Differential identities for parametric correlation functions in disordered systems

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We derive a family of differential identities for parametric correlation functions in disordered systems by casting them as first- or second-order Ward identities of an associated matrix model. We show that this approach allows for a systematic classification of such identities, and provides a template for deriving higher-order results. We also reestablish and generalize some identities of this type which had been derived previously using a different method.

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I. INTRODUCTION

Modeling by ensembles of random Hamiltonians is a common tool in the study of disordered or chaotic quantum systems [1,2]. Either or both spectral and wave-function statistics may be of interest in any given setting. Spectral statistics is usually studied *via* the correlation functions of the density of states (DoS) of a Hamiltonian H,

$$\rho(E;H) = -\frac{1}{\pi}\,\mathrm{Im}\,\mathrm{Tr}\,G^R(E;H) \equiv -\frac{1}{\pi}\,\mathrm{Im}\,\mathrm{Tr}\,\frac{1}{E-H+i0}, \tag{1}$$

where $G^{R(A)}$ is used to denote retarded (advanced) Green's function $(E-H\pm i0)^{-1}$. The simplest of such correlation functions is the two-point spectral correlation function

$$R_0(E_1,E_2) = \langle \rho(E_1;H)\rho(E_2;H)\rangle_H, \tag{2}$$

where the symbol $\langle\cdots\rangle_H$ is used to denote averaging with respect to the chosen ensemble $\mathcal{P}_H[H]$ of random Hamiltonians H. The properties of wave functions usually manifest themselves when physical objects of interest involve matrix elements of operators, as exemplified by various response functions, such as the density response,

$$N_0(E_1,E_2) = \frac{1}{\pi^2}\langle \mathrm{Tr}[\mathrm{Im}\,G^R(E_1;H)\mathrm{Im}\,G^R(E_2;H)]\rangle_H. \tag{3}$$

Quite generally, measuring the response of a physical system to an external perturbation is a primary tool to study the system's properties. The effect of an external perturbation on a quantum system can usually be formalized as an additive change in the Hamiltonian,

$$H \mapsto H(\lambda) = H + \lambda V. \tag{4}$$

The perturbation V can be either deterministic or stochastic. For the bulk of this paper we concentrate on the case where V is stochastic, and, moreover, Gaussian-distributed, commenting on other cases where appropriate. The multiplier λ is introduced as a convenient tool to tune the strength of the perturbation while keeping the distribution $\mathcal{P}_V[V]$ of V fixed [3].

One is thus naturally led to define parametric generalizations of the correlation functions [4–6]. Concentrating on the two functions introduced above, their parametric counterparts (symmetrized for convenience) are

$$R(E_1,E_2;\lambda) = \langle \rho[E_1;H(-\lambda/2)]\rho[E_2;H(\lambda/2)]\rangle_{H;V} \tag{5}$$

and

$$N(E_1,E_2;\lambda) = \frac{1}{\pi^2}\langle \mathrm{Tr}\{\mathrm{Im}\,G^R[E_1;H(-\lambda/2)] \times \mathrm{Im}\,G^R[E_2;H(\lambda/2)]\}\rangle_{H;V}, \tag{6}$$

where the averaging over V is explicitly reflected in the notation.

Parametric correlation functions in disordered and chaotic systems have been extensively studied using diagrammatic, field-theoretic, and, in random matrix theory models, orthogonal polynomials techniques [5–11]. The interest in these functions has been recently renewed in connection with the study of the parametric response functions in the time domain—often referred to in the literature as fidelity response [12].

To the extent that random matrix modeling [13] may be applicable [14,15], spectra and wave functions are statistically uncorrelated. Even so, once the perturbation is applied, the spectrum and the eigenfunctions of the perturbed Hamiltonian depend in a complicated nonlinear way on both the spectrum and the eigenfunctions of the original Hamiltonian, and thus acquire nontrivial statistical correlations with them. As a result, explicit calculations of the parametric correlators can often be rather lengthy and cumbersome, relying on nontrivial technical devices, such as the supersymmetric nonlinear σ-model [1], or the theory of orthogonal polynomials.

In light of this, valuable qualitative insight as well as a reduction of the calculational effort is achieved whenever different types of parametric correlation functions can be related to each other. From the experimental point of view, spectral properties are often easier to access than the wave-function-dependent properties, thus putting a premium on any analytical tool which would allow the latter to be inferred from the former.

Differential relations connecting different types of parametric correlation functions were first discovered in [8], and later generalized and further explored in [9,14,16] using field-theoretic representations of correlation functions *via* supersymmetric nonlinear σ models [1]. An important role in the derivation and structure of the identities discovered in Refs. [8,9,14,16] is played by the notion of universality. Briefly, assuming $\Omega = E_2 - E_1$ and λ are sufficiently small, all correlation functions are universal "fast" functions of $\omega = \Omega/\Delta$ and $x = \lambda\sqrt{C(0)}/\Delta$, while the mean level spacing Δ and level velocity variance $C(0)$ are "slow" system-specific functions of $\bar{E} = (E_1+E_2)/2$ and parameters characterizing the Hamiltonian of the system [5,14]. The distinction between "fast" and "slow" is controlled by large parameters, for example, matrix size N in the random matrix theory context, or inverse mean free time τ^{-1} in disordered systems. In the latter case, as well as in chaotic systems, there is also a hierarchy of intermediate scales controlled by the spectrum of the diffusion operator, or, more generally, Perron-Frobenius operator [1,17].

The purpose of the present communication is to provide an alternative—arguably simpler—way to derive and classify the differential relations between various parametric correlation functions. The approach suggested here clarifies the connection between the phenomenon of universality and the existence of such identities. Moreover, we identify a class of identities which are fulfilled exactly rather than in the universal approximation in which the "slow" dependencies described above are neglected. Overall, this point of view serves to elucidate and, to some extent, demystify, the origin of these relations by identifying them as Ward identities of a type of a two-matrix model. A partial summary of this approach has been given previously in [18].

The following principal results are obtained. In terms of the resolvents,

$$G_1 \equiv G(z_1;-\lambda/2) = \frac{1}{z_1 - H(-\lambda/2)},$$

$$G_2 \equiv G(z_2;\lambda/2) = \frac{1}{z_2 - H(\lambda/2)}, \tag{7}$$

as functions of complex "energies" $z_{1,2}$ we show that, for example, in the unitary symmetry class,

$$\partial_1\langle[\mathrm{Tr}\, G_1]^2\, \mathrm{Tr}\, G_2\rangle_{H;V} + \partial_2\langle[\mathrm{Tr}\, G_2]^2\, \mathrm{Tr}\, G_1\rangle_{H;V} + 2\partial_1\partial_2\langle\mathrm{Tr}[G_1G_2]\rangle_{H;V} = -8\gamma\frac{\partial}{\partial(\lambda^2)}\langle\mathrm{Tr}\, G_1\, \mathrm{Tr}\, G_2\rangle_{H;V}, \tag{8}$$

where $\partial_{1,2} \equiv \partial/\partial z_{1,2}$. This identity is fulfilled identically for any "energies" z_1, z_2 and any distribution of H, provided the matrix elements of the perturbation V have a Gaussian distribution with variance $1/\gamma$. Note that while the right-hand side (rhs), as well as the last term on the let-hand side (lhs), are closely related to the derivatives of the functions R and N, respectively, as defined above, the remaining terms represent higher-order spectral correlation functions. As mentioned before, this identity connects purely spectral correlation functions to the density correlation function (the last term in the lhs), the latter incorporating statistical properties of the eigenfunctions as well as those of the eigenvalues.

If the energies are chosen real, $z_{2,1} = E \pm \Omega/2 \pm i0$, so that $G_1 \equiv G^A(E-\Omega/2;-\lambda/2)$ and $G_2 \equiv G^R(E+\Omega/2;\lambda/2)$, and Ω is small compared to the smallest of the large parameters controlling the "slow" dependencies, a further class of identities can be obtained in the universal approximation. To present them, it is convenient to employ slightly different definitions of the rescaled correlation functions,

$$\Delta^2 R(E_1,E_2;\lambda) = 1 + \mathrm{Re}\, k(\omega,x),$$

$$\Delta^2 N(E_1,E_2;\lambda) = \mathrm{Re}\, n(\omega,x). \tag{9}$$

The functions $k(\omega,x)$ and $n(\omega,x)$ introduced above are related to the resolvent averages *via*

$$\langle \mathrm{Tr}\, G_1^A\, \mathrm{Tr}\, G_2^R\rangle = \mathrm{Tr}\, G_0^A\, \mathrm{Tr}\, G_0^R + (2\pi^2/\Delta^2)k(\omega,x),$$

$$\langle \mathrm{Tr}[G_1^A G_2^R]\rangle = \frac{2\pi^2}{\Delta^2}n(\omega,x). \tag{10}$$

In the same approximation, the average of the resolvent is an entirely "slow" function, and thus can be treated as a constant, $\langle G^{A,R}(E;\lambda)\rangle = G_0^{A,R}$, and averages containing only advanced or only retarded resolvents decouple,

$$\langle G^R(E_1;\lambda_1)G^R(E_2;\lambda_2)\rangle = (G_0^R)^2, \tag{11}$$

etc.

In terms of these functions, we recover the identity

$$2\frac{\partial}{\partial(x^2)}k(\omega;x) = \frac{\partial^2}{\partial\omega^2}n(\omega;x) \tag{12}$$

which was first found in [8]. It is obtained in the present approach under the assumption that the correlation functions are invariant under infinitesimal shifts of H. This assumption is equivalent to an idealization in which all "slow" dependencies are neglected, and it is thus an approximate symmetry of the system, as expressed in Eq. (30). Formally, such a shift is equivalent to a shift in H_0 defined as the "reference" Hamiltonian, whose meaning in the main classes of systems can be described as follows. In the case of invariant random matrix theory, the "reference" Hamiltonian is the $H=0$ origin of the distribution of H. In the case of a typical model of a disordered metal, the "reference" is the band Hamiltonian H_0 in the absence of the disorder potential. Finally, if one is interested in the properties of an individual chaotic system, the "reference" Hamiltonian is the Hamiltonian H_0 of the system, while averaging is understood as the averaging over an energy interval (for the unity of notation, we retain the symbol $\langle\cdots\rangle_{H;V}$ to denote averaging in this case). Higher-order identities of the same type can be derived systematically.

Among other results is the identity given by Eq. (34) which is a counterpart to Eq. (12). It connects density response functions of second and fourth order. Remarkably, the identity (in the unitary case)

$$1 + k(\omega;0) = \lim_{x\to 0}\omega^2\frac{\partial}{\partial x^2}n(\omega;x) \tag{13}$$

first obtained in [16] by direct computation within the random matrix limit of the nonlinear σ model is shown to follow from Eq. (34) as a special case. It immediately follows that Eq. (13) possesses a broader range of validity than that implied by the derivation in [16], restricted only by the validity of Eq. (30) (see the discussion below). Finally, we discuss how deterministic (including finite rank) perturbations can be accommodated in the same framework, obtaining, for example, an exact identity given by Eq. (49).

II. WARD IDENTITIES IN A MATRIX MODEL

A. General formulation

Consider a generic parametric correlation function

$$\mathcal{C}[J;\lambda] = \langle J \rangle_{H;V}, \qquad (14)$$

where J is any combination of the resolvents $G(z;\pm\lambda/2)$, and $\mathcal{C}$ is a function of the corresponding set of "energies" z. For example, the spectral correlation function is recovered by setting $J=F\equiv \mathrm{Tr}\, G_1 \,\mathrm{Tr}\, G_2$, while density response is obtained from $J=\mathcal{N}\equiv \mathrm{Tr}[G_1G_2]$. As will become evident form the derivation below, it is not necessary for J to be an invariant, although all the specific examples which we consider will fall in this class.

It is now convenient to "promote" $H(\pm\lambda/2)$ to integration variables, by writing, for any function J of the two matrices $H(\pm\lambda/2)$,

$$J[H(-\lambda/2),H(\lambda/2)] = \int dH_1 dH_2 \delta[H_1 - H(-\lambda/2)] \times \delta[H_2 - H(\lambda/2)]\, J[H_1,H_2]. \qquad (15)$$

Here the matrix δ function is understood as a product of scalar δ functions, one per each independent component of the matrix, and dH is the corresponding flat measure on the space of independent components of H. Fourier transforming the δ functions, we rewrite the general correlation function as

$$\mathcal{C}[J;\lambda] = \Bigg\langle \int dH_1 dH_2 \int d\Lambda_1 d\Lambda_2 \times \exp(i\, \mathrm{Tr}\{\Lambda_1[H_1 - H + (\lambda/2)V]\}) \times \exp(i\,\mathrm{Tr}\{\Lambda_2[H_2 - H - (\lambda/2)V]\}) J[H_1,H_2] \Bigg\rangle_{H;V}, \qquad (16)$$

where the flat measures $d\Lambda_{1,2}$ are assumed to include (possibly infinite) normalizing factors.

Equation (16) can be viewed as a field-theoretic correlation function in which H, $H_{1,2}$, $\Lambda_{1,2}$, and V play the role of fields, and

$$\mathcal{S} = i\,\mathrm{Tr}\{\Lambda_1[H_1 - H + (\lambda/2)V] + \Lambda_2[H_2 - H - (\lambda/2)V]\} \qquad (17)$$

is the "action." For reasons that will become obvious shortly, it is convenient to treat the distributions of H and V (not yet specified) as parts of the measure of integration with respect to the corresponding variables rather than include them explicitly in the "action."

The expressions (16) and (17) remain somewhat formal until the structure of the matrices H, $H_{1,2}$, and $\Lambda_{1,2}$ is elaborated. Concentrating on the pure symmetry classes, H and $H_{1,2}$ are Hermitian matrices in the unitary symmetry class, real symmetric matrices in the orthogonal symmetry class, and matrices composed of real quaternions in the symplectic symmetry class [13]. As shown in the Appendix, the same structure in each symmetry class is inherited by $\Lambda_{1,2}$.

Let us now explore the behavior of the correlation functions under infinitesimal shifts,

$$H_1 \mapsto \mathcal{T}_1 H_1 = H_1 + \delta H_1, \quad H_2 \mapsto \mathcal{T}_2 H_2 = H_2 + \delta H_2,$$

$$H \mapsto \mathcal{T} H = H + \delta H, \qquad (18)$$

where δH and $\delta_{H_{1,2}}$ are arbitrary infinitesimal matrices of the same symmetry as H, H_1, and H_2. Matrix differential operators $\mathbf{d}_{1,2}$ and $\mathbf{d}$ acting on $H_{1,2}$, and H, respectively, are the generators of these translations, so that

$$\mathrm{Tr}\{\mathbf{d}_{1,2}\delta H_{1,2}\} = \mathcal{T}_{1,2} - \mathbb{1}, \qquad (19)$$

and similarly for $\mathbf{d}$. Explicit index structure of the matrix operators $\mathbf{d}$ is given in the Appendix for all three main symmetry classes.

The flat measures dH_1 and dH_2 are invariant under additive shifts. Utilizing the arbitrariness of the infinitesimal matrices δH_1 and δH_2, we find a pair of identities

$$\langle \mathbf{d}_1 J + J\mathbf{d}_1\mathcal{S}\rangle_{H;V;\mathcal{S}} = \langle \mathbf{d}_2 J + J\mathbf{d}_2\mathcal{S}\rangle_{H;V;\mathcal{S}} = 0, \qquad (20)$$

where for compactness the subscript $\mathcal{S}$ indicates the integral over $H_{1,2}$ and $\Lambda_{1,2}$ weighted by $e^{\mathcal{S}}$. Note also that, since only the invariance of the dH_1 and dH_2 measures is exploited, a version of these identities is valid for any values of Λ_1, Λ_2, and V,

$$\langle (\mathbf{d}_1 J + J\mathbf{d}_1\mathcal{S})e^{\mathcal{S}}\rangle_{H_1;H_2;H} = \langle (\mathbf{d}_2 J + J\mathbf{d}_2\mathcal{S})e^{\mathcal{S}}\rangle_{H_1;H_2;H} = 0. \qquad (21)$$

One can also form invariants by tracing $\mathbf{d}_j J$ with, for example, $\mathbf{d}_j\mathcal{S}$ with respect to the indices of $\mathbf{d}$ (irrespective of any possible index structure of J). One such invariant will be useful below in deriving second-order identities: Setting $\delta H_1 = -\delta H_2$, and utilizing the fact that $\mathcal{S}$ is a linear function of $H_{1,2}$ and H, we find

$$\langle \mathrm{Tr}[(\mathbf{d}_1\mathcal{S} - \mathbf{d}_2\mathcal{S})(\mathbf{d}_1 - \mathbf{d}_2)]J + J\,\mathrm{Tr}[(\mathbf{d}_1\mathcal{S} - \mathbf{d}_2\mathcal{S})^2]\rangle_{H;V;\mathcal{S}} = 0. \qquad (22)$$

Let us now apply the shift operators twice, setting $\delta H_1 = -\delta H_2$ both times. Again using the linearity of $\mathcal{S}$ in $H_{1,2}$ and H, and contracting with respect to the indices of $\mathbf{d}_1$ and $\mathbf{d}_2$, we find

$$\langle \mathrm{Tr}[(\mathbf{d}_1 - \mathbf{d}_2)^2]J + 2\,\mathrm{Tr}[(\mathbf{d}_1 - \mathbf{d}_2)J(\mathbf{d}_1\mathcal{S} - \mathbf{d}_2\mathcal{S})] + J\,\mathrm{Tr}[(\mathbf{d}_1\mathcal{S} - \mathbf{d}_2\mathcal{S})^2]\rangle_{H;V;\mathcal{S}} = 0. \qquad (23)$$

Using Eq. (22), the cross term can be eliminated, resulting in the following identity:

$$\langle \mathrm{Tr}[(\mathbf{d}_1 - \mathbf{d}_2)^2]J\rangle_{H;V;\mathcal{S}} = \langle J\,\mathrm{Tr}[(\mathbf{d}_1\mathcal{S} - \mathbf{d}_2\mathcal{S})^2]\rangle_{H;V;\mathcal{S}}. \qquad (24)$$

Equation (24) is one of the principal results reported in this work. Its explicit form in the case when $J=F$

$\equiv \mathrm{Tr}\, G_1 \,\mathrm{Tr}\, G_2$, in the unitary symmetry class, is

$$\langle \partial_1 [\mathrm{Tr}\, G_1]^2 \,\mathrm{Tr}\, G_2 + \partial_2 [\mathrm{Tr}\, G_2]^2 \,\mathrm{Tr}\, G_1 + 2\partial_1\partial_2 \,\mathrm{Tr}[G_1 G_2]\rangle_{H;V;\mathcal{S}} = \langle \mathrm{Tr}\, G_1 \,\mathrm{Tr}\, G_2 \,\mathrm{Tr}(\Lambda_1 - \Lambda_2)^2\rangle_{H;V;\mathcal{S}}. \quad (25)$$

In the symplectic (orthogonal) case this identity takes a somewhat different form,

$$\Big\langle \tfrac{1}{2}\partial_1 [\mathrm{Tr}\, G_1]^2 \,\mathrm{Tr}\, G_2 + \tfrac{1}{2}\partial_2 [\mathrm{Tr}\, G_2]^2 \,\mathrm{Tr}\, G_1 + \left(\pm \tfrac{1}{2}\partial_1^2 + 2\partial_1\partial_2 \pm \tfrac{1}{2}\partial_2^2\right)\mathrm{Tr}[G_1 G_2]\Big\rangle_{H;V;\mathcal{S}} = \langle \mathrm{Tr}\, G_1 \,\mathrm{Tr}\, G_2 \,\mathrm{Tr}(\Lambda_1 - \Lambda_2)^2\rangle_{H;V;\mathcal{S}}. \quad (26)$$

The key feature of these results is the fact that they connect purely spectral parametric correlation functions (those involving only powers of $\mathrm{Tr}\, G_j$) to the correlation functions such as $\mathrm{Tr}[G_1 G_2]$ which involve eigenfunctions. In this general form, however, Eqs. (25) and (26) are not yet very useful or informative since the rhs effectively involves averaging with respect to a different distribution of V compared to the lhs. Explicitly, assuming V is distributed according to some (not necessarily invariant) distribution $\mathcal{P}_V[V]$, the average in the rhs, after integrating out the Lagrange multipliers $\Lambda_{1,2}$, is taken with respect to $\tilde{\mathcal{P}}_V = -\frac{4}{\lambda^2}[\mathrm{Tr}(\frac{\partial}{\partial V})^2]\mathcal{P}_V[V]$.

The structure of the identity simplifies, however, in the important special case of Gaussian-distributed V. Assuming

$$\mathcal{P}_V[V] = e^{-(\gamma/2)\mathrm{Tr}\, V^2}, \quad (27)$$

the rhs of Eq. (25) takes the form

$$-8\gamma \frac{\partial}{\partial(\lambda^2)}\langle \mathrm{Tr}\, G_1 \,\mathrm{Tr}\, G_2\rangle_{H;V;\mathcal{S}}, \quad (28)$$

thus connecting energy and parameter (i.e., λ^2) derivatives of the parametric spectral correlations to the energy derivatives of the parametric response function. Note that Eq. (27) implies $C(0) = 1/\gamma$, and therefore

$$x^2 = \frac{\lambda^2}{\gamma\Delta^2}. \quad (29)$$

Crucially, Eq. (24) and its explicit "incarnations" for different forms of J and different ensembles—e.g, Eq. (25)—are exact identities, valid for any distribution of H, and for arbitrary values of z_1, z_2, and λ. In the context of disordered electronic systems, for example, this means the absence of any semiclassical restrictions on $z_1 - z_2$. It is clear that higher-order identities of the same type can be derived (e.g., by repeatedly applying powers of $\mathrm{Tr}[(\mathbf{d}_1 - \mathbf{d}_2)^2]$). It is immediately obvious that such higher-order identities would necessarily involve not only higher derivatives, but also higher powers of the resolvents.

A notable feature of these identities is that they are not homogeneous in the degree of the energy derivatives, and therefore in the powers of the resolvents. This circumstance has a serious implication when comparing this class of identities to those which can be obtained in supersymmetric σ models. An increase in the powers of the Green's functions requires, generically, a corresponding increase in the dimensionality of the (super-) matrix fields. Thus, taken together, this family of identities, although generated from the same matrix field theory, Eq. (17), cannot be said to characterize any finite-dimensional σ model. In every specific case, nevertheless, each identity could be expressed using a σ model of the correspondingly pre-enlarged dimensionality of the target space.

B. Universal approximation

Another class of identities can be established based on the universality property. As described above, the universality of parametric correlation functions is understood as the property that the correlation functions are universal "fast" functions of the scaled variables, and "slow" functions of all the microscopic characteristics of the system [5]. We now formally treat the "slow" dependencies as absent, i.e., treat the correlation functions as invariant under infinitesimal changes in the "reference" Hamiltonian. Although in the derivation which will be presented below this assumption is shown to be a sufficient condition for the validity of Eq. (12), it is difficult to envision a mechanism whereby this is not also a necessary condition.

To proceed, let us now consider the effect of simultaneous infinitesimal shifts in H_1, H_2, and H, i.e., the application of the operator $\mathbf{d}_1 + \mathbf{d}_2 + \mathbf{d}$. It is obvious that the "action" is invariant, $(\mathbf{d}_1 + \mathbf{d}_2 + \mathbf{d})\mathcal{S} = 0$. The effect of $\mathbf{d}_1 + \mathbf{d}_2 + \mathbf{d}$ on J is the same as that of $\mathbf{d}_1 + \mathbf{d}_2$. Unlike the $\mathbf{d}_1 - \mathbf{d}_2$ case considered above, however, the distribution $\mathcal{P}_H[H]$ is no longer invariant under the action of such a shift. Applying the universality assumption as discussed above, the terms arising from the noninvariance of $\mathcal{P}_H[H]$ can be ignored, resulting in

$$\langle \mathrm{Tr}(\mathbf{d}_1 + \mathbf{d}_2)^2 J\rangle_{H;V;\mathcal{S}} \approx 0. \quad (30)$$

Substituting this result into Eq. (24), we obtain

$$-4\langle \mathrm{Tr}[(\mathbf{d}_1\mathbf{d}_2)]J\rangle_{H;V;\mathcal{S}} = \langle J\,\mathrm{Tr}[(\mathbf{d}_1\mathcal{S} - \mathbf{d}_2\mathcal{S})^2]\rangle_{H;V;\mathcal{S}}. \quad (31)$$

This identity is the second key result of this study. It should be emphasized that, in order to ensure the validity of the universality assumption, the energy variables in J can no longer be chosen arbitrarily. Rather, the scale for $z_1 - z_2$ is now restricted to being smaller than the smallest of the parameters which control the "slow" dependencies. Substituting the explicit form of $J = F$, setting $z_{2,1} = E \pm \Omega/2 \pm i0$ so that G_1 (respectively, G_2) becomes the advanced (retarded) Green's function of a real argument, and using Eq. (28) together with the definitions (10) and (29), we find Eq. (12) which had been previously established [8,14] using σ-model representations of the correlation functions.

As evident from its derivation, Eq. (31) is valid—assuming validity of Eq. (30)—for any index structure of J. In particular, choosing

$$J = \mathcal{N} \equiv \mathrm{Tr}[G_1 G_2] \quad (32)$$

and observing that

$$\mathrm{Tr}[\mathbf{d}_1\mathbf{d}_2]\mathrm{Tr}[G_1G_2] = \begin{cases} \frac{1}{2}(\mathrm{Tr}[G_1G_2])^2 + \frac{1}{2}\mathrm{Tr}[(G_1G_2)^2] & \text{(orthogonal)}, \\ (\mathrm{Tr}[G_1G_2])^2 & \text{(unitary)}, \\ \frac{1}{2}(\mathrm{Tr}[G_1G_2])^2 - \frac{1}{2}\mathrm{Tr}[(G_1G_2)^2] & \text{(symplectic)} \end{cases} \tag{33}$$

we find another Ward identity

$$-2\gamma\frac{\partial}{\partial(\lambda^2)}\langle\mathrm{Tr}[G_1G_2]\rangle_{H;V;\mathcal{S}} = \begin{cases} \frac{1}{2}\langle(\mathrm{Tr}[G_1G_2])^2\rangle_{H;V;\mathcal{S}} + \frac{1}{2}\langle\mathrm{Tr}[(G_1G_2)^2]\rangle_{H;V;\mathcal{S}} & \text{(orthogonal)}, \\ \langle(\mathrm{Tr}[G_1G_2])^2\rangle_{H;V;\mathcal{S}} & \text{(unitary)}, \\ \frac{1}{2}\langle(\mathrm{Tr}[G_1G_2])^2\rangle_{H;V;\mathcal{S}} - \frac{1}{2}\langle\mathrm{Tr}[(G_1G_2)^2]\rangle_{H;V;\mathcal{S}} & \text{(symplectic)}. \end{cases} \tag{34}$$

We will now use this general identity to reestablish and generalize another result which holds in the $\lambda \to 0$ limit, first established in [16]. Setting again $z_{2,1} = E \pm \Omega/2 \pm i0$, we note that

$$G_1 - G_2 = G_1(\Omega - \lambda V)G_2. \tag{35}$$

We now evaluate the averages of the squares of the traces of both sides. Expanding the square of the trace of the lhs as

$$\langle(\mathrm{Tr}[G_1 - G_2])^2\rangle_{H;V;\mathcal{S}} = \langle(\mathrm{Tr}[G_1])^2 + (\mathrm{Tr}[G_2])^2 - 2\,\mathrm{Tr}[G_1]\mathrm{Tr}[G_2]\rangle_{H;V;\mathcal{S}}, \tag{36}$$

we equate it, in the $\lambda \to 0$ limit, to the square of the trace of the rhs. The latter is evaluated (in the unitary case) using Eq. (34),

$$\lim_{\lambda\to 0}\Omega^2\langle(\mathrm{Tr}[G_1G_2])^2\rangle_{H;V;\mathcal{S}} = -2\gamma\Omega^2\lim_{\lambda\to 0}\frac{\partial}{\partial(\lambda^2)}\langle\mathrm{Tr}[G_1G_2]\rangle_{H;V;\mathcal{S}}. \tag{37}$$

Now, for $\lambda = 0$, the traces of the retarded and advanced Green's functions are related *via*

$$\mathrm{Tr}\,G^A = \mathrm{Tr}\,G^R + 2i\pi\rho, \tag{38}$$

where ρ is the density of states defined previously, and $\langle\rho\rangle_H = 1/\Delta$. Using also the fact that the averages containing only advanced or only retarded functions decouple, we find that the rhs of Eq. (36) reduces in the $\lambda \to 0$ limit to

$$-(4\pi^2/\Delta^2)(1+k)_{\lambda=0}, \tag{39}$$

while the rhs of Eq. (37) is immediately recognized in the unitary case as

$$-2\gamma\Omega^2\lim_{\lambda\to 0}\frac{\partial}{\partial(\lambda^2)}\frac{2\pi^2 n}{\Delta^2}, \tag{40}$$

where the definitions of k and n have been given in Eq. (10). In the orthogonal (symplectic) cases Eq. (40) is modified as follows:

$$-4\gamma\Omega^2\lim_{\lambda\to 0}\frac{\partial}{\partial(\lambda^2)}\frac{2\pi^2 n}{\Delta^2} \pm \Omega^2\langle\mathrm{Tr}[(G_1G_2)^2]\rangle_{H;V;\mathcal{S}}. \tag{41}$$

In order to evaluate the extra term arising in these two ensembles, we use Eq. (35) twice more in the $\lambda \to 0$ limit. Taking the average of its trace, we recover a well-known result

$$\langle\mathrm{Tr}[G_1^AG_2^R]\rangle_H = \frac{2\pi i}{\Omega\Delta}, \tag{42}$$

and averaging the trace of its square we find

$$\Omega^2\langle\mathrm{Tr}[(G_1^AG_2^R)^2]\rangle_H = -2\langle\mathrm{Tr}[G_1^AG_2^R]\rangle_H = -\frac{4\pi i}{\Omega\Delta}. \tag{43}$$

Thus, using again Eq. (29), we find

$$1 + k(\omega;0) = \lim_{x\to 0}\omega^2\frac{\partial}{\partial(x^2)}n(\omega;x) \tag{44}$$

in the unitary case, and

$$1 + k(\omega;0) = \lim_{x\to 0}2\omega^2\frac{\partial}{\partial(x^2)}n(\omega;x) \pm \frac{i}{\omega} \tag{45}$$

in the orthogonal (symplectic) case. The real part of these identities has been given as Eq. (7) of Ref. [16] where it was obtained by direct evaluation of the corresponding correlation functions in the random matrix limit [23]. The present derivation shows that these identities have a broader range of validity, limited only by Eq. (30).

It is important to emphasize that the analytical tools employed here are not suited to the analysis of the validity of the universality condition such as Eq. (30). Formal proofs of the universality of correlation functions can be partially achieved at a "physical level of rigor" using σ-model analysis, and in the random matrix theory using recent progress in asymptotic methods for solving Riemann-Hilbert problems (see, e.g, [21]). In general, there does not yet exist an algorithm for proving universality of correlation functions in any specific system, and thus the results which follow from Eq. (31) [in contrast to those stemming from Eq. (24)] are valid to the extent that the validity of Eq. (30) can be established. Note that the present scheme easily allows a systematic way to account for the effect of deviations from Eq. (30) on these identities by including the terms arising from the action of $\mathbf{d}$ on $\mathcal{P}_H[H]$. In the field-theoretic analogy, such terms represent the "anomaly" generated by the noninvariance of the measure $\mathcal{P}_H[H]dH$ under translations of H. Explicit evaluation of such terms, however, requires methods which go beyond the symmetry analysis presented here.

As remarked above, in disordered and chaotic systems there is an intermediate hierarchy of scales, controlled by the (gapped) spectrum $\{\gamma_n\}$ of the Perron-Frobenius operator. The correlation functions can be represented as expansions in powers of γ_n^{-1}. Thus, the possible corrections arising form deviations from Eq. (30) would arise as the result of the perturbation in γ_n, therefore appearing at the subleading order γ_n^{-2}. A manifestation of this phenomenon has been observed in a different context as the "accidental" cancellation of the leading order contributions to the third moment of global conductance in diffusive systems [22]. (In such systems, the scale of the spectrum γ_n is set by the single parameter—dimensionless conductance g.) Indeed, applying $\mathrm{Tr}[\mathbf{d}_H^2]$ to (nonparametric) $\langle \mathrm{Tr}\, G_1^A\, \mathrm{Tr}\, G_2^R\rangle$ and using Eqs. (30) and (42), we find

$$\frac{1}{\Delta}\partial_\omega\langle\{[\mathrm{Tr}\, G_1^A]^2\, \mathrm{Tr} G_2^R - [\mathrm{Tr}\, G_2^R]^2\, \mathrm{Tr}\, G_2^A\}\rangle \approx \frac{8\pi i}{\Delta^3\omega^3}. \quad (46)$$

The integral of the lhs is precisely proportional to the nontrivial part of the average $\langle\rho^3\rangle = -(1/\pi^3)\langle \mathrm{Tr}[(G^R - G^A)^3]\rangle$, which thus appears independent of g in the approximation of Eq. (30), corresponding to the absence of the leading $1/g$ contributions observed in [22].

C. Deterministic perturbations

Let us now consider an example where the perturbation V is deterministic rather than stochastic. In this section we switch the notation to $V = \Phi$ [i.e., formally, $\mathcal{P}_V[V] = \delta(V - \Phi)$] to avoid confusion with the stochastic case considered elsewhere in this paper. Forming an invariant by taking the trace of the shift operators with Φ, and setting $J = F$, we note that

$$\mathrm{Tr}[(\mathbf{d}_1 - \mathbf{d}_2)\Phi]F = \partial_2\, \mathrm{Tr}\, G_1\, \mathrm{Tr}[G_2\Phi] - \partial_1\, \mathrm{Tr}[G_1\Phi]\mathrm{Tr}\, G_2. \quad (47)$$

Specializing to the case of disordered systems, it is usually sufficient to assume that Φ is diagonal in the coordinate representation. Again setting $z_{2,1} = E \pm \Omega/2 \pm i0$, imposing the standard restrictions on Ω, and using the first-order identity (21), we find

$$\int d\mathbf{r}_1 d\mathbf{r}_2\left(2\frac{\partial}{\partial\lambda} + \frac{\Phi(\mathbf{r}_1) + \Phi(\mathbf{r}_2)}{\Delta}\partial_\omega\right)k(\mathbf{r}_1,\mathbf{r}_2;\omega,\lambda) = 0, \quad (48)$$

where a coordinate-dependent generalization of k is defined *via*

$$\langle G^A(\mathbf{r}_1,\mathbf{r}_1;E - \Omega/2;-\lambda/2)G^R(\mathbf{r}_2,\mathbf{r}_2;E + \Omega/2;\lambda/2)\rangle = G_0^A(\mathbf{r}_1,\mathbf{r}_1)G_0^R(\mathbf{r}_2,\mathbf{r}_2) + (2\pi^2/\Delta^2)k(\mathbf{r}_1,\mathbf{r}_2;\omega,\lambda). \quad (49)$$

While this result is expressed, for convenience, in terms of the rescaled function k, the universality assumption implicit in Eq. (30) has not been used. Thus, this result belongs, in the present classification, to the class of exact identities, and, moreover, in contrast to Eqs. (12) and (34), it is a first-order Ward identity.

A similar relation has been derived in [9] using σ-model analysis in a more restricted setting where Φ is not only diagonal in the coordinate representation, but also local on the scale of the mean free path. However, that result [Eq. (3) in [9]] differs from Eq. (48) by the presence of an extra term, second order in Φ. Given that Eq. (48) does not involve any approximations apart from those inherent in the definitions of the rescaled correlation function k, it is likely that the discrepancy can be, at least in part, attributed to the fact that in [9] the expansion of the σ model action in powers of the local potential Φ has been truncated at the second order (cf. analysis of local parametric perturbations in Ref. [20]).

III. DISCUSSION

Given the original provenance of some of the identities discussed here in semiclassical statistical field theories [8], a discussion of the relation of the present approach to this type of analysis appears to be in order. Originally, these identities were established either by direct computation in the random matrix limit [8,16], or, in a more general setting, by utilizing a discrete symmetry of the σ model [see Eq. (19) in [9]]. On the other hand, the present approach embeds these results in a larger family of Ward identities in a two-matrix model, thus tying them to a continuous symmetry. At the same time, the fact that the σ-model representation of some of these identities would have required enlarging the target space compared to the "standard" 4×4 (or 8×8, depending on symmetry requirements [1]) cases indicates that these are not Ward identities of the standard finite-dimensional σ models. Indeed, in a study of σ-model Ward identities of Zuk's [24] type in the parametric case performed in Ref. [25], it is clearly seen that all such identities involve powers of ω and x rather than derivatives with respect to these parameters, so that identities of this type are, in essence, dual to the identities of the type in Eqs. (24) and (31). The identities (44) and (45) are of "hybrid" character in this classification, since the auxiliary identities (42) and (43) involved in their derivation starting from Eq. (34) can be obtained alternatively as Ward identities of the σ model [25].

It seems appropriate to conclude with a conjecture regarding a possible route towards reconciling these two approaches to deriving identities for parametric correlation functions. The infinitesimal transformation in Eq. (18) involves configuration space dependence. It is thus not surprising that it does not have direct analogs in a σ model description in which the fields are functions on the phase space which could be viewed as a semiclassical truncation of a direct product of configuration space with itself. This is underscored in the cases when the fields are constants on this space as happens in the random matrix theory context, or in the so-called 0-mode limit for disordered and chaotic systems [1]. There is, however, a way to represent an infinitesimal shift in the Hamiltonian as a gauge transformation in the grand σ model involving a continuous set of frequencies rather than a fixed $\Omega = z_1 - z_2$—alternatively, a set of σ model fields which depend on two times as well as phase space coordinates. In this approach, shifts in the Hamiltonians are mapped to continuous transformations with accompanying Ward identities. The standard σ models can be loosely thought of as "slices" of this "grand" σ model at fixed frequencies, thus—possibly—engendering a reduction of continuous symmetries to discrete ones.

In summary, we have shown that differential identities connecting various types of parametric correlation functions are Ward identities of a two-matrix model. We have shown a systematic way to derive these identities, obtaining new results as well as classifying, generalizing (and in some cases amending) the results which had been obtained previously. Some of these identities are exact, and valid in very general settings. Another class of identities relies on the "universal approximation" which can be cast as an approximate Ward identity of this two-matrix model. Under the conditions of random matrix universality, the identity is exact in the scaling limit. The relation of these results to the discrete and continuous symmetries of nonlinear σ models commonly used for the analysis of random matrix, disordered, and chaotic systems remains to be fully explored.

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APPENDIX

For completeness, we present the technical details of the Fourier decomposition of matrix δ functions, and the action of the differential matrix operators $\mathbf{d}$.

1. Orthogonal symmetry class

The Hamiltonians are constrained by $H^T=H$. Each such $n\times n$ matrix contains n diagonal degrees of freedom $H_{\alpha\alpha}$, and $n(n-1)/2$ off-diagonal degrees of freedom which can be parametrized by $H_{\alpha\beta}$ with $\alpha<\beta$. The Fourier decomposition of a δ-function of H takes the form

$$\delta(H) = \int \prod_{\alpha\leq\beta} \frac{d\tilde{\Lambda}_{\alpha\beta}}{2\pi} \exp\left(i \sum_{\alpha\leq\beta} \tilde{\Lambda}_{\alpha\beta} H_{\alpha\beta}\right). \tag{A1}$$

Extending $\tilde{\Lambda}$ to $\alpha>\beta$ by imposing $\tilde{\Lambda}_{\alpha\beta}=\tilde{\Lambda}_{\beta\alpha}$, and identifying

$$\Lambda_{\alpha\beta} = \tilde{\Lambda}_{\alpha\alpha}\delta_{\alpha\beta} + \tfrac{1}{2}\tilde{\Lambda}_{\alpha\beta}(1-\delta_{\alpha\beta}), \tag{A2}$$

we bring the expression in the exponent to the compact form $i\,\mathrm{Tr}\,\Lambda H$, with $\Lambda^T=\Lambda$ so that both Λ and H belong to the same symmetry class. The integration measure in terms of Λ is

$$\prod_{\alpha} \frac{d\Lambda_{\alpha\alpha}}{2\pi} \prod_{\alpha<\beta} \frac{d\Lambda_{\alpha\beta}}{\pi}. \tag{A3}$$

The independent components of the differential operator $\mathbf{d}$ are exhausted by $\frac{\partial}{\partial H_{\alpha\beta}}$ for $\alpha\leq\beta$. The partial derivatives are evaluated with respect to the independent components of H, so that for $\alpha\neq\beta$, $\frac{\partial}{\partial H_{\alpha\beta}}\,\mathrm{Tr}[AH]=A_{\alpha\beta}+A_{\beta\alpha}$.

Symmetrized expressions are obtained by extending $\mathbf{d}$ as follows:

$$\mathbf{d}_{\alpha\beta} = \delta_{\alpha\beta}\partial_{\alpha\alpha} + \tfrac{1}{2}(1-\delta_{\alpha\beta})(\partial_{\alpha\beta}+\partial_{\beta\alpha}), \tag{A4}$$

where the operators $\partial_{\alpha\beta}$ are defined to act on the $\alpha\beta$ element of the matrix H as if it were an independent variable, $\partial_{\alpha\beta}\,\mathrm{Tr}[AH]=A_{\beta\alpha}$.

It is easy to check by direct computation that for any Hermitian matrices A, B satisfying $A^T=A$, $B^T=B$, we have identically

$$\mathrm{Tr}[A\mathbf{d}]\mathrm{Tr}[BH] = \mathrm{Tr}[AB]. \tag{A5}$$

In particular,

$$\mathrm{Tr}[A\mathbf{d}]\mathrm{Tr}[G(H)] = \mathrm{Tr}[AG(H)G(H)]. \tag{A6}$$

For a general pair of matrices, the result (A5) takes the form $\mathrm{Tr}[\bar{A}B]$ where $\bar{A}=(A+A^T)/2$. Similarly, for any pair of matrices A and B,

$$\mathrm{Tr}[\mathbf{d}AHB] = \tfrac{1}{2}\mathrm{Tr}\,A\;\mathrm{Tr}\,B + \tfrac{1}{2}\mathrm{Tr}[A^TB]. \tag{A7}$$

Equations (A5)–(A7) and their analogs are used repeatedly in the calculations.

2. Unitary symmetry class

In a similar way,

$$\delta(H) = \int \prod_{\alpha} \frac{d\tilde{\Lambda}_{\alpha\alpha}}{2\pi} \prod_{\alpha<\beta} \frac{d\tilde{\Lambda}^R_{\alpha\beta} d\tilde{\Lambda}^I_{\alpha\beta}}{(2\pi)^2} \exp\Bigg(i\sum_{\alpha} \tilde{\Lambda}_{\alpha\alpha}H_{\alpha\alpha} + i\sum_{\alpha<\beta} (\tilde{\Lambda}^R_{\alpha\beta}\,\mathrm{Re}\,H_{\alpha\beta} + \tilde{\Lambda}^I_{\alpha\beta}\,\mathrm{Im}\,H_{\alpha\beta})\Bigg). \tag{A8}$$

The exponent acquires a compact $\mathrm{Tr}[\Lambda H]$ form if, for $\alpha \leq\beta$,

$$\Lambda_{\alpha\beta} = \delta_{\alpha\beta}\tilde{\Lambda}_{\alpha\alpha} + \tfrac{1}{2}(1-\delta_{\alpha\beta})(\tilde{\Lambda}^R_{\alpha\beta} + i\tilde{\Lambda}^I_{\alpha\beta}) \tag{A9}$$

and the Hermiticity condition $\Lambda^T=\Lambda^*$ is imposed to complete the definition for $\alpha>\beta$. The operator $\mathbf{d}$ is now defined as

$$\mathbf{d}_{\alpha\beta} = \delta_{\alpha\beta}\partial_{\alpha\alpha} + \tfrac{1}{2}(1-\delta_{\alpha\beta})(\partial^R_{\alpha\beta} + \partial^R_{\beta\alpha} + i\partial^I_{\alpha\beta} - i\partial^I_{\beta\alpha}), \tag{A10}$$

where symbols $\partial^{R,I}_{\alpha\beta}$ act on the real (imaginary) part of the $\alpha\beta$ matrix element (obviously, no such distinction is needed for the diagonal elements), e.g.,

$$\partial^I_{\alpha\beta}\,\mathrm{Tr}[AH] = iA_{\beta\alpha}. \tag{A11}$$

Note that, given a Hermitian matrix A,

$$\exp\{\mathrm{Tr}[A\mathbf{d}]\} = \exp\Bigg(\sum_{\alpha} A_{\alpha\alpha}\partial_{\alpha\alpha} + \sum_{\alpha\neq\beta} (\mathrm{Re}\,A)_{\beta\alpha}\partial^R_{\beta\alpha} + (\mathrm{Im}\,A)_{\beta\alpha}\partial^I_{\beta\alpha}\Bigg), \tag{A12}$$

thus effecting a shift of the symmetric part of H by a symmetric matrix, and a shift of the antisymmetric part of H by an antisymmetric matrix.

The validity of Eqs. (A5) and (A6) and their analogs now follows by direct computation, while the analog of Eq. (A7) takes the form

$$\mathrm{Tr}[\mathbf{d}AHB] = \mathrm{Tr}\,A\;\mathrm{Tr}\,B. \tag{A13}$$

3. Symplectic symmetry class

As usual, parametrizing symplectic $2N\times 2N$ matrices with real quaternions, we have $H_{\alpha\beta}=h^{(0)}_{\alpha\beta}e_0+\Sigma_{a=1}^3 h^{(i)}_{\alpha\beta}e_a$, where e_0 is a 2×2 unit matrix, and $e_1=i\sigma_3$, $e_2=i\sigma_2$, $e_3=i\sigma_1$, with σ_j being the usual Pauli matrices. We have,

$$\delta(H) = \int \prod_{\alpha} \frac{d\tilde{\Lambda}^{(0)}_{\alpha\alpha}}{2\pi} \prod_{a=1}^{3} \prod_{\alpha\leqslant\beta} \frac{d\tilde{\Lambda}^{(a)}_{\alpha\beta}}{2\pi} \times \exp\left(i\sum_{\alpha} \tilde{\Lambda}_{\alpha\alpha} h^{(0)}_{\alpha\alpha} + i \sum_{\alpha<\beta} \sum_{a=0}^{3} (\tilde{\Lambda}^{(a)}_{\alpha\beta} h^{(a)}_{\alpha\beta}) \right). \tag{A14}$$

The form $\mathrm{Tr}[\Lambda H]$ is achieved by identifying

$$\Lambda_{\alpha\beta} = \tfrac{1}{2}\delta_{\alpha\beta}\tilde{\Lambda}^{(0)}_{\alpha\alpha}e_0 + \tfrac{1}{4}(1-\delta_{\alpha\beta})\sum_{a=0}^{3} \tilde{\Lambda}^{(a)}_{\alpha\beta}e_a \tag{A15}$$

for $\alpha\leqslant\beta$, and imposing $\Lambda^\dagger=\Lambda$ to complete the definition for $\alpha>\beta$.

Similarly, in the obvious notation,

$$\mathbf{d}_{\alpha\beta} = \frac{1}{2}\delta_{\alpha\beta}\partial^{(0)}_{\alpha\alpha} + \frac{1}{4}(1-\delta_{\alpha\beta})\left((\partial^{(0)}_{\alpha\beta} + \partial^{(0)}_{\beta\alpha})e_0 - \sum_{a=1}^{3} (\partial^{(a)}_{\beta\alpha} - \partial^{(a)}_{\alpha\beta})e_a \right). \tag{A16}$$

Equation (A5) now holds for a pair of quaternion-real matrices, and for a general pair of quaternion matrices A and B the following relation holds:

$$\mathrm{Tr}[\mathbf{d}AHB] = \tfrac{1}{2}\mathrm{Tr}\,A\;\mathrm{Tr}\,B - \tfrac{1}{2}\mathrm{Tr}[\bar{A}^T B], \tag{A17}$$

where $(\bar{A}^T)_{\alpha\beta}=a^{(0)}_{\beta\alpha}e_0-\Sigma_{a=1}^3 a^{(i)}_{\beta\alpha}e_a$ if A is defined *via* $A_{\alpha\beta}=a^{(0)}_{\alpha\beta}e_0+\Sigma_{a=1}^3 a^{(i)}_{\alpha\beta}e_a$.

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