_{22} = 0.4, \\ E_1 &= 1, \quad E_2 = 2.2. \end{aligned}$$

Furthermore, let $\bar{\alpha} = \mathbb{E}\{\alpha(k)\} = 0.9$ and suppose that the randomly varying nonlinearities are given by

$$\begin{aligned} & g_1(x(k)) = g_2(x(k)) \\ = & \begin{bmatrix} 0.05x_1(k) - \tanh(0.05x_1(k)) & 0.2x_2(k) \end{bmatrix}^T \\ & h_1(x(k)) = h_2(x(k)) \\ = & \begin{bmatrix} -0.1x_1(k) & \tanh(0.1x_1(k)) \end{bmatrix}^T \end{aligned}$$

It can be readily seen that (3) is satisfied with $B_{11} = B_{12} = \text{diag}\{0.1, 0.2\}$ and $B_{21} = B_{22} = \text{diag}\{0.1, 0.1\}$.

The saturation functions $\sigma(C_i x(k))$ $(i = 1, 2)$ are described as follows:

$$\begin{aligned} & \sigma(C_i x(k)) \\ = & \begin{cases} C_i x(k), & \text{if } -v_{C_i x(k),\max} \le C_i x(k) \le v_{C_i x(k),\max}; \\ v_{C_i x(k),\max}, & \text{if } C_i x(k) > v_{C_i x(k),\max}; \\ -v_{C_i x(k),\max}, & \text{if } C_i x(k)(k) < -v_{C_i x(k),\max} \end{cases} \end{aligned}$$

where the saturation values are taken as $v_{C_1 x(k),\max} = v_{C_2 x(k),\max} = 0.5$ and $L = 0.3$, $L_1 = 0.7$.

With the above parameters, the fault detection filter design problem can be solved by using Algorithm *OFDFD*. For the three different cases of transition probability matrices, the locally optimized index $J_{\min}$ and the corresponding filter gains are summarized in Table I. It can be observed from Table I that, the more known knowledge in the transition probability matrix we have, the better fault detection performance the filter can achieve.

For the simulation purpose, we consider the initial value $x(0) = \begin{bmatrix} 0.2 & -0.5 \end{bmatrix}^T$ and $\hat{x}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ with $k = 0, 1, \ldots, 300$. The exogenous disturbance input is $w(k) = 10^{-4} \sin(5k) v(k)$ where $v(k)$ is a uniformly distributed noise over $[-0.5, 0.5]$. The fault signal $f(k)$ is given as follows:

$$f(k) = \begin{cases} 1, & 100 \le k \le 200 \\ 0, & \text{else}. \end{cases}$$

To demonstrate the mode switches, we take the transition probability matrix $\hat{\Psi}_1$ as an example and let $\theta(0) = 2$. The corresponding evolution functions $\bar{J}(\tilde{r}) = \left\{ \sum_{l=0}^{k} \tilde{r}^T(l) \tilde{r}(l) \right\}^{\frac{1}{2}}$ for both the faulty case and fault free case are shown in Figs. 1–3, respectively. The selected thresholds $\bar{J}_{th} = \sup_{f=0} \mathbb{E}\{\sum_{k=0}^{300} \tilde{r}^T(k) \tilde{r}(k)\}^{\frac{1}{2}}$ are obtained in all cases which are listed in Table II. Also, the time steps required for successfully detecting the faults are calculated and outlined in Table II. Obviously, the more knowledge about the transition probabilities we have, the faster the fault detection process would be.

V. Conclusion

In this paper, the fault detection problem has been investigated for discrete-time Markovian jump systems with randomly varying nonlinearities and sensor saturation. The transition probability matrix is allowed to have partially unknown entries, while the cases with completely known or completely unknown transition probabilities have also been investigated as two special cases. Two energy norm indices have been used for the fault detection problem in order to account for, respectively, the restraint of disturbance and the sensitivity of faults. A locally optimized fault detection filter has been designed such that 1) the fault detection dynamics is stochastically stable; 2) the effect from the exogenous disturbance on the residual is attenuated with respect to a minimized $\mathcal{H}_\infty$-norm; and 3) the sensitivity of the residual to the fault is enhanced in terms of a maximized $\mathcal{H}_\infty$-norm. A simulation example has been exploited to demonstrate the effectiveness of the theoretical results presented in this paper. It should be noted that one of the future research topics would be to investigate the globally optimal tradeoff between the restraint on disturbances and the sensitivity to faults in the filter design for the fault detection problems.

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Hongli Dong (M'11) received the B.E. degree in computer science and technology from Heilongjiang Institute of Science and Technology, Harbin, China, in 2000, and the M.E. degree in control theory and engineering from Northeast Petroleum University, Daqing, China, in 2003. She is currently working toward the Ph.D. degree in control science and engineering from the Harbin Institute of Technology, Harbin. From July 2009 to January 2010, she was a Research Assistant in the Department of Applied Mathematics, the City University of Hong Kong. From October 2010 to January 2011, she was a Research Assistant in the Department of Mechanical Engineering, the University of Hong Kong. She is now a Visiting Scholar in the Department of Information Systems and Computing, Brunel University, UK.

She is currently a Lecturer at Northeast Petroleum University. Her current research interests include robust control and networked control systems. She is an active reviewer for many international journals.

Zidong Wang (SM'03) was born in Jiangsu, China, in 1966. He received his B.Sc. degree in Mathematics in 1986 from Suzhou University, Suzhou, China, the M.Sc. degree in Applied Mathematics in 1990 and the Ph.D. degree in Electrical and Computer Engineering in 1994, both from Nanjing University of Science and Technology, Nanjing, China.

Dr. Wang is now a Professor of Dynamical Systems and Computing at Brunel University in the UK. His research interests include dynamical systems, signal processing, bioinformatics, control theory and applications. He has published more than 120 papers in refereed international journals.

Dr. Wang is currently serving as an Associate Editor for 12 international journals including *IEEE Transactions on Automatic Control*, *IEEE Transactions on Neural Networks*, *IEEE Transactions on Signal Processing*, *IEEE Transactions on Systems, Man, and Cybernetics - Part C* and *IEEE Transactions on Control Systems Technology.*

Huijun Gao (SM'06) was born in Heilongjiang Province, China, in 1976. He received the M.S. degree in electrical engineering from Shenyang University of Technology, Shengyang, China, in 2001, and the Ph.D. degree in control science and engineering from Harbin Institute of Technology, Harbin, China, in 2005.

Dr. Gao is now a Professor at Harbin Institute of Technology, China. His research interests include network-based control, robust control/filter theory, model reduction, time-delay systems, and multidimensional systems, and their applications. He has published more than 80 papers in refereed international journals.

Dr. Gao is an Associate Editor or member of editorial board for several journals, such as *Automatica*, *IEEE Transactions on Systems, Man, and Cybernetics - Part B*, *IEEE Transactions on Industrial Electronics*, *IEEE Transactions on Fuzzy Systems*, *IEEE Transactions on Circuits and Systems*.