Spectral Density of Complex Networks with a Finite Mean Degree

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Abstract

In order to clarify the statistical features of complex networks, the spectral density of adjacency matrices has often been investigated. Adopting a static model introduced by Goh, Kahng and Kim, we analyse the spectral density of complex scale free networks. For that purpose, we utilize the replica method and effective medium approximation (EMA) in statistical mechanics. As a result, we identify a new integral equation which determines the asymptotic spectral density of scale free networks with a finite mean degree p. In the limit $p \to \infty$, known asymptotic formulae are rederived. Moreover, the $1/p$ corrections to known results are analytically calculated by a perturbative method.

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1 Introduction

The theory of complex networks, which originates in Erdös and Rényi's study on random graphs[1], has dramatically developed in the last decade. Complex networks in biological and social systems were extensively investigated and universal features were elucidated. Moreover, as Erdös and Rényi's random graphs could not reproduce some of those features, new theoretical models of complex networks were introduced[2, 3].

One of the newly discovered features of complex networks is the scale free property, which means that the degree (the number of the nodes directly connected to each node) distribution obeys a power law. The scale free property was originally explained by Barabási and Albert in terms of a growing network model[3]. Later Goh, Kahng and Kim showed that a static network model with the scale free property could also be constructed[4].

The connection patterns of the nodes in a complex network are represented by the adjacency matrix. The spectral (eigenvalue) density of adjacency matrices is used to measure the statistical features of large complex networks. For example, as explained in the following, a power law behaviour can be observed. Let us denote by N the size of the adjacency matrix. In the limit $N \to \infty$, we obtain an asymptotic form of the spectral density. When the mean value p of the degree becomes large, Erdös and Rényi's random graphs tend to have the asymptotic spectral density with the semicircle shape, as expected from a classical result of random matrix theory[5]. On the other hand, the tail of the asymptotic spectral density for scale free networks has a power law behaviour. This power law behaviour was analytically confirmed by Dorogovtsev et al. for tree-like scale free graphs[6, 7]. Rodgers et al. also found a similar power law for Goh, Kahng and Kim's static model[8].

Rodgers et al. analysed the spectral density of the static model in the limit $p \to \infty$[8, 9]. In their analysis, the replica method in statistical physics was employed. A similar usage of the replica method can be traced back to Rodgers and Bray's study of sparse random matrices[10]. Recently Semerjian and Cugliandolo[11] proposed a scheme called the effective medium approximation (EMA) and elegantly rederived the results in [10]. This scheme was also applied to sparse sample covariance matrices and analogous results were obtained[12].

In this paper, we extend the works in [8] and [11] by treating complex networks with a finite mean degree p. For that purpose, we make use of the replica method and EMA for the analysis of Goh, Kahng and Kim's static

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model. Consequently, in the asymptotic limit $N \to \infty$, we obtain a new integral equation which determines the asymptotic spectral density with a finite p. We show that known results originally found in [8] are rederived in the limit $p \to \infty$. Moreover, using the new integral equation, we evaluate the $1/p$ corrections to the known results.

This paper is organised as follows. In §2, the replica method is introduced for Goh, Kahng and Kim's static model of scale free networks. In §3, using EMA, we derive a nonlinear integral equation which determines the asymptotic spectral density with a mean degree p. In the limit $p \to \infty$, the results in [8] are rederived. In §4, a perturbative method is developed to calculate the $1/p$ corrections. In §5 and §6, perturbation terms are analytically evaluated around the band center and in the tail region.

2 Replica Method

Goh, Kahng and Kim's static model of complex networks is constructed by the following procedure[4]. Let us suppose that there are N nodes and consider the asymptotic limit $N \to \infty$. Each node j is assigned a probability P_j, which is normalised as $\sum_{j=1}^{N} P_j = 1$. In order to realize a power-law degree distribution with an exponent $\lambda = (1 + \alpha)/\alpha$, we assume that

$$P_j = \frac{j^{-\alpha}}{\sum_{j=1}^{N} j^{-\alpha}} \sim (1 - \alpha)N^{\alpha-1}j^{-\alpha}, \quad 0 < \alpha < 1. \tag{2.1}$$

In one step of the procedure, nodes j and l are chosen with probabilities P_j and P_l, respectively. Then the nodes j and l are connected by an edge, if j does not coincide with l and the nodes are not already connected. Repeating such a step $pN/2$ times, one obtains a network with a mean degree p. The probability that a pair of nodes j and l are connected by an edge is

$$f_{jl} = 1 - (1 - 2P_j P_l)^{pN/2} \sim 1 - \mathrm{e}^{-pNP_j P_l}. \tag{2.2}$$

In this paper, we are interested in the spectral density of the adjacency matrix of the network. The adjacency matrix J is symmetric ($J_{jl} = J_{lj}$) and the elements are independently distributed. The p.d.f.(probability distribution function) of each element is given by

$$\mathcal{P}_{jl}(J_{jl}) = (1 - f_{jl})\delta(J_{jl}) + f_{jl}\delta(J_{jl} - 1), \quad j < l. \tag{2.3}$$

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Denoting the average over the above probability distribution by brackets $\langle \cdot \rangle$, we define the spectral density of J as

$$\rho(\mu) = \left\langle \frac{1}{N} \sum_{j=1}^{N} \delta(\mu - \mu_j) \right\rangle, \tag{2.4}$$

where $\mu_1, \mu_2, \cdots, \mu_N$ are the eigenvalues of J and $\delta(x)$ is Dirac's delta function.

In order to analyse the asymptotic behaviour of the spectral density in the limit $N \to \infty$, the partition function

$$Z = \int \prod_{j=1}^{N} \mathrm{d}\phi_j \, \exp\left(\frac{i}{2}\mu \sum_{j=1}^{N} \phi_j^2 - \frac{i}{2} \sum_{jl} J_{jl}\phi_j\phi_l \right) \tag{2.5}$$

is useful. In terms of Z, the spectral density can be rewritten as

$$
\begin{aligned}
\rho(\mu) &= \frac{1}{N\pi}\mathrm{Im}\mathrm{Tr}\left\langle \{J - (\mu + i\epsilon)I\}^{-1} \right\rangle \\
&= \frac{2}{N\pi}\mathrm{Im}\frac{\partial}{\partial\mu}\langle \ln Z(\mu + i\epsilon) \rangle.
\end{aligned} \tag{2.6}
$$

Here I is an identity matrix and ϵ is a positive infinitesimal number. Moreover, the relation

$$\lim_{n \to 0} \frac{\ln\langle Z^n \rangle}{n} = \langle \ln Z \rangle \tag{2.7}$$

can be employed to find

$$\rho(\mu) = \lim_{n \to 0} \frac{2}{Nn\pi}\mathrm{Im}\frac{\partial}{\partial\mu}\ln\langle \{Z(\mu)\}^n \rangle. \tag{2.8}$$

Hence we need to evaluate the average of the n-th power of Z. In terms of the replica variables

$$\vec{\phi}_j = (\phi_j^{(1)}, \phi_j^{(2)}, \cdots, \phi_j^{(n)}) \tag{2.9}$$

and the corresponding measures

$$\mathrm{d}\vec{\phi}_j = \mathrm{d}\phi_j^{(1)}\mathrm{d}\phi_j^{(2)} \cdots \mathrm{d}\phi_j^{(n)}, \tag{2.10}$$

it can be written as

$$\langle Z^n \rangle = \int \prod_{j=1}^{N} \mathrm{d}\vec{\phi}_j \, \exp\left(\frac{i}{2}\mu \sum_{j=1}^{N} \vec{\phi}_j^2 \right) \left\langle \exp\left(-\frac{i}{2} \sum_{jl} J_{jl}\vec{\phi}_j \cdot \vec{\phi}_l \right) \right\rangle, \tag{2.11}$$

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where the average is evaluated as

$$\left\langle \exp\left(-\frac{i}{2}\sum_{jl}^{N}J_{jl}\vec{\phi}_{j}\cdot\vec{\phi}_{l}\right)\right\rangle = \prod_{j<l}^{N}\left(\int \mathrm{d}J_{jl}\ \mathcal{P}_{jl}(J_{jl})\mathrm{e}^{-iJ_{jl}\vec{\phi}_{j}\cdot\vec{\phi}_{l}}\right)$$

$$= \prod_{j<l}^{N}\left\{1+f_{jl}\left(\mathrm{e}^{-i\vec{\phi}_{j}\cdot\vec{\phi}_{l}}-1\right)\right\}. \qquad (2.12)$$

Then one obtains an asymptotic estimate in the limit $N\to\infty$ as

$$\ln\left\langle \exp\left(-\frac{i}{2}\sum_{jl}^{N}J_{jl}\vec{\phi}_{j}\cdot\vec{\phi}_{l}\right)\right\rangle \sim pN\sum_{j<l}^{N}P_{j}P_{l}\left(\mathrm{e}^{-i\vec{\phi}_{j}\cdot\vec{\phi}_{l}}-1\right). \qquad (2.13)$$

It was shown in [13] that the remainder term of this asymptotic estimate was $O(1)$ for $0<\alpha<1/2$, $O((\ln N)^2)$ for $\alpha=1/2$ and $O(N^{2-(1/\alpha)}\ln N)$ for $1/2<\alpha<1$. Moreover we define

$$\tilde{c}_{j}(\vec{\phi}) = \delta(\vec{\phi}-\vec{\phi}_{j}), \qquad (2.14)$$

so that

$$\left\langle \exp\left(-\frac{i}{2}\sum_{jl}^{N}J_{jl}\vec{\phi}_{j}\cdot\vec{\phi}_{l}\right)\right\rangle$$

$$\sim \exp\left\{\frac{pN}{2}\sum_{jl}^{N}P_{j}P_{l}\int \mathrm{d}\vec{\phi}\int \mathrm{d}\vec{\psi}\ \tilde{c}_{j}(\vec{\phi})\tilde{c}_{l}(\vec{\psi})\left(\mathrm{e}^{-i\vec{\phi}\cdot\vec{\psi}}-1\right)\right\}. \qquad (2.15)$$

Then, introducing auxiliary functions $c_{j}(\vec{\phi})$, we find

$$\langle Z^{n}\rangle \sim \int\prod_{j=1}^{N}\mathrm{d}\vec{\phi}_{j}\ \exp\left\{\frac{i}{2}\mu\sum_{j=1}^{N}\int \mathrm{d}\vec{\phi}\ \tilde{c}_{j}(\vec{\phi})\vec{\phi}^{2}\right\}$$

$$\times \ \exp\left\{\frac{pN}{2}\sum_{jl}^{N}P_{j}P_{l}\int \mathrm{d}\vec{\phi}\int \mathrm{d}\vec{\psi}\ \tilde{c}_{j}(\vec{\phi})\tilde{c}_{l}(\vec{\psi})\left(\mathrm{e}^{-i\vec{\phi}\cdot\vec{\psi}}-1\right)\right\}$$

$$= \ \int\prod_{j=1}^{N}\mathrm{d}\vec{\phi}_{j}\int\prod_{j=1}^{N}\mathcal{D}c_{j}(\vec{\phi})\prod_{j=1}^{N}\prod_{\vec{\phi}}\delta(c_{j}(\vec{\phi})-\tilde{c}_{j}(\vec{\phi}))\mathrm{e}^{S_{1}+S_{2}}, \qquad (2.16)$$

where

$$S_{1} = \frac{i}{2}\mu\sum_{j=1}^{N}\int \mathrm{d}\vec{\phi}\ c_{j}(\vec{\phi})\vec{\phi}^{2} \qquad (2.17)$$

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and

$$S_2 = \frac{pN}{2} \sum_{jl}^{N} P_j P_l \int \mathrm{d}\vec{\phi} \int \mathrm{d}\vec{\psi} \; c_j(\vec{\phi}) \, c_l(\vec{\psi}) \left(\mathrm{e}^{-i\vec{\phi}\cdot\vec{\psi}} - 1 \right). \tag{2.18}$$

Here the functional integration is taken over $c_j(\vec{\phi})$ satisfying

$$\int \mathrm{d}\vec{\phi} \; c_j(\vec{\phi}) = 1. \tag{2.19}$$

Let us note

$$\begin{aligned}
& \int \prod_{j=1}^{N} \mathrm{d}\vec{\phi}_j \prod_{j=1}^{N} \prod_{\vec{\phi}} \delta(c_j(\vec{\phi}) - \tilde{c}_j(\vec{\phi})) \\
= \; & \int \prod_{j=1}^{N} \mathrm{d}\vec{\phi}_j \int \prod_{j=1}^{N} \mathcal{D}a_j(\vec{\phi}) \, \exp\left[2\pi i \sum_{j=1}^{N} \int \mathrm{d}\vec{\phi} \; a_j(\vec{\phi}) \left\{ c_j(\vec{\phi}) - \tilde{c}_j(\vec{\phi}) \right\} \right] \\
= \; & \int \prod_{j=1}^{N} \mathcal{D}a_j(\vec{\phi}) \, \exp\left[\sum_{j=1}^{N} \left\{ 2\pi i \int \mathrm{d}\vec{\phi} \; a_j(\vec{\phi}) c_j(\vec{\phi}) - F_j \right\} \right], \tag{2.20}
\end{aligned}$$

where

$$\begin{aligned}
F_j &= -\ln \int \mathrm{d}\vec{\phi}_j \, \exp\left\{ -2\pi i \int \mathrm{d}\vec{\phi} \; a_j(\vec{\phi}) \tilde{c}_j(\vec{\phi}) \right\} \\
&= -\ln \int \mathrm{d}\vec{\phi}_j \, \exp\left\{ -2\pi i a_j(\vec{\phi}_j) \right\}. \tag{2.21}
\end{aligned}$$

In the limit $N \to \infty$, the functional integral (2.20) is dominated by the contribution from the neighbourhood of the extremum satisfying

$$\frac{\delta}{\delta a_j(\vec{\phi})} \left\{ 2\pi i \int \mathrm{d}\vec{\phi} \; a_j(\vec{\phi}) c_j(\vec{\phi}) - F_j \right\} = 2\pi i c_j(\vec{\phi}) - 2\pi i \mathrm{e}^{-2\pi i a_j(\vec{\phi}) + F_j} = 0, \tag{2.22}$$

so that

$$-\int \mathrm{d}\vec{\phi} \; c_j(\vec{\phi}) \ln c_j(\vec{\phi}) = 2\pi i \int \mathrm{d}\vec{\phi} \; a_j(\vec{\phi}) c_j(\vec{\phi}) - F_j. \tag{2.23}$$

We thus obtain

$$\int \prod_{j=1}^{N} \mathrm{d}\vec{\phi}_j \prod_{j=1}^{N} \prod_{\vec{\phi}} \delta(c_j(\vec{\phi}) - \tilde{c}_j(\vec{\phi})) \sim \exp\left\{ -\sum_{j=1}^{N} \int \mathrm{d}\vec{\phi} \; c_j(\vec{\phi}) \ln c_j(\vec{\phi}) \right\}, \tag{2.24}$$

from which it follows that

$$\langle Z^n \rangle \sim \int \prod_{j=1}^{N} \mathcal{D}c_j(\vec{\phi}) \; e^{S_0+S_1+S_2} \tag{2.25}$$

with

$$S_0 = -\sum_{j=1}^{N} \int d\vec{\phi} \; c_j(\vec{\phi}) \ln c_j(\vec{\phi}). \tag{2.26}$$

3   Integral Equation

In the limit $N \to \infty$, the dominant contribution to the functional integral (2.25) again comes from the neighbourhood of the extremum. In order to work out the extremum, we replace $c_j(\vec{\phi})$ with a Gaussian ansatz

$$c_j(\vec{\phi}) = \frac{1}{(2\pi i \sigma_j)^{n/2}} \exp\left(-\frac{\vec{\phi}^2}{2i\sigma_j}\right). \tag{3.1}$$

Then the extremum condition can be written as

$$\frac{\partial}{\partial \sigma_k}(S_0 + S_1 + S_2) = 0. \tag{3.2}$$

This variational scheme based on the Gaussian ansatz is called the effective medium approximation (EMA)[11].

Using the identity

$$\int d\vec{\phi} \; \vec{\phi}^2 c_j(\vec{\phi}) = ni\sigma_j, \tag{3.3}$$

we can readily find

$$S_0 = \frac{n}{2} \sum_{j=1}^{N} \ln(2\pi i \sigma_j) + \frac{Nn}{2}, \quad S_1 = -\frac{\mu n}{2} \sum_{j=1}^{N} \sigma_j, \tag{3.4}$$

so that

$$\frac{\partial S_0}{\partial \sigma_k} = \frac{n}{2\sigma_k}, \quad \frac{\partial S_1}{\partial \sigma_k} = -\frac{\mu n}{2}. \tag{3.5}$$

It is straightforward to obtain

$$S_2 = -\frac{Np}{2}\left\{1 - \sum_{jl}^{N} P_j P_l \int d\vec{\phi} d\vec{\psi} \frac{\exp\left(-\frac{\vec{\phi}^2}{2i\sigma_j} - \frac{\vec{\psi}^2}{2i\sigma_l} - i\vec{\phi}\cdot\vec{\psi}\right)}{(2\pi i \sigma_j)^{n/2}(2\pi i \sigma_l)^{n/2}}\right\}$$

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$$
= -\frac{Np}{2}\left[1 - \sum_{jl}^{N} P_j P_l \int d\vec{\psi} \frac{1}{(2\pi i \sigma_l)^{n/2}} \exp\left\{-\frac{1 - \sigma_j \sigma_l}{2i\sigma_l}\vec{\psi}^2\right\}\right]
$$

$$
= -\frac{Np}{2}\left\{1 - \sum_{jl}^{N} \frac{P_j P_l}{(1 - \sigma_j \sigma_l)^{n/2}}\right\}. \tag{3.6}
$$

Therefore

$$
\frac{\partial S_2}{\partial \sigma_k} = \frac{Npn}{2} P_k \sum_{j=1}^{N} \frac{P_j \sigma_j}{(1 - \sigma_j \sigma_k)^{(n/2)+1}}. \tag{3.7}
$$

Then the extremum condition (3.2) yields

$$
\mu - \frac{1}{\sigma_k} - NpP_k \sum_{j=1}^{N} \frac{P_j \sigma_j}{1 - \sigma_j \sigma_k} = 0 \tag{3.8}
$$

in the limit $n \to 0$. Putting (2.1) into (3.8), we find

$$
0 = \mu - \frac{1}{\sigma_k} - p(1 - \alpha)^2 \left(\frac{k}{N}\right)^{-\alpha} \frac{1}{N} \sum_{j=1}^{N} \frac{(j/N)^{-\alpha}\sigma_j}{1 - \sigma_j \sigma_k}. \tag{3.9}
$$

In the limit $N \to \infty$, it follows that

$$
\sigma(x)x^{-\alpha} = \frac{1}{\mu x^{\alpha} - p(1 - \alpha)^2 \int_0^1 \frac{y^{-\alpha}\sigma(y)}{1 - \sigma(x)\sigma(y)}dy} \tag{3.10}
$$

with $x = k/N$. The function $\sigma(x)$ is determined by this integral equation.

The spectral density can be evaluated as

$$
\begin{aligned}
\rho(\mu) &= \lim_{n \to 0} \frac{2}{Nn\pi} \mathrm{Im} \frac{\partial}{\partial \mu} \ln\langle\{Z(\mu)\}^n\rangle \\
&\sim \lim_{n \to 0} \frac{2}{Nn\pi} \mathrm{Im} \frac{\partial}{\partial \mu}(S_0 + S_1 + S_2) \\
&= -\frac{1}{N\pi} \mathrm{Im} \sum_{j=1}^{N} \sigma_j, \tag{3.11}
\end{aligned}
$$

so that

$$
\rho(\mu) = -\frac{1}{\pi} \mathrm{Im} \int_0^1 \sigma(x)dx \tag{3.12}
$$

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in the limit $N \to \infty$. The asymptotic spectral density is thus written in terms of the solution of the integral equation (3.10).

Using (2.1) and (2.2), we generated adjacency matrices J with $N = 1000$ and evaluated the spectral density (NUMERICAL) by numerical diagonalization. On the other hand, the EMA spectral density (EMA) was calculated by solving (3.9) with $N = 1000$ by numerical iteration and by putting the solution into (3.11). We show the both results for $p = 10$ and $\alpha = 0.5$ in Figure 1. The agreement seems good enough to demonstrate the validity of EMA.
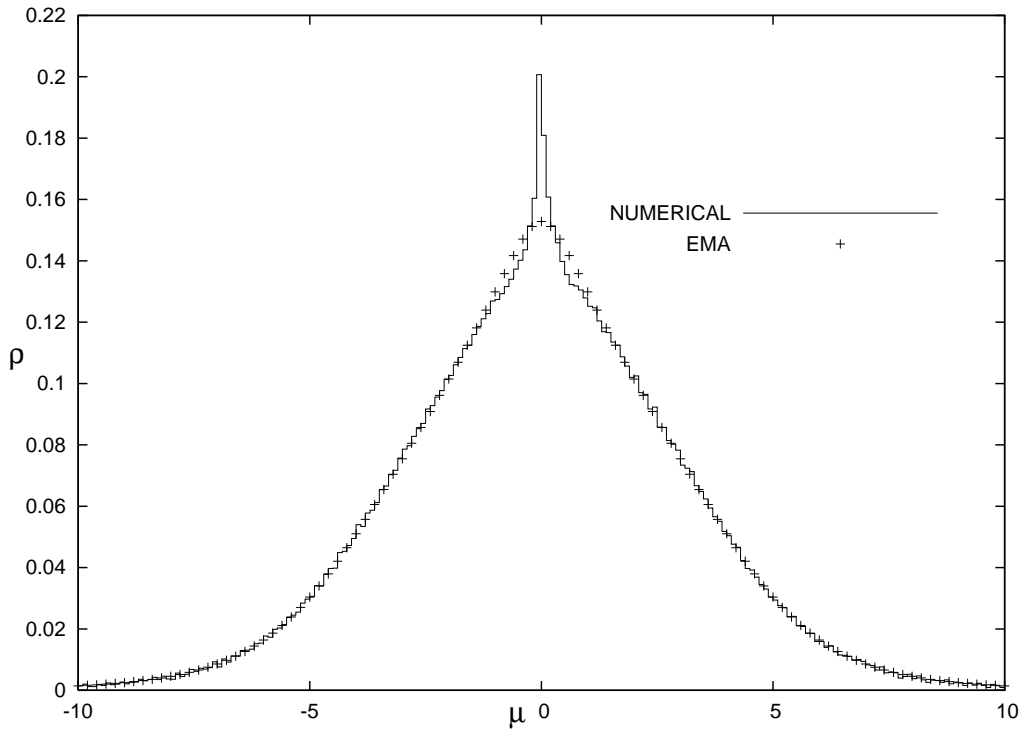

Figure 1: Comparison of a numerically generated spectral density (average over 100 samples, $N = 1000$) and an EMA solution ($p = 10, \alpha = 0.5$).

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4 Perturbation Theory

In order to analytically evaluate the EMA spectral density, we start from the integral equation (3.10) and develop a perturbation theory for large p. Introducing scalings

$$\sigma(x) \rightarrow \frac{1}{\sqrt{p}}s(x), \quad \mu \rightarrow \sqrt{p}m, \tag{4.1}$$

we obtain

$$s(x)x^{-\alpha} = \frac{1}{mx^\alpha - (1-\alpha)^2 \int_0^1 \dfrac{y^{-\alpha}s(y)}{1 - \dfrac{1}{p}s(x)s(y)}\mathrm{d}y}. \tag{4.2}$$

Let us expand $s(x)$ in terms of $1/p$ as

$$s(x) = s_0(x) + \frac{1}{p}s_1(x) + O\left(\frac{1}{p^2}\right), \tag{4.3}$$

so that the unperturbed term $s_0(x)$ satisfies

$$s_0(x)x^{-\alpha} = \frac{1}{mx^\alpha - (1-\alpha)^2 \int_0^1 y^{-\alpha}s_0(y)\mathrm{d}y}. \tag{4.4}$$

This integral equation for the unperturbed term was originally found in [8].

Then, expanding the RHS of (4.2) in terms of $1/p$, we find the relation

$$\left\{s_0(x) + \frac{1}{p}s_1(x)\right\} x^{-\alpha}$$

$$\sim \quad s_0(x)x^{-\alpha} + \frac{(1-\alpha)^2}{p}\{s_0(x)\}^2 x^{-2\alpha} \int_0^1 \left[s_1(y) + s_0(x)\{s_0(y)\}^2\right] y^{-\alpha}\mathrm{d}y \tag{4.5}$$

in the limit $p \rightarrow \infty$. Therefore, the perturbation term $s_1(x)$ can be evaluated as

$$\begin{aligned}
s_1(x)x^{-\alpha} &= (1-\alpha)^2\{s_0(x)\}^2 x^{-2\alpha} \int_0^1 s_1(y)y^{-\alpha}\mathrm{d}y \\
&+ (1-\alpha)^2\{s_0(x)\}^3 x^{-2\alpha} \int_0^1 \{s_0(y)\}^2 y^{-\alpha}\mathrm{d}y, \tag{4.6}
\end{aligned}$$

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so that

$$\int_0^1 s_1(x)\mathrm{d}x = (1-\alpha)^2 I_1(D+J_1). \tag{4.7}$$

Here

$$I_1 = \int_0^1 \{s_0(x)\}^2 x^{-\alpha}\mathrm{d}x, \quad J_1 = \int_0^1 \{s_0(x)\}^3 x^{-\alpha}\mathrm{d}x \tag{4.8}$$

and

$$D = \int_0^1 s_1(x)x^{-\alpha}\mathrm{d}x. \tag{4.9}$$

Integrating the both sides of (4.6) over x from 0 to 1, we find

$$\begin{aligned} D &= (1-\alpha)^2 D \int_0^1 \{s_0(x)\}^2 x^{-2\alpha}\mathrm{d}x + (1-\alpha)^2 I_1 \int_0^1 \{s_0(x)\}^3 x^{-2\alpha}\mathrm{d}x \\ &= (1-\alpha)^2 D I_2 + (1-\alpha)^2 I_1 J_2, \end{aligned} \tag{4.10}$$

where

$$I_2 = \int_0^1 \{s_0(x)\}^2 x^{-2\alpha}\mathrm{d}x, \quad J_2 = \int_0^1 \{s_0(x)\}^3 x^{-2\alpha}\mathrm{d}x. \tag{4.11}$$

Therefore D can be evaluated as

$$D = \frac{(1-\alpha)^2 I_1 J_2}{1 - (1-\alpha)^2 I_2}, \tag{4.12}$$

so that

$$\int_0^1 s_1(x)\mathrm{d}x = (1-\alpha)^2 I_1 \left\{ \frac{(1-\alpha)^2 I_1 J_2}{1 - (1-\alpha)^2 I_2} + J_1 \right\}. \tag{4.13}$$

Let us summarise the scheme of the perturbation. First we evaluate $s_0(x)$ by solving the integral equation (4.4). Next we calculate

$$\mathrm{Im} \int_0^1 s_0(x)\mathrm{d}x \tag{4.14}$$

and the other integrals I_1, J_1, I_2 and J_2. Then, using the relation (4.13), we are able to compute

$$\mathrm{Im} \int_0^1 s_1(x)\mathrm{d}x. \tag{4.15}$$

In terms of the integrals (4.14) and (4.15), the asymptotic spectral density is expressed as

$$\rho(\mu) \sim -\frac{1}{\pi\sqrt{p}} \left\{ \mathrm{Im} \int_0^1 s_0(x)\mathrm{d}x + \frac{1}{p}\mathrm{Im} \int_0^1 s_1(x)\mathrm{d}x + O\left(\frac{1}{p^2}\right) \right\} \tag{4.16}$$

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in the limit $N \to \infty$.

In the following sections, we evaluate the small and large m expansions of the integrals (4.14) and (4.15), so that (4.16) gives the asymptotic behaviour of the EMA spectral density around the band center and in the tail region.

5   Band Center

Let us evaluate the EMA spectral density around the band center, by calculating the small m expansions of the integrals (4.14) and (4.15). Noting the dependence of $s_0(x)$ on m, we expand it as

$$s_0(x) = g_0(x) + mg_1(x) + m^2 g_2(x) + O\left(m^3\right). \tag{5.1}$$

In order to compute the central value $g_0(x)$, we put $m = 0$ in (4.4) and find

$$g_0(x)x^{-\alpha} = -\frac{1}{(1-\alpha)^2 \int_0^1 y^{-\alpha}g_0(y)\mathrm{d}y}. \tag{5.2}$$

Integrating the both sides in x from 0 to 1, we obtain

$$\int_0^1 y^{-\alpha}g_0(y)\mathrm{d}y = -i\frac{1}{1-\alpha}, \tag{5.3}$$

so that

$$g_0(x) = -i\frac{1}{1-\alpha}x^{\alpha}. \tag{5.4}$$

Let us next evaluate $g_1(x)$ and $g_2(x)$. We expand (4.4) in terms of m and find

$$
\begin{aligned}
&\left\{g_0(x) + mg_1(x) + m^2 g_2(x)\right\} x^{-\alpha} \\
\sim\quad & g_0(x)x^{-\alpha}\left[1 - mg_0(x)x^{-\alpha}\left\{x^{\alpha} - (1-\alpha)^2 \int_0^1 g_1(y)y^{-\alpha}\mathrm{d}y\right\}\right. \\
+\quad & m^2 g_0(x)x^{-\alpha}(1-\alpha)^2 \int_0^1 g_2(y)y^{-\alpha}\mathrm{d}y \\
+\quad & \left. m^2\{g_0(x)\}^2 x^{-2\alpha}\left\{x^{\alpha} - (1-\alpha)^2 \int_0^1 g_1(y)y^{-\alpha}\mathrm{d}y\right\}^2\right]
\end{aligned}
\tag{5.5}
$$

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in the limit $m \to 0$. The linear term in m reads

$$
\begin{aligned}
g_1(x)x^{-\alpha} &= -\{g_0(x)\}^2 x^{-2\alpha}\left\{x^\alpha - (1-\alpha)^2 \int_0^1 g_1(y)y^{-\alpha}dy\right\} \\
&= \frac{x^\alpha}{(1-\alpha)^2} - \int_0^1 g_1(y)y^{-\alpha}dy.
\end{aligned} \tag{5.6}
$$

We again integrate the both sides in x from 0 to 1 and find

$$
\int_0^1 g_1(y)y^{-\alpha}dy = \frac{1}{2(1-\alpha)^2(1+\alpha)}, \tag{5.7}
$$

so that

$$
g_1(x)x^{-\alpha} = \frac{x^\alpha}{(1-\alpha)^2} - \frac{1}{2(1-\alpha)^2(1+\alpha)}. \tag{5.8}
$$

The quadratic term in m is similarly extracted from (5.5) as

$$
\begin{aligned}
g_2(x)x^{-\alpha} &= \{g_0(x)\}^2 x^{-2\alpha}(1-\alpha)^2 \int_0^1 g_2(y)y^{-\alpha}dy \\
&+ \{g_0(x)\}^3 x^{-3\alpha}\left\{x^\alpha - (1-\alpha)^2 \int_0^1 g_1(y)y^{-\alpha}dy\right\}^2 \\
&= -\int_0^1 g_2(y)y^{-\alpha}dy + i\frac{1}{(1-\alpha)^3}\left\{x^\alpha - \frac{1}{2(1+\alpha)}\right\}^2. \tag{5.9}
\end{aligned}
$$

Integrating the both sides in x from 0 to 1 yields

$$
\int_0^1 g_2(y)y^{-\alpha}dy = i\frac{1}{2(1-\alpha)^3}\left\{\frac{1}{1+2\alpha} - \frac{3}{4}\frac{1}{(1+\alpha)^2}\right\}, \tag{5.10}
$$

from which it follows that

$$
g_2(x)x^{-\alpha} = i\frac{1}{2(1-\alpha)^3}\left\{2x^{2\alpha} - \frac{2x^\alpha}{1+\alpha} + \frac{5}{4}\frac{1}{(1+\alpha)^2} - \frac{1}{1+2\alpha}\right\}. \tag{5.11}
$$

Thus we obtain the small m expansion of $s_0(x)$:

$$
\begin{aligned}
s_0(x)x^{-\alpha} &= -i\frac{1}{1-\alpha} + \frac{m}{(1-\alpha)^2}\left\{x^\alpha - \frac{1}{2(1+\alpha)}\right\} \\
&+ i\frac{m^2}{2(1-\alpha)^3}\left\{2x^{2\alpha} - \frac{2x^\alpha}{1+\alpha} + \frac{5}{4}\frac{1}{(1+\alpha)^2} - \frac{1}{1+2\alpha}\right\} + O\left(m^3\right),
\end{aligned} \tag{5.12}
$$

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so that

$$\mathrm{Im} \int_0^1 s_0(x)\mathrm{d}x = -\frac{1}{1-\alpha^2} + \frac{m^2(1+5\alpha+18\alpha^2+20\alpha^3+16\alpha^4)}{8(1-\alpha^2)^3(1+2\alpha)(1+3\alpha)} + O\left(m^4\right),$$

(5.13)

which yields the spectral density in the limit $p \to \infty$. This result for the unperturbed term was first derived in [8].

We are now in a position to evaluate the perturbation term. The integral D can be computed from (4.12) as

$$
\begin{aligned}
D &= -i\frac{1}{2(1-\alpha)^3(1+\alpha)^2} + \frac{m(2+4\alpha+5\alpha^2)}{2(1-\alpha)^4(1+\alpha)^3(1+2\alpha)}\\
&\quad + 3i\frac{m^2(5+35\alpha+128\alpha^2+252\alpha^3+256\alpha^4+144\alpha^5)}{16(1-\alpha)^5(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)} + O\left(m^3\right).
\end{aligned}
$$

(5.14)

Then it follows from (4.7) that

$$
\begin{aligned}
\mathrm{Im} \int_0^1 s_1(x)\mathrm{d}x &= -\frac{1+2\alpha+2\alpha^2}{2(1+2\alpha)(1-\alpha^2)^3}\\
&\quad + \frac{3m^2(5+55\alpha+314\alpha^2+1034\alpha^3+2068\alpha^4+2648\alpha^5+1920\alpha^6+768\alpha^7)}{16(1+2\alpha)^2(1-\alpha^2)^5(1+7\alpha+12\alpha^2)}\\
&\quad + O\left(m^4\right).
\end{aligned}
$$

(5.15)

As mentioned in the Introduction, Rodgers and Bray[10] analysed sparse random matrices and derived a formula for the asymptotic spectral density

$$\rho(\mu) \sim \frac{2}{\pi(\mu_c)^2}\{(\mu_c)^2 - \mu^2\}^{1/2}\left[1 + \frac{1}{p}\left\{1 - 4\frac{\mu^2}{(\mu_c)^2}\right\} + O\left(\frac{1}{p^2}\right)\right]$$

(5.16)

with $\mu_c^2 = 4\{p + 1 + O(1/p)\}$. Sparse random matrices in [10] give the adjacency matrices of Erdös and Rényi's random graphs, which correspond to the limit $\alpha \to 0$ of Goh, Kahng and Kim's complex networks. From (5.13) and (5.15), we obtain

$$\mathrm{Im} \int_0^1 s_0(x)\mathrm{d}x = -1 + \frac{1}{8}m^2 + O\left(m^4\right)$$

(5.17)

and

$$\mathrm{Im} \int_0^1 s_1(x)\mathrm{d}x = -\frac{1}{2} + \frac{15}{16}m^2 + O\left(m^4\right)$$

(5.18)

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in the limit $\alpha \to 0$. It follows that

$$\rho(\mu) \sim \frac{1}{\pi\sqrt{p}} \left[1 - \frac{m^2}{8} + O\left(m^4\right) + \frac{1}{p}\left\{ \frac{1}{2} - \frac{15}{16}m^2 + O\left(m^4\right) \right\} + O\left(\frac{1}{p^2}\right) \right]$$

(5.19)

in agreement with (5.16), as expected.

6 Tail Region

In order to analyse the EMA spectral density in the tail region, we need to work out the asymptotic formulae in the limit $m \to \infty$ for the integrals (4.14) and (4.15).

To begin with, we define $c = a - ib$ (with real a and b) as

$$c = (1 - \alpha)^2 \int_0^1 s_0(x)x^{-\alpha}\mathrm{d}x$$

(6.1)

and rewrite (4.4) as

$$s_0(x)x^{-\alpha} = \frac{1}{mx^\alpha - c}.$$

(6.2)

Integrating over x from 0 to 1, one obtains

$$\frac{c}{(1 - \alpha)^2} = \int_0^1 \frac{1}{mx^\alpha - c}\mathrm{d}x.$$

(6.3)

This integral equation determines the parameter c. In [8], asymptotic formulae

$$\begin{aligned}
a &\sim \frac{1 - \alpha}{m}, \\
b &\sim \pi\frac{(1 - \alpha)^{(1+\alpha)/\alpha}}{\alpha}\frac{1}{m^{(2-\alpha)/\alpha}}
\end{aligned}$$

(6.4)

were derived in the limit $m \to \infty$. It follows that

$$\begin{aligned}
\mathrm{Im}\int_0^1 s_0(x)\mathrm{d}x &= \mathrm{Im}\int_0^1 \frac{x^\alpha}{mx^\alpha - c}\mathrm{d}x \\
&\sim -2\pi\frac{(1 - \alpha)^{1/\alpha}}{\alpha}\frac{1}{m^{(2+\alpha)/\alpha}},
\end{aligned}$$

(6.5)

which gives the tail behaviour of the spectral density in the limit $p \to \infty$.

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Let us evaluate the $1/p$ correction to this asymptotic formula. Using the expressions

$$I_1 = \int_0^1 \frac{x^\alpha}{(mx^\alpha - c)^2}\mathrm{d}x, \quad I_2 = \int_0^1 \frac{1}{(mx^\alpha - c)^2}\mathrm{d}x, \tag{6.6}$$

$$J_1 = \int_0^1 \frac{x^{2\alpha}}{(mx^\alpha - c)^3}\mathrm{d}x, \quad J_2 = \int_0^1 \frac{x^\alpha}{(mx^\alpha - c)^3}\mathrm{d}x \tag{6.7}$$

and (6.4), we can derive the asymptotic formulae

$$\mathrm{Re}\, I_1 \;\sim\; \frac{1}{1-\alpha}\frac{1}{m^2},$$
$$\mathrm{Im}\, I_1 \;\sim\; -\pi\frac{(1-\alpha)^{(1-\alpha)/\alpha}}{\alpha^2}\frac{1}{m^{2/\alpha}}, \tag{6.8}$$

$$\mathrm{Re}\, J_1 \;\sim\; \frac{1}{1-\alpha}\frac{1}{m^3},$$
$$\mathrm{Im}\, J_1 \;\sim\; -\frac{\pi}{2}\frac{(1+\alpha)\,(1-\alpha)^{(1-\alpha)/\alpha}}{\alpha^3}\frac{1}{m^{(2+\alpha)/\alpha}}, \tag{6.9}$$

$$\mathrm{Im}\, I_2 \sim -\pi\frac{(1-\alpha)^{(1-\alpha)/\alpha}}{\alpha^2}\frac{1}{m^{2(1-\alpha)/\alpha}} \tag{6.10}$$

and

$$\mathrm{Im}\, J_2 \sim -\frac{\pi}{2}\frac{(1-\alpha)^{(1-\alpha)/\alpha}}{\alpha^3}\frac{1}{m^{(2-\alpha)/\alpha}} \tag{6.11}$$

in the limit $m \to \infty$. Moreover, omitting logarithmic factors of the form $\ln m$, we find the asymptotic estimates

$$\mathrm{Re}\, I_2 \sim \begin{cases} O(m^{-2}), & \alpha \le 1/2, \\ O(m^{2(\alpha-1)/\alpha}), & \alpha > 1/2 \end{cases} \tag{6.12}$$

and

$$\mathrm{Re}\, J_2 \sim \begin{cases} O(m^{-3}), & \alpha \le 1/2, \\ O(m^{(\alpha-2)/\alpha}), & \alpha > 1/2. \end{cases} \tag{6.13}$$

Putting the above asymptotic formulae into (4.13), we obtain

$$\mathrm{Im}\int_0^1 s_1(x)\mathrm{d}x \sim -\pi\frac{(1+\alpha)\,(1-\alpha)^{1/\alpha}}{\alpha^3}\frac{1}{m^{(2+3\alpha)/\alpha}}. \tag{6.14}$$

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7   Discussion

Using the replica method and effective medium approximation (EMA) in statistical physics, we evaluated the spectral density for the adjacency matrix of Goh, Kahng and Kim's model of complex networks with a finite mean degree p. The EMA result was compared with numerically generated spectral density. In the limit $p \to \infty$, known results derived in [8] were reproduced. Moreover, perturbative analytic formulae were presented for the EMA solution in the forms of $1/p$ expansions. As shown in Figure 1, although the agreement of the EMA spectral density with numerical data was fairly good, a significant discrepancy was observed around the band center. In order to improve the agreement, it seems necessary to develop more accurate schemes, such as a non-perturbative technique[10] or single defect approximation[11, 12, 14]. The discrepancy around the band center might be related to the occurrence of the eigenvector localisation. In connection to the localisation, we expect that further studies on the local properties of the spectra, such as the distribution of the eigenvalue spacings[15] or the largest eigenvalues[16], will be illuminating.

Acknowledgements

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